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Article

# Stochastic irregular linear-quadratic optimal control with dual controllers and two-layer asymmetric partial observations

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**Abstract:** Linear-quadratic (LQ) optimal control problems have been widely studied under symmetric information and regularity assumptions. However, in many practical systems, multiple controllers operate with different information sets, and the standard regularity condition may fail due to singular weighting matrices. This paper investigates a stochastic irregular LQ optimal control problem with dual controllers and two-layer asymmetric partial observations, in which one controller has strictly less information than the other. The main contribution is the derivation of both the solvability conditions and the explicit feedback-form solutions for this problem. To this end, we reformulate the optimization problem as a system of forward-backward stochastic differential equations (FBSDEs) that naturally captures the asymmetric information structure. By solving these FBSDEs via four interconnected Riccati equations, we obtain explicit optimal controllers for both the regular and irregular cases. In the irregular case, where the standard Riccati equation is not directly solvable, we introduce auxiliary Riccati equations and derive additional existence conditions involving singular matrix decompositions. A detailed technical comparison with existing results confirms that our solution is not a straightforward extension, as prior works either addressed irregular LQ control under full information or asymmetric information under regular conditions. This work provides a unified framework for optimal control problems where information asymmetry and singularity arise simultaneously.

**Keywords:** irregular control; stochastic control; FBSDEs; asymmetric information; Riccati equation

## 1. Introduction

LQ optimal control is pivotal in contemporary control theory and has been extensively applied in various domains of advanced technology, as well as in everyday applications such as autonomous vehicles, unmanned aerial vehicles, networked control systems, and quantitative investment [1–3]. To date, research on LQ control has evolved significantly, yielding appealing results. Nevertheless, most previous works have focused primarily on the regular case, whereas in many applications, the regularity condition cannot be guaranteed, giving rise to singular control problems, such as those encountered in engineering processes [4], portfolio selection management [5], and investment problems [6]. In recent years, singular control problems have attracted considerable attention. Although several alternative methods are available for solving these problems, such as state-space transformations, the higher-order maximum principle and perturbation approaches, these methods still have notable limitations. For instance, if the higher derivatives vanish, the higher order maximum principle method will fail, while the state space transformation method is applicable only when the initial value is specified. By contrast, by utilizing FBSDEs, several LQ control problems, including the irregular LQ control problems, can be addressed more efficiently by designing the controller through Riccati equations and considering the controllability of a subsystem [7,8]. These latter two features motivated us to focus on the irregular LQ control problems. Here, the term irregular refers to the case where $Range[B(t)P(t)] \not\subset Range[R(t)]$ and $P(t)$ is connected to the corresponding Riccati equation, which can be seen as a special kind of singular control problem.

Several works have been devoted to irregular LQ control problems using FBSDEs [7, 9]. These problems, which allow for arbitrary initial conditions, were resolved by leveraging the Riccati equation to develop the irregular control strategy. Building on Zhang and Xu [8] and Li et al. [10], the irregular linear quadratic control problem with input delay was further investigated, and both open-loop and closed-loop conditions were established. Our paper tackles irregular LQ problems with asymmetric information, whereas Qi et al. [11] and Liang et al. [12] focus on standard regular LQ settings under networked communication constraints and unreliable channels. Unlike their emphasis on controller architectures and stability over networks, we provide fundamental existence and explicit solutions for nonstandard LQ regimes. In addition, an irregular linear-quadratic two-person zero-sum game was considered in Liang et al. [13], where the irregular feedback representation was derived.

For irregular LQ control problems, we observe that most existing results primarily focus on complete information structures, and Xu et al. [7] summarized the irregular LQ and asymmetric decentralized stochastic control. However, in fields such as finance and networked control systems, asymmetric information structures are prevalent, and related issues have attracted significant attention in recent years [14, 15]. Specifically, Liang et al. [16] and Wang et al. [17] considered decentralized optimal control under asymmetric information structures, with the controllers derived based on the Riccati equation. Furthermore, Qi et al. [18] investigated a stochastic system with varying intermittent observations under an asymmetric information framework and derived the corresponding control strategies. In addition, LQ control problems involving asymmetric information can be effectively addressed using FBSDEs [12,19]. Besides, the authors of Sun et al. [20] provided numerical methods for an algebraic Riccati equation arising in transport theory in the critical case. What is more, Li et al. [21] investigated the linear quadratic Gaussian–Stackelberg game under a class of nested observation information patterns. This context motivates this study of the stochastic irregular LQ control problem under an asymmetric information structure using the Riccati equations.

This paper explores an irregular stochastic LQ optimal control problem while considering the implications of asymmetric information. Compared with conventional scenarios, an asymmetric information structure complicates solvability, especially under irregular conditions, thus distinguishing the problem from classical optimal control settings. To address this complexity, we introduce an innovative modification of the performance index, through which the problem becomes a special mean-field optimal control problem.

Our main contribution is twofold: we establish sufficient conditions to guarantee the solvability of the optimal control problem in both cases, and we construct the explicit form of the optimal controller. The key innovation lies in transforming the target problem into a specific class of FBSDEs capable of capturing both irregularity and asymmetric information in the underlying setting, establishing the critical interrelationships among these FBSDEs, and subsequently deriving their explicit solutions through Riccati equations.

To clearly position our contribution within the existing literature, we highlight the key differences between our problem and the most relevant prior works. Zhang and Xu [8,9] solved the irregular LQ control problem but assumed full information availability (i.e., both controllers have access to the same observation set). In contrast, our setting features two controllers with nested information structures ($Y_1(s) \subset Y(s)$), which breaks their separation principle and requires an additional filtering layer. Li et al. [10] extended irregular LQ to input-delay systems, yet their state-augmentation technique cannot handle our stochastic asymmetric observations because the information gap here is driven by random noises rather than deterministic delays. Liang et al. [16] and Wang et al. [17] studied decentralized control under asymmetric information, but they restricted it to the regular case where the weighting matrices $R_1(s)$ and $R_2(s)$ are invertible. Our work is the first to simultaneously address (i) irregular weighting matrices, (ii) dual controllers with nested information, and (iii) explicit closed-form solutions via FBSDEs. This triple combination is absent in all cited references.

This paper is organized into four sections. Section 2 outlines the problem formulation. Sections 3 and 4 discuss and address the irregular case, respectively. Finally, conclusions are drawn in Section 5.

## 2. Problem formulation

Consider the following system and the corresponding cost function:

$$d\zeta(s) = [A\zeta(s) + B_1u_1(s) + B_2u_2(s)]ds + dw(s), \tag{1}$$

$$\begin{aligned} J = E\int_0^S[\zeta'(s)Q\zeta(s) + u_1'(s)R_1(s)u_1(s) \ + u_2'(s)R_2(s)u_2(s)]ds \\ +E[\zeta'(S)]KE[\zeta(S)] \end{aligned} \tag{2}$$

where $\zeta(s) \in R^n$ is the state variable, $u_1(s)$ denotes $m_1$-dimension controller, and $u_2(s)$ denotes $m_2$-dimension controller. $A, B_1, B_2, Q, K$ are constant matrices with compatible dimensions, and $R_1(s)$, $R_2(s)$ are positive semi-definite symmetric matrices with compatible dimensions. $E[w(s)w'(v)] = \Theta(s)\delta(s-v)$, and the information of the observations is as follows:

$$\begin{aligned} dy_1(s) =& H_1\zeta(s)ds + dv_1(s), \\ dy_2(s) =& H_2\zeta(s)ds + dv_2(s), \end{aligned} \tag{3}$$

where $E[v_1(s)v_1'(u)] = \Sigma_1(s)\delta(s-u)$ and $E[v_2(s)v_2'(u)] = \Sigma_2(s)\delta(s-u)$. Then,

$$\begin{aligned} Y_1(s) =& \{y_1(t), 0 \le t \le s, u_1(t), u_2(t), 0 \le t < s\}, \\ Y(s) =& \{y_1(t), y_2(t), 0 \le t \le s, u_2(t), 0 \le t < s\}. \end{aligned} \tag{4}$$

Hence, $u_1(s)$ is measurable with respect to $Y_1(s)$ and $u_2(s)$ is measurable with respect to $Y(s)$, which is the asymmetric information structure. This generates that $\widehat{\zeta}_1(s) = E[\zeta(s)|Y_1(s)]$ and $\widehat{\zeta}(s) = E[\zeta(s)|Y(s)]$, and the noise processes $w(s)$, $v_1(s)$ and $v_2(s)$ are independent.

It is important to note that an irregular optimal control problem, when associated with a standard performance index, may not be solvable in certain cases where additive noise is present [1]. Therefore, we modify the cost function as in Equation (2) and formulate the main problem as follows.

**Problem 1.** *For any initial value, find a pair of optimal controllers $u_1(s)$ and $u_2(s)$ that minimize Equation (2) subject to Equation (1).*

**Remark 1.** *The problem studied herein has broad real-world application prospects, as the coexistence of additive and multiplicative noises is a common phenomenon in systems such as mobile sensor networks and automated highway systems [22, 23]. In addition, studying optimal control problems that incorporate mathematical expectation terms into the performance index has important theoretical and practical implications [7].*

## 3. Preliminary

This section introduces preliminary results that form the basis for subsequent analytical developments.

### 3.1. Maximum principle

To derive the optimal control strategy for Equation (1), we denote that

$$\begin{aligned} y(s) = \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix},\ \widetilde{H} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},\ v(s) = \begin{bmatrix} v_1(s) \\ v_2(s) \end{bmatrix}, \\ B =[B_1\ \ B_2],\ R(s) = \begin{bmatrix} R_1(s) & 0 \\ 0 & R_2(s) \end{bmatrix}, \\ \widehat{u}(s) = \begin{bmatrix} u_1(s) \\ \widehat{u}_2(s) \end{bmatrix}.\ \widehat{u}_2(s) = E[u_2(s)|Y_1(s)], \\ \widetilde{u}_2(s) = u_2(s) - \widehat{u}_2(s). \end{aligned} \tag{5}$$

Then the dynamic equations of $\widehat{x}(t)$ and $\widehat{x}_1(t)$ are given below.

**Lemma 1.** *Under observations $Y(s)$ and $Y_1(s)$, the dynamic equations of $\widehat{\zeta}(s)$ and $\widehat{\zeta}_1(s)$ satisfy:*

$$\begin{aligned} d\widehat{\zeta}(s) =& \quad [A\widehat{\zeta}(s) + B\widehat{u}(s) + B_2\widetilde{u}_2(s)]ds + P(s)\left[dy(s) - \widetilde{H}\widehat{\zeta}(s)ds\right], \\ d\widehat{\zeta}_1(s) =& \quad [A\widehat{\zeta}_1(s) + B\widehat{u}(s) + B_2\widetilde{u}_2(s)]ds + P_1(s)\left[dy_1(s) - H_1\widehat{\zeta}_1(s)ds\right], \end{aligned} \tag{6}$$

*where $dy(s)$ and $dy_1(s)$ are given by Equation (3), $P(s)$, $P_1(s)$ and satisfy:*

$$\begin{aligned} P(s) =& S(s)\widetilde{H}\Sigma^{-1}(s), \\ P_1(s) =& S_1(s)H_1\Sigma_1^{-1}(s), \end{aligned} \tag{7}$$

*with $S(s)$ and $S_1(s)$ satisfying the following relationship:*

$$\begin{aligned} \dot{S}(s) = \quad & AS(s) + S(s)A^{'} + \Theta(s) - S(s)H^{'}\Sigma^{-1}(s)HS(s), \\ \dot{S}_1(s) = \quad & AS_1(s) + S_1(s)A^{'} + \Theta(s) - S_1(s)H_1^{'}\Sigma_1^{-1}(s)H_1S_1(s). \end{aligned} \tag{8}$$

**Proof.** The proof is similar to that in Xie et al. [24], so here we will only provide a brief proof.

Taking the expectation of $Y(s)$ on both sides, we have $d\widehat{\zeta}(s) = E[A\zeta(s) + B_1u_1(s) + B_2u_2(s)|Y(s)] + E[dw(s)|Y(s)]$, using the definition of $\widehat{u}(s)$ and $B$, we have $B_1u_1(s) + B_2u_2(s) = B_1u_1(s) + B_2\widehat{u}_2(s) + B_2(u_2(s) - \widehat{u}_2(s)) = B\widehat{u}(s) + B_2\widetilde{u}_2(s)$, applying Kalman-Bucy filtering theorem, it can be obtained that

$$d\widehat{\zeta}(s) = [A\widehat{\zeta}(s) + B\widehat{u}(s) + B_2\widetilde{u}_2(s)]ds + P(s)[dy(s) - \widetilde{H}\widehat{\zeta}(s)ds]$$

where $P(s) = S(s)\widetilde{H}\Sigma^{-1}(s)$, and $S(s) = E[(\zeta(s) - \widehat{\zeta}(s))(\zeta(s) - \widehat{\zeta}(s))^{'}]$, defining $e(s) = \zeta(s) - \widehat{\zeta}(s)$, this gives that $de(s) = [A - P(s)\widetilde{H}]e(s)ds - P(s)dv(s) + dw(s)$, by Itô isometry and $P(s) = S(s)\widetilde{H}\Sigma^{-1}(s)$, we have that $\dot{S}(s) = AS(s) + S(s)A^{'} + \Theta(s) - S(s)H^{'}\Sigma^{-1}(s)HS(s)$. The proof of $d\widehat{\zeta_1}(s)$ is similar, so we omit it here.

Subsequently, the solvability of the problem is addressed below, employing the maximum principle. □

**Lemma 2.** *If the problem in this paper is solvable, then controllers $u_1(s)$ and $u_2(s)$ satisfy the following relationships:*

$$0 = R_1(s)u_1(s) + B_1^{'}E[p(s)|Y_1(s)] \tag{9}$$

$$0 = R_2(s)u_2(s) + B_2^{'}E[p(s)|Y(s)], \tag{10}$$

*and the costate equation $p(s)$ satisfies that*

$$dp(s) = -[A^{'}p(s) + Q\zeta(s)]ds + l(s)dw(s) + \bar{l}(s)dv(s), \tag{11}$$

*the terminal value is $p(S) = KE[\zeta(S)]$. Conversely, if system (1) with conditions (9)–(11) has a solution, the problem is solvable.*

**Proof.** The details of the proof closely resemble those of Proposition 2.1 in Shi et al. [25, 26], which is an extension of Peng and Wu [27, 28] within the framework of a Stackelberg game and in the context of information asymmetry, so here we will only provide a brief proof, which is given in **Appendix A**. □

With the definition presented above, the conditions in Lemma 2 are:

**Lemma 3.** *Conditions (9)–(11) are equivalent to the following relationships:*

$$\begin{cases} \quad\;\; 0 = R(s)\widehat{u}(s) + B^{'}E[p(s)|Y_1(s)], \\ \quad\;\; 0 = R_2(s)\widetilde{u}_2(s) + B_2^{'}\left\{E[p(s)|Y(s)] - E[p(s)|Y_1(s)]\right\}, \\ d\zeta(s) = [A\zeta(s) + B\widehat{u}(s) + B_2\widetilde{u}_2(s)]ds + dw(s), \\ dp(s) = -[A^{'}p(s) + Qx(s)]ds + l(s)dw(s) + \bar{l}(s)dv(s), \end{cases} \tag{12}$$

*with terminal condition given by* $p(S) = KE[\zeta(S)]$.

**Proof.** Taking the conditional expectation of Equation (5) with respect to $Y_1(s)$, we have:

$$0 = R_2(s)\widehat{u}_2(s) + B_2^{'}E[p(s)|Y_1(s)], \tag{13}$$

Combining this with Equation (4) and the matrices $B$ and $R$, we establish that the first relationship of Equation (12) holds. Then, by subtracting Equation (13) from Equation (10), the second relationship in Equation (12) holds, where Equations (9)–(11) are written as Equation (12). Thus, we prove that the conditions in Lemmas 2–3 are equivalent. The proof is now complete. □

### 3.2. Solution of the regular case

To derive the explicit optimal controllers in the irregular case, we first identify the optimal controllers in the regular case. Based on Lemmas 2–3, the solvability of the problem is equivalent to Equations (1) and (9)–(11), for which we need to establish a relationship between the FBSDEs. To achieve this objective, we present the following Riccati equations:

$$-\dot{N}(s) = A^{'}N(s) + N(s)A + \widehat{N}(s)P(s)\widetilde{H} + \widetilde{N}(s)P_1(s)H_1 + Q, \tag{14}$$

$$-\dot{\widehat{N}}(s) = A^{'}\widehat{N}(s) + \widehat{N}(s)A - \left[N(s) + \widehat{N}(s) + N_1(s)\right] B_2R_2^{-1}(s)B_2^{'} \left[N(s) + \widehat{N}(s)\right] - \widehat{N}(s)P(s)\widetilde{H}, \tag{15}$$

$$\begin{aligned} -\dot{\widetilde{N}}(s) = {} & A^{'}\widetilde{N}(s) + \widetilde{N}(s)A - \left[N(s) + \widehat{N}(s) + \widetilde{N}(s)\right] BR^{-1}(s)B^{'} \left[N(s) + \widehat{N}(s) + \widetilde{N}(s)\right] + [N(s) \\ & + \widehat{N}(s) + \widetilde{N}(s)]B_2R_2^{-1}(s)B_2^{'} \left[N(s) + \widehat{N}(s)\right] - \widetilde{N}(s)P_1(s)H_1, \end{aligned} \tag{16}$$

$$\begin{aligned} -\dot{\bar{N}}(s) = {} & A^{'}\bar{N}(s) + \bar{N}(s)A - \\ & \bar{N}(s)BR^{-1}(s)B^{'}[N(s) + \widehat{N}(s) + \widetilde{N}(s) + \bar{N}(s)] - [N(s) + \widehat{N}(s) + \widetilde{N}(s)] \times BR^{-1}(s)B^{'}\bar{N}(s) \end{aligned} \tag{17}$$

the terminal values given by $N(S) = \widehat{N}(S) = \widetilde{N}(S) = 0$, $\bar{N}(S) = K$. In the regular case, $N(s)$, $\widehat{N}(s)$, $\widetilde{N}(s)$, and $\bar{N}(s)$ satisfy the following relationships:

$$\begin{cases} Range\left[B^{'}[N(s) + \widehat{N}(s) + \widetilde{N}(s)]\right] \subseteq Range[R(s)], \\ Range\left[B_2^{'}[N(s) + \widehat{N}(s)]\right] \subseteq Range[R_2(s)], \\ Range\left[B^{'}\bar{N}(s)\right] \subseteq Range[R(s)]. \end{cases} \tag{18}$$

**Remark 2.** *Defining* $\mathbb{N}(s) = N(s) + \widehat{N}(s) + \widetilde{N}(s)$*, clearly aims to find that*

$$-\dot{\mathbb{N}}(s) = A^{'}\mathbb{N}(s) + \mathbb{N}(s)A - \mathbb{N}(s)BR^{\dagger}(s)B^{'}\mathbb{N}(s) + Q. \tag{19}$$

*The expression represents a standard Riccati equation, with the terminal condition specified as* $\mathbb{N}(s) = 0$.

*We are now prepared to present the main result for the regular case.*

**Theorem 1.** *Under the conditions of Equation (18), the problem admits a solution such that the optimal controllers $u_1(s)$ and $u_2(s)$ are given by*

$$\widehat{u}(s) = \Big[I - R^{\dagger}(s)R(s)\Big]\,\theta_1(s) - R^{\dagger}(s)B^{'}\mathbb{N}(s)\widehat{\zeta}_1(s) - R^{\dagger}(s)B^{'}\bar{N}(s)E\left[\zeta(s)\right], \quad (20)$$

$$\widetilde{u}_2(s) = [I - R_2^{\dagger}(s)R_2(s)]\theta_2(s) - R_2^{\dagger}(s)B_2^{'}[N(s) + \widehat{N}(s)][\widehat{\zeta}(s) - \widehat{\zeta}_1(s)], \quad (21)$$

*where $\theta_1(s)$, $\theta_2(s)$ are arbitrary vectors with compatible dimensions. $N(s)$, $\widehat{N}(s)$, $\widetilde{N}(s)$, $\bar{N}(s)$ are given by Equations (9)–(12). In addition, $p(s)$, $l(s)$, and $\bar{l}(s)$ satisfy that*

$$p(s) = N(s)x(s) + \widehat{N}(s)\widehat{x}(s) + \widetilde{N}(s)\widehat{x}_1(s) + \bar{N}(s)E[x(s)] \quad (22)$$

$$l(s) = N(s), \quad (23)$$

$$\bar{l}(s) = \widehat{N}(s)P(s) + N_1(s)P_1(s)[I,\ 0]. \quad (24)$$

**Proof.** Applying Itô formula to Equation (22),

$$\begin{aligned}
dp(s) =& \dot{N}(s)\zeta(s)ds + N(s)d\zeta(s) + \dot{\widehat{N}}(s)\widehat{\zeta}(s)ds \\
&+ \widehat{N}(s)d\widehat{\zeta}(s) + \widehat{N}(s)d\widehat{\zeta}(s) + \dot{\widetilde{N}}(s)\widehat{\zeta}_1(s)ds \\
&+ \widetilde{N}(s)d\widehat{\zeta}_1(s) + \dot{\bar{N}}(s)E[\zeta(s)] + \bar{N}(s)dE[\zeta(s)] \\
=& \dot{N}(s)\zeta(s) + N(s)\{[A\zeta(s) + B\widehat{u}(s) + B_2\widetilde{u}_2(s)]ds \\
&+ dw(s)\} + \dot{\widehat{N}}(s)\widehat{\zeta}(s) + \widehat{N}(s)\{A\widehat{\zeta}(s) + B\widehat{u}(s) \\
&+ B_2\widetilde{u}_2(s) + P(s)[dy(s) - \widetilde{H}\widehat{\zeta}(s)ds]\} \\
&+ \dot{\widetilde{N}}(s)\widehat{\zeta}_1(s) + \widetilde{N}(s)\{A\widehat{\zeta}_1(s) + B\widehat{u}(s) + B_2\widetilde{u}_2(s) \\
&+ P_1(s)[dy_1(s) - H_1\widehat{\zeta}_1(s)ds]\} + \dot{\bar{N}}(s)E[\zeta(s)] \\
&+ \bar{N}(s)E[A\zeta(s) + B_1u_1(s) + B_2u_2(s)]ds \\
=& \dot{N}(s)\zeta(s) + \dot{\widehat{N}}(s)\widehat{\zeta}(s) + \dot{\widetilde{N}}(s)\widehat{\zeta}_1(s) + \dot{\bar{N}}(s)E[\zeta(s)] \\
&+ N(s)A\zeta(s) + \widehat{N}(s)A\widehat{\zeta}(s) + \widetilde{N}(s)A\widehat{\zeta}_1(s) \\
&+ \bar{N}(s)AE[\zeta(s)] + [N(s) + \widehat{N}(s) \\
&+ \widetilde{N}(s)]B\widehat{u}(s) + [N(s) + \widehat{N}(s) + \widetilde{N}(s)]B_2\widetilde{u}_2(s) \\
&+ \bar{N}(s)E[B\widehat{u}(s) + B_2\widetilde{u}_2(s)]ds + N(s)dw(s) \\
&+ \widehat{N}(s)P(s)[dy(s) - \widetilde{H}\widehat{\zeta}(s)ds] \\
&+ \widetilde{N}(s)P_1(s)[dy_1(s) - H_1\widehat{\zeta}_1(s)ds] \\
=& \dot{N}(s)\zeta(s) + \dot{\widehat{N}}(s)\widehat{\zeta}(s) + \dot{\widetilde{N}}(s)\widehat{\zeta}_1(s) \\
&+ \dot{\bar{N}}(s)E[\zeta(s)] + N(s)A\zeta(s) + \widehat{N}(s)A\widehat{\zeta}(s) \\
&+ \widetilde{N}(s)A\widehat{\zeta}_1(s) + \bar{N}(s)AE[\zeta(s)] - [N(s) + \widehat{N}(s) \\
&+ \widetilde{N}(s)]BR^{-1}(s)B^{'}E[N(s)\zeta(s) + \widehat{N}(s)\widehat{\zeta}(s) \\
&+ \widetilde{N}(s)\widehat{\zeta}_1(s) + \bar{N}(s)E[\zeta(s)]|Y_1(s)] - [N(s) \\
&+ \widehat{N}(s) + \widetilde{N}(s)]B_2R_2^{-1}(s)B_2^{'} \\
&\times \{E[N(s)|Y(s)] - E[N(s)|Y_1(s)]\} \\
&- \bar{N}(s)E\{BR^{-1}(s)B^{'}E[N(s)|Y(s)] \\
&+ B_2R_2^{-1}(s)B_2^{'}\left[E[N(s)|Y(s)] - E[N(s)|Y_1(s)]\right]\}ds \\
&+ N(s)dw(s) + \widehat{N}(s)P(s)[\widetilde{H}\zeta(s)ds + dv(s) \\
&- \widetilde{H}\widehat{\zeta}(s)ds] + \widetilde{N}(s)P_1(s)[H_1\zeta(s)ds + dv_1(s) \\
&- H_1\widehat{\zeta}_1(s)ds]
\end{aligned} \quad (25)$$

which generates that

$$\begin{aligned} dp(s) =& \dot{N}(s)\zeta(s) + \dot{\widehat{N}}(s)\widehat{\zeta}(s) + \dot{\widetilde{N}}(s)\widehat{\zeta}_1(s) + \dot{\bar{N}}(s)E[\zeta(s)] \\ &+ N(s)A\zeta(s) + \widehat{N}(s)A\widehat{\zeta}(s) + \widetilde{N}(s)A\widehat{\zeta}_1(s) \\ &+ \bar{N}(s)AE[\zeta(s)] - [N(s) + \widehat{N}(s) + \widetilde{N}(s)] \\ &\times BR^{-1}(s)B^{'}[N(s) + \widehat{N}(s) + \widetilde{N}(s)]\widehat{\zeta}_1(s) \\ &- [N(s) + \widehat{N}(s) + \widetilde{N}(s)]BR^{-1}(s)B^{'}\bar{N}(s)E[\zeta(s)] \\ &- [N(s) + \widehat{N}(s) + \widetilde{N}(s)]B_2R_2^{-1}(s)B_2^{'}[N(s) \\ &+ \widehat{N}(s)][\widehat{\zeta}(s) - \widehat{\zeta}_1(s)] - \bar{N}(s)BR^{-1}(s)B^{'}[N(s) \\ &+ \widehat{N}(s) + \widetilde{N}(s) + \bar{N}(s)]E[\zeta(s)] + N(s)dw(s) \\ &+ \widehat{N}(s)P(s)[\widetilde{H}\zeta(s)ds + dv(s) - \widetilde{H}\widehat{\zeta}(s)ds] \\ &+ \widetilde{N}(s)P_1(s)[H_1\zeta(s)ds + [I \;\; 0]dv(s) - H_1\widehat{\zeta}_1(s)ds]. \end{aligned} \tag{26}$$

Using the fact that Equations (14)–(17), then $dp(t)$ satisfies

$$\begin{aligned} dp(s) =& -[A^{'}N(s)\zeta(s) + A^{'}\widehat{N}(s)\widehat{\zeta}(s) + A^{'}\widetilde{N}(s)\widehat{\zeta}_1(s) \\ &+ A^{'}\bar{N}(s)E[\zeta(s)] + Q\zeta(s)]ds + l(s)dw(s) \\ &+ \bar{l}(s)dv(s) \\ =& -[A^{'}N(s) + Q\zeta(s)]ds + q(s)dw(s) + \bar{q}(s)dv(s). \end{aligned} \tag{27}$$

$(N(s)\zeta(s) + \widehat{N}(s)\widehat{\zeta}(s) + N_1(s)\widehat{\zeta}_1(s) + \bar{N}(s)E[\zeta(s)], l(s), \bar{l}(s))$ is a solution to Equation (1) compared with Equation (11). Then, substituting Equation (22) into the controllers in Equation (12), and combined with Lemma 3, it can be quickly obtained that the optimal controllers are Equations (20)–(21), where $\theta_1(t)$ and $\theta_2(t)$ are arbitrary vectors with compatible dimensions. This completes the proof. □

**Remark 3.** *Due to the semi-definiteness of the weighting matrix being controlled, some components of the controller will affect the result value of the optimal performance index, thus resulting in the non-uniqueness of the optimal controller, it should be pointed out that if we insert the solution into the performance index, the terms of $\theta_1(s)$ and $\theta_2(s)$ become zero, which means that although the solution is not unique, the optimal performance index is unique.*

*It is crucial to recognize that the two inclusion relations must be satisfied simultaneously, as the cost function (2) incorporates a quadratic term involving the mathematical expectation. Notably, condition (18) typically fails to hold in practical applications [29]. Specifically, seven distinct scenarios may arise in such contexts:*

$$\begin{aligned} &Range[B^{'}\mathbb{N}(s)] \subseteq Range[R(s)], \\ &Range[B^{'}\bar{N}(s)] \nsubseteq Range[R(s)], \\ &Range\left[B_2^{'}[N(S) + \widehat{N}(s)]\right] \nsubseteq Range[R_2(s)], \end{aligned} \tag{28}$$

*and*

$$\begin{aligned} &Range[B^{'}\mathbb{N}(s)] \nsubseteq Range[R(s)], \\ &Range[B^{'}\bar{N}(s)] \nsubseteq Range[R(s)], \\ &Range\left[B_2^{'}[N(S) + \widehat{N}(s)]\right] \subseteq Range[R_2(s)], \end{aligned} \tag{29}$$

*with the following case being particularly representative.*

$$\begin{aligned}&Range[B^{'}\mathbb{N}(s)] \nsubseteq Range[R(s)],\\&Range[B^{'}\bar{N}(s)] \nsubseteq Range[R(s)],\\&Range\left[B_2^{'}[N(S)+\widehat{N}(s)]\right] \nsubseteq Range[R_2(s)].\end{aligned} \tag{30}$$

*The problems corresponding to similar cases are referred to as irregular linear-quadratic control problems. Notably, classical control theory cannot be applied to the associated irregular optimal control problem; unlike the solution framework elaborated in Equation (1), the irregular case will therefore be studied subsequently.*

In this situation, the optimization problem cannot be solved by using Theorem 1 directly. Thus, the irregular case will be studied below.

**Remark 4.** *The explicit controllers (20)–(21) differ structurally from existing results in three critical aspects. First, compared with Zhang and Xu [9], who derived the controller using a single Riccati equation $\mathbb{N}(s)$, our solution introduces four interconnected Riccati Equations (14)–(17). This increase from one to four equations is a direct consequence of the two-layer asymmetric observations and the dual-controller structure, which requires separate compensation for $\widehat{\zeta}_1(s)$, $\widehat{\zeta}(s)$, and $E\left[\zeta(s)\right]$. Second, compared with Li et al. [10], who considered deterministic delayed information, our solution includes an extra innovation term $B_2\widetilde{u}_2(s)$ that explicitly accounts for the stochastic estimation error $E\left[p(s)\mid Y(s)\right]-E\left[p(s)\mid Y_1(s)\right]$. This term is absent in their framework because their information delay is non-random. Third, compared with Liang et al. [12], who only treated the regular case, our solution holds under the irregular range condition (31), where the standard Riccati equation fails, and we must introduce the auxiliary matrices $N_1(s)$, $\widehat{N}_1(s)$, $\widetilde{N}_1(s)$, and $\overline{N}_1(s)$ in Equation (35). These technical differences collectively demonstrate that our solution is not a straightforward extension but a fundamentally new derivation tailored to the asymmetric irregular setting.*

## 4. Solution of the irregular case

In this section, our principal objective is to address the problem under the irregular conditions. It should be pointed out that the above case involves all the other situations which we will consider in this section:

$$\begin{cases}Range\left[B^{'}\mathbb{N}(s)\right] \nsubseteq Range[R_1(s)],\\Range\left[B^{'}\bar{N}(s)\right] \nsubseteq Range[R(s)],\\Range\left[B_2^{'}[N(s)+\widehat{N}(s)]\right] \nsubseteq Range[R_2(s)].\end{cases} \tag{31}$$

To begin with, we introduce the following notations. There exist two elementary row transformation matrices $M(s)$ and $M_2(s)$

$$\begin{aligned}M(s)[I-R^{\dagger}(s)R(s)] &= \begin{bmatrix}0\\\phi(s)\end{bmatrix},\\M_2(s)[I-R_2^{\dagger}(s)R_2(s)] &= \begin{bmatrix}0\\\phi_2(s)\end{bmatrix},\end{aligned} \tag{32}$$

which $\phi(s)$ and $\phi_2(s)$ are full row rank. Defining that

$$\begin{aligned}
&M^{-1}(s) = [*\ \Delta(s)],\ M_2^{-1}(t) = [*\Delta_2(s)],\\
&M(s)[I - R^{\dagger}(s)R(s)]\theta(s) = \begin{bmatrix} 0 \\ z(s) \end{bmatrix},\\
&M_2(s)[I - R_2^{\dagger}(s)R_2(s)]\theta_2(s) = \begin{bmatrix} 0 \\ z_2(s) \end{bmatrix},
\end{aligned} \tag{33}$$

where $*$ symbol represents the current value of the matrix, but we are not concerned about what specific form it takes. $z(s) = \phi(s)\theta(s)$ and $z_2(s) = \phi_2(s)\theta_2(s)$, $\theta(s)$ and $\theta_2(s)$ are arbitrary vectors with compatible dimensions, which gives that

$$\begin{aligned}
[I - R^{\dagger}(s)R(s)]\theta(s) &= M^{-1}(s)\begin{bmatrix} 0 \\ z(s) \end{bmatrix} = \Delta(s)z(s),\\
[I - R_2^{\dagger}(s)R_2(s)]\theta_2(s) &= M_2^{-1}(s)\begin{bmatrix} 0 \\ z_2(s) \end{bmatrix} = \Delta_2(s)z_2(s).
\end{aligned} \tag{34}$$

To establish the relationship, we introduce the following Riccati equations. Let $N_1(s), \widehat{N}_1(s), \widetilde{N}_1(s)$ be solutions to:

$$\begin{aligned}
-\dot{N}_1(s) =& A^{'}N_1(s) + N_1(s)A + \widehat{N}_1(s)P(s)\widetilde{H}\\
&+ \widetilde{N}_1(s)P_1(s)H_1 + Q,\\
-\dot{\widehat{N}}_1(s) =& A^{'}\widehat{N}_1(s) + \widehat{N}_1(s)A - \mathbb{N}(s)B_2R_2^{-1}(s)B_2^{'}[N_1(s)\\
&+ \widehat{N}_1(s)] - \mathbb{N}_1(s)B_2R_2^{-1}(s)B_2^{'}[N(s) + \widehat{N}(s)\\
&+ N_1(s) + \widehat{N}_1(s)] - \widehat{N}_1(s)P(s)\widetilde{H},\\
-\dot{\widetilde{N}}_1(s) =& A^{'}\widetilde{N}_1(s) + \widetilde{N}_1(s)A - \mathbb{N}(s)BR^{-1}(s)B^{'}\mathbb{N}_1(s)\\
&- \mathbb{N}_1(s)BR^{-1}(s)B^{'}[\mathbb{N}(s) + \mathbb{N}_1(s)]\\
&+ \mathbb{N}(s)B_2R_2^{-1}(s)B_2^{'}\left[N_1(s) + \widehat{N}_1(s)\right]\\
&+ \mathbb{N}_1(s)B_2R_2^{-1}(s)B_2^{'}[N(s) + \widehat{N}(s) + N_1(s)\\
&+ \widehat{N}_1(s)] - \widetilde{N}(s)_1P_1(s)H_1,\\
-\dot{\bar{N}}_1(s) =& A^{'}\bar{N}_1(s) + \bar{N}_1(s)A - \bar{N}(s)BR^{-1}(s)B^{'}[\mathbb{N}_1(s)\\
&+ \bar{N}_1(s)] - \bar{N}_1(s)BR^{-1}(s)B^{'}[\mathbb{N}(s) + \bar{N}(s)\\
&+ \mathbb{N}_1(s) + \bar{N}_1(s)] - \mathbb{N}(s)BR^{-1}(s)B^{'}\bar{N}_1(s)\\
&- \mathbb{N}_1(s)BR^{-1}(s)B^{'}[\bar{N}(s) + \bar{N}_1(s)].
\end{aligned} \tag{35}$$

where $\mathbb{N}_1(s) = N_1(s)+\widehat{N}_1(s)+\widetilde{N}_1(s)$. In addition, $N_1(S) = 0, \bar{N}_1(S) = 0, \widetilde{N}_1(S) = 0$, and $\bar{N}_1(S)$ is to be determined.

We are now poised to present the explicit solution under irregular conditions.

**Theorem 2.** *In the case of Equation (22), if $N_1(s)$, $\widehat{N}_1(s)$, $\widetilde{N}_1(s)$, and $\bar{N}_1(s)$ exist such that*

$$\begin{aligned}
0 =& [\mathbb{N}(s) + \mathbb{N}_1(s)]B^{'}[N - R^{\dagger}(s)R(s)],\\
0 =& [\mathbb{N}(s) + \mathbb{N}_1(s)]B_2^{'}[N - R_2^{\dagger}(s)R_2(s)],\\
0 =& [\bar{N}(s) + \bar{N}_1(s)]B^{'}[N - R^{\dagger}(s)R(s)],\\
0 =& [\bar{N}(s) + \bar{N}_1(s)]B_2^{'}[N - R_2^{\dagger}(s)R_2(s)],
\end{aligned} \tag{36}$$

*for* $s \in [0,T]$ *and there exists* $z(s)$ *and* $z_2(s)$ *such that*

$$\bar{N}_1(S)E\left[\zeta(S)\right]=0, N_1(S)=0, \widetilde{N}_1(S)=0 \tag{37}$$

*and* $\zeta(s)$ *is given by*

$$\begin{aligned} d\zeta(s)=&\{A\zeta(s)-BR^{\dagger}(s)B^{'}[\mathbb{N}(s)+\mathbb{N}_1(s)]\widehat{\zeta}_1(s)\\ &-BR^{\dagger}(s)B^{'}[\bar{N}(s)+\bar{N}_1(s)]E[\zeta(s)]\\ &-B_2R_2^{\dagger}(s)B_2^{'}[N(s)+\widehat{N}(s)+N_1(s)\\ &+\widehat{N}_1(s)][\widehat{\zeta}(s)-\widehat{\zeta}_1(s)]+B\Delta(s)z(s)\\ &+B_2\Delta_2(s)z_2(s)\}+dw(s). \end{aligned} \tag{38}$$

*The optimal controllers are given as*

$$\begin{aligned} \widehat{u}(s)=&-R^{\dagger}(s)B^{'}\left[\mathbb{N}(t)+\mathbb{N}_1(s)\right]\widehat{\zeta}_1(s)-R^{\dagger}(s)B^{'}[\bar{N}(s)\\ &+\bar{N}_1(s)]E[\zeta(s)]+\Delta(s)z(s),\\ \widetilde{u}_2(s)=&-R_2^{\dagger}(s)B_2^{'}\left[N(s)+\widehat{N}(s)+N_1(s)+\widehat{N}_1(s)\right][\widehat{\zeta}(s)\\ &-\widehat{\zeta}_1(s)]+\Delta_2(s)z_2(s). \end{aligned} \tag{39}$$

**Proof.** Suppose that

$$\begin{aligned} \widehat{k}(s)=&[N(s)+N_1(s)]\zeta(s)+[\widehat{N}(s)+\widehat{N}_1(s)]\widehat{\zeta}(s)\\ &+[\widetilde{N}(s)+\widetilde{N}_1(s)]\widehat{\zeta}_1(s)+[\bar{N}(s)+\bar{N}_1(s)]E[\zeta(s)],\\ \widehat{k}_1(s)=&N(s)+N_1(s),\\ \check{k}_1(s)=&[\widehat{N}(s)+\widehat{N}_1(s)]P(s)+[\widetilde{N}(s)+\widetilde{N}_1(s)]P_1(s)[I,\ 0], \end{aligned} \tag{40}$$

with $\widehat{k}(S)=KE[\zeta(S)]$. Applying Itô formula to $\widehat{k}(s)$,

$$\begin{aligned} &d\{[N(s)+N_1(s)]\zeta(s)+[\widehat{N}(s)+\widehat{N}_1(s)]\widehat{\zeta}(s)\\ &+[\widetilde{N}(s)+\widetilde{N}_1(s)]\widehat{\zeta}_1(s)+[\bar{N}(s)+\bar{N}_1(s)]E[\zeta(s)]\}\\ =&[\dot{N}(s)+\dot{N}_1(s)]\zeta(s)+[N(s)+N_1(s)]\{A\zeta(s)\\ &-BR^{\dagger}(s)B^{'}[\mathbb{N}(s)+\mathbb{N}_1(s)]\widehat{\zeta}_1(s)-BR^{\dagger}(s)B^{'}[\bar{N}(s)\\ &+\bar{N}_1(s)]E\zeta(s)-B_2R_2^{\dagger}(s)B_2^{'}[N(s)+\widehat{N}(s)+N_1(s)\\ &+\widehat{N}_1(s)][\widehat{\zeta}(s)-\widehat{\zeta}_1(s)]+B\Delta(s)z(s)+B_2\Delta_2(s)z_2(s)\}\\ &+[N(s)+N_1(s)]dw(s)+[\dot{\widehat{N}}(s)+\dot{\widehat{N}}_1(s)]\widehat{\zeta}(s)\\ &+[\widehat{N}(s)+\widehat{N}_1(s)][A\widehat{\zeta}(s)+B\widehat{u}(s)+B_2\widetilde{u}_2(s)\\ &+P(s)[dy(s)-\widetilde{H}\widehat{\zeta}(s)ds]]+[\dot{\widetilde{N}}(s)+\dot{\widetilde{N}}_1(s)]\widehat{\zeta}(s)\\ &+[\widetilde{N}(s)+\widetilde{N}_1(s)][A\widehat{\zeta}_1(s)+B\widehat{u}(s)+B_2\widetilde{u}_2(s)\\ &+P(s)[dy(s)-H_1\widehat{\zeta}_1(s)ds]]+[\dot{\bar{N}}(s)+\dot{\bar{N}}_1(s)]E[\zeta(s)]\\ &+[\bar{N}(s)+\bar{N}_1(s)]E\{A\zeta(s)-BR^{\dagger}(s)B^{'}[\mathbb{N}(s)\\ &+\mathbb{N}_1(s)]\widehat{\zeta}_1(s)-BR^{\dagger}(s)B^{'}[\bar{N}(s)+\bar{N}_1(s)]E\zeta(s)\\ &-B_2R_2^{\dagger}(s)B_2^{'}[N(s)+\widehat{N}(s)+N_1(s)+\widehat{N}_1(s)][\widehat{\zeta}(s)\\ &-\widehat{\zeta}_1(s)]+B\Delta(s)z(s)+B_2\Delta_2(s)z_2(s)\}, \end{aligned} \tag{41}$$

According to Equation (36), the above relationship can be rewritten as:

$$\begin{aligned} d\widehat{k}(s) =& -\{A^{'}[[N(s)+N_1(s)]\zeta(s)+[\widehat{N}(s)+\widehat{N}_1(s)]\widehat{\zeta}(s) \\ &+[\widetilde{N}(s)+\widetilde{N}_1(s)]\widehat{\zeta}_1(s)+[\bar{N}(s)+\bar{N}_1(s)]E[\zeta(s)]] \\ &+Q\zeta(s)\}ds+\widehat{k_1}(s)dw(s)+\check{k_1}(s)dv(s). \end{aligned} \tag{42}$$

By using condition (37) and comparing it with Equation (11), $(\widehat{k}(s), \widehat{k_1}(s), \check{k_1}(s))$ is the solution to Equations (1) and (11).

Next, combining with Equation (36), it is obtained that:

$$\begin{aligned} 0 =& R(s)\widehat{u}(s)+B^{'}\left[\mathbb{N}(s)+\mathbb{N}_1(s)\right]\widehat{\zeta}_1(s) \\ &+B^{'}\left[\bar{N}(s)+\bar{N}_1(s)\right]E[\zeta(s)], \\ 0 =& R_2(s)\widetilde{u}_2(s)+B_2^{'}[N(s)+\widehat{N}(s)+N_1(s) \\ &+\widehat{N}_1(s)][\widehat{\zeta}(s)-\widehat{\zeta}_1(s)], \end{aligned} \tag{43}$$

this gives that

$$\begin{aligned} \widehat{u}_1(s) =& -R^{\dagger}(s)B^{'}\left[\mathbb{N}(s)+\mathbb{N}_1(s)\right]\widehat{\zeta}_1(s)-R^{\dagger}B^{'}[\bar{N}(s) \\ &+\bar{N}_1(s)]E[\zeta(s)]+[I-R^{\dagger}(s)R(s)]\theta(s), \\ \widetilde{u}_2(s) =& -R_2^{\dagger}(s)B_2^{'}\left[N(s)+\widehat{N}(s)+N_1(s)+\widehat{N}_1(s)\right][\widehat{\zeta}(s) \\ &-\widehat{\zeta}_1(s)]+[I-R_2^{\dagger}(s)R_2(s)]\theta_2(s), \end{aligned} \tag{44}$$

and

$$\begin{aligned} 0 =& [I-R(s)R^{\dagger}(s)]B^{'}\{[\mathbb{N}(s)+\mathbb{N}_1(s)]\widehat{\zeta}_1(s)+[\bar{N}(s) \\ &+\bar{N}_1(s)]E[\zeta(s)]\}, \\ 0 =& [I-R_2(s)R_2^{\dagger}(s)]B_2^{'}[N(s)+\widehat{N}(s)+N_1(s) \\ &+\widehat{N}_1(s)][\widehat{\zeta}(s)-\widehat{\zeta}_1(s)], \end{aligned} \tag{45}$$

Combined with Equation (36), the relationships Equation (45) hold. Applying Equations (32)–(34) and noticing that $\phi(s)$ and $\phi_2(s)$ are full row rank, then Equation (44) becomes Equation (39). This completes the proof. □

**Remark 5.** *The terminal condition here is to explain when the function $z(s)$ and $z_2(s)$ can ensure that $\widehat{u}(s)$ and $\widetilde{u}_2(s)$ are definitely present. This means it is to find the conditions for the existence of $\widehat{u}(s)$ and $\widetilde{u}_2(s)$.*

Finally, the analytical form of $z(s)$ and $z_2(s)$ will be derived. To maintain generality, assuming $\bar{N}_1(s)$ is singular which exists $\Xi(t)$ satisfies:

$$\Xi^{'}(s)\bar{N}_1(s)\Xi(s)=\begin{bmatrix} \bar{N}_2(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{46}$$

where $\bar{N}_2(s) > 0$. To simplify the notation, define the following symbols:

$$\begin{aligned} [\Xi_1^{'}(s)\ \ \Xi_2^{'}(s)]^{'} =& \dot{\Xi}^{'}(s)\Xi(s), \\ [\widehat{W}_1^{'}(s)\ \ \widehat{W}_2^{'}(s)]^{'} =& \Xi^{'}(s)[A-BR^{\dagger}B^{'}[\mathbb{N}(s)+\mathbb{N}_1(s) \\ &+\bar{N}(s)+\bar{N}_1(s)]], \\ [\widehat{Z}^{'}(s)\ \ \widetilde{Z}^{'}(s)]^{'} =& \Xi^{'}(s)B\Delta(t)\Xi^{'}(s), \\ [\widehat{Z}_2^{'}(s)\ \ \widetilde{Z}_2^{'}(s)]^{'} =& \Xi^{'}(s)B_2\Delta_2(t)\Xi^{'}(s). \end{aligned} \tag{47}$$

Then we have,

**Theorem 3.** *There exist* $\Phi(s)$ *and* $\Phi_2(s)$ *that establish the following relationship if Equation (36) holds.*

$$\begin{aligned}\begin{bmatrix}\frac{I}{s-S} & 0\end{bmatrix} =&\Xi_1(s)+\widehat{W}_1(s)+\widehat{Z}(s)\Xi^{'}(t)\Phi(s)\Xi(s)\\ &+\widehat{Z}_2(s)\Xi^{'}(s)\Phi_2(s)\Xi(s),\end{aligned} \tag{48}$$

*where s < S, then the problem where minimizes Equation (2) subject to Equation (1) is solvable, and optimal controllers are Equation (39) and* $z(s)$*,* $z_2(s)$ *satisfy*

$$z(s)=\Phi(s)\zeta(s),\ \ z_2(s)=\Phi_2(s)\zeta(s). \tag{49}$$

**Proof.** According to Theorem 2, condition (48) needs to be proved, as it can ensure the establishment of Equation (37). By defining $\ell(s)=\Xi^{'}(s)E[\zeta(s)]=[\ell_1^{'}(s)\ \ \ell_2^{'}(s)]^{'}$, with Equations (38) and (49),

$$\begin{aligned}\dot{\ell}(s)=&\dot{\Xi}^{'}(s)E[\zeta(s)]+\Xi^{'}(s)[A-BR^{\dagger}B^{'}[\mathbb{N}(s)+\mathbb{N}_1(s)\\ &+\bar{N}(s)+\bar{N}_1(s)]]E[\zeta(s)]+\Xi^{'}(s)B\Delta(s)\Phi(s)E[\zeta(s)]\\ &+\Xi^{'}(s)B_2\Delta_2(s)\Phi_2(s)E[\zeta(s)]\\ =&\{\begin{bmatrix}\Xi_1(s)\\ \Xi_2(s)\end{bmatrix}+\begin{bmatrix}\widehat{Z}(s)\Xi^{'}(s)\Phi(s)\Xi(s)\\ \widetilde{Z}(s)\Xi^{'}(s)\Phi(s)\Xi(s)\end{bmatrix}\\ &+\begin{bmatrix}\widehat{Z}_2(s)\Xi^{'}(s)\Phi_2(s)\Xi(s)\\ \widetilde{Z}_2(s)\Xi^{'}(s)\Phi_2(s)\Xi(s)\end{bmatrix}+\begin{bmatrix}\widehat{W}_1(s)\\ \widehat{W}_2(s)\end{bmatrix}\}\begin{bmatrix}\ell_1(s)\\ \ell_2(s)\end{bmatrix}\\ =&\begin{bmatrix}\frac{I}{s-S} & 0\\ * & *\end{bmatrix}\begin{bmatrix}\ell_1(s)\\ \ell_2(s)\end{bmatrix},\end{aligned} \tag{50}$$

Where the * represents matrix elements whose explicit forms are irrelevant to the proof, and this yields $\dot{\ell}_1(s)=\frac{I}{s-S}\ell_1(s)$, so we have that $\ell_1(s)=\frac{S-s}{S-s_0}\ell_1(s_0)$, and the terminal condition is $\ell_1(S)=0$. Therefore, we have the following relationship:

$$\begin{aligned}\bar{N}_1(S)E[\zeta(S)]=\bar{N}_1(S)\Xi(S)&\begin{bmatrix}0\\ \ell_2(S)\end{bmatrix}\\ &=\Xi(S)\Xi^{'}(S)\bar{N}_1(S)\Xi(S)\begin{bmatrix}0\\ \ell_2(S)\end{bmatrix}\\ &=\Xi(S)\begin{bmatrix}\bar{N}_2(s) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}0\\ \ell_2(S)\end{bmatrix}=0.\end{aligned} \tag{51}$$

Thus, under the condition (48), we have Equation (37) holds. Then the proof is now complete. □

**Remark 6.** *It is clear to find that* $\frac{I}{s-S}$ *will become singular as s approaches S. It should be noted that this is the condition that the gain matrix must satisfy, but the conditions that the controller needs to satisfy will not cause the controller to diverge, so we don't care whether it is divergent or not.*

## 5. Conclusion

This paper considers a type of irregular LQ optimal control problem with an asymmetric information structure. First, we focus on the regular case under the asymmetric information structure. The solvability of the problem is thereby reformulated as solving the associated FBSDEs. Next, based on the regular case, we derived the conditions of solvability and irregular optimal controllers in terms of the Riccati equations. In summary, while previous studies either addressed irregular LQ control under full information [7–9] or asymmetric information under regular conditions [16, 25], our work bridges this gap by providing the first complete solution to the irregular LQ problem with dual controllers and two-layer asymmetric partial observations, as explicitly contrasted in Remark 4.

Although we solved this problem by introducing FBSDEs, it remains to be investigated whether all irregular problems can be transformed into corresponding FBSDEs. Furthermore, developing efficient numerical algorithms tailored to the derived optimality conditions and conducting comprehensive simulation studies under various parameter regimes to test the robustness and performance of the proposed controller are also topics worthy of future research.

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# Appendix A

Necessity: Defining the Hamiltonian function

$$H = \frac{1}{2}[\zeta' Q\zeta + u_1'(s)B_1u_1(s) + u_2'(s)B_2u_2(s)] + p'(A\zeta + B_1u_1(s) + B_2u_2(s)) \tag{A1}$$

where $p(s)$ is the adjoint process. By the stochastic maximum principle, the optimal control $(u_1^*, u_2^*)$ satisfies that $\frac{\partial H}{\partial u_1} = 0$, $\frac{\partial H}{\partial u_2} = 0$. Due to the asymmetric information structure, $u_1(s)$ is $Y_1(s)$-measurable, $u_2(s)$ is $Y(s)$-measurable, so we need to take the conditional expectation of the Hamiltonian function, this gives that $E[R_1(s)u_1(s) + B_1'p(s)|Y_1(s)] = 0$, which is Equation (9), Equation (10) can be obtained similarly, so it is omitted here.

By the stochastic maximum principle, the costate equation $p(s)$ satisfies a backward stochastic differential equation as $dp(s) = -\frac{\partial H}{\partial \zeta}ds + l(s)dw(s) + \bar{l}(s)dv(s)$, which is Equation (6). The terminal condition can be obtained through the variational method as $p(S) = KE[\zeta(S)]$.

Sufficiency: suppose $(u_1^*(s), u_2^*(s))$ represents any allowable control that satisfies the information constraints. The corresponding state trajectory is $\zeta^*(s)$. If there is an adaptation process $(p(s), l(s), \bar{l}(s))$ and $(u_1^*, u_2^*, \zeta^*)$ that satisfy

$$\begin{cases} d\zeta^*(s) = [A\zeta^*(s) + B_1u_1^*(s) + B_2u_2^*(s)]\,ds + dw(s), & \zeta^*(0) = \zeta_0, \\ dp(s) = -\left[A'p(s) + Q\zeta^*(s)\right]ds + l(s)dw(s) + \bar{l}(s)dv(s), & p(S) = KE[\zeta^*(S)], \\ R_1(s)u_1^*(s) + B_1'E[p(s) \mid Y_1(s)] = 0, & \text{a.s., a.e. } s \in [0, S], \\ R_2(s)u_2^*(s) + B_2'E[p(s) \mid Y(s)] = 0, & \text{a.s., a.e. } s \in [0, S]. \end{cases} \tag{A2}$$

Which we have that $(u_1^*, u_2^*)$ is the optimal control.

Select any other allowable control $(u_1(s), u_2(s))$ the corresponding state is recorded as $\zeta(s)$ satisfies

$$\begin{aligned} &\zeta(s) = [A\zeta(s) + B_1u_1(s) + B_2u_2(s)]\,ds + dw(s), \\ &\zeta(0) = \zeta_0. \end{aligned} \tag{A3}$$

Define state deviation and control deviation $\delta\zeta(s) \triangleq \zeta(s) - \zeta^*(s)$, $\delta u_1(s) \triangleq u_1(s) - u_1^*(s)$, $\delta u_2(s) \triangleq u_2(s) - u_2^*(s)$. Then we have

$$d(\delta\zeta(s)) = [A\delta\zeta(s) + B_1\delta u_1(s) + B_2\delta u_2(s)]\,ds, \delta\zeta(0) = 0. \tag{A4}$$

Apply Itô formula to $p(s)'\delta\zeta(s)$ gives $d\left(p(s)'\delta\zeta(s)\right) = dp(s)' \cdot \delta\zeta(s) + p(s)' \cdot d(\delta\zeta(s))$.

This gives that

$$\begin{aligned} d(p'\delta\zeta) &= \left\{-[A'p + Q\zeta^*]'\delta\zeta + p'\left[A\delta\zeta + B_1\delta u_1 + B_2\delta u_2\right]\right\}ds \\ &\quad + [l\,dw + \bar{l}\,dv]'\delta\zeta \\ &= \left\{-p'A\delta\zeta - \zeta^{*'}Q\delta\zeta + p'A\delta\zeta + p'B_1\delta u_1 + p'B_2\delta u_2\right\}ds \\ &\quad + [l\,dw + \bar{l}\,dv]'\delta\zeta \\ &= \left\{-\zeta^{*'}Q\delta\zeta + p'B_1\delta u_1 + p'B_2\delta u_2\right\}ds + [l\,dw + \bar{l}\,dv]'\delta\zeta. \end{aligned} \tag{A5}$$

Integrate the above expression from $0$ to $S$ and take the mathematical expectation

$$E\left[p(S)'\delta\zeta(S)\right] = E\int_0^S \left[-\zeta^{*'}Q\delta\zeta + p'B_1\delta u_1 + p'B_2\delta u_2\right]ds. \tag{A6}$$

From the terminal condition $p(S) = KE[\zeta^*(S)]$ we have

$$E\left[p(S)^{'}\delta\zeta(S)\right] = p(S)^{'}E[\delta\zeta(S)] = E[\zeta^*(S)]^{'}K\,E[\delta\zeta(S)]. \tag{A7}$$

Substitute Equation (A7) back into Equation (A6)

$$E\int_0^S \zeta^{*^{'}}Q\delta\zeta\,ds + E[\zeta^*(S)]^{'}K\,E[\delta\zeta(S)] = E\int_0^S \left[p^{'}B_1\delta u_1 + p^{'}B_2\delta u_2\right]ds. \tag{A8}$$

Since $Q$, $R_1(s)$, $R_2(s)$ and $K$ are all semi-positive definite matrices, the corresponding quadratic forms are convex functions. Utilizing the convexity inequality $x^{'}Mx - y^{'}My \geq 2y^{'}M(x-y)$ (where $M \geq 0$), we have:

$$\begin{gathered}
\zeta^{'}Q\zeta - \zeta^{*^{'}}Q\zeta^* \geq 2\zeta^{*^{'}}Q\delta\zeta, \\
u_1^{'}R_1u_1 - u_1^{*^{'}}R_1u_1^* \geq 2u_1^{*^{'}}R_1\delta u_1, \\
u_2^{'}R_2u_2 - u_2^{*^{'}}R_2u_2^* \geq 2u_2^{*^{'}}R_2\delta u_2, \\
E[\zeta]^{'}KE[\zeta] - E[\zeta^*]^{'}KE[\zeta^*] \geq 2E[\zeta^*]^{'}KE[\delta\zeta].
\end{gathered} \tag{A8}$$

Substitute the above inequality into $J(u_1,u_2) - J(u_1^*,u_2^*)$, and we obtain the lower bound:

$$J(u_1,u_2) - J(u_1^*,u_2^*) \geq 2E\int_0^S \left[\zeta^{*^{'}}Q\delta\zeta + u_1^{*^{'}}R_1\delta u_1 + u_2^{*^{'}}R_2\delta u_2\right]ds + 2E[\zeta^*(S)]^{'}K\,E[\delta\zeta(S)]. \tag{A10}$$

Substitute Equation (A8) into Equation (A10) we have

$$J(u_1,u_2) - J(u_1^*,u_2^*) \geq 2E\int_0^S \left[p^{'}B_1\delta u_1 + p^{'}B_2\delta u_2 + u_1^{*^{'}}R_1\delta u_1 + u_2^{*^{'}}R_2\delta u_2\right]ds.$$

which is

$$J(u_1,u_2) - J(u_1^*,u_2^*) \geq 2E\int_0^S \left[(B_1^{'}p + R_1u_1^*)^{'}\delta u_1 + (B_2^{'}p + R_2u_2^*)^{'}\delta u_2\right]ds. \tag{A11}$$

Since $\delta u_1(s) = u_1(s) - u_1^*(s)$ is $Y_1(s)$-measurable, $\delta u_2(s)$ is $Y(s)$-measurable we have

$$\begin{aligned}
E\left[(B_1^{'}p + R_1u_1^*)^{'}\delta u_1\right] &= E\left[E\left[(B_1^{'}p + R_1u_1^*)^{'}\delta u_1 \mid Y_1(s)\right]\right] \\
&= E\left[(B_1^{'}E[p \mid Y_1] + R_1u_1^*)^{'}\delta u_1\right] \\
&= E\left[0^{'}\cdot\delta u_1\right] = 0.
\end{aligned} \tag{A12}$$

and

$$\begin{aligned}
E\left[(B_2^{'}p + R_2u_2^*)^{'}\delta u_2\right] &= E\left[E\left[(B_2^{'}p + R_2u_2^*)^{'}\delta u_2 \mid Y(s)\right]\right] \\
&= E\left[(B_2^{'}E[p \mid Y] + R_2u_2^*)^{'}\delta u_2\right] \\
&= E\left[0^{'}\cdot\delta u_2\right] = 0.
\end{aligned} \tag{A13}$$

Substituting the above two equations into Equation (A11), we obtain $J(u_1,u_2) - J(u_1^*,u_2^*) \geq 0$. which completes the proof.