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Article

Enriched polynomial contractions and a neural fixed-point solver for fractional boundary value problems

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Abstract: We introduce two classes of operators on normed spaces, the enriched polynomial contractions and the almost enriched polynomial contractions. Both classes contain the enriched contractions of Berinde and Păcurar and the polynomial-type contractions of Jleli, Pacurar and Samet as special cases. In Banach spaces, we prove existence, uniqueness, and convergence theorems for the fixed points, first under continuity of the operator and then under the weaker assumption of Picard continuity, with the fixed points approximated by Krasnoselskii-type iteration. Several classical results, including the Banach contraction principle and the enriched contraction theorem, are recovered as special cases, and several worked examples are given. As the main application, we consider a nonlinear Caputo fractional boundary value problem of order $2 < \mathcal{N} \leq 3$. Writing it as a fixed-point equation and assuming an explicit Lipschitz condition, we prove that it has a unique solution. We then examine the constructive side of this result numerically: on a manufactured problem with a known exact solution, we run the associated fractional-quadrature Krasnoselskii iteration, check that its measured geometric rate stays below the certified contraction factor, and recover the same solution with a neural fixed-point solver, a network trained to satisfy the discretized fixed-point equation of the same operator, and we relate the accuracy of both solvers to the underlying quadrature error. The numerical results are consistent with the theory and illustrate its use on a class of nonlinear fractional models of this form.

Keywords: enriched polynomial contraction; almost enriched polynomial contraction; Banach space; fixed point; Krasnoselskii iteration; Caputo fractional differential equation; neural fixed-point solver

1. Introduction and preliminaries

A great many nonlinear problems, integral equations, differential and fractional differential equations, and evolution equations among them, become easier to handle once they are recast as the search for a fixed point,

$$\Delta\, \mathring{\alpha} = \mathring{\alpha}.$$

Posed this way, the twin questions of whether a solution exists and whether it is the only one turn into questions about a single operator: Δ sends the space $\mathbb{G}$ into itself, and a solution $\mathring{\alpha} \in \mathbb{G}$ is simply a point that Δ leaves unmoved.

The cornerstone of the subject is the Banach fixed point theorem [1], which guarantees a unique fixed point whenever Δ contracts distances. In precise terms, if $\Delta : \mathbb{G} \to \mathbb{G}$ acts on a complete metric space $(\mathbb{G}, \sigma)$ and obeys

$$\sigma(\Delta\mathring{\alpha}, \Delta\mathring{\lambda}) \leq \theta\, \sigma(\mathring{\alpha}, \mathring{\lambda}), \quad \forall\, \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}, \tag{1}$$

with $0 \leq \theta < 1$, then a unique fixed point $\mathring{\alpha}^*$ exists, and the Picard sequence $\mathring{\alpha}_{n+1} = \Delta\mathring{\alpha}_n$ started from any $\mathring{\alpha}_0 \in \mathbb{G}$ converges to it.

The theorem has since been generalized in several ways: by weakening the contraction inequality, by replacing the metric with a more general distance, and by passing from single-valued to multi-valued maps. Each generalization enlarges the class of nonlinear problems that fixed point methods can treat, which is important in applied mathematics, physics, and engineering, where the hypotheses of the contraction principle are often not satisfied. Fixed point methods therefore remain central both to existence theory and to numerical schemes.

For example, Berinde [2,3] gave approximation results for weak contractions and for Ćirić almost contractions, Boyd and Wong [4] treated nonlinear contractions, and Ćirić [5] proved a well-known extension of the contraction principle. Alraddadi et al. [6] constructed fractals from enriched $\mathcal{Z}$-contractions with applications to iterated function systems, and Reich [7] studied fixed points of contractive functions.

Another direction generalizes the notion of distance itself. Branciari [8] studied contraction mappings in generalized spaces of Banach–Caccioppoli type, Czerwik [9] weakened the triangle inequality to define b-metric spaces, and Cho and Saadat [10] proved fixed point theorems in generalized D-metric spaces. Two further generalizations of the metric were given by Jleli and Samet [11] and by Mustafa and Sims [12]. These frameworks extend fixed point theory to additional nonlinear problems in applied mathematics and computational modelling.

Nadler [13] extended contraction results to set-valued operators, and later authors continued this line. Berinde [14] studied enriched nonexpansive mappings using a retraction–displacement condition, Öztürk and Radenović [15] proved Banach-type theorems in hemi-metric spaces, and Petrov [16] obtained a fixed point theorem for maps that shrink the perimeters of triangles.

Berinde and Păcurar [17] introduced enriched contractions as a generalization of the Banach contraction. This class is wider than the standard contractive one, and it also includes certain nonexpansive operators that the classical definition excludes. They showed that an enriched contraction on a Banach space has a unique fixed point, and that this fixed point can be computed by a Krasnoselskii-type iteration.

Let $(\mathbb{G}, \|\cdot\|)$ be a normed space and $\Delta : \mathbb{G} \to \mathbb{G}$ an operator. Then Δ is an enriched contraction if there exist constants $\mathfrak{c} \geq 0$ and $\theta \in [0, \mathfrak{c}+1)$ such that

$$\left\|\mathfrak{c}(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \Delta\mathring{\lambda}\right\| \leq \theta\, \|\mathring{\alpha} - \mathring{\lambda}\|, \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

This inequality covers both ordinary contractions and some nonexpansive maps, while still forcing the fixed point to be unique. For this reason it applies in many different settings.

We will use the averaging construction throughout. Let $\mathcal{K}$ be a nonempty convex subset of $\mathbb{G}$, let $\omega \in (0,1]$, and let $\Delta : \mathcal{K} \to \mathcal{K}$. The averaged mapping $\Delta_\omega : \mathcal{K} \to \mathcal{K}$ is defined by

$$\Delta_\omega(\mathring{\alpha}) = (1-\omega)\mathring{\alpha} + \omega\Delta(\mathring{\alpha}).$$

The maps Δ and Δ_ω have the same set of fixed points, so iterating Δ_ω instead of Δ does not change the limit. This is what makes averaging useful in convergence proofs and in the numerical treatment of fixed point problems.

Banach's theorem has one structural limitation. Any map satisfying Equation (1) is continuous on $(\mathbb{G}, \sigma)$, so the theorem cannot be applied to discontinuous operators. The first fixed point result that removed the continuity requirement is due to Kannan [18], who considered maps $\Delta : \mathbb{G} \to \mathbb{G}$ satisfying

$$\sigma(\Delta\mathring{\alpha}, \Delta\mathring{\lambda}) \leq \theta\big[\sigma(\mathring{\alpha}, \Delta\mathring{\alpha}) + \sigma(\mathring{\lambda}, \Delta\mathring{\lambda})\big], \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G},$$

with $\theta \in [0, \frac{1}{2})$. After Kannan, many fixed point theorems were proved without a continuity assumption; see the literature [7,19] and the references there.

Berinde and Păcurar [20] later introduced a larger class, the almost enriched contractions (also called weak enriched contractions), which contains Kannan-type operators and several other generalizations. In the same paper they treated fixed point problems and variational inequality problems in Banach spaces using Krasnoselskii-type schemes designed for these operators. The definition is as follows.

Definition 1 ([20])**.** *An operator $\Delta : \mathbb{G} \to \mathbb{G}$ is called an almost enriched contraction on a normed space $(\mathbb{G}, \|\cdot\|)$ if there exist constants $\mathfrak{c} \geq 0$, $\theta \in (0,1)$, and $\mathfrak{L} \geq 0$ such that*

$$\|\mathfrak{c}(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \Delta\mathring{\lambda}\| \leq \theta\|\mathring{\alpha} - \mathring{\lambda}\| + \mathfrak{L}\,\|\mathfrak{c}(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \mathring{\lambda}\|, \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

Theorem 1. *Let $(\mathbb{G}, \|\cdot\|)$ be a Banach space and let $\Delta : \mathbb{G} \to \mathbb{G}$ be a $(\mathfrak{c}, \theta, \mathfrak{L})$-almost enriched contraction. Then the following hold:*

1. *The set of fixed points of Δ is nonempty, i.e.,* $\mathrm{Fix}(\Delta) \neq \emptyset$.
2. *For any initial point $\mathring{\alpha}_0 \in \mathbb{G}$ and $\omega \in (0,1)$, the Krasnoselskii iteration*

$$\mathring{\alpha}_{n+1} = (1-\omega)\mathring{\alpha}_n + \omega\Delta\mathring{\alpha}_n, \quad n \geq 0,$$

converges to some $\mathring{\alpha}^* \in$ $\mathrm{Fix}(\Delta)$.

An almost enriched contraction need not have a unique fixed point; it may have several (Example 3 [20]). More on the definitions and properties of enriched contractions can be found in previous studies [21–23] and the references there. The class remains an active subject: weak enriched contractions have been approximated through Kirk-type iterations [24], the equivalence and convergence of Picard, Mann, and Ishikawa schemes built on higher order averaged mappings have been analyzed [25], and the notion has been carried to fuzzy Banach spaces, with applications to fractals and to market equilibrium models [26].

A different generalization was given by Jleli et al. [27], who introduced polynomial contractions. Here the continuity hypothesis is replaced by the weaker Picard continuity, which still suffices to recover the Banach fixed point theorem as a special case. They also defined a more general class of almost polynomial-type operators and proved corresponding existence and uniqueness results. Polynomial and almost polynomial contractions provide a common framework for fixed point problems in various functional settings and apply to operators that the earlier conditions do not cover.

Definition 2. *Let $(\mathbb{G}, \sigma)$ be a metric space and $\Delta : \mathbb{G} \to \mathbb{G}$ a mapping. The mapping Δ is called a polynomial contraction if there exists a constant $\theta \in (0,1)$ and a finite collection of functions $b_j : \mathbb{G} \times \mathbb{G} \to [0,\infty)$, for $j = 0, \ldots, q$, such that*

$$\sum_{j=0}^{q} b_j(\Delta\mathring{\alpha}, \Delta\mathring{\lambda})\, \sigma^j(\Delta\mathring{\alpha}, \Delta\mathring{\lambda}) \leq \theta \sum_{j=0}^{q} b_j(\mathring{\alpha}, \mathring{\lambda})\, \sigma^j(\mathring{\alpha}, \mathring{\lambda}), \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}. \tag{2}$$

Definition 3. *Let $(\mathbb{G}, \sigma)$ be a metric space with a self-mapping Δ. The mapping Δ is called an almost polynomial contraction if there exists a constant $\theta \in (0,1)$, a finite collection $\{\mathfrak{L}_j\}_{j=0}^{q} \subset (0,\infty)$, and functions $b_j : \mathbb{G} \times \mathbb{G} \to [0,\infty)$, for $j = 0, \ldots, q$, such that*

$$\begin{gathered} \sum_{j=0}^{q} b_j(\Delta\mathring{\alpha}, \Delta\mathring{\lambda})\, \sigma^j(\Delta\mathring{\alpha}, \Delta\mathring{\lambda}) \leq \theta \sum_{j=0}^{q} b_j(\mathring{\alpha}, \mathring{\lambda})\big(\sigma^j(\mathring{\alpha}, \mathring{\lambda}) + \mathfrak{L}_j\, \sigma^j(\mathring{\lambda}, \Delta\mathring{\alpha})\big), \\ \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}. \end{gathered} \tag{3}$$

The present paper builds on two of these developments: the enriched contractions of Berinde and Păcurar [17,20] and the polynomial contractions of Jleli et al. [27]. The first extends the classical contraction principles to enriched mappings; the second generalizes Banach's theorem using polynomial conditions and a weaker continuity requirement. We combine the two. We introduce enriched polynomial contractions and almost (or weak) enriched polynomial contractions and study them in a single framework that contains both as special cases. Combining the enriched and polynomial structures enlarges the class of maps that can be treated in one theory. As an application, we apply the framework to a Caputo fractional differential equation, prove existence and uniqueness of a solution under standard hypotheses, and test the constructive part of the argument numerically, including a neural-network solver built on the fixed-point operator.

2. Main results

We first define the enriched polynomial contractions.

Definition 4. *Consider* $(\mathbb{G}, \|\cdot\|)$ *as a normed space and a mapping* $\Delta : \mathbb{G} \to \mathbb{G}$*. Then* Δ *is called an enriched polynomial mapping if there exist* $\mathfrak{c} \geq 0$*,* $\theta \in [0,1)$ *and a set of mappings* $b_j : \mathbb{G} \times \mathbb{G} \to [0, \infty)$ *for* $j = 0, \cdots, q$*, such that*

$$\sum_{j=0}^{q} b_j \left(\frac{\mathfrak{c}\mathring{\alpha} + \Delta\mathring{\alpha}}{\mathfrak{c}+1}, \frac{\mathfrak{c}\mathring{\lambda} + \Delta\mathring{\lambda}}{\mathfrak{c}+1} \right) \|\mathfrak{c}(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \Delta\mathring{\lambda}\|^j \leq \theta(1+\mathfrak{c})^q \sum_{j=0}^{q} b_j(\mathring{\alpha}, \mathring{\lambda}) \|\mathring{\alpha} - \mathring{\lambda}\|^j, \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}. \tag{4}$$

To specify the constants in inequality (4), we refer to the mapping Δ as a $(\mathfrak{c}, \theta, \{b_j\}_{j=1}^{q})$-enriched polynomial contraction.

Remark 1. *Every polynomial contraction* Δ *in the sense of the study by Jleli et al. [27], with contraction constant* c *and associated sequence of functions* $\{b_j\}_{j=1}^{q}$*, can be viewed as a* $(0, c, \{b_j\}_{j=1}^{q})$*-enriched polynomial contraction. More precisely,* Δ *satisfies inequality* (4) *with* $\mathfrak{c} = 0$ *and* $\theta = c \in [0,1)$*, together with the same sequence of functions* $\{b_j\}_{j=1}^{q}$*. The verification is a direct substitution, and the evaluation points match on both sides. For* $\mathfrak{c} = 0$ *the averaged arguments in inequality* (4) *reduce to*

$$\frac{\mathfrak{c}\mathring{\alpha} + \Delta\mathring{\alpha}}{\mathfrak{c}+1} = \Delta\mathring{\alpha}, \qquad \frac{\mathfrak{c}\mathring{\lambda} + \Delta\mathring{\lambda}}{\mathfrak{c}+1} = \Delta\mathring{\lambda},$$

and the factor $(1+\mathfrak{c})^q$ *equals* 1*, so inequality* (4) *becomes*

$$\sum_{j=0}^{q} b_j(\Delta\mathring{\alpha}, \Delta\mathring{\lambda}) \, \|\Delta\mathring{\alpha} - \Delta\mathring{\lambda}\|^j \leq \theta \sum_{j=0}^{q} b_j(\mathring{\alpha}, \mathring{\lambda}) \, \|\mathring{\alpha} - \mathring{\lambda}\|^j,$$

which is the defining inequality of the study by Jleli et al. [27] for the metric induced by the norm: on the left the functions b_j *are evaluated at* $(\Delta\mathring{\alpha}, \Delta\mathring{\lambda})$ *and on the right at* $(\mathring{\alpha}, \mathring{\lambda})$*, exactly as there. For* $\mathfrak{c} > 0$ *the left-hand evaluation points move from* $(\Delta\mathring{\alpha}, \Delta\mathring{\lambda})$ *to* $(\Delta_\omega\mathring{\alpha}, \Delta_\omega\mathring{\lambda})$ *with* $\omega = \frac{1}{1+\mathfrak{c}}$*; this is deliberate, since the proofs below iterate* Δ_ω *rather than* Δ*, and it is what makes the reduction* (6) *in the proof of Theorem 2 an identity rather than an estimate. The inclusion of the enriched contractions of Berinde and Păcurar, where no evaluation points occur because the coefficients are constant, is recorded separately in Remark 3.*

In the rest of the section, we study the fixed points of these operators. We first consider the case in which Δ_ω is continuous and determine the existence and behaviour of the fixed points of Δ.

Theorem 2. *Consider* $(\mathbb{G}, \|\cdot\|)$*, a Banach space with a self mapping* Δ *which is an enriched polynomial contraction, together with the following additional conditions for* $\omega = \frac{1}{1+\mathfrak{c}}$*:*

1. Δ_ω *is continuous;*

2. *there exist an index $j \in \{1, \cdots, q\}$ and a constant $B_j > 0$ for which*

$$b_j(\mathring{\alpha}, \mathring{\lambda}) \geq B_j, \quad \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

Then,

1. *Δ possesses a unique fixed point $\mathring{\alpha}^* \in \mathbb{G}$;*
2. *for every $\mathring{\alpha}_0 \in \mathbb{G}$, the iterative scheme*

$$\mathring{\alpha}_{n+1} = (1-\omega)\mathring{\alpha}_n + \omega\Delta\mathring{\alpha}_n = \Delta_\omega\mathring{\alpha}_n, \qquad n \geq 0, \tag{5}$$

converges to the unique fixed point $\mathring{\alpha}^*$.

Proof. Using the inequality (4), for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$, we obtain

$$\sum_{j=0}^{q} b_j\Big((1-\omega)\mathring{\alpha} + \omega\Delta\mathring{\alpha},\, (1-\omega)\mathring{\lambda} + \omega\Delta\mathring{\lambda}\Big) \left\|\left(\tfrac{1}{\omega} - 1\right)(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \Delta\mathring{\lambda}\right\|^j$$
$$\leq \frac{\theta}{\omega^q} \sum_{j=0}^{q} b_j(\mathring{\alpha}, \mathring{\lambda}) \|\mathring{\alpha} - \mathring{\lambda}\|^j,$$

or, equivalently,

$$\sum_{j=0}^{q} b_j(\Delta_\omega\mathring{\alpha}, \Delta_\omega\mathring{\lambda}) \|\Delta_\omega\mathring{\alpha} - \Delta_\omega\mathring{\lambda}\|^j \leq \theta \sum_{j=0}^{q} b_j(\mathring{\alpha}, \mathring{\lambda}) \|\mathring{\alpha} - \mathring{\lambda}\|^j. \tag{6}$$

We first argue that Δ_ω admits a fixed point. Let $\mathring{\alpha}_0 \in \mathbb{G}$ be the initial guess for the sequence in sequence (5) and set $(\mathring{\alpha}, \mathring{\lambda}) = (\mathring{\alpha}_0, \mathring{\alpha}_1)$ in inequality (6) to obtain

$$\sum_{j=0}^{q} b_j\left(\Delta_\omega\mathring{\alpha}_0, \Delta_\omega\mathring{\alpha}_1\right) \|\Delta_\omega\mathring{\alpha}_0 - \Delta_\omega\mathring{\alpha}_1\|^j \leq \theta \sum_{j=0}^{q} b_j\left(\mathring{\alpha}_0, \mathring{\alpha}_1\right) \|\mathring{\alpha}_0 - \mathring{\alpha}_1\|^j.$$

That is,

$$\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_1, \mathring{\alpha}_2\right) \|\mathring{\alpha}_1 - \mathring{\alpha}_2\|^j \leq \theta \sum_{j=0}^{q} b_j\left(\mathring{\alpha}_0, \mathring{\alpha}_1\right) \|\mathring{\alpha}_0 - \mathring{\alpha}_1\|^j. \tag{7}$$

Again applying inequality (6) with $(\mathring{\alpha}, \mathring{\lambda}) = (\mathring{\alpha}_1, \mathring{\alpha}_2)$, we have

$$\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_2, \mathring{\alpha}_3\right) \|\mathring{\alpha}_2 - \mathring{\alpha}_3\|^j \leq \theta \sum_{j=0}^{q} b_j\left(\mathring{\alpha}_1, \mathring{\alpha}_2\right) \|\mathring{\alpha}_1 - \mathring{\alpha}_2\|^j. \tag{8}$$

Combining inequality (7) with inequality (8) gives

$$\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_2, \mathring{\alpha}_3\right) \|\mathring{\alpha}_2 - \mathring{\alpha}_3\|^j \leq \theta^2 \sum_{j=0}^{q} b_j\left(\mathring{\alpha}_0, \mathring{\alpha}_1\right) \|\mathring{\alpha}_0 - \mathring{\alpha}_1\|^j.$$

Proceeding by induction, we obtain

$$\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_n, \mathring{\alpha}_{n+1}\right) \|\mathring{\alpha}_n - \mathring{\alpha}_{n+1}\|^j \leq \theta^n \sum_{j=0}^{q} b_j\left(\mathring{\alpha}_0, \mathring{\alpha}_1\right) \|\mathring{\alpha}_0 - \mathring{\alpha}_1\|^j, \quad n \geq 0. \tag{9}$$

Fix an index $j \in \{1, \cdots, q\}$ and a constant $B_j > 0$ as in condition (2). To keep this index separate from the summation variable, we write the latter as i from now on, and we abbreviate

$$S_0 = \sum_{i=0}^{q} b_i\left(\mathring{\alpha}_0, \mathring{\alpha}_1\right) \|\mathring{\alpha}_0 - \mathring{\alpha}_1\|^i$$

for the sum appearing on the right of inequality (9). If $\mathring{\alpha}_1 = \mathring{\alpha}_0$, then $\mathring{\alpha}_0$ is already a fixed point of Δ_ω, the sequence (5) is constant, and there is nothing to prove about its convergence; assume therefore $\mathring{\alpha}_1 \neq \mathring{\alpha}_0$, in which case $S_0 \geq B_j\|\mathring{\alpha}_0 - \mathring{\alpha}_1\|^j > 0$. Every term of the sum on the left of inequality (9) is nonnegative, so keeping only the term $i = j$ and applying condition (2) gives

$$B_j\|\mathring{\alpha}_n - \mathring{\alpha}_{n+1}\|^j \leq \theta^n S_0, \quad n \geq 0.$$

Taking j-th roots, we arrive at

$$\|\mathring{\alpha}_n - \mathring{\alpha}_{n+1}\| \leq \tau^n \varsigma_j, \quad n \geq 0, \qquad \tau := \theta^{\frac{1}{j}} \in (0, 1), \tag{10}$$

where

$$\varsigma_j = \left[B_j^{-1} S_0\right]^{\frac{1}{j}}; \tag{11}$$

the j-th root converts the mixed sum S_0, whose terms carry different powers of length, into a quantity comparable with the displacement on the left of inequality (10). Using the triangle inequality together with Equations (10) and (11), for all $n \geq 0$ and $m \geq 1$ we get

$$\begin{aligned}\|\mathring{\alpha}_n - \mathring{\alpha}_{n+m}\| &\leq \|\mathring{\alpha}_n - \mathring{\alpha}_{n+1}\| + \|\mathring{\alpha}_{n+1} - \mathring{\alpha}_{n+2}\| + \cdots + \|\mathring{\alpha}_{n+m-1} - \mathring{\alpha}_{n+m}\| \\ &\leq \varsigma_j\left(\tau^n + \tau^{n+1} + \cdots + \tau^{n+m-1}\right) = \varsigma_j\, \tau^n \frac{1 - \tau^m}{1 - \tau} \leq \varsigma_j \frac{\tau^n}{1 - \tau},\end{aligned}$$

which implies

$$\|\mathring{\alpha}_n - \mathring{\alpha}_{n+m}\| \leq \left(\frac{\varsigma_j}{1 - \tau}\right) \tau^n \to 0 \text{ as } n \to \infty.$$

Hence the sequence $\{\mathring{\alpha}_n\}$ is Cauchy. Since $\mathbb{G}$ is complete, there exists $\mathring{\alpha}^* \in \mathbb{G}$ with $\lim\limits_{n\to\infty} \|\mathring{\alpha}_n - \mathring{\alpha}^*\| = 0$. By the continuity of Δ_ω,

$$\lim_{n\to\infty} \|\mathring{\alpha}_{n+1} - \Delta_\omega \mathring{\alpha}^*\| = \lim_{n\to\infty} \|\Delta_\omega \mathring{\alpha}_n - \Delta_\omega \mathring{\alpha}^*\| = 0,$$

so that $\Delta_\omega \mathring{\alpha}^* = \mathring{\alpha}^*$; that is, $\mathring{\alpha}^*$ is a fixed point of Δ_ω. To establish uniqueness, suppose $\mathring{\lambda}^* \in \mathbb{G}$ is another fixed point of Δ_ω, with $\|\mathring{\alpha}^* - \mathring{\lambda}^*\| > 0$. Applying inequality (6) with $(\mathring{\alpha}, \mathring{\lambda}) = \left(\mathring{\alpha}^*, \mathring{\lambda}^*\right)$ gives

$$\sum_{i=0}^{q} b_i\left(\mathring{\alpha}^*, \mathring{\lambda}^*\right) \|\mathring{\alpha}^* - \mathring{\lambda}^*\|^i \leq \theta \sum_{i=0}^{q} b_i\left(\mathring{\alpha}^*, \mathring{\lambda}^*\right) \|\mathring{\alpha}^* - \mathring{\lambda}^*\|^i. \tag{12}$$

By condition (2), the common sum in inequality (12) satisfies $\sum\limits_{i=0}^{q} b_i\left(\mathring{\alpha}^*, \mathring{\lambda}^*\right) \|\mathring{\alpha}^* - \mathring{\lambda}^*\|^i \geq B_j\|\mathring{\alpha}^* - \mathring{\lambda}^*\|^j > 0$. Dividing inequality (12) by this positive quantity gives

$1 \leq \theta$, which contradicts $\theta \in [0,1)$. Therefore $\mathring{\alpha}^*$ is the unique fixed point of Δ_ω and, consequently, the unique fixed point of Δ. $\square$

Remark 2. *A comment on condition (2) of Theorem 2 is in order. The lower bound $b_j \geq B_j > 0$ is a technical hypothesis needed for the proof; it is not part of Definition 4, and we have kept it out of the definition on purpose. It enters the argument at exactly two points: it converts the summed estimate* (9) *into the displacement bound* (10)*, and it makes the quantity dividing* (12) *positive in the uniqueness step. Some nondegeneracy of this kind cannot be dispensed with entirely: if all the functions b_j vanish identically, then inequality* (4) *holds for every self mapping of $\mathbb{G}$, including fixed-point-free translations, so the bare inequality* (4) *carries no fixed point information. Definition 4 thus delimits the class of operators in the generality of the study by Jleli et al. [27], while Theorem 2 applies to the subclass singled out by condition (2).*

Several special cases of Theorem 2 are worth spelling out.

Proposition 1. *Consider a normed space $\mathbb{G}$ and a self mapping Δ which is an enriched polynomial contraction as in Definition 4. Assume in addition that:*

1. $b_0 \equiv 0$*, that is,* $b_0(\mathring{\alpha}, \mathring{\lambda}) = 0$ *for all* $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$*;*
2. *for all $j \in \{1, \cdots, q\}$ there exists $C_j > 0$ with $b_j(\mathring{\alpha}, \mathring{\lambda}) \leq C_j$ for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$;*
3. *there exist $j \in \{1, \cdots, q\}$ and $B_j > 0$ with $b_j(\mathring{\alpha}, \mathring{\lambda}) \geq B_j$ for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$.*

Then Δ is continuous on $\mathbb{G}$.

Proof. Let $\{\mathring{\alpha}_n\} \subset \mathbb{G}$ satisfy

$$\lim_{n\to\infty} \|\mathring{\alpha}_n - \mathring{\alpha}\| = 0 \tag{13}$$

for some $\mathring{\alpha} \in \mathbb{G}$. Using condition 1 of the Proposition 1 and inequality (6) with $(\mathring{\alpha}, \mathring{\lambda}) = (\mathring{\alpha}_n, \mathring{\alpha})$, we obtain

$$\sum_{i=1}^{q} b_i\left(\Delta_\omega \mathring{\alpha}_n, \Delta_\omega \mathring{\alpha}\right) \|\Delta_\omega \mathring{\alpha}_n - \Delta_\omega \mathring{\alpha}\|^i \leq \theta \sum_{i=1}^{q} b_i\left(\mathring{\alpha}_n, \mathring{\alpha}\right) \|\mathring{\alpha}_n - \mathring{\alpha}\|^i, \quad n \geq 0,$$

which, by conditions 2 and 3 of the Proposition 1, applied with the index j provided by condition 3, implies

$$B_j \|\Delta_\omega \mathring{\alpha}_n - \Delta_\omega \mathring{\alpha}\|^j \leq \theta \sum_{i=1}^{q} C_i \|\mathring{\alpha}_n - \mathring{\alpha}\|^i, \quad n \geq 0. \tag{14}$$

Letting $n \to \infty$ in inequality (14) and invoking Equation (13) gives $\lim\limits_{n\to\infty} \|\Delta_\omega \mathring{\alpha}_n - \Delta_\omega \mathring{\alpha}\| = 0$. So Δ_ω is continuous, and since Δ is recovered from the continuous Δ_ω by scaling and translation, Δ is continuous as well. $\square$

The following result follows from Theorem 2 and Proposition 1.

Corollary 1. *Let $(\mathbb{G}, \|\cdot\|)$ be a Banach space and $\Delta : \mathbb{G} \to \mathbb{G}$ an enriched polynomial contraction. Suppose that:*

1. $b_0 \equiv 0$;
2. *for all $j \in \{1, \cdots, q\}$ there exists $C_j > 0$ with $b_j(\mathring{\alpha}, \mathring{\lambda}) \leq C_j$ for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$;*
3. *there exist $j \in \{1, \cdots, q\}$ and $B_j > 0$ with $b_j(\mathring{\alpha}, \mathring{\lambda}) \geq B_j$ for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$.*

Then Δ possesses a unique fixed point $\mathring{\alpha}^* \in \mathbb{G}$, and for every $\mathring{\alpha}_0 \in \mathbb{G}$ the sequence $\{\mathring{\alpha}_n\}$ defined by $\mathring{\alpha}_{n+1} = (1-\omega)\mathring{\alpha}_n + \omega\Delta\mathring{\alpha}_n$, $n \geq 0$, converges to $\mathring{\alpha}^*$.

The next result follows directly from Corollary 1.

Corollary 2. *Let $\mathbb{G}$ be a Banach space and Δ a self mapping. Suppose there exist $\mathfrak{c} \geq 0$, $\theta \in (0,1)$ and a finite sequence $\{b_j\}_{j=1}^q \subset (0,\infty)$ such that*

$$\sum_{j=1}^{q} b_j \|\mathfrak{c}(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \Delta\mathring{\lambda}\|^j \leq \theta(1+\mathfrak{c})^q \sum_{j=1}^{q} b_j \|\mathring{\alpha} - \mathring{\lambda}\|^j, \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}. \tag{15}$$

Then Δ possesses a unique fixed point $\mathring{\alpha}^* \in \mathbb{G}$, and for every $\mathring{\alpha}_0 \in \mathbb{G}$ the sequence $\{\mathring{\alpha}_n\}$ given by sequence (5) *converges to $\mathring{\alpha}^*$.*

Remark 3. *Corollary 2 contains the fixed point theorem of Berinde and Păcurar. Indeed, taking $q = 1$ and $b_1 = 1$, inequality* (15) *reduces to*

$$\|\mathfrak{c}(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \Delta\mathring{\lambda}\| \leq \theta(1+\mathfrak{c})\|\mathring{\alpha} - \mathring{\lambda}\|, \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

An example illustrating Theorem 2 follows.

Example 1. *Let $\mathbb{G} = \{\mathring{\alpha}_1, \mathring{\alpha}_2, \mathring{\alpha}_3, \mathring{\alpha}_4\} \subset \mathbb{R}$ and $\Delta : \mathbb{G} \to \mathbb{R}$ be defined by*

$$\Delta\mathring{\alpha}_1 = \mathring{\alpha}_1, \quad \Delta\mathring{\alpha}_2 = 3\mathring{\alpha}_3 - 2\mathring{\alpha}_2, \quad \Delta\mathring{\alpha}_3 = 3\mathring{\alpha}_4 - 2\mathring{\alpha}_3, \quad \Delta\mathring{\alpha}_4 = 3\mathring{\alpha}_1 - 2\mathring{\alpha}_4.$$

Equip $\mathbb{G}$ with the norm $\|\mathring{\alpha}_m - \mathring{\alpha}_n\| = 1$ if $m \neq n$ and 0 if $m = n$, and define $b_0 : \mathbb{G} \times \mathbb{G} \to [0,\infty)$ by

$$\begin{aligned}
&b_0(\mathring{\alpha}_m, \mathring{\alpha}_n) = b_0(\mathring{\alpha}_n, \mathring{\alpha}_m), \quad b_0(\mathring{\alpha}_m, \mathring{\alpha}_m) = 0, \\
&b_0(\mathring{\alpha}_1, \mathring{\alpha}_2) = b_0(\mathring{\alpha}_2, \mathring{\alpha}_3) = 3, \quad b_0(\mathring{\alpha}_1, \mathring{\alpha}_3) = b_0(\mathring{\alpha}_3, \mathring{\alpha}_4) = 2, \\
&b_0(\mathring{\alpha}_1, \mathring{\alpha}_4) = 1, \quad b_0(\mathring{\alpha}_2, \mathring{\alpha}_4) = 6.
\end{aligned}$$

The fixed point theorem for polynomial contractions of Jleli et al. [27] cannot be applied here, because $b_0(\Delta\mathring{\alpha}, \Delta\mathring{\lambda})$ is not defined: when $\mathring{\alpha}$ and $\mathring{\lambda}$ differ from $\mathring{\alpha}_1$, the images $\Delta\mathring{\alpha}$ and $\Delta\mathring{\lambda}$ need not belong to $\mathbb{G}$. Choosing $\mathfrak{c} = 2$, we obtain $\omega = \frac{1}{3}$ and the operator

$$\Delta_{\frac{1}{3}}\mathring{\alpha}_1 = \mathring{\alpha}_1, \quad \Delta_{\frac{1}{3}}\mathring{\alpha}_2 = \mathring{\alpha}_3, \quad \Delta_{\frac{1}{3}}\mathring{\alpha}_3 = \mathring{\alpha}_4, \quad \Delta_{\frac{1}{3}}\mathring{\alpha}_4 = \mathring{\alpha}_1.$$

We claim that

$$b_0(\Delta_{\frac{1}{3}}\mathring{\alpha}, \Delta_{\frac{1}{3}}\mathring{\lambda}) + \|\Delta_{\frac{1}{3}}\mathring{\alpha} - \Delta_{\frac{1}{3}}\mathring{\lambda}\| \leq \tfrac{3}{4}\left(b_0(\mathring{\alpha}, \mathring{\lambda}) + \|\mathring{\alpha} - \mathring{\lambda}\|\right), \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}, \tag{16}$$

so that Δ is an enriched polynomial contraction in the sense of Definition 4 with $q = 1$, $b_1 \equiv 1$, and $\theta = \frac{3}{4}$. If $\mathring{\alpha} = \mathring{\lambda}$ or $(\mathring{\alpha}, \mathring{\lambda}) = (\mathring{\alpha}_1, \mathring{\alpha}_4)$, then inequality (16) *is immediate. By symmetry it remains to verify inequality* (16) *for $\mathring{\alpha}_m, \mathring{\alpha}_n$ with $1 \leq m < n \leq 4$ and $(m, n) \neq (1, 4)$. **Table 1** lists the relevant values and confirms inequality* (16)*. Hence all conditions of Theorem 2 hold (condition 2 with $B_1 = 1$), and Δ has the unique fixed point $\mathring{\alpha}^* = \mathring{\alpha}_1$, as predicted.*

Table 1. Values of $b_0\left(\Delta_{\frac{1}{3}}\mathring{\alpha}_m, \Delta_{\frac{1}{3}}\mathring{\alpha}_n\right) + \|\Delta_{\frac{1}{3}}\mathring{\alpha}_m - \Delta_{\frac{1}{3}}\mathring{\alpha}_n\|$ and $b_0\left(\mathring{\alpha}_m, \mathring{\alpha}_n\right) + \|\mathring{\alpha}_m - \mathring{\alpha}_n\|$.

(m, n)	$b_0\left(\Delta_{\frac{1}{3}}\mathring{\alpha}_m, \Delta_{\frac{1}{3}}\mathring{\alpha}_n\right) + \|\Delta_{\frac{1}{3}}\mathring{\alpha}_m - \Delta_{\frac{1}{3}}\mathring{\alpha}_n\|$	$b_0\left(\mathring{\alpha}_m, \mathring{\alpha}_n\right) + \|\mathring{\alpha}_m - \mathring{\alpha}_n\|$
$(1, 2)$	3	4
$(1, 3)$	2	3
$(2, 3)$	3	4
$(2, 4)$	3	7
$(3, 4)$	2	3

Berinde and Păcurar's theorem does not apply either, since $\frac{\|\Delta_{\frac{1}{3}}\mathring{\alpha}_1 - \Delta_{\frac{1}{3}}\mathring{\alpha}_2\|}{\|\mathring{\alpha}_1 - \mathring{\alpha}_2\|} = \frac{\|\mathring{\alpha}_1 - \mathring{\alpha}_3\|}{\|\mathring{\alpha}_1 - \mathring{\alpha}_2\|} = 1 > \theta$. Moreover, with $\mathcal{D}(\mathring{\alpha}, \mathring{\lambda}) = \|\mathring{\alpha} - \mathring{\lambda}\| + b_0(\mathring{\alpha}, \mathring{\lambda})$, inequality (16) *reads $\mathcal{D}(\Delta_{\frac{1}{3}}\mathring{\alpha}, \Delta_{\frac{1}{3}}\mathring{\lambda}) \leq \frac{3}{4}\mathcal{D}(\mathring{\alpha}, \mathring{\lambda})$ for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$. Nevertheless $\mathcal{D}$ is not a metric on $\mathbb{G}$, as **Table 1** shows: $\mathcal{D}\left(\mathring{\alpha}_2, \mathring{\alpha}_4\right) = 7 > 6 = \mathcal{D}\left(\mathring{\alpha}_2, \mathring{\alpha}_1\right) + \mathcal{D}\left(\mathring{\alpha}_1, \mathring{\alpha}_4\right)$.*

We now weaken the continuity demand placed on Δ_ω in Theorem 2.

Definition 5. *Let $\mathbb{G}$ be a normed space. A self mapping Δ on $\mathbb{G}$ is called a Picard continuous operator if, for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$,*

$$\lim_{n\to\infty} \|\Delta^n\mathring{\alpha} - \mathring{\lambda}\| = 0 \implies \lim_{n\to\infty} \|\Delta\left(\Delta^n\mathring{\alpha}\right) - \Delta\mathring{\lambda}\| = 0,$$

where $\Delta^0\mathring{\alpha} = \mathring{\alpha}$ and $\Delta^{n+1}\mathring{\alpha} = \Delta\left(\Delta^n\mathring{\alpha}\right)$, $n \geq 0$.

A continuous map $\Delta : \mathbb{G} \to \mathbb{G}$ is always Picard continuous; the reverse implication is false, as the next example demonstrates.

Example 2. *Let $\mathbb{G} = [m, n]$ with $m < n$, and define $\Delta : \mathbb{G} \to \mathbb{G}$ by*

$$\Delta\mathring{\alpha} = \begin{cases} m & \text{if } m \leq \mathring{\alpha} < n, \\ \frac{m+n}{2} & \text{if } \mathring{\alpha} = n, \end{cases}$$

with the usual norm $\|\mathring{\alpha} - \mathring{\lambda}\| = |\mathring{\alpha} - \mathring{\lambda}|$. Clearly Δ is not continuous at n. Nonetheless it is Picard continuous in the sense of Definition 5: for every $\mathring{\alpha} \in \mathbb{G}$ we have $\Delta^k\mathring{\alpha} = m$ for all $k \geq 2$, so if $\lim_{k\to\infty} \|\Delta^k\mathring{\alpha} - \mathring{\lambda}\| = 0$ then $\mathring{\lambda} = m$ and $\lim_{k\to\infty} \|\Delta\left(\Delta^k\mathring{\alpha}\right) - \Delta\mathring{\lambda}\| = \|\Delta m - \Delta m\| = 0$.

Theorem 3. *Let $\mathbb{G}$ be a Banach space and Δ an enriched polynomial contraction on it. Assume that:*

1. *Δ_ω is Picard continuous, where $\omega = \frac{1}{1+\mathfrak{c}}$;*
2. *there exist $j \in \{1, \cdots, q\}$ and $B_j > 0$ with $b_j(\mathring{\alpha}, \mathring{\lambda}) \geq B_j$ for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$.*

Then Δ possesses exactly one fixed point $\mathring{\alpha}^*$, which is an element of $\mathbb{G}$, and for any $\mathring{\alpha}_0 \in \mathbb{G}$ the iterative sequence $\{\mathring{\alpha}_n\}$ defined in sequence (5) *converges to $\mathring{\alpha}^*$.*

Proof. We first show that the fixed point set of Δ_ω is nonempty. Fix $\mathring{\alpha}_0 \in \mathbb{G}$ and let $\mathring{\alpha}_n = \Delta_\omega^n \mathring{\alpha}_0, n \geq 0$. As in the proof of Theorem 2, $\{\mathring{\alpha}_n\}$ is Cauchy, so by completeness there exists $\mathring{\alpha}^* \in \mathbb{G}$ with $\lim\limits_{n\to\infty} \|\Delta_\omega^n \mathring{\alpha}_0 - \mathring{\alpha}^*\| = 0$. By the Picard continuity of Δ_ω,

$$\lim_{n\to\infty} \|\Delta_\omega^{n+1}\mathring{\alpha}_0 - \Delta_\omega \mathring{\alpha}^*\| = 0,$$

and since $\{\Delta_\omega^{n+1}\mathring{\alpha}_0\}_{n\geq 0}$ is the sequence $\{\mathring{\alpha}_n\}$ shifted by one index, it also converges to $\mathring{\alpha}^*$; limits being unique, $\Delta_\omega \mathring{\alpha}^* = \mathring{\alpha}^*$. So $\mathring{\alpha}^*$ is a fixed point of Δ_ω, and therefore of Δ; the remaining claims go through exactly as in Theorem 2. ☐

Remark 4. *Two observations clarify what Picard continuity does and does not deliver here. First, the hypothesis is invoked along the iterative sequence only: once $\{\mathring{\alpha}_n\}$ is known to converge to $\mathring{\alpha}^*$, Definition 5 transfers this convergence to the image sequence, and the shift argument in the proof does the rest. No behaviour of Δ_ω away from the orbit of $\mathring{\alpha}_0$ is used. This is precisely why the hypothesis is weaker than continuity in a useful way: it can hold for discontinuous operators, such as the one in Example 3 below, to which Theorem 2 does not apply. Second, Picard continuity need not be verified from the definition in practice. Proposition 2 below shows that an inequality involving only norms and the defining constants, namely inequality* (21)*, already implies it, and for operators that are continuous, such as the integral operator of Section 4, it holds automatically, so nothing has to be checked in either situation.*

Theorem 3 is illustrated by the next example.

Example 3. *Let $\mathbb{G} = \mathbb{R}$ with the usual norm and $\Delta : \mathbb{G} \to \mathbb{G}$ given by*

$$\Delta\mathring{\alpha} = \begin{cases} 1 - 3\mathring{\alpha} & \text{if } \mathring{\alpha} \neq 1, \\ -3\mathring{\alpha} & \text{if } \mathring{\alpha} = 1. \end{cases}$$

Setting $\mathfrak{c} = 3$ gives $\omega = \frac{1}{4}$ and the self operator

$$\Delta_{\frac{1}{4}}\mathring{\alpha} = \begin{cases} \frac{1}{4} & \text{if } \mathring{\alpha} \neq 1, \\ 0 & \text{if } \mathring{\alpha} = 1. \end{cases}$$

Although $\Delta_{\frac{1}{4}}$ is not continuous, it is Picard continuous (see Example 2). Define $b_0 : \mathbb{G} \times \mathbb{G} \to [0, \infty)$ by

$$b_0(\mathring{\alpha}, \mathring{\lambda}) = \tfrac{5}{6}\left(\mathring{\alpha}\left|\mathring{\alpha} - \tfrac{1}{4}\right| + \mathring{\lambda}\left|\mathring{\lambda} - \tfrac{1}{4}\right|\right), \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

We claim that

$$b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| \leq \tfrac{1}{2}\left(b_0(\mathring{\alpha}, \mathring{\lambda}) + \|\mathring{\alpha} - \mathring{\lambda}\|\right), \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}, \quad (17)$$

that is, Δ is an enriched polynomial contraction in the sense of Definition 4 with $q = 1$, $b_1 \equiv 1$, and $\theta = \frac{1}{2}$. Using the symmetry of b_0, we consider four cases.
Case 1: $\mathring{\alpha} \neq 1$ and $\mathring{\lambda} \neq 1$. Here we obtain $b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| = b_0\left(\frac{1}{4}, \frac{1}{4}\right) + \|\frac{1}{4} - \frac{1}{4}\| = 0$, which yields inequality (17)*.*
Case 2: $\mathring{\alpha} \neq 1$ and $\mathring{\lambda} = 1$. Then $b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| = b_0\left(\frac{1}{4}, 0\right) + \|\frac{1}{4} - 0\| = \frac{1}{4}$, while $\frac{1}{2}\left(b_0(\mathring{\alpha}, 1) + \|\mathring{\alpha} - 1\|\right) = \frac{5}{12}\mathring{\alpha}\left|\mathring{\alpha} - \frac{1}{4}\right| + \frac{5}{16} + \frac{|\mathring{\alpha}-1|}{2} \geq \frac{1}{4}$, which leads to inequality (17)*.*
Case 3: $\mathring{\alpha} = 1$ and $\mathring{\lambda} \neq 1$; by switching the roles of $\mathring{\alpha}$ and $\mathring{\lambda}$ in Case 2 we again obtain inequality (17)*.*
Case 4: $\mathring{\alpha} = \mathring{\lambda} = 1$. Here $b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| = b_0(0,0) + \|0 - 0\| = 0$, which implies inequality (17)*.*
Thus inequality (17) *holds throughout $\mathbb{G}$ and Theorem 3 applies. The map $\Delta_{\frac{1}{4}}$ has the single fixed point $\mathring{\alpha}^* = \frac{1}{4}$, which is then a fixed point of Δ as well, in line with Theorem 3. Theorem 2, by contrast, is unavailable here, since Δ is discontinuous.*

3. Almost enriched polynomial contractions

Combining the enriched contractions of Berinde and Păcurar [17, 20] with the polynomial contractions of Jleli et al. [27], we now introduce the almost enriched polynomial contractions.

Definition 6. *A self mapping Δ on a normed space $\mathbb{G}$ is called an almost enriched polynomial contraction if there exist $\mathfrak{c} \geq 0$, $\theta \in (0,1)$, a collection of constants $\{\mathfrak{L}_j\}_{j=0}^{q} \subset (0,\infty)$, and functions $b_j : \mathbb{G} \times \mathbb{G} \to [0,\infty)$ for $j = 0,\ldots,q$, such that for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$,*

$$\begin{aligned} &\sum_{j=0}^{q} b_j\left(\frac{\mathfrak{c}\mathring{\alpha} + \Delta\mathring{\alpha}}{1+\mathfrak{c}}, \frac{\mathfrak{c}\mathring{\lambda} + \Delta\mathring{\lambda}}{1+\mathfrak{c}}\right) \|\mathfrak{c}(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \Delta\mathring{\lambda}\|^j \\ &\leq \theta \sum_{j=0}^{q} b_j(\mathring{\alpha}, \mathring{\lambda}) \left[(1+\mathfrak{c})^q \|\mathring{\alpha} - \mathring{\lambda}\|^j + \mathfrak{L}_j \|(1+\mathfrak{c})(\mathring{\lambda} - \mathring{\alpha}) + \mathring{\alpha} - \Delta\mathring{\alpha}\|^j\right]. \end{aligned} \tag{18}$$

It will help to recall what is meant by a Krasnoselskii operator.

Definition 7. *A self mapping Δ on a normed space $\mathbb{G}$ is called a Krasnoselskii operator if*

1. *the set of fixed points of Δ is nonempty;*
2. *for every $\mathring{\alpha}_0 \in \mathbb{G}$, the Krasnoselskii sequence defined in sequence* (5) *converges, and its limit lies in the fixed point set of Δ.*

The section's main theorem is the following.

Theorem 4. *Let $(\mathbb{G}, \|\cdot\|)$ be a Banach space with an almost enriched polynomial contraction Δ, and assume that:*

1. *Δ_ω is Picard continuous for $\omega = \frac{1}{1+\mathfrak{c}}$;*
2. *there exist $j \in \{1,\cdots,q\}$ and $B_j > 0$ with $b_j(\mathring{\alpha}, \mathring{\lambda}) \geq B_j$ for all $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$.*

Then Δ is a Krasnoselskii operator.

Proof. Applying inequality (18), we obtain

$$\sum_{j=0}^{q} b_j\Big((1-\omega)\mathring{\alpha}+\omega\Delta\mathring{\alpha},\ (1-\omega)\mathring{\lambda}+\omega\Delta\mathring{\lambda}\Big)\left\|\left(\tfrac{1}{\omega}-1\right)(\mathring{\alpha}-\mathring{\lambda})+\Delta\mathring{\alpha}-\Delta\mathring{\lambda}\right\|^j$$
$$\leq \theta\sum_{j=0}^{q} b_j(\mathring{\alpha},\mathring{\lambda})\left[\frac{\|\mathring{\alpha}-\mathring{\lambda}\|^j}{\omega^q}+\mathfrak{L}_j\left\|\tfrac{\mathring{\lambda}-\mathring{\alpha}}{\omega}+\mathring{\alpha}-\Delta\mathring{\alpha}\right\|^j\right],$$

or equivalently,

$$\begin{gathered}\sum_{j=0}^{q} b_j(\Delta_\omega\mathring{\alpha},\Delta_\omega\mathring{\lambda})\|\Delta_\omega\mathring{\alpha}-\Delta_\omega\mathring{\lambda}\|^j\leq\\ \theta\sum_{j=0}^{q} b_j(\mathring{\alpha},\mathring{\lambda})\left[\|\mathring{\alpha}-\mathring{\lambda}\|^j+\mathfrak{L}_j\|\mathring{\lambda}-\Delta_\omega\mathring{\alpha}\|^j\right].\end{gathered} \tag{19}$$

Let $\mathring{\alpha}_0\in\mathbb{G}$ be fixed and let $\{\mathring{\alpha}_n\}\subset\mathbb{G}$ be the Krasnoselskii sequence $\mathring{\alpha}_{n+1}=(1-\omega)\mathring{\alpha}_n+\omega\Delta\mathring{\alpha}_n$, $n\geq 0$, with $\omega=\frac{1}{1+\mathfrak{c}}$, that is, the sequence (5) for which inequality (19) is available. Applying inequality (19) with $(\mathring{\alpha},\mathring{\lambda})=(\mathring{\alpha}_0,\mathring{\alpha}_1)$,

$$\sum_{j=0}^{q} b_j\left(\Delta_\omega\mathring{\alpha}_0,\Delta_\omega\mathring{\alpha}_1\right)\|\Delta_\omega\mathring{\alpha}_0-\Delta_\omega\mathring{\alpha}_1\|^j$$
$$\leq\theta\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_0,\mathring{\alpha}_1\right)\left[\|\mathring{\alpha}_0-\mathring{\alpha}_1\|^j+\mathfrak{L}_j\|\mathring{\alpha}_1-\Delta_\omega\mathring{\alpha}_0\|^j\right],$$

that is, since $\mathring{\alpha}_1=\Delta_\omega\mathring{\alpha}_0$ gives $\|\mathring{\alpha}_1-\Delta_\omega\mathring{\alpha}_0\|=0$,

$$\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_1,\mathring{\alpha}_2\right)\|\mathring{\alpha}_1-\mathring{\alpha}_2\|^j\leq\theta\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_0,\mathring{\alpha}_1\right)\|\mathring{\alpha}_0-\mathring{\alpha}_1\|^j. \tag{20}$$

Likewise, inequality (19) with $(\mathring{\alpha},\mathring{\lambda})=(\mathring{\alpha}_1,\mathring{\alpha}_2)$ gives

$$\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_2,\mathring{\alpha}_3\right)\|\mathring{\alpha}_2-\mathring{\alpha}_3\|^j\leq\theta\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_1,\mathring{\alpha}_2\right)\|\mathring{\alpha}_1-\mathring{\alpha}_2\|^j,$$

which, with inequality (20), yields

$$\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_2,\mathring{\alpha}_3\right)\|\mathring{\alpha}_2-\mathring{\alpha}_3\|^j\leq\theta^2\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_0,\mathring{\alpha}_1\right)\|\mathring{\alpha}_0-\mathring{\alpha}_1\|^j.$$

At every step of this scheme the choice $\mathring{\lambda}=\mathring{\alpha}_{n+1}=\Delta_\omega\mathring{\alpha}_n$ makes the additional terms on the right of inequality (19) vanish, because $\|\mathring{\alpha}_{n+1}-\Delta_\omega\mathring{\alpha}_n\|=0$ holds by the definition of the sequence itself, not by an extra assumption; the mechanism used for the first step above therefore repeats at each n. Proceeding by induction,

$$\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_n,\mathring{\alpha}_{n+1}\right)\|\mathring{\alpha}_n-\mathring{\alpha}_{n+1}\|^j\leq\theta^n\sum_{j=0}^{q} b_j\left(\mathring{\alpha}_0,\mathring{\alpha}_1\right)\|\mathring{\alpha}_0-\mathring{\alpha}_1\|^j,\quad n\geq 0,$$

which, by assumption 2 of Theorem 4, implies

$$\|\mathring{\alpha}_n - \mathring{\alpha}_{n+1}\| \le \tau^n \varsigma_j, \quad n \ge 0,$$

with $\tau = \theta^{\frac{1}{j}}$ and ς_j as in inequality (11), arguing exactly as in the passage from inequality (9) to inequality (10), the trivial case $\mathring{\alpha}_1 = \mathring{\alpha}_0$ being settled first as before. Arguing as in the proof of Theorem 2, $\{\mathring{\alpha}_n\}$ is Cauchy, and by completeness there exists $\mathring{\alpha}^* \in \mathbb{G}$ with $\lim_{n\to\infty} \|\mathring{\alpha}_n - \mathring{\alpha}^*\| = 0$. Since Δ_ω is Picard continuous, $\lim_{n\to\infty} \|\mathring{\alpha}_{n+1} - \Delta_\omega \mathring{\alpha}^*\| = 0$, so that $\mathring{\alpha}^* = \Delta_\omega \mathring{\alpha}^*$, limits being unique. The fixed point set of Δ_ω, hence that of Δ, is therefore nonempty, and the sequence (5) issued from an arbitrary $\mathring{\alpha}_0$ converges to a member of that set, which is what Definition 7 requires of a Krasnoselskii operator. $\square$

Remark 5. *Along the specific sequence* (5) *the additional terms in inequality* (18) *drop out, so the convergence mechanism of Theorem 4 coincides with that of Theorem 3. The difference between the two classes lies in the structure of the fixed point set, not in the iteration. Under the hypotheses of Theorem 3, the fixed point is unique, whereas an almost enriched polynomial contraction may admit several: the map of Example 4 below satisfies all hypotheses of Theorem 4 and has exactly two fixed points, so it cannot satisfy inequality* (4) *together with condition 2 of Theorem 2 for any choice of parameters. Definition 6 therefore describes a strictly larger class, and the price paid for the extra room is the loss of uniqueness. This is the same trade observed between contractions and almost contractions [3] and between enriched contractions and their weak counterparts [20], and it is the reason for keeping both definitions.*

A few concrete consequences of Theorem 4 follow.

Proposition 2. *Let* $(\mathbb{G}, \|\cdot\|)$ *be a normed space and* $\Delta : \mathbb{G} \to \mathbb{G}$ *a mapping. Suppose there exist* $\mathfrak{c} \ge 0$*,* $\theta \in (0,1)$ *and finite sequences* $\{b_j\}_{j=1}^q, \{\mathfrak{L}_j\}_{j=1}^q \subset (0,\infty)$ *such that*

$$\sum_{j=1}^q b_j \|\mathfrak{c}(\mathring{\alpha} - \mathring{\lambda}) + \Delta\mathring{\alpha} - \Delta\mathring{\lambda}\|^j$$
$$\le \theta \sum_{j=1}^q b_j \left[(1+\mathfrak{c})^q \|\mathring{\alpha} - \mathring{\lambda}\|^j + \mathfrak{L}_j \|(1+\mathfrak{c})(\mathring{\lambda} - \mathring{\alpha}) + \mathring{\alpha} - \Delta\mathring{\alpha}\|^j\right], \tag{21}$$
$$\forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

Then Δ_ω *is a Picard continuous operator for* $\omega = \frac{1}{1+\mathfrak{c}}$.

Proof. Let $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$ satisfy

$$\lim_{k\to\infty} \|\Delta_\omega{}^k \mathring{\alpha} - \mathring{\lambda}\| = 0, \tag{22}$$

where $\Delta_\omega := (1-\omega)I_d + \omega\Delta$ for $\omega = \frac{1}{1+\mathfrak{c}}$. Applying inequality (21) with $(\mathring{\alpha}, \mathring{\lambda}) =$

$\left(\Delta_\omega{}^k\mathring{\alpha},\mathring{\lambda}\right)$, we obtain

$$\sum_{j=1}^{q} b_j\|\Delta_\omega{}^{k+1}\mathring{\alpha}-\Delta_\omega\mathring{\lambda}\|^j \le \theta\sum_{j=1}^{q} b_j\left[\|\Delta_\omega^k\mathring{\alpha}-\mathring{\lambda}\|^j+\mathfrak{L}_j\|\mathring{\lambda}-\Delta_\omega^{k+1}\mathring{\alpha}\|^j\right],$$

and hence

$$\|\Delta_\omega\left(\Delta_\omega{}^k\mathring{\alpha}\right)-\Delta_\omega\mathring{\lambda}\| \le \frac{\theta}{b_1}\sum_{j=1}^{q} b_j\left[\|\Delta_\omega^k\mathring{\alpha}-\mathring{\lambda}\|^j+\mathfrak{L}_j\|\mathring{\lambda}-\Delta_\omega{}^{k+1}\mathring{\alpha}\|^j\right].$$

Letting $k\to\infty$ and using Equation (22) yields $\lim_{k\to\infty}\|\Delta_\omega\left(\Delta_\omega{}^k\mathring{\alpha}\right)-\Delta_\omega\mathring{\lambda}\|=0$. Thus Δ_ω is Picard continuous. □

Putting Theorem 4 together with Proposition 2 yields the next corollary.

Corollary 3. *Let $\Delta:\mathbb{G}\to\mathbb{G}$ be a mapping on a Banach space $(\mathbb{G},\|\cdot\|)$, and suppose there exist $\mathfrak{c}\ge 0$, $\theta\in(0,1)$, and finite sequences $\{b_j\}_{j=1}^q$ and $\{\mathfrak{L}_j\}_{j=1}^q\subset(0,\infty)$ such that inequality* (21) *holds for all $\mathring{\alpha},\mathring{\lambda}\in\mathbb{G}$. Then Δ is a Krasnoselskii operator.*

Proof. Inequality (21) is the special case of inequality (18) with $b_0\equiv 0$ and b_j constant for all $j\in\{1,\dots,q\}$. The conclusion then follows from Theorem 4 together with Proposition 2. □

Remark 6. *With $q=1$, $b_1=1$, and $\mathfrak{L}_1=\frac{\ell}{\theta}$ for some $\ell>0$, inequality* (21) *collapses to inequality* (3)*, so Corollary 3 contains Berinde's fixed point theorem [20] as a special case.*

Two examples close the section. The first shows that, even under the hypotheses of Theorem 4, a map can have more than one fixed point.

Example 4. *Let $\mathbb{G}=\{\mathring{\alpha}_1,\mathring{\alpha}_2,\mathring{\alpha}_3\}\subset\mathbb{R}$ be a normed space with $\|\mathring{\alpha}_m-\mathring{\alpha}_n\|=|m-n|$, and let $\Delta:\mathbb{G}\to\mathbb{R}$ be given by*

$$\Delta\mathring{\alpha}_1=\mathring{\alpha}_1,\quad \Delta\mathring{\alpha}_2=\mathring{\alpha}_2,\quad \Delta\mathring{\alpha}_3=2\mathring{\alpha}_1-\mathring{\alpha}_3.$$

Setting $\mathfrak{c}=1$ gives $\omega=\frac{1}{2}$ and

$$\Delta_{\frac{1}{2}}\mathring{\alpha}_1=\mathring{\alpha}_1,\quad \Delta_{\frac{1}{2}}\mathring{\alpha}_2=\mathring{\alpha}_2,\quad \Delta_{\frac{1}{2}}\mathring{\alpha}_3=\mathring{\alpha}_1.$$

We assert that Δ satisfies inequality (21) *with $q=2$, $b_1=b_2=1$, $\theta=\frac{2}{3}$, and $\mathfrak{L}_1=\mathfrak{L}_2=\frac{1}{2}$; equivalently,*

$$\begin{aligned}\|\Delta_{\frac{1}{2}}\mathring{\alpha}-\Delta_{\frac{1}{2}}\mathring{\lambda}\|+\|\Delta_{\frac{1}{2}}\mathring{\alpha}-\Delta_{\frac{1}{2}}\mathring{\lambda}\|^2 \le \tfrac{2}{3}\Big[&\left(\|\mathring{\alpha}-\mathring{\lambda}\|+\|\mathring{\alpha}-\mathring{\lambda}\|^2\right)\\ &+\tfrac{1}{2}\left(\|\mathring{\lambda}-\Delta_{\frac{1}{2}}\mathring{\alpha}\|+\|\mathring{\lambda}-\Delta_{\frac{1}{2}}\mathring{\alpha}\|^2\right)\Big]\end{aligned}\tag{23}$$

for all $\mathring{\alpha},\mathring{\lambda}\in\mathbb{G}$. If $\mathring{\alpha}=\mathring{\lambda}$ or $(\mathring{\alpha},\mathring{\lambda})\in\{(\mathring{\alpha}_1,\mathring{\alpha}_3),(\mathring{\alpha}_3,\mathring{\alpha}_1)\}$, then inequality (23) *is immediate.* ***Table 2*** *lists the two sides of inequality* (23) *for all $m,n\in\{1,2,3\}$ with*

$m \neq n$ and $(m,n) \notin \{(1,3),(3,1)\}$, confirming inequality (23)*. Corollary 3 therefore applies: the fixed point set of Δ is nonempty, and in fact equals $\{\mathring{\alpha}_1, \mathring{\alpha}_2\}$, just as the conclusion predicts.*

Table 2. The two sides of inequality (23) for the admissible pairs (m,n).

(m,n)	$\\|\Delta_{\frac{1}{2}}\mathring{\alpha} - \Delta_{\frac{1}{2}}\mathring{\lambda}\\| + \\|\Delta_{\frac{1}{2}}\mathring{\alpha} - \Delta_{\frac{1}{2}}\mathring{\lambda}\\|^2$	$\left(\\|\mathring{\alpha} - \mathring{\lambda}\\| + \\|\mathring{\alpha} - \mathring{\lambda}\\|^2\right) + \frac{1}{2}\left(\\|\mathring{\lambda} - \Delta_{\frac{1}{2}}\mathring{\alpha}\\| + \\|\mathring{\lambda} - \Delta_{\frac{1}{2}}\mathring{\alpha}\\|^2\right)$
$(1,2)$	2	3
$(2,1)$	2	3
$(2,3)$	2	3
$(3,2)$	2	3

The final example returns to Theorem 4, this time with a unique fixed point.

Example 5. *Let $\mathbb{G} = \mathbb{R}$ with the usual norm, and define $\Delta : \mathbb{G} \to \mathbb{G}$ by*

$$\Delta\mathring{\alpha} = \begin{cases} 1 - 3\mathring{\alpha} & \text{if } \mathring{\alpha} \neq 1, \\ -3\mathring{\alpha} & \text{if } \mathring{\alpha} = 1. \end{cases}$$

Setting $\mathfrak{c} = 3$ gives $\omega = \frac{1}{4}$ and the self operator

$$\Delta_{\frac{1}{4}}\mathring{\alpha} = \begin{cases} \frac{1}{4} & \text{if } \mathring{\alpha} \neq 1, \\ 0 & \text{if } \mathring{\alpha} = 1. \end{cases}$$

Recall that $\Delta_{\frac{1}{4}}$ is Picard continuous (see Example 2). Consider $b_0 : \mathbb{G} \times \mathbb{G} \to [0,\infty)$ defined by

$$b_0(\mathring{\alpha}, \mathring{\lambda}) = \left|4\mathring{\alpha}^2 - 3\mathring{\alpha} + \tfrac{1}{2}\right| + \left|4\mathring{\lambda}^2 - 3\mathring{\lambda} + \tfrac{1}{2}\right|, \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

We claim that

$$b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| \leq \frac{1}{2}\left(2b_0(\mathring{\alpha}, \mathring{\lambda}) + \|\mathring{\alpha} - \mathring{\lambda}\| + \|\mathring{\lambda} - \Delta_{\frac{1}{4}}\mathring{\alpha}\|\right), \quad \forall \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}, \tag{24}$$

so that Δ is an almost enriched polynomial contraction in the sense of Definition 6 with $q = 1$, $b_1 \equiv 1$, $\mathfrak{L}_0 = \mathfrak{L}_1 = 1$, and $\theta = \frac{1}{2}$. We examine four cases.
Case 1: $\mathring{\alpha} \neq 1$ and $\mathring{\lambda} \neq 1$. Then $b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| = b_0\left(\frac{1}{4}, \frac{1}{4}\right) + 0 = 0$, so inequality (24) *holds.*
Case 2: $\mathring{\alpha} \neq 1$ and $\mathring{\lambda} = 1$. Then

$$\begin{aligned} b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| &= b_0\left(\tfrac{1}{4}, 0\right) + \|\tfrac{1}{4} - 0\| \\ &= \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4} = \|\mathring{\lambda} - \Delta_{\frac{1}{4}}\mathring{\alpha}\| \\ &\leq \tfrac{1}{2}\left(2b_0(\mathring{\alpha}, \mathring{\lambda}) + \|\mathring{\alpha} - \mathring{\lambda}\| + \|\mathring{\lambda} - \Delta_{\frac{1}{4}}\mathring{\alpha}\|\right), \end{aligned}$$

so inequality (24) *holds.*

Case 3: $\mathring{\alpha} = 1$ *and* $\mathring{\lambda} \neq 1$. *Then* $b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| = b_0\left(0, \frac{1}{4}\right) + \frac{1}{4} = \frac{3}{4}$, *while* $b_0(1, \mathring{\lambda}) = \frac{3}{2} + \left|4\mathring{\lambda}^2 - 3\mathring{\lambda} + \frac{1}{2}\right| \geq \frac{3}{2}$. *Hence*

$$\begin{aligned} b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| = \tfrac{3}{4} \leq \tfrac{3}{2} \leq \tfrac{1}{2}\left[2b_0(\mathring{\alpha}, \mathring{\lambda})\right] \\ \leq \tfrac{1}{2}\left(2b_0(\mathring{\alpha}, \mathring{\lambda}) + \|\mathring{\alpha} - \mathring{\lambda}\| + \|\mathring{\lambda} - \Delta_{\frac{1}{4}}\mathring{\alpha}\|\right), \end{aligned}$$

so inequality (24) *holds.*
Case 4: $\mathring{\alpha} = \mathring{\lambda} = 1$. *Then* $b_0(\Delta_{\frac{1}{4}}\mathring{\alpha}, \Delta_{\frac{1}{4}}\mathring{\lambda}) + \|\Delta_{\frac{1}{4}}\mathring{\alpha} - \Delta_{\frac{1}{4}}\mathring{\lambda}\| = b_0(0,0) + 0 = 1 = \|\mathring{\lambda} - \Delta_{\frac{1}{4}}\mathring{\alpha}\| \leq \frac{1}{2}\left(2b_0(\mathring{\alpha}, \mathring{\lambda}) + \|\mathring{\alpha} - \mathring{\lambda}\| + \|\mathring{\lambda} - \Delta_{\frac{1}{4}}\mathring{\alpha}\|\right)$, *so inequality* (24) *holds.*
Inequality (24) *holds across* $\mathbb{G}$, *condition 2 of Theorem 4 is met with* $j = 1$ *and* $B_1 = 1$, *and the theorem applies. Its conclusion is borne out, since* $\mathring{\alpha}^* = \frac{1}{4}$ *is a fixed point of* Δ.

4. Application to a Caputo fractional boundary value problem

We now apply the main results to the existence and uniqueness of solutions of a boundary value problem for a Caputo fractional differential equation. Fractional differential models remain a broad and active area, with many directions still open on both the analytical and the computational side [28], and boundary value problems of Caputo type in particular continue to generate existence, uniqueness, and stability results [29]. Throughout, $\mathbb{G} = \mathfrak{C}([0,1], \mathbb{R})$ denotes the Banach space of continuous functions on $[0,1]$ with values in $\mathbb{R}$, carrying the norm

$$\|\mathring{\alpha}\| = \max_{\mathring{\zeta} \in [0,1]} |\mathring{\alpha}(\mathring{\zeta})|.$$

Definition 8 ([30,31])**.** *The Caputo fractional derivative of order* $\mathcal{N} > 0$ *of a function* μ *on* $[a,b]$ *is*

$$\begin{gathered} \left({}^c\mathfrak{D}^{\mathcal{N}}_{a+}\right)\mu(\mathring{\zeta}) = \tfrac{1}{\Gamma(k-\mathcal{N})}\int_a^{\mathring{\zeta}} (\mathring{\zeta} - \daleth)^{k-\mathcal{N}-1}\mu^{(k)}(\daleth)\, d\daleth, \\ k - 1 \leq \mathcal{N} < k,\ k = [\mathcal{N}] + 1, \end{gathered} \tag{25}$$

where Γ *is the gamma function and* $[\mathcal{N}]$ *is the integer part of* $\mathcal{N}$.

Consider the fractional boundary value problem

$$\begin{aligned} &{}^c\mathfrak{D}^{\mathcal{N}}_{0+}\mathring{\alpha}(\mathring{\zeta}) = h(\mathring{\zeta}, \mathring{\alpha}(\mathring{\zeta})), \qquad \mathring{\zeta} \in [0,1],\ 2 < \mathcal{N} \leq 3, \\ &\mathring{\alpha}(0) = c_0, \quad \mathring{\alpha}'(0) = c, \quad \mathring{\alpha}''(1) = c_1, \end{aligned} \tag{26}$$

where $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $c_0, c, c_1 \in \mathbb{R}$.

Definition 9 ([32])**.** *A function* $\mathring{\alpha} \in \mathbb{G}$ *is a solution of Equation* (26) *if its Caputo derivative of order* $\mathcal{N}$ *exists on* $[0,1]$ *and* ${}^c\mathfrak{D}^{\mathcal{N}}_{0+}\mathring{\alpha}(\mathring{\zeta}) = h(\mathring{\zeta}, \mathring{\alpha}(\mathring{\zeta}))$ *for* $\mathring{\zeta} \in [0,1]$, *together with* $\mathring{\alpha}(0) = c_0$, $\mathring{\alpha}'(0) = c$, *and* $\mathring{\alpha}''(1) = c_1$.

We record the standard representation lemma in the form we shall use.

Lemma 1 ([32]). *Let $2 < \mathcal{N} \le 3$ and let $\mu : [0,1] \to \mathbb{R}$ be continuous. A function $\mathring{\alpha} \in \mathbb{G}$ solves the linear problem*

$$ {}^{c}\mathfrak{D}_{0+}^{\mathcal{N}} \mathring{\alpha}(\mathring{\zeta}) = \mu(\mathring{\zeta}), \qquad \mathring{\alpha}(0) = c_0,\ \mathring{\alpha}'(0) = c,\ \mathring{\alpha}''(1) = c_1, \tag{27} $$

if and only if $\mathring{\alpha}$ satisfies the fractional integral equation

$$ \begin{gathered} \mathring{\alpha}(\mathring{\zeta}) = \tfrac{1}{\Gamma(\mathcal{N})} \int_0^{\mathring{\zeta}} (\mathring{\zeta} - \beth)^{\mathcal{N}-1} \mu(\beth)\, d\beth - \\ \tfrac{\mathring{\zeta}^2}{2\Gamma(\mathcal{N}-2)} \int_0^1 (1-\beth)^{\mathcal{N}-3} \mu(\beth)\, d\beth + c_0 + c\,\mathring{\zeta} + \tfrac{c_1}{2}\mathring{\zeta}^2. \end{gathered} \tag{28} $$

Proof. For $2 < \mathcal{N} \le 3$ one has $k = 3$, and the general solution of ${}^{c}\mathfrak{D}_{0+}^{\mathcal{N}} \mathring{\alpha} = \mu$ is $\mathring{\alpha}(\mathring{\zeta}) = \tfrac{1}{\Gamma(\mathcal{N})} \int_0^{\mathring{\zeta}} (\mathring{\zeta} - \beth)^{\mathcal{N}-1} \mu(\beth)\, d\beth + d_0 + d_1\mathring{\zeta} + d_2\mathring{\zeta}^2$ for constants d_0, d_1, d_2. The conditions $\mathring{\alpha}(0) = c_0$ and $\mathring{\alpha}'(0) = c$ give $d_0 = c_0$ and $d_1 = c$. Differentiating twice, $\mathring{\alpha}''(\mathring{\zeta}) = \tfrac{1}{\Gamma(\mathcal{N}-2)} \int_0^{\mathring{\zeta}} (\mathring{\zeta} - \beth)^{\mathcal{N}-3} \mu(\beth)\, d\beth + 2d_2$, so $\mathring{\alpha}''(1) = c_1$ yields $2d_2 = c_1 - \tfrac{1}{\Gamma(\mathcal{N}-2)} \int_0^1 (1-\beth)^{\mathcal{N}-3} \mu(\beth)\, d\beth$. Substituting d_2 gives Equation (28), and the converse is a direct computation. □

In view of Lemma 1, define the operator $\Delta : \mathbb{G} \to \mathbb{G}$ by

$$ \begin{aligned} \Delta\mathring{\alpha}(\mathring{\zeta}) \;=\; & \frac{1}{\Gamma(\mathcal{N})} \int_0^{\mathring{\zeta}} (\mathring{\zeta} - \beth)^{\mathcal{N}-1} h(\beth, \mathring{\alpha}(\beth))\, d\beth \\ & - \frac{\mathring{\zeta}^2}{2\Gamma(\mathcal{N}-2)} \int_0^1 (1-\beth)^{\mathcal{N}-3} h(\beth, \mathring{\alpha}(\beth))\, d\beth + c_0 + c\,\mathring{\zeta} + \frac{c_1}{2}\mathring{\zeta}^2, \end{aligned} \tag{29} $$

and for $\omega \in (0,1]$, its averaged version

$$ \begin{aligned} \Delta_\omega\mathring{\alpha}(\mathring{\zeta}) =& \frac{\omega}{\Gamma(\mathcal{N})} \int_0^{\mathring{\zeta}} (\mathring{\zeta} - \beth)^{\mathcal{N}-1} h(\beth, \mathring{\alpha}(\beth))\, d\beth \\ & - \frac{\omega\mathring{\zeta}^2}{2\Gamma(\mathcal{N}-2)} \int_0^1 (1-\beth)^{\mathcal{N}-3} h(\beth, \mathring{\alpha}(\beth))\, d\beth \\ & + \omega\Big(c_0 + c\,\mathring{\zeta} + \tfrac{c_1}{2}\mathring{\zeta}^2\Big) + (1-\omega)\mathring{\alpha}(\mathring{\zeta}). \end{aligned} $$

By Lemma 1 the solutions of Equation (26) are precisely the fixed points of Δ, and Δ shares its fixed point set with Δ_ω. The main existence-and-uniqueness result is the following.

Theorem 5. *Suppose $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $L > 0$ such that*

$$ |h(\mathring{\zeta}, \mathring{\alpha}_1) - h(\mathring{\zeta}, \mathring{\alpha}_2)| \le L\,|\mathring{\alpha}_1 - \mathring{\alpha}_2|, \qquad \forall\, \mathring{\zeta} \in [0,1],\ \mathring{\alpha}_1, \mathring{\alpha}_2 \in \mathbb{R}, \tag{30} $$

and that

$$ 0 < \theta := L\left(\frac{1}{\Gamma(\mathcal{N}+1)} + \frac{1}{2\Gamma(\mathcal{N}-1)}\right) < 1. \tag{31} $$

Then Equation (26) has a unique solution $\mathring{\alpha}^* \in \mathbb{G}$, and for every $\mathring{\alpha}_0 \in \mathbb{G}$ the Picard iteration $\mathring{\alpha}_{n+1} = \Delta\mathring{\alpha}_n$, that is, the Krasnoselskii scheme (5) with $\omega = 1$, converges uniformly to $\mathring{\alpha}^*$.

Proof. We show that Δ in Equation (29) is a Banach contraction with constant θ. For $\mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}$ and $\mathring{\zeta} \in [0,1]$, the affine terms in Equation (29) cancel, so using Equation (30),

$$\begin{aligned}
|\Delta\mathring{\alpha}(\mathring{\zeta}) - \Delta\mathring{\lambda}(\mathring{\zeta})| &\leq \frac{1}{\Gamma(\mathcal{N})}\int_0^{\mathring{\zeta}} (\mathring{\zeta} - \gimel)^{\mathcal{N}-1} |h(\gimel, \mathring{\alpha}(\gimel)) - h(\gimel, \mathring{\lambda}(\gimel))|\, d\gimel \\
&\quad + \frac{\mathring{\zeta}^2}{2\Gamma(\mathcal{N}-2)}\int_0^1 (1-\gimel)^{\mathcal{N}-3} |h(\gimel, \mathring{\alpha}(\gimel)) - h(\gimel, \mathring{\lambda}(\gimel))|\, d\gimel \\
&\leq L\, \|\mathring{\alpha} - \mathring{\lambda}\| \left[\frac{\mathring{\zeta}^{\mathcal{N}}}{\Gamma(\mathcal{N}+1)} + \frac{\mathring{\zeta}^2}{2\Gamma(\mathcal{N}-1)} \right] \\
&\leq L \left(\frac{1}{\Gamma(\mathcal{N}+1)} + \frac{1}{2\Gamma(\mathcal{N}-1)} \right) \|\mathring{\alpha} - \mathring{\lambda}\|,
\end{aligned}$$

where the third line follows from the exact evaluations

$$\int_0^{\mathring{\zeta}} (\mathring{\zeta} - \gimel)^{\mathcal{N}-1}\, d\gimel = \frac{\mathring{\zeta}^{\mathcal{N}}}{\mathcal{N}}, \qquad \int_0^1 (1-\gimel)^{\mathcal{N}-3}\, d\gimel = \frac{1}{\mathcal{N}-2},$$

the second integral being finite because $\mathcal{N} - 3 > -1$, together with the Gamma function identities $\mathcal{N}\,\Gamma(\mathcal{N}) = \Gamma(\mathcal{N}+1)$ and $(\mathcal{N}-2)\,\Gamma(\mathcal{N}-2) = \Gamma(\mathcal{N}-1)$, which absorb the factors $1/\mathcal{N}$ and $1/(\mathcal{N}-2)$ into the Gamma functions; the fourth line then uses $\mathring{\zeta} \leq 1$. Taking the supremum over $\mathring{\zeta} \in [0,1]$ gives

$$\|\Delta\mathring{\alpha} - \Delta\mathring{\lambda}\| \leq \theta\, \|\mathring{\alpha} - \mathring{\lambda}\|, \qquad \forall\, \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

Thus Δ satisfies inequality (15) with $\mathfrak{c} = 0$, $q = 1$, $b_1 = 1$, and contraction constant $\theta \in (0,1)$; that is, Δ is a $(0, \theta, \{1\})$-enriched polynomial contraction. By Corollary 2, Δ has a unique fixed point $\mathring{\alpha}^* \in \mathbb{G}$, which by Lemma 1 is the unique solution of Equation (26), and the Picard iteration $\mathring{\alpha}_{n+1} = \Delta\mathring{\alpha}_n$ converges to it. ☐

Remark 7. *Theorem 5 is phrased for Δ itself, that is, the case $\mathfrak{c} = 0$ ($\omega = 1$) of the framework. The averaged operator is covered too: for any $\omega \in (0,1]$, equivalently $\mathfrak{c} = \frac{1}{\omega} - 1 \geq 0$, combining $\Delta_\omega = (1-\omega)I + \omega\Delta$ with the contraction estimate above gives*

$$\|\Delta_\omega\mathring{\alpha} - \Delta_\omega\mathring{\lambda}\| \leq \big[(1-\omega) + \omega\,\theta\big]\, \|\mathring{\alpha} - \mathring{\lambda}\|, \qquad \forall\, \mathring{\alpha}, \mathring{\lambda} \in \mathbb{G}.$$

Since $\theta < 1$, the factor $\theta(\omega) := (1-\omega) + \omega\theta$ is strictly less than 1 for every $\omega \in (0,1]$, so the Krasnoselskii iteration (5) converges for all such ω, and $\omega = 1$ gives the smallest constant $\theta(1) = \theta$. Thus the enriched relaxation does not prevent convergence, but for this operator, whose kernel is positive and already contractive, it does not improve the rate. This is confirmed numerically in the next section. Two further points deserve to be stated plainly. First, Theorem 5 is the $\mathfrak{c} = 0$, $q = 1$ instance of the framework: its proof verifies inequality (15) and concludes through Corollary 2, so the application occupies the simplest corner of Definition 4, while the estimate above describes how the remaining members $\mathfrak{c} > 0$ of the same family act on this operator. Second, the fact that no enrichment is needed here is structural. Enrichment helps when Δ fails to contract because of a strongly negative component, as with the multiplier -3 in Examples 3 and

5, which averaging folds back toward the identity; under Equation (31) *the operator* (29) *is already a contraction, and mixing in a multiple of the identity can only slow the iteration down. For the same reason, the Picard continuity refinement of Theorems 3 and 4 is not called upon here, since Δ_ω is continuous. Locating fractional models whose solution operators do require $\mathfrak{c} > 0$ or $q > 1$ is, in our view, the natural next step for this line of work, and we formulate it as an open problem in Section 6.*

5. Numerical illustration and a neural approximation

The result of the previous section is constructive: the proof gives the solution as the limit of the Krasnoselskii sequence for Δ. We now examine three questions. How does the iteration behave in practice? How does the certified factor θ compare with the measured rate? And does a mesh-free neural solver produce the same solution? We study all three on a problem with a known exact solution, so that the errors reported below are exact.

5.1. A test problem with a known solution

We use the method of manufactured solutions. Fix the order $\mathcal{N} = \frac{5}{2}$ and prescribe the target

$$\mathring{\alpha}^*(\mathring{\zeta}) = \mathring{\zeta}^3 + \mathring{\zeta}^2 + 1, \qquad \mathring{\zeta} \in [0,1].$$

Since the Caputo derivative of order $\mathcal{N} \in (2,3]$ annihilates constants, linear and quadratic terms, only the cubic term survives, giving ${}^c\mathfrak{D}^{\mathcal{N}}_{0+}\mathring{\alpha}^*(\mathring{\zeta}) = \frac{\Gamma(4)}{\Gamma(4-\mathcal{N})}\,\mathring{\zeta}^{\,3-\mathcal{N}}$. We then define the right-hand side

$$h(\mathring{\zeta},\mathring{\alpha}) = L\sin(\mathring{\alpha}) + b(\mathring{\zeta}), \qquad b(\mathring{\zeta}) = {}^c\mathfrak{D}^{\mathcal{N}}_{0+}\mathring{\alpha}^*(\mathring{\zeta}) - L\sin\big(\mathring{\alpha}^*(\mathring{\zeta})\big),$$

so that $\mathring{\alpha}^*$ solves Equation (26) exactly. Because sin is 1-Lipschitz, h obeys Equation (30) with constant L; we take $L = \frac{1}{2}$. The boundary data inherited from $\mathring{\alpha}^*$ are $c_0 = 1$, $c = 0$, $c_1 = 8$. With $\mathcal{N} = \frac{5}{2}$ the constant in Equation (31) is

$$\theta = L\left(\frac{1}{\Gamma(\mathcal{N}+1)} + \frac{1}{2\Gamma(\mathcal{N}-1)}\right) = 0.5 \times 0.865091 = 0.432545 < 1,$$

which is below 1, so the hypotheses of Theorem 5 are in force and the problem has a unique solution.

5.2. A fractional-quadrature Krasnoselskii solver

We mesh $[0,1]$ uniformly, taking $\mathring{\zeta}_i = i/M$ for $i = 0,\ldots,M$ with $M = 400$, and approximate the two integrals in Equation (29) by product-integration rules. The Riemann–Liouville integral is approximated by the product-trapezoidal weights of Diethelm, Ford and Freed [33], which integrate piecewise-linear data against the weakly singular kernel exactly; the boundary integral $\int_0^1 (1-\beth)^{\mathcal{N}-3} g(\beth)\, d\beth$ is treated by the analogous weights for the kernel $(1-\beth)^{\mathcal{N}-3}$, whose integrable endpoint singularity is resolved in closed form on each subinterval. This yields a discrete operator $\Delta_h(\mathring{\alpha}) = K\,g + p$, where $g_i = h(\mathring{\zeta}_i, \mathring{\alpha}_i)$, the matrix K collects the quadrature weights, and p is the affine term carrying the boundary data. Applying Δ_h to $\mathring{\alpha}^*$ reproduces it to within 8.9×10^{-5} at $M = 400$, which is the discretization error and confirms that the discrete operator is accurate.

The Krasnoselskii iteration (5) [34] is $\mathring{\alpha}^{(n+1)} = (1-\omega)\mathring{\alpha}^{(n)} + \omega\,\Delta_h(\mathring{\alpha}^{(n)})$, with $\mathring{\alpha}^{(0)} \equiv 0$. The iterates converge quickly to $\mathring{\alpha}^*$: the pointwise error reaches the discretization floor, and for $\omega = 1$ the residual falls below 10^{-6} after six steps.

5.3. Observed convergence rate versus the certified bound

Theorem 5 guarantees geometric convergence with rate at most θ. To test this, we track the distance $\|\mathring{\alpha}^{(n)} - \mathring{\alpha}_h^*\|_\infty$ to the numerically exact fixed point $\mathring{\alpha}_h^*$ of Δ_h. In **Figure 1a**, plotted on a logarithmic scale for $\omega \in \{1, 0.7, 0.4\}$, each error curve is a straight line, hence geometric, and the steepest one corresponds to $\omega = 1$. Fitting the slopes gives the empirical per-step factors $\hat{\rho}(\omega)$, which are compared in **Figure 1b** with the theoretical value $\theta(\omega) = (1-\omega) + \omega\theta$ for the averaged operator (Remark 7).

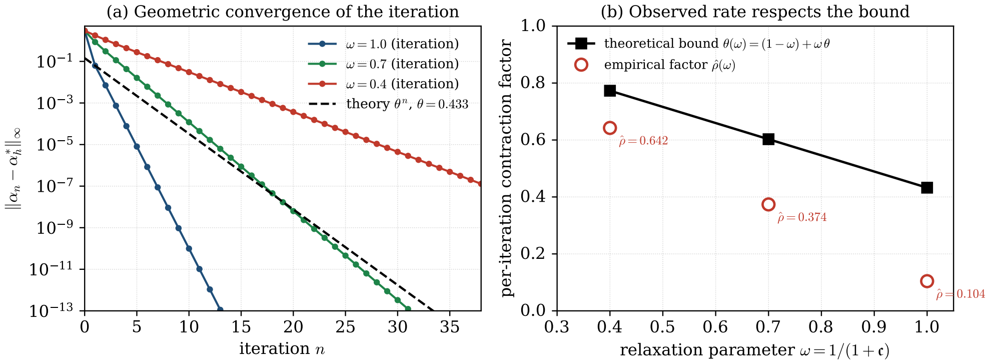

Figure 1. **(a)** Geometric convergence of the Krasnoselskii iteration for three values of ω; the dashed envelope is θ^n with $\theta = 0.433$. **(b)** The empirical per-step factor $\hat{\rho}(\omega)$ (circles) lies below the certified bound $\theta(\omega) = (1-\omega) + \omega\theta$ (squares) for every ω, and $\omega = 1$ gives the fastest convergence.

Two observations are worth making. First, the measured factors are strictly smaller than the certified bound, $\hat{\rho}(\omega) \leq \theta(\omega)$, for every ω we tested: $\hat{\rho}(1) \approx 0.10$ against $\theta(1) = \theta \approx 0.43$, and $\hat{\rho}(0.7) \approx 0.37$ against 0.60. The bound is therefore correct but not tight. The reason is that θ is computed by taking absolute values inside both integrals of Equation (29) and adding them, while the two integral terms actually have opposite signs and partly cancel, so the true contraction factor of Δ is smaller than the triangle-inequality value. The slope of the error curve equals $\ln \hat{\rho}$, and $\ln \hat{\rho} \leq \ln \theta$, so Theorem 5 gives an upper bound on the rate and not the exact rate. Second, $\hat{\rho}(\omega)$ increases as ω decreases, so $\omega = 1$ (that is, $\mathfrak{c} = 0$, ordinary successive approximation) is optimal for this problem. This is again a property of the operator. Averaging improves the rate only when the map fails to contract in a reflecting direction, as in the discrete Examples 3 and 5, where the negative multiplier makes Δ_ω contractive only for suitable ω. Here the fractional solution operator already has a positive, contractive kernel, so relaxation gives no improvement, in agreement with Remark 7.

5.4. A neural fixed-point solver

As an independent, mesh-free check, we also approximate the solution by a neural network. Residual-based training of networks for differential equations originates in physics-informed learning [35] and has grown into a broad literature [36]; for fractional equations, where automatic differentiation does not supply the fractional derivative, the operator has been provided through discrete quadrature [37], Monte Carlo sampling [38], and spectral representations [39]. The solver used here is of a related but distinct kind, and we call it a neural fixed-point solver rather than a fractional physics-informed network: no fractional derivative of the network is formed at any stage. Instead, the network is trained to satisfy the discrete fixed-point equation $\mathring{\alpha} = \Delta_h(\mathring{\alpha})$ of Section 5.2, so the fractional structure enters through the quadrature matrix K only. The solution is represented by a small multilayer perceptron $\mathring{\alpha}_\vartheta : [0,1] \to \mathbb{R}$ with three hidden layers of width 64 and tanh activations, trained on the collocation points $\{\mathring{\zeta}_i\}$ by minimizing the residual

$$\mathcal{L}(\vartheta) = \frac{1}{M+1}\sum_{i=0}^{M}\Big(\mathring{\alpha}_\vartheta(\mathring{\zeta}_i) - \Delta_h(\mathring{\alpha}_\vartheta)(\mathring{\zeta}_i)\Big)^2.$$

Training a network to satisfy a fixed-point equation rather than to fit data is the viewpoint taken by deep equilibrium models [40], which we adopt here in a purely numerical role. Using the integral operator Δ has a practical advantage. The boundary conditions are contained in the affine part p, so no separate boundary-loss term is required; every value of Δ satisfies $\mathring{\alpha}(0) = c_0$, $\mathring{\alpha}'(0) = c$, and $\mathring{\alpha}''(1) = c_1$ automatically. We train with Adam for 6000 steps and then apply a short L-BFGS refinement.

After training, the network reaches a residual $\mathcal{L} \approx 5 \times 10^{-8}$ and reproduces the exact solution with a maximum error of 1.0×10^{-3} and an L^2-error of 2.3×10^{-4}.

The relation between the discretization error and the network error deserves a comment, since the loss involves the discrete operator Δ_h and not Δ itself. Writing $\mathring{\alpha}_h^*$ for the fixed point of Δ_h, the triangle inequality splits the error of the trained network $\mathring{\alpha}_\vartheta$ into

$$\|\mathring{\alpha}_\vartheta - \mathring{\alpha}^*\| \;\le\; \|\mathring{\alpha}_\vartheta - \mathring{\alpha}_h^*\| + \|\mathring{\alpha}_h^* - \mathring{\alpha}^*\|.$$

The second term is the discretization error of the quadrature, about 8.9×10^{-5} at $M = 400$, and no amount of training can reduce it, because the loss never sees Δ. The first term is controlled by the training residual: on the grid, Δ_h is a contraction whose factor agrees with $\theta \approx 0.4325$ up to the quadrature perturbation, so

$$\|\mathring{\alpha}_\vartheta - \mathring{\alpha}_h^*\| \;\le\; \frac{\|\mathring{\alpha}_\vartheta - \Delta_h(\mathring{\alpha}_\vartheta)\|}{1-\theta}.$$

With the final root-mean-square residual $\sqrt{\mathcal{L}} \approx 2.2 \times 10^{-4}$, this bound gives roughly 3.9×10^{-4} in the discrete L^2 sense, consistent with the measured L^2-error of 2.3×10^{-4}. Both numbers still sit above the 8.9×10^{-5} floor set by Δ_h, so at the reported training level the network error is dominated by the residual term rather than by the quadrature; training long enough to push $\sqrt{\mathcal{L}}$ below that floor would sharpen the solution of the discrete equation without bringing the network any closer to the continuous solution, because the loss never sees Δ. In this sense, the discretization error marks the useful limit of training, and the network error cannot fall below it. The remaining gap up to the maximum error of 1.0×10^{-3} reflects the behaviour of $\mathring{\alpha}_\vartheta$ between collocation points, which the loss does not constrain. A finer mesh or a higher-order quadrature would lower the attainable floor for both solvers at the usual cost.

Table 3 reports the accuracy and cost of the two solvers. For a one-dimensional problem, the classical iteration is more accurate and much cheaper, as expected. The value of the neural solver is that it is mesh-free and differentiable and reaches the same solution from a different formulation. The analytical fixed point, the quadrature iteration, and the network agree, which provides a consistency check on Theorem 5.

Table 3. Accuracy and cost of the two solvers on the manufactured problem ($\mathcal{N} = 2.5$, $M = 400$, double precision). Errors are measured against the exact solution $\mathring{\alpha}^*$.

Method	Max error	L^2 error	Budget	Time (s)
Krasnoselskii iteration ($\omega = 1$)	8.9×10^{-5}	4.7×10^{-5}	6 iterations	0.008
Neural fixed-point solver	1.0×10^{-3}	2.3×10^{-4}	$6,000$ Adam + L-BFGS	19.7

Every computation was performed in double precision, and the accompanying code regenerates the figures and the table.

6. Conclusions and future directions

We introduced two classes of enriched polynomial mappings on normed spaces, the enriched polynomial contractions and the almost enriched polynomial contractions. For both classes, we proved fixed point theorems under continuity and, more generally, under Picard continuity, and we approximated the fixed points by Krasnoselskii-type iteration. Several known results, including the enriched contraction theorem of Berinde and Păcurar, are special cases. The examples cover maps with several fixed points and discontinuous maps, which fall outside the classical contraction setting.

As the main application, we wrote a nonlinear Caputo fractional boundary value problem of order $2 < \mathcal{N} \leq 3$ as a fixed-point problem for an integral operator and used an explicit Lipschitz condition to prove existence and uniqueness of the solution. The class of problems covered by this result is the one delimited by the hypotheses (30) and (31): one-dimensional Caputo problems with a globally Lipschitz nonlinearity whose constant is small enough, and on this class the solution operator is already a contraction, so the application draws on the $\mathfrak{c} = 0$, $q = 1$ part of the theory. We then tested this result numerically. On a manufactured problem with a known exact solution, we ran the corresponding fractional-quadrature Krasnoselskii iteration, checked that its observed geometric rate stays below the certified contraction factor, and examined the effect of the relaxation parameter. We also trained a neural fixed-point solver on the residual of the discretized operator and related the accuracy of both solvers to the underlying quadrature error. The analytical solution, the quadrature iteration, and the network output agree, in line with the theorem and with the constructive character of its proof.

Several problems remain open. The one we consider most pressing, in view of

Remark 7, is to identify fractional models whose solution operators fail to contract without enrichment, for instance through strongly negative feedback terms of the kind that drive Examples 3 and 5; a model of that type would engage the $\mathfrak{c} > 0$ and $q > 1$ range of the theory, where the present application does not reach. The framework could also be extended to cyclic contractions in the sense of Kirk, and enriched polynomial versions of the Kannan and Chatterjea contractions could be defined and studied. A fuzzy version of the theory would be a further generalization, for which the fuzzy enriched contractions [26] are a natural starting point. On the numerical side, it would be useful to analyze the fractional-quadrature operator at higher order, to apply the neural fixed-point solver to multidimensional and inverse fractional problems, and to derive a posteriori error estimates based on the gap between the certified factor and the observed rate reported here.

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