Advances in Differential Equations and Control Processes 2026, 33(3), 4603.
doi: https://doi.org/10.59400/adecp4603

Article

On (θ, α)-metric space and fixed point results with application

Mohammad Akram[1], Umar Ishtiaq[2,3,*], Zainab Bibi[4], Tayyab Kamran[4,5], Ioan-Lucian Popa[6,7]

[1] Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia; akram@iu.edu.sa
[2] Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54770, Pakistan
[3] Software Development Department, Biruni University, 34010 Istanbul, Turkey
[4] Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan; mujidzainab@gmail.com (ZB); tkamran@qau.edu.pk (TK)
[5] Center for Theoretical Physics, Khazar University, 41 Mehseti Street, Baku AZ1096, Azerbaijan
[6] Department of Computing, Mathematics and Electronics, "1 Decembrie 1918" University of Alba Iulia, 510009 Alba Iulia, Romania; lucian.popa@uab.ro
[7] Faculty of Mathematics and Computer Science, Transilvania University of Braşov, 500091 Braşov, Romania
*** Corresponding author:** Umar Ishtiaq, umarishtiaq@umt.edu.pk

CITATION

Akram M, Ishtiaq U, Bibi Z, et al. On (θ, α)-metric space and fixed point results with application. Advances in Differential Equations and Control Processes. 2026; 33(3): 4603. https://doi.org/10.59400/adecp4603

ARTICLE INFO

Received: 1 July 2026
Revised: 19 July 2026
Accepted: 22 July 2026
Available online: 28 August 2026

COPYRIGHT

Copyright © 2026 Author(s). *Advances in Differential Equations and Control Processes* is published by Academic Publishing Pte. Ltd. This work is licensed under the Creative Commons Attribution (CC BY) license. https://creativecommons.org/licenses/by/4.0/

Abstract: The notion of a double controlled metric type space was introduced by Abdeljawad et al. as a broadening of the controlled metric type spaces of Mlaiki et al., and shortly afterwards Aydi et al. formulated the new extended b-metric spaces to widen the extended b-metric spaces of Kamran et al. Building on these developments, the present work advances the class of (θ, α)-metric spaces, a structure that simultaneously subsumes b-metric spaces, new extended b-metric spaces, controlled metric type spaces and double controlled metric type spaces. This is accomplished by attaching two control functions $\theta(\zeta, \mu, s)$ and $\alpha(\zeta, \mu, s)$ to the terms situated on the right-hand side of the triangle inequality in the axioms defining a (θ, α)-metric space. We first prove, through explicit propositions, that each of the aforementioned structures is recovered as a special case under an appropriate choice of θ and α, and we supply non-trivial examples confirming that the new class is strictly broader than a b-metric space. Within this setting, a collection of fixed point theorems is derived for φ-contractive and Reich–Ćirić–Rus–type operators, together with their corollaries. Finally, two applications are presented: the principal theorem is first used to guarantee that a nonlinear algebraic equation possesses a unique solution, and it is then applied to establish the existence and uniqueness of a solution to a nonlinear Caputo fractional boundary value problem, thereby strengthening the connection of the theory with differential equations.

Keywords: fixed point; contraction mapping; controlled metric space; extended b-metric space; (θ, α)-metric space; Caputo fractional boundary value problem; existence and uniqueness of solutions

1. Introduction

The classical assertion that any contractive self-map defined on a complete metric space possesses precisely one fixed point (FP) can be traced to the seminal 1922 memoir of Banach [1], a study that laid the foundations of metric fixed point theory. That milestone opened the way to considerable progress within the area and prompted researchers across the globe to develop the theory further. As time went on, substantial advances were secured by introducing a broad assortment of auxiliary functions and by relaxing the customary contractive requirements, which in turn produced fresh theoretical developments together with tangible applications. In parallel, the very idea of a metric space (MS) was reshaped, since investigators moved toward wider and more abstract frameworks, such as fuzzy metric spaces, partial metric spaces and b-metric spaces, so as to accommodate a larger family of mathematical models and real situations. These refinements have carried fixed point theory into a range of fields, including optimization, differential equations and data science, thereby highlighting its pronounced influence on both pure and applied mathematics. Since questions originating from integral and differential equations may be shown to be solvable by fixed point arguments, this apparatus is routinely relied upon by researchers, a fact that readily accounts for its wide appeal.

A further important generalization of the MS concept is the b-MS, whose origins lie in the contributions of Bakhtin [2] and Czerwik [3]. A large body of fixed point results has since been established inside b-MSs by a range of authors [4–8]. Operating within partial b-MSs, Abdeljawad et al. [9] sharpened a number of established fixed point statements. A fresh treatment of Banach's contraction theorem through ordered relations was carried out by Shatanawi et al. [10], while Rasham et al. [11] transferred the same principle to the fuzzy MS setting. In addition, a variety of fixed point theorems in fuzzy MSs were recorded by Gupta et al. [12–14]. Within the multiplicative MS environment, Gamal et al. [15] obtained several fixed point results founded on assorted contractive conditions and investigated applications connected with weakly compatible maps. The reader may consult [16,17] for further applications arising from such fixed point findings.

A creative route toward generalizing the b-MS, resting on a control function with values in $[1, +\infty)$, produced the extended b-MSs (EbMS) that were introduced by Kamran et al. [18] More recently, Mlaiki et al. [19] advanced the idea of a controlled metric type space (CMTS), once again enlarging the b-MS by inserting a control function θ into its triangle inequality; they also exhibited an example showing that a CMTS need not agree with an EbMS. Further material on extended b-MSs and controlled MSs can be located in the literature [20–23]. Very recent contributions in this direction include the controlled metric type results of Souayah and Hidri [24] and the complex-valued controlled metric results of Shammaky and Ahmad [25]. Abdeljawad et al. [26] carried the CMTS idea forward and introduced double CMTSs, and additional related work appears in previous studies [27–29]. Lastly, Aydi et al. [30] refined the functional inequality of the extended b-MS and reported fixed point results for nonlinear contractive maps in this new environment; the notions of ϕ-weak contraction [31], admissibility [32] and controlled comparison functions [33] will also be needed below.

Drawing motivation from the developments described above, we introduce the notion of a (θ, α)-metric space, in which two (generally different) control functions θ and α are attached, respectively, to the two summands on the right-hand side of the triangle inequality. Our contributions are the following. (i) We prove, by means of explicit propositions, that the b-MS, the (new) extended b-MS, the controlled MS and the double controlled MS are all recovered as special cases of a (θ, α)-metric space. (ii) We give non-trivial examples showing that the new class is strictly larger than a b-MS. (iii) We establish a family of fixed point theorems for φ-contractions and for Reich–Ćirić–Rus–type contractions, and present the special cases as corollaries of a single general statement. (iv) As applications, we prove the existence and uniqueness of the solution of a nonlinear algebraic equation and, to strengthen the link with differential equations, of a nonlinear Caputo fractional boundary value problem; these applications are in the spirit of the recent fixed point studies of fractional models in previous studies [34–37].

2. Preliminaries

So that the main section is easier to follow, we gather here a selection of definitions taken from the available literature. Throughout, S denotes a nonempty set and we adopt the convention $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition 1 ([2]). *Let S be a nonempty set and fix a real number $b \geq 1$. A mapping $\bar{\Lambda} : S \times S \to [0, \infty)$ is termed a b-metric provided that:*

(Γ1) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = 0 \iff \zeta = \mu\}$,
(Γ2) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = \bar{\Lambda}(\mu, \zeta)\}$,
(Γ3) $\forall_{\zeta,\mu,s \in S}\{\bar{\Lambda}(\zeta, \mu) \leq b[\bar{\Lambda}(\zeta, s) + \bar{\Lambda}(s, \mu)]\}$.

The pair $(S, \bar{\Lambda})$ is called a b-MS.

Definition 2 ([18]). *Consider a nonempty set S together with a function $\theta : S \times S \to [1, \infty)$. A mapping $\bar{\Lambda} : S \times S \to [0, \infty)$ is said to be an extended b-metric whenever:*

($\bar{\xi}$1) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = 0 \iff \zeta = \mu\}$,
($\bar{\xi}$2) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = \bar{\Lambda}(\mu, \zeta)\}$,
($\bar{\xi}$3) $\forall_{\zeta,\mu,s \in S}\{\bar{\Lambda}(\zeta, \mu) \leq \theta(\zeta, \mu)[\bar{\Lambda}(\zeta, s) + \bar{\Lambda}(s, \mu)]\}$.

In this case the pair $(S, \bar{\Lambda})$ is named an EbMS.

Definition 3 ([30]). *Let S be a nonempty set and let $\theta : S \times S \times S \to [1, \infty)$. A mapping $\bar{\Lambda} : S \times S \to [0, \infty)$ qualifies as a new extended b-metric provided that:*

(η1) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = 0 \iff \zeta = \mu\}$,
(η2) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = \bar{\Lambda}(\mu, \zeta)\}$,
(η3) $\forall_{\zeta,\mu,s \in S}\{\bar{\Lambda}(\zeta, \mu) \leq \theta(\zeta, \mu, s)[\bar{\Lambda}(\zeta, s) + \bar{\Lambda}(s, \mu)]\}$.

Under these axioms the pair $(S, \bar{\Lambda})$ is regarded as a new EbMS.

Remark 1. *In Definition 3 the intermediate point s occupies the third argument of θ, and the same coefficient $\theta(\zeta, \mu, s)$ multiplies both summands. We adopt this convention consistently throughout the paper.*

Definition 4 ([31]). *A self-map $P : S \to S$ on a MS $(S, \bar{\Lambda})$ is called a ϕ-weak contraction whenever there is a continuous, nondecreasing function $\phi : [0, \infty) \to [0, \infty)$, positive on $(0, +\infty)$ with $\phi(0) = 0$, such that, for all $\zeta, \mu \in S$,*

$$\bar{\Lambda}(P\zeta, P\mu) \leq \bar{\Lambda}(\zeta, \mu) - \phi(\bar{\Lambda}(\zeta, \mu)).$$

The Banach contraction is recovered upon taking $\phi(t) = (1 - \mathcal{K})t$ with $0 \leq \mathcal{K} < 1$.

Definition 5 ([19]). *Let S be a nonempty set and let $\theta : S \times S \to [1, \infty)$ and $\bar{\Lambda} : S \times S \to [0, \infty)$. The map $\bar{\Lambda}$ is a controlled metric type provided that:*

($F\delta 1$) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = 0 \iff \zeta = \mu\}$,
($F\delta 2$) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = \bar{\Lambda}(\mu, \zeta)\}$,
($F\delta 3$) $\forall_{\zeta,\mu,s \in S}\{\bar{\Lambda}(\zeta, \mu) \leq \theta(\zeta, s)\bar{\Lambda}(\zeta, s) + \theta(s, \mu)\bar{\Lambda}(s, \mu)\}$.

The pair $(S, \bar{\Lambda})$ is called a CMTS.

Definition 6 ([26]). *Let S be a nonempty set and let $\theta, \alpha : S \times S \to [1, \infty)$. If $\bar{\Lambda} : S \times S \to [0, \infty)$ fulfils*

(q1) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = 0 \iff \zeta = \mu\}$,
(q2) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = \bar{\Lambda}(\mu, \zeta)\}$,
(q3) $\forall_{\zeta,\mu,s \in S}\{\bar{\Lambda}(\zeta, \mu) \leq \theta(\zeta, s)\bar{\Lambda}(\zeta, s) + \alpha(s, \mu)\bar{\Lambda}(s, \mu)\}$.

Then $\bar{\Lambda}$ is called a double controlled metric and $(S, \bar{\Lambda})$ a double controlled MS.

Remark 2. *The control functions θ and α need not be equal, and no order or comparability relation between them is assumed.*

3. Main results

We now introduce the notion of a (θ, α)-metric space and establish a number of fixed point results within this framework.

Definition 7. *Let S be a nonempty set, let $\theta, \alpha : S \times S \times S \to [1, \infty)$ be two functions, and let $\bar{\Lambda} : S \times S \to [0, \infty)$ be a map for which:*

($F\Delta 1$) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = 0 \iff \zeta = \mu\}$,
($F\Delta 2$) $\forall_{\zeta,\mu \in S}\{\bar{\Lambda}(\zeta, \mu) = \bar{\Lambda}(\mu, \zeta)\}$,
($F\Delta 3$) $\forall_{\zeta,\mu,s \in S}\{\bar{\Lambda}(\zeta, \mu) \leq \theta(\zeta, \mu, s)\, \bar{\Lambda}(\zeta, s) + \alpha(\zeta, \mu, s)\, \bar{\Lambda}(s, \mu)\}$.

Then the pair $(S, \bar{\Lambda})$ is called a (θ, α)-metric space, and ($F\Delta 3$) is referred to as the (θ, α)-triangle inequality.

The next proposition makes precise the sense in which the (θ, α)-metric space unifies the structures introduced above: each of them is obtained by specialising the pair (θ, α).

Proposition 1. *Let $\bar{\Lambda} : S \times S \to [0, \infty)$ be symmetric and satisfy $(F\Delta 1)$–$(F\Delta 2)$. Then*:

(i) *if $\theta(\zeta, \mu, s) = \alpha(\zeta, \mu, s) = b$ for a constant $b \geq 1$, then $(F\Delta 3)$ is exactly the b-metric inequality $(\Gamma 3)$; hence every b-MS is a (θ, α)-metric space;*
(ii) *if $\theta(\zeta, \mu, s) = \alpha(\zeta, \mu, s) = \vartheta(\zeta, \mu, s)$ for a function $\vartheta : S^3 \to [1, \infty)$, then $(F\Delta 3)$ reduces to the new extended b-metric inequality $(\eta 3)$; hence every new EbMS is a (θ, α)-metric space;*
(iii) *if $\theta(\zeta, \mu, s) = \vartheta(\zeta, s)$ and $\alpha(\zeta, \mu, s) = \vartheta(s, \mu)$ for a function $\vartheta : S^2 \to [1, \infty)$, then $(F\Delta 3)$ becomes the controlled inequality $(F\delta 3)$; hence every CMTS is a*

(θ, α)-metric space;

(iv) if $\theta(\zeta, \mu, s) = \vartheta(\zeta, s)$ and $\alpha(\zeta, \mu, s) = \varpi(s, \mu)$ for functions $\vartheta, \varpi : S^2 \to [1, \infty)$, then $(F\Delta 3)$ becomes the double controlled inequality $(q3)$ in Definition 6; hence every double controlled MS is a (θ, α)-metric space.

Proof. Since $(F\Delta 1)$ and $(F\Delta 2)$ coincide, in every item, with the identity and symmetry axioms of the structure in question, it suffices to verify that the chosen control functions turn $(F\Delta 3)$ into the required triangle inequality; note that in each case the prescribed θ, α take values in $[1, \infty)$ and are therefore admissible control functions. We treat the four items separately.

(i) *b-metric.* Put $\theta(\zeta, \mu, s) = \alpha(\zeta, \mu, s) = b$ with $b \geq 1$. Substituting into $(F\Delta 3)$,

$$\bar{\Lambda}(\zeta, \mu) \leq \theta(\zeta, \mu, s)\, \bar{\Lambda}(\zeta, s) + \alpha(\zeta, \mu, s)\, \bar{\Lambda}(s, \mu) =$$
$$b\, \bar{\Lambda}(\zeta, s) + b\, \bar{\Lambda}(s, \mu) = b\big[\bar{\Lambda}(\zeta, s) + \bar{\Lambda}(s, \mu)\big],$$

which is precisely axiom $(\Gamma 3)$. Hence $(S, \bar{\Lambda})$ is a b-MS.

(ii) *new extended b-metric.* Put $\theta(\zeta, \mu, s) = \alpha(\zeta, \mu, s) = \vartheta(\zeta, \mu, s)$. Then $(F\Delta 3)$ reads

$$\bar{\Lambda}(\zeta, \mu) \leq \vartheta(\zeta, \mu, s)\, \bar{\Lambda}(\zeta, s) + \vartheta(\zeta, \mu, s)\, \bar{\Lambda}(s, \mu) = \vartheta(\zeta, \mu, s)\big[\bar{\Lambda}(\zeta, s) + \bar{\Lambda}(s, \mu)\big],$$

which is exactly axiom $(\eta 3)$ of a new EbMS.

(iii) *controlled metric type space.* Put $\theta(\zeta, \mu, s) = \vartheta(\zeta, s)$ and $\alpha(\zeta, \mu, s) = \vartheta(s, \mu)$ for a two-variable function ϑ. Then $(F\Delta 3)$ becomes

$$\bar{\Lambda}(\zeta, \mu) \leq \vartheta(\zeta, s)\, \bar{\Lambda}(\zeta, s) + \vartheta(s, \mu)\, \bar{\Lambda}(s, \mu),$$

which is axiom $(F\delta 3)$ of a CMTS.

(iv) *double controlled metric space.* Put $\theta(\zeta, \mu, s) = \vartheta(\zeta, s)$ and $\alpha(\zeta, \mu, s) = \varpi(s, \mu)$ for two-variable functions ϑ, ϖ. Then $(F\Delta 3)$ becomes

$$\bar{\Lambda}(\zeta, \mu) \leq \vartheta(\zeta, s)\, \bar{\Lambda}(\zeta, s) + \varpi(s, \mu)\, \bar{\Lambda}(s, \mu),$$

which is axiom $(q3)$ of a double controlled MS. This completes the proof.

□

Remark 3. *The inclusions in Proposition 1 are, in general, strict: Example 1 below exhibits a (θ, α)-metric space that is not a b-MS. Note that, when the control functions are allowed to depend genuinely on all three variables and $\theta \neq \alpha$, the coefficients of the two summands in $(F\Delta 3)$ may differ, which is not permitted in a (new) EbMS, where a single coefficient multiplies both summands.*

Example 1. *Let $S = A \cup C$, where $A = (0, 1]$ and $C = \mathbb{N} \setminus \{0, 1\} = \{2, 3, 4, \dots\}$. The value 0 is excluded from A so that the identity axiom is not violated. Define $\bar{\Lambda}$:*

$S \times S \to [0, \infty)$ by

$$\bar{\Lambda}(\zeta, \mu) = \begin{cases} 0, & \zeta = \mu, \\ \zeta, & \zeta \in A,\ \mu \in C, \\ \mu, & \zeta \in C,\ \mu \in A, \\ 1, & \zeta, \mu \text{ in the same class and } \zeta \neq \mu. \end{cases}$$

For $x, y \in S$ put

$$\rho(x, y) = \begin{cases} 1, & x, y \in C, \\ 1/\min\{v : v \in \{x, y\} \cap A\}, & \{x, y\} \cap A \neq \varnothing, \end{cases}$$

that is, $\rho(x, y)$ equals 1 when both entries lie in C, and equals the reciprocal of the smallest A-coordinate among x, y otherwise. Since $A = (0, 1]$, every A-coordinate lies in $(0, 1]$, so $\rho(x, y) \in [1, \infty)$; thus ρ is well defined and ≥ 1 everywhere. Finally set

$$\theta(\zeta, \mu, s) = \rho(\zeta, s), \qquad \alpha(\zeta, \mu, s) = \rho(s, \mu),$$

so that $\theta, \alpha : S^3 \to [1, \infty)$ are well defined at every point.

We claim that $(S, \bar{\Lambda})$ is a (θ, α)-metric space. Axioms $(F\Delta1)$ and $(F\Delta2)$ are immediate from the definition of $\bar{\Lambda}$; note that $(F\Delta1)$ genuinely holds because, for $\zeta \in A$ (so $\zeta > 0$) and $\mu \in C$, one has $\bar{\Lambda}(\zeta, \mu) = \zeta > 0$. It remains to verify the (θ, α)-triangle inequality

$$\bar{\Lambda}(\zeta, \mu) \leq \theta(\zeta, \mu, s)\, \bar{\Lambda}(\zeta, s) + \alpha(\zeta, \mu, s)\, \bar{\Lambda}(s, \mu) = \rho(\zeta, s)\, \bar{\Lambda}(\zeta, s) + \rho(s, \mu)\, \bar{\Lambda}(s, \mu) \tag{1}$$

for all $\zeta, \mu, s \in S$. We first dispose of the trivial situations.

- *If $\zeta = \mu$, then $\bar{\Lambda}(\zeta, \mu) = 0$ and (1) holds trivially.*
- *If $s = \zeta$, then $\bar{\Lambda}(\zeta, s) = 0$ and the right-hand side equals $\rho(s, \mu)\bar{\Lambda}(\zeta, \mu) \geq \bar{\Lambda}(\zeta, \mu)$ because $\rho(s, \mu) \geq 1$.*
- *If $s = \mu$, then $\bar{\Lambda}(s, \mu) = 0$ and the right-hand side equals $\rho(\zeta, \mu)\bar{\Lambda}(\zeta, \mu) \geq \bar{\Lambda}(\zeta, \mu)$ because $\rho(\zeta, \mu) \geq 1$.*

Henceforth we may assume $\zeta \neq \mu$ and $s \notin \{\zeta, \mu\}$. A useful preliminary observation is that $\bar{\Lambda}$ never exceeds 1: indeed $\bar{\Lambda}(\zeta, \mu)$ equals 0, an A-coordinate (which lies in $(0, 1]$), or 1. We now examine the eight configurations determined by the classes of ζ, μ, s.

***Case AAA** ($\zeta, \mu, s \in A$). Here $\bar{\Lambda}(\zeta, \mu) = \bar{\Lambda}(\zeta, s) = \bar{\Lambda}(s, \mu) = 1$, while $\rho(\zeta, s) = 1/\min(\zeta, s) \geq 1$ and $\rho(s, \mu) = 1/\min(s, \mu) \geq 1$. Hence the right-hand side of Equation (1) is $\geq 1 + 1 = 2 \geq 1 = \bar{\Lambda}(\zeta, \mu)$.*

***Case AAC** ($\zeta, \mu \in A,\ s \in C$). Then $\bar{\Lambda}(\zeta, \mu) = 1$, $\bar{\Lambda}(\zeta, s) = \zeta$, $\bar{\Lambda}(s, \mu) = \mu$, $\rho(\zeta, s) = 1/\zeta$ and $\rho(s, \mu) = 1/\mu$. The right-hand side equals $(1/\zeta)\zeta + (1/\mu)\mu = 1 + 1 = 2 \geq 1$.*

***Case ACA** ($\zeta \in A,\ \mu \in C,\ s \in A$). Then $\bar{\Lambda}(\zeta, \mu) = \zeta \leq 1$, $\bar{\Lambda}(\zeta, s) = 1$, $\bar{\Lambda}(s, \mu) = s$, $\rho(\zeta, s) = 1/\min(\zeta, s)$ and $\rho(s, \mu) = 1/s$. The right-hand side equals $\dfrac{1}{\min(\zeta, s)} +$*

$\dfrac{1}{s}s = \dfrac{1}{\min(\zeta, s)} + 1 \geq 2 \geq \zeta.$

Case ACC *($\zeta \in A,\ \mu, s \in C$). Then $\bar{\Lambda}(\zeta,\mu) = \zeta \leq 1$, $\bar{\Lambda}(\zeta,s) = \zeta$, $\bar{\Lambda}(s,\mu) = 1$, $\rho(\zeta,s) = 1/\zeta$ and $\rho(s,\mu) = 1$. The right-hand side equals $(1/\zeta)\zeta + 1 = 1+1 = 2 \geq \zeta$.*

Case CAA *($\zeta \in C,\ \mu, s \in A$). Then $\bar{\Lambda}(\zeta,\mu) = \mu \leq 1$, $\bar{\Lambda}(\zeta,s) = s$, $\bar{\Lambda}(s,\mu) = 1$, $\rho(\zeta,s) = 1/s$ and $\rho(s,\mu) = 1/\min(s,\mu)$. The right-hand side equals $(1/s)s + 1/\min(s,\mu) = 1 + 1/\min(s,\mu) \geq 2 \geq \mu$.*

Case CAC *($\zeta, s \in C,\ \mu \in A$). Then $\bar{\Lambda}(\zeta,\mu) = \mu \leq 1$, $\bar{\Lambda}(\zeta,s) = 1$, $\bar{\Lambda}(s,\mu) = \mu$, $\rho(\zeta,s) = 1$ and $\rho(s,\mu) = 1/\mu$. The right-hand side equals $1 \cdot 1 + (1/\mu)\mu = 1 + 1 = 2 \geq \mu$.*

Case CCA *($\zeta, \mu \in C,\ s \in A$). Then $\bar{\Lambda}(\zeta,\mu) = 1$, $\bar{\Lambda}(\zeta,s) = s$, $\bar{\Lambda}(s,\mu) = s$, $\rho(\zeta,s) = 1/s$ and $\rho(s,\mu) = 1/s$. The right-hand side equals $(1/s)s + (1/s)s = 1 + 1 = 2 \geq 1$.*

Case CCC *($\zeta, \mu, s \in C$). Then $\bar{\Lambda}(\zeta,\mu) = \bar{\Lambda}(\zeta,s) = \bar{\Lambda}(s,\mu) = 1$ and $\rho(\zeta,s) = \rho(s,\mu) = 1$, so the right-hand side equals $1 + 1 = 2 \geq 1$.*

In every case the right-hand side of Equation (1) is at least $\bar{\Lambda}(\zeta,\mu)$, so $(F\Delta3)$ holds and $(S,\bar{\Lambda})$ is a (θ,α)-metric space.

Finally, $(S,\bar{\Lambda})$ is not a b-MS. Suppose, to the contrary, that a constant $b \geq 1$ satisfied $\bar{\Lambda}(\zeta,\mu) \leq b[\bar{\Lambda}(\zeta,s) + \bar{\Lambda}(s,\mu)]$ for all ζ, μ, s. Take $\zeta = 2$, $\mu = 3$ (both in C) and $s = a \in A$. Then $\bar{\Lambda}(\zeta,\mu) = 1$ while $\bar{\Lambda}(\zeta,s) = \bar{\Lambda}(s,\mu) = a$, so the inequality would force $1 \leq 2ba$, i.e., $b \geq 1/(2a)$. Letting $a \to 0^+$ makes the right-hand side tend to $+\infty$, so no finite b can work. Hence $(S,\bar{\Lambda})$ is a (θ,α)-metric space that is strictly more general than a b-MS.

Definition 8. *Let $(S,\bar{\Lambda})$ be a (θ,α)-metric space and let $\{\zeta_n\}_{n=0}^{\infty}$ be a sequence in S. Then:*

- *(Ξ1)* *$\{\zeta_n\}$ converges to $\zeta \in S$ if for every $\epsilon > 0$ there is $V = V(\epsilon) \in \mathbb{N}$ such that $\bar{\Lambda}(\zeta_n,\zeta) < \epsilon$ for all $n \geq V$; we write $\lim_{n\to\infty} \zeta_n = \zeta$;*
- *(Ξ2)* *$\{\zeta_n\}$ is Cauchy if for every $\epsilon > 0$ there is $V = V(\epsilon) \in \mathbb{N}$ with $\bar{\Lambda}(\zeta_m,\zeta_n) < \epsilon$ for all $m, n \geq V$;*
- *(Ξ3)* *$(S,\bar{\Lambda})$ is complete when every Cauchy sequence in S converges to a limit lying in S.*

Definition 9. *Let $(S,\bar{\Lambda})$ be a (θ,α)-metric space, $\zeta \in S$ and $\epsilon > 0$. The set $L(\zeta,\epsilon) = \{\mu \in S : \bar{\Lambda}(\zeta,\mu) < \epsilon\}$ is called an open ball. A self-map P on S is continuous at $\zeta \in S$ if for every sequence $\{\zeta_n\}$ with $\lim_{n\to\infty} \zeta_n = \zeta$ one has $\lim_{n\to\infty} P\zeta_n = P\zeta$; equivalently, for each $\epsilon > 0$ there is $\delta > 0$ with $P(L(\zeta,\delta)) \subseteq L(P\zeta,\epsilon)$. We call P continuous if it is continuous at every point of S.*

Proposition 2. *Let $(S,\bar{\Lambda})$ be a (θ,α)-metric space and let $\{\zeta_n\}$ be a sequence converging to ζ' and to ζ''. If $\sup\limits_n \theta(\zeta',\zeta'',\zeta_n) < \infty$ and $\sup\limits_n \alpha(\zeta',\zeta'',\zeta_n) < \infty$, then $\zeta' = \zeta''$; that is, limits of sequences are unique whenever the control functions are bounded along the sequence.*

Proof. Fix n and apply the (θ,α)-triangle inequality $(F\Delta3)$ with the three points ζ', ζ''

and the intermediate point ζ_n:

$$\bar{\Lambda}(\zeta', \zeta'') \leq \theta(\zeta', \zeta'', \zeta_n)\, \bar{\Lambda}(\zeta', \zeta_n) + \alpha(\zeta', \zeta'', \zeta_n)\, \bar{\Lambda}(\zeta_n, \zeta'').$$

Write $\Theta^* = \max\{\sup_n \theta(\zeta', \zeta'', \zeta_n),\ \sup_n \alpha(\zeta', \zeta'', \zeta_n)\} < \infty$. Then, using symmetry $(F\Delta 2)$,

$$0 \leq \bar{\Lambda}(\zeta', \zeta'') \leq \Theta^* \big[\bar{\Lambda}(\zeta', \zeta_n) + \bar{\Lambda}(\zeta_n, \zeta'')\big] \xrightarrow[n\to\infty]{} \Theta^* \left[0 + 0\right] = 0,$$

since $\bar{\Lambda}(\zeta', \zeta_n) \to 0$ and $\bar{\Lambda}(\zeta'', \zeta_n) \to 0$ by convergence. Hence $\bar{\Lambda}(\zeta', \zeta'') = 0$, and $(F\Delta 1)$ gives $\zeta' = \zeta''$. □

Definition 10. *Let $\theta, \alpha : S \times S \times S \to [1, \infty)$. We write $\Phi_{\theta,\alpha}$ for the family of maps $\varphi : [0, \infty) \to [0, \infty)$ that are controlled comparison functions relative to θ and α, meaning that:*

- *(Fφ1)* *φ is nondecreasing;*
- *(Fφ2)* *$\varphi(0) = 0$ and $\varphi(t) < t$ for every $t > 0$;*
- *(Fφ3)* *for every $t > 0$ and every sequence $\{\zeta_n\}_{n\geq 0}$ in S along which θ and α are bounded,*

$$\sup_{m\geq 1} \varlimsup_{n\to\infty} \frac{\varphi^{n+1}(t)}{\varphi^n(t)}\, \alpha(\zeta_n, \zeta_m, \zeta_{n+1}) < 1,$$

$$\sum_{n=1}^{\infty} \varphi^n(t)\, \theta(\zeta_n, \zeta_m, \zeta_{n+1}) \prod_{j=1}^{n-1} \alpha(\zeta_j, \zeta_m, \zeta_{j+1}) < +\infty,$$

where φ^n denotes the n-th iterate of φ.

Remark 4. *The condition $\varphi(t) < t$ in $(F\varphi 2)$ is used repeatedly in the fixed point arguments below; it is consistent with $(F\varphi 3)$, since the summability there already forces $\varphi^n(t) \to 0$. A prototypical member of $\Phi_{\theta,\alpha}$, when θ, α are bounded along the orbits considered, is $\varphi(t) = \kappa t$ with $0 \leq \kappa < 1$: then $\varphi^n(t) = \kappa^n t$, the ratio $\varphi^{n+1}(t)/\varphi^n(t) = \kappa$, and the two requirements in $(F\varphi 3)$ hold as soon as $\kappa \cdot \sup \alpha < 1$ and the control products grow sub-geometrically.*

Throughout the sequel we use the following orbit-control hypothesis.

(C_0) For a given $\zeta_0 \in S$ with Picard orbit $\mathcal{O} = \{\zeta_n = P^n\zeta_0 : n \geq 0\}$ and limit ζ' (when the orbit converges), the control functions are bounded on the orbit together with its limit: there is a constant $\Theta^* < \infty$ such that $\theta(\zeta, \mu, s) \leq \Theta^*$ and $\alpha(\zeta, \mu, s) \leq \Theta^*$ for all $\zeta, \mu, s \in \mathcal{O} \cup \{\zeta', P\zeta'\}$. In particular $\theta(\zeta_k, \zeta_m, \zeta_{k+1}) \leq \Theta^*$ and $\alpha(\zeta_k, \zeta_m, \zeta_{k+1}) \leq \Theta^*$ for all $k, m \in \mathbb{N}$.

Theorem 1. *Let $(S, \bar{\Lambda})$ be a complete (θ, α)-metric space, let $\varphi \in \Phi_{\theta,\alpha}$, and let P: $S \to S$ be a continuous self-map satisfying (C_0) and*

$$\bar{\Lambda}(P\zeta, P\mu) \leq \varphi\big(M(\zeta, \mu)\big), \qquad M(\zeta, \mu) = \max\{\bar{\Lambda}(\zeta, \mu), \bar{\Lambda}(\zeta, P\zeta), \bar{\Lambda}(\mu, P\mu)\}, \tag{2}$$

for all $\zeta, \mu \in S$. Then P possesses a unique fixed point in S.

Proof. Fix $\zeta_0 \in S$ and set $\zeta_i = P^i\zeta_0$ for $i \geq 0$. If $\zeta_{i_0} = \zeta_{i_0+1}$ for some i_0, then $\zeta_{i_0} = P\zeta_{i_0}$ is a fixed point and the existence part is finished; we therefore assume $\zeta_i \neq \zeta_{i+1}$, and hence $\bar{\Lambda}(\zeta_i, \zeta_{i+1}) > 0$, for every $i \geq 0$.

Step 1. The consecutive distances satisfy $\bar{\Lambda}(\zeta_i, \zeta_{i+1}) \leq \varphi^i(\bar{\Lambda}(\zeta_0, \zeta_1))$ and tend to 0. Applying the contraction (2) with $\zeta = \zeta_{i-1}$ and $\mu = \zeta_i$ (so that $P\zeta = \zeta_i$ and $P\mu = \zeta_{i+1}$),

$$\bar{\Lambda}(\zeta_i, \zeta_{i+1}) = \bar{\Lambda}(P\zeta_{i-1}, P\zeta_i) \leq \varphi\big(M(\zeta_{i-1}, \zeta_i)\big), \tag{3}$$

where, using $\bar{\Lambda}(\zeta_{i-1}, P\zeta_{i-1}) = \bar{\Lambda}(\zeta_{i-1}, \zeta_i)$ and $\bar{\Lambda}(\zeta_i, P\zeta_i) = \bar{\Lambda}(\zeta_i, \zeta_{i+1})$,

$$M(\zeta_{i-1}, \zeta_i) = \max\{\bar{\Lambda}(\zeta_{i-1}, \zeta_i),\ \bar{\Lambda}(\zeta_{i-1}, \zeta_i),\ \bar{\Lambda}(\zeta_i, \zeta_{i+1})\} = \max\{\bar{\Lambda}(\zeta_{i-1}, \zeta_i),\ \bar{\Lambda}(\zeta_i, \zeta_{i+1})\}.$$

Suppose, for contradiction, that this maximum equalled $\bar{\Lambda}(\zeta_i, \zeta_{i+1})$ for some i. Then Equation (3) and $(F\varphi 2)$ would give

$$0 < \bar{\Lambda}(\zeta_i, \zeta_{i+1}) \leq \varphi\big(\bar{\Lambda}(\zeta_i, \zeta_{i+1})\big) < \bar{\Lambda}(\zeta_i, \zeta_{i+1}),$$

which is impossible. Hence $M(\zeta_{i-1}, \zeta_i) = \bar{\Lambda}(\zeta_{i-1}, \zeta_i)$ for every $i \geq 1$, and Equation (3) becomes $\bar{\Lambda}(\zeta_i, \zeta_{i+1}) \leq \varphi\big(\bar{\Lambda}(\zeta_{i-1}, \zeta_i)\big)$. Since φ is nondecreasing, iterating this inequality yields

$$\bar{\Lambda}(\zeta_i, \zeta_{i+1}) \leq \varphi\big(\bar{\Lambda}(\zeta_{i-1}, \zeta_i)\big) \leq \varphi^2\big(\bar{\Lambda}(\zeta_{i-2}, \zeta_{i-1})\big) \leq \cdots \leq \varphi^i\big(\bar{\Lambda}(\zeta_0, \zeta_1)\big), \qquad i \geq 0. \tag{4}$$

Because every factor θ and α is ≥ 1, the summability in $(F\varphi 3)$ implies in particular that $\sum\limits_{n\geq 1} \varphi^n(t) < \infty$ for $t = \bar{\Lambda}(\zeta_0, \zeta_1)$; consequently $\varphi^n(\bar{\Lambda}(\zeta_0, \zeta_1)) \to 0$, and Equation (4) gives

$$\lim_{i\to\infty} \bar{\Lambda}(\zeta_i, \zeta_{i+1}) = 0. \tag{5}$$

Step 2. The telescoping estimate. We claim that, for all $m > n \geq 1$,

$$\bar{\Lambda}(\zeta_n, \zeta_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \alpha(\zeta_j, \zeta_m, \zeta_{j+1})\right) \theta(\zeta_k, \zeta_m, \zeta_{k+1})\, \bar{\Lambda}(\zeta_k, \zeta_{k+1}), \tag{6}$$

with the convention that the empty product ($k = n$) equals 1. To see how Equation (6) arises, apply $(F\Delta 3)$ to $\bar{\Lambda}(\zeta_n, \zeta_m)$ with intermediate point ζ_{n+1}, then to the resulting $\bar{\Lambda}(\zeta_{n+1}, \zeta_m)$ with intermediate point ζ_{n+2}, and so on:

$$\begin{aligned}
\bar{\Lambda}(\zeta_n, \zeta_m) &\leq \theta(\zeta_n, \zeta_m, \zeta_{n+1})\bar{\Lambda}(\zeta_n, \zeta_{n+1}) + \alpha(\zeta_n, \zeta_m, \zeta_{n+1})\bar{\Lambda}(\zeta_{n+1}, \zeta_m) \\
&\leq \theta(\zeta_n, \zeta_m, \zeta_{n+1})\bar{\Lambda}(\zeta_n, \zeta_{n+1}) + \alpha(\zeta_n, \zeta_m, \zeta_{n+1})\theta(\zeta_{n+1}, \zeta_m, \zeta_{n+2})\bar{\Lambda}(\zeta_{n+1}, \zeta_{n+2}) \\
&\quad + \alpha(\zeta_n, \zeta_m, \zeta_{n+1})\alpha(\zeta_{n+1}, \zeta_m, \zeta_{n+2})\bar{\Lambda}(\zeta_{n+2}, \zeta_m) \\
&\leq \cdots \\
&\leq \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \alpha(\zeta_j, \zeta_m, \zeta_{j+1})\right) \theta(\zeta_k, \zeta_m, \zeta_{k+1})\bar{\Lambda}(\zeta_k, \zeta_{k+1}) \\
&\quad + \left(\prod_{j=n}^{m-1} \alpha(\zeta_j, \zeta_m, \zeta_{j+1})\right) \bar{\Lambda}(\zeta_m, \zeta_m),
\end{aligned}$$

and the last term vanishes because $\bar{\Lambda}(\zeta_m, \zeta_m) = 0$. Formally, Equation (6) is proved by induction on the gap $d = m - n \geq 1$. For $d = 1$ (i.e., $m = n + 1$) the right-hand side is the single term $\theta(\zeta_n, \zeta_{n+1}, \zeta_{n+1})\bar{\Lambda}(\zeta_n, \zeta_{n+1}) \geq \bar{\Lambda}(\zeta_n, \zeta_{n+1})$, since $\theta \geq 1$; so the base case holds. For the inductive step, assume Equation (6) holds for every pair whose gap equals $d - 1$. Applying $(F\Delta 3)$ to $\bar{\Lambda}(\zeta_n, \zeta_m)$ with intermediate point ζ_{n+1},

$$\bar{\Lambda}(\zeta_n, \zeta_m) \leq \theta(\zeta_n, \zeta_m, \zeta_{n+1})\bar{\Lambda}(\zeta_n, \zeta_{n+1}) + \alpha(\zeta_n, \zeta_m, \zeta_{n+1})\bar{\Lambda}(\zeta_{n+1}, \zeta_m).$$

The pair (ζ_{n+1}, ζ_m) has gap $m - (n + 1) = d - 1$, so the induction hypothesis applies to $\bar{\Lambda}(\zeta_{n+1}, \zeta_m)$; multiplying that bound by $\alpha(\zeta_n, \zeta_m, \zeta_{n+1})$ absorbs the factor into the product $\prod_{j=n}^{k-1} \alpha$, and the first term above supplies the missing $k = n$ summand. This reproduces Equation (6) for the pair (ζ_n, ζ_m) and proves the claim.

Step 3. The sequence $\{\zeta_i\}$ is Cauchy. Insert the bound Equation (4) into Equation (6) and use $\prod\limits_{j=n}^{k-1} \alpha \leq \prod\limits_{j=1}^{k-1} \alpha$ (all factors are ≥ 1):

$$\bar{\Lambda}(\zeta_n, \zeta_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=1}^{k-1} \alpha(\zeta_j, \zeta_m, \zeta_{j+1}) \right) \theta(\zeta_k, \zeta_m, \zeta_{k+1})\, \varphi^k\big(\bar{\Lambda}(\zeta_0, \zeta_1)\big) =: \sum_{k=n}^{m-1} u_k.$$

Writing $T_p = \sum\limits_{k=1}^{p} u_k$, this says $\bar{\Lambda}(\zeta_n, \zeta_m) \leq T_{m-1} - T_{n-1}$, so it suffices to prove that the series $\sum\limits_{k\geq 1} u_k$ converges. By (C_0) we have $\theta(\zeta_k, \zeta_m, \zeta_{k+1}) \leq \Theta^*$, hence $u_k \leq \Theta^* w_k$, where

$$w_k = \left(\prod_{j=1}^{k-1} \alpha(\zeta_j, \zeta_m, \zeta_{j+1}) \right) \varphi^k\big(\bar{\Lambda}(\zeta_0, \zeta_1)\big).$$

The successive ratio of the majorant is

$$\frac{w_{k+1}}{w_k} = \alpha(\zeta_k, \zeta_m, \zeta_{k+1})\, \frac{\varphi^{k+1}\big(\bar{\Lambda}(\zeta_0, \zeta_1)\big)}{\varphi^k\big(\bar{\Lambda}(\zeta_0, \zeta_1)\big)},$$

and $(F\varphi 3)$, applied to the orbit $\{\zeta_n\}$ (along which θ, α are bounded by (C_0)), gives $\sup\limits_{m\geq 1} \varlimsup\limits_{k\to\infty} w_{k+1}/w_k < 1$. By the ratio test the series $\sum\limits_k w_k$ converges, uniformly in m; therefore $\sum\limits_k u_k \leq \Theta^* \sum\limits_k w_k < \infty$. Consequently $T_{m-1} - T_{n-1} \to 0$ as $n, m \to \infty$, whence $\bar{\Lambda}(\zeta_n, \zeta_m) \to 0$ and $\{\zeta_i\}$ is Cauchy.

Step 4. Existence of a fixed point. By completeness there exists $\zeta' \in S$ with $\bar{\Lambda}(\zeta_i, \zeta') \to 0$, i.e., $\zeta_i \to \zeta'$. Since P is continuous, $P\zeta_i \to P\zeta'$, that is $\zeta_{i+1} \to P\zeta'$. Thus the sequence $\{\zeta_{i+1}\}$ converges both to ζ' (being a shift of $\{\zeta_i\}$) and to $P\zeta'$. By (C_0) the control functions are bounded on $\mathcal{O} \cup \{\zeta', P\zeta'\}$, so $\sup\limits_i \theta(\zeta', P\zeta', \zeta_{i+1}) \leq \Theta^*$ and $\sup\limits_i \alpha(\zeta', P\zeta', \zeta_{i+1}) \leq \Theta^*$; Proposition 2 then yields $P\zeta' = \zeta'$. Hence ζ' is a fixed point of P.

Step 5. Uniqueness. Let ζ' and s be fixed points with $\zeta' \neq s$, so $\bar{\Lambda}(\zeta', s) > 0$. Since $P\zeta' = \zeta'$ and $Ps = s$, we have $\bar{\Lambda}(\zeta', P\zeta') = \bar{\Lambda}(s, Ps) = 0$, whence $M(\zeta', s) =$

$\max\{\bar{\Lambda}(\zeta', s), 0, 0\} = \bar{\Lambda}(\zeta', s)$. Applying Equation (2) and $(F\varphi 2)$,

$$\bar{\Lambda}(\zeta', s) = \bar{\Lambda}(P\zeta', Ps) \leq \varphi\big(M(\zeta', s)\big) = \varphi\big(\bar{\Lambda}(\zeta', s)\big) < \bar{\Lambda}(\zeta', s),$$

a contradiction. Therefore $\zeta' = s$, and the fixed point is unique. □

Corollary 1. *Let $(S, \bar{\Lambda})$ be a complete (θ, α)-metric space, $\varphi \in \Phi_{\theta,\alpha}$, and $P : S \to S$ a continuous self-map satisfying (C_0) and $\bar{\Lambda}(P\zeta, P\mu) \leq \varphi(\bar{\Lambda}(\zeta, \mu))$ for all $\zeta, \mu \in S$. Then P has a unique fixed point.*

Proof. For all $\zeta, \mu \in S$ we have $\bar{\Lambda}(\zeta, \mu) \leq \max\{\bar{\Lambda}(\zeta, \mu), \bar{\Lambda}(\zeta, P\zeta), \bar{\Lambda}(\mu, P\mu)\} = M(\zeta, \mu)$. Since φ is nondecreasing, $\varphi(\bar{\Lambda}(\zeta, \mu)) \leq \varphi(M(\zeta, \mu))$, and therefore

$$\bar{\Lambda}(P\zeta, P\mu) \leq \varphi(\bar{\Lambda}(\zeta, \mu)) \leq \varphi(M(\zeta, \mu)),$$

so the contraction (2) of Theorem 1 is satisfied. The conclusion follows from Theorem 1. □

Theorem 2. *Let $(S, \bar{\Lambda})$ be a complete (θ, α)-metric space and suppose there exist $r, a \in (0, 1]$ and $h \in [0, 1)$ such that a continuous self-map $P : S \to S$ satisfies (C_0) and:*

(D_1) for all $\zeta, \mu \in S$,

$$\bar{\Lambda}(P\zeta, P\mu) \leq r\,\theta(\zeta, \mu, P\mu)\,\bar{\Lambda}(\zeta, \mu) + a\,\alpha(\zeta, P\zeta, P\mu)\,\bar{\Lambda}(\zeta, P\zeta) + h\,\bar{\Lambda}(\mu, P\mu); \quad (7)$$

(D_2) writing $\lambda = \dfrac{r + a}{1 - h}$ and $\zeta_j = P^j\zeta_0$, the limits $\lim_{k\to\infty} \theta(\zeta_k, \zeta_m, \zeta_{k+1})$ exist, are finite and positive for every m, and

$$\sup_{m\geq 1} \overline{\lim_{k\to\infty}}\ \lambda\,\theta(\zeta_k, \zeta_{k+1}, \zeta_{k+2})\,\alpha(\zeta_k, \zeta_{k+1}, \zeta_{k+2})\,\alpha(\zeta_k, \zeta_m, \zeta_{k+1}) < 1. \quad (8)$$

Then P has a fixed point in S.

Proof. Fix $\zeta_0 \in S$, set $\zeta_j = P^j\zeta_0$, and assume $\zeta_j \neq \zeta_{j+1}$ (otherwise a fixed point already exists), so that $\bar{\Lambda}(\zeta_j, \zeta_{j+1}) > 0$ for all j.

Step 1. A one-step recursion. Take $\zeta = \zeta_{i-1}$ and $\mu = \zeta_i$ in Equation (7). Then $P\zeta_{i-1} = \zeta_i$ and $P\zeta_i = \zeta_{i+1}$, so that $\bar{\Lambda}(\zeta_{i-1}, P\zeta_{i-1}) = \bar{\Lambda}(\zeta_{i-1}, \zeta_i)$ and $\bar{\Lambda}(\zeta_i, P\zeta_i) = \bar{\Lambda}(\zeta_i, \zeta_{i+1})$; hence

$$\bar{\Lambda}(\zeta_i, \zeta_{i+1}) \leq r\,\theta(\zeta_{i-1}, \zeta_i, \zeta_{i+1})\bar{\Lambda}(\zeta_{i-1}, \zeta_i) + a\,\alpha(\zeta_{i-1}, \zeta_i, \zeta_{i+1})\bar{\Lambda}(\zeta_{i-1}, \zeta_i) + h\,\bar{\Lambda}(\zeta_i, \zeta_{i+1}).$$

Collecting the terms containing $\bar{\Lambda}(\zeta_i, \zeta_{i+1})$ and dividing by $1 - h > 0$,

$$\bar{\Lambda}(\zeta_i, \zeta_{i+1}) \leq \frac{r\,\theta(\zeta_{i-1}, \zeta_i, \zeta_{i+1}) + a\,\alpha(\zeta_{i-1}, \zeta_i, \zeta_{i+1})}{1 - h}\,\bar{\Lambda}(\zeta_{i-1}, \zeta_i).$$

Since $\theta, \alpha \geq 1$ we have $r\theta + a\alpha \leq (r + a)\theta\alpha$, and therefore

$$\bar{\Lambda}(\zeta_i, \zeta_{i+1}) \leq \lambda\,\theta(\zeta_{i-1}, \zeta_i, \zeta_{i+1})\,\alpha(\zeta_{i-1}, \zeta_i, \zeta_{i+1})\,\bar{\Lambda}(\zeta_{i-1}, \zeta_i), \qquad \lambda = \frac{r + a}{1 - h}. \quad (9)$$

Step 2. The sequence $\{\zeta_i\}$ is Cauchy. Estimate Equation (6) of Theorem 1 was derived using only the (θ, α)-triangle inequality, hence it is available here as well: for $m > n \geq 1$,

$$\bar{\Lambda}(\zeta_n, \zeta_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=1}^{k-1} \alpha(\zeta_j, \zeta_m, \zeta_{j+1}) \right) \theta(\zeta_k, \zeta_m, \zeta_{k+1})\, \bar{\Lambda}(\zeta_k, \zeta_{k+1}) =: \sum_{k=n}^{m-1} P_k,$$

where again $\prod_{j=n}^{k-1} \alpha \leq \prod_{j=1}^{k-1} \alpha$ was used. It suffices to prove $\sum_{k \geq 1} P_k < \infty$. Since $\bar{\Lambda}(\zeta_k, \zeta_{k+1}) > 0$, we may form the ratio

$$\frac{P_{k+1}}{P_k} = \alpha(\zeta_k, \zeta_m, \zeta_{k+1})\, \frac{\theta(\zeta_{k+1}, \zeta_m, \zeta_{k+2})}{\theta(\zeta_k, \zeta_m, \zeta_{k+1})}\, \frac{\bar{\Lambda}(\zeta_{k+1}, \zeta_{k+2})}{\bar{\Lambda}(\zeta_k, \zeta_{k+1})}.$$

By Equation (9),

$$\frac{\bar{\Lambda}(\zeta_{k+1}, \zeta_{k+2})}{\bar{\Lambda}(\zeta_k, \zeta_{k+1})} \leq \lambda\, \theta(\zeta_k, \zeta_{k+1}, \zeta_{k+2})\, \alpha(\zeta_k, \zeta_{k+1}, \zeta_{k+2}).$$

Moreover, by (D_2) the limit $\lim_k \theta(\zeta_k, \zeta_m, \zeta_{k+1})$ exists, is finite and positive, so the quotient $\theta(\zeta_{k+1}, \zeta_m, \zeta_{k+2})/\theta(\zeta_k, \zeta_m, \zeta_{k+1}) \to 1$ as $k \to \infty$. Combining these facts,

$$\varlimsup_{k\to\infty} \frac{P_{k+1}}{P_k} \leq \sup_{m\geq 1} \varlimsup_{k\to\infty} \lambda\, \theta(\zeta_k, \zeta_{k+1}, \zeta_{k+2})\, \alpha(\zeta_k, \zeta_{k+1}, \zeta_{k+2})\, \alpha(\zeta_k, \zeta_m, \zeta_{k+1}) < 1$$

by Equation (8). The ratio test therefore gives $\sum_k P_k < \infty$ (uniformly in m), so $\bar{\Lambda}(\zeta_n, \zeta_m) \to 0$ as $n, m \to \infty$ and $\{\zeta_i\}$ is Cauchy.

Step 3. Existence of a fixed point. By completeness $\zeta_i \to \zeta'$ for some $\zeta' \in S$. As in Step 4 of Theorem 1, continuity of P gives $\zeta_{i+1} = P\zeta_i \to P\zeta'$, while $\zeta_{i+1} \to \zeta'$; since θ, α are bounded on $\mathcal{O} \cup \{\zeta', P\zeta'\}$ by (C_0), 2 yields $P\zeta' = \zeta'$. Thus ζ' is a fixed point of P. $\square$

Corollary 2. *Under the hypotheses of 2, assume in addition that* $\varlimsup_{i\to\infty} \theta(P^i\zeta, P^i\mu, P^i s) < \frac{1}{r}$ *for all* $\zeta, \mu, s \in S$. *Then the fixed point of* P *is unique.*

Proof. Let ζ' and μ' be fixed points, so $P\zeta' = \zeta'$, $P\mu' = \mu'$ and $\bar{\Lambda}(\zeta', P\zeta') = \bar{\Lambda}(\mu', P\mu') = 0$. Taking $\zeta = \zeta'$, $\mu = \mu'$ in Equation (7), the second and third summands vanish and

$$\bar{\Lambda}(\zeta', \mu') = \bar{\Lambda}(P\zeta', P\mu') \leq r\, \theta(\zeta', \mu', P\mu')\, \bar{\Lambda}(\zeta', \mu').$$

Because ζ', μ' are fixed points, $P^i\zeta' = \zeta'$, $P^i\mu' = \mu'$ and $P^i\mu' = \mu'$ for every i, so $\theta(\zeta', \mu', P\mu') = \varlimsup_i \theta(P^i\zeta', P^i\mu', P^i\mu') < 1/r$ by hypothesis. Hence $\bar{\Lambda}(\zeta', \mu') \leq r \cdot \frac{1}{r}\, \bar{\Lambda}(\zeta', \mu')$ would be strict, i.e., $\bar{\Lambda}(\zeta', \mu') < \bar{\Lambda}(\zeta', \mu')$, unless $\bar{\Lambda}(\zeta', \mu') = 0$. Therefore $\zeta' = \mu'$. $\square$

Example 2. *Let* $S = \{0, 1, 2, 3, \dots\}$ *and define* $P : S \to S$ *by* $P\zeta = \sqrt{\zeta}$ *if* ζ *is a perfect square with* $\zeta \neq 1$, *and* $P\zeta = 0$ *otherwise. Define the distance on the two* *variables ζ, μ only:*

$$\bar{\Lambda}(\zeta,\mu) = \begin{cases} 0, & \zeta = \mu, \\ 1, & \textit{exactly one of } \zeta,\mu \textit{ is even (the other odd)}, \\ \max\{\zeta,\mu\}, & \zeta \neq \mu \textit{ and } \zeta,\mu \textit{ have the same parity}, \end{cases}$$

and let $\theta, \alpha : S^3 \to [1,\infty)$ be

$$\theta(\zeta,\mu,s) = \begin{cases} \zeta+\mu+s, & (\zeta,\mu,s) \neq (0,0,0), \\ 1, & (\zeta,\mu,s) = (0,0,0), \end{cases}$$

$$\alpha(\zeta,\mu,s) = \begin{cases} \zeta+\mu+2s, & (\zeta,\mu,s) \neq (0,0,0), \\ 1, & (\zeta,\mu,s) = (0,0,0). \end{cases}$$

We show that $(S,\bar{\Lambda})$ is a complete (θ,α)-metric space and that P satisfies the φ-contraction Equation (2) *of Theorem 1 with $\varphi(t) = \frac{1}{2}t$, namely*

$$\bar{\Lambda}(P\zeta,P\mu) \leq \tfrac{1}{2}\, M(\zeta,\mu), \qquad M(\zeta,\mu) = \max\{\bar{\Lambda}(\zeta,\mu), \bar{\Lambda}(\zeta,P\zeta), \bar{\Lambda}(\mu,P\mu)\}, \tag{10}$$

for all $\zeta,\mu \in S$; consequently P has the unique fixed point $\zeta = 0$.

$(S,\bar{\Lambda})$ is a (θ,α)-metric space. Axioms $(F\Delta 1)$ and $(F\Delta 2)$ are clear from the definition of $\bar{\Lambda}$. For $(F\Delta 3)$ one checks, case by case according to the parities of ζ,μ,s, that $\bar{\Lambda}(\zeta,\mu) \leq \theta(\zeta,\mu,s)\bar{\Lambda}(\zeta,s) + \alpha(\zeta,\mu,s)\bar{\Lambda}(s,\mu)$; the verification is routine because $\theta(\zeta,\mu,s) = \zeta+\mu+s$ and $\alpha(\zeta,\mu,s) = \zeta+\mu+2s$ dominate the value $\max\{\zeta,\mu\}$ *that $\bar{\Lambda}$ can take, while in the mixed-parity case the left-hand side is* 1 *and the right-hand side is at least* 1.

Completeness. Let $\{\zeta_i\}$ be a Cauchy sequence. Taking $\epsilon = \frac{1}{2}$ in (Ξ2)*, there is i_0 with $\bar{\Lambda}(\zeta_i,\zeta_m) < \frac{1}{2}$ for all $m \geq i \geq i_0$. But $\bar{\Lambda}$ takes values in $\{0\} \cup \{1\} \cup \{2,3,\dots\}$, so $\bar{\Lambda}(\zeta_i,\zeta_m) < \frac{1}{2}$ forces $\bar{\Lambda}(\zeta_i,\zeta_m) = 0$, i.e., $\zeta_i = \zeta_m$. Thus $\{\zeta_i\}$ is eventually constant, say equal to ζ', and $\zeta_i \to \zeta' \in S$. Hence $(S,\bar{\Lambda})$ is complete.*

A key identity. If ζ is a perfect square with $\zeta \geq 4$, then ζ and $\sqrt{\zeta}$ have the same parity (a square is even iff its root is even), so

$$\bar{\Lambda}(\zeta,P\zeta) = \bar{\Lambda}(\zeta,\sqrt{\zeta}) = \max\{\zeta,\sqrt{\zeta}\} = \zeta, \qquad \textit{whence} \qquad M(\zeta,\mu) \geq \zeta. \tag{11}$$

Moreover in this case $\sqrt{\zeta} \geq 2$; note also that $P\zeta$ never takes the value 1*, since $P\zeta = \sqrt{\zeta} = 1$ would require $\zeta = 1$, which is excluded.*

Verification of Equation (10)*. Write $a = P\zeta$ and $b = P\mu$; each of a,b equals* 0 *or a value $\sqrt{\cdot} \geq 2$. If $a = b$ then $\bar{\Lambda}(P\zeta,P\mu) = 0$ and Equation* (10) *is trivial, so assume $a \neq b$. There are two remaining cases.*

Case 1: a and b have opposite parity. *Then $\bar{\Lambda}(a,b) = 1$, and we must show $M(\zeta,\mu) \geq 2$. Since $a \neq b$ have opposite parity, at least one of them is odd; an odd value of P is ≥ 3 (recall P omits the value* 1*), say $a = \sqrt{\zeta}$ is odd with $a \geq 3$. Then $\zeta = a^2 \geq 9$ is a perfect square ≥ 4, so Equation* (11) *gives $M(\zeta,\mu) \geq \zeta \geq 9 \geq 2$. Hence*

$\bar{\Lambda}(P\zeta, P\mu) = 1 \leq \frac{1}{2} \cdot 2 \leq \frac{1}{2}M(\zeta, \mu)$.

Case 2: a and b have the same parity. *Then $\bar{\Lambda}(a, b) = \max\{a, b\}$; say $a > b$, so $a \geq 2$ and hence $a = \sqrt{\zeta}$ with $\zeta = a^2 \geq 4$ a perfect square. By Equation (11), $M(\zeta, \mu) \geq \zeta = a^2$. Since $a \geq 2$ implies $a \leq \frac{1}{2}a^2$, we obtain*

$$\bar{\Lambda}(P\zeta, P\mu) = \max\{a, b\} = a \leq \tfrac{1}{2}a^2 \leq \tfrac{1}{2}M(\zeta, \mu).$$

Thus Equation (10) holds for all $\zeta, \mu \in S$; the extremal case $(\zeta, \mu) = (0, 4)$ gives $\bar{\Lambda}(P0, P4) = \bar{\Lambda}(0, 2) = 2 = \frac{1}{2} \cdot 4 = \frac{1}{2}M(0, 4)$, so equality is attained and the constant $\frac{1}{2}$ is sharp. Finally, $P^i\zeta_0 = 0$ for all sufficiently large i (repeated square-rooting of a non-square eventually reaches 0), so along every orbit θ, α are eventually equal to 1 and are, in particular, bounded; hence (C_0) holds and $\varphi(t) = \frac{1}{2}t \in \Phi_{\theta,\alpha}$. By Theorem 1, P has a unique fixed point, namely $\zeta = 0$.

We conclude the section with a result for γ-admissible mappings. Recall that $P : S \to S$ is *γ-admissible* if there is $\gamma : S \times S \to [0, \infty)$ with $\gamma(\zeta, \mu) \geq 1 \Rightarrow \gamma(P\zeta, P\mu) \geq 1$, and that P is a *γ-φ contraction* if $\gamma(\zeta, \mu)\, \bar{\Lambda}(P\zeta, P\mu) \leq \varphi(\bar{\Lambda}(\zeta, \mu))$ for all $\zeta, \mu \in S$, for some $\varphi \in \Phi_{\theta,\alpha}$.

Theorem 3. *Let $(S, \bar{\Lambda})$ be a complete (θ, α)-metric space and let $P : S \to S$ be a continuous γ-φ contraction for some $\varphi \in \Phi_{\theta,\alpha}$, satisfying (C_0). Assume P is γ-admissible and that there is $\zeta_0 \in S$ with $\gamma(\zeta_0, P\zeta_0) \geq 1$. Then P has a fixed point; if, moreover, $\gamma(a, b) \geq 1$ for every pair of fixed points a, b, the fixed point is unique.*

Proof. Set $\zeta_n = P^n\zeta_0$; we may assume $\zeta_n \neq \zeta_{n+1}$ for all n, so $\bar{\Lambda}(\zeta_n, \zeta_{n+1}) > 0$.

Step 1. Admissibility propagates along the orbit. By hypothesis $\gamma(\zeta_0, \zeta_1) = \gamma(\zeta_0, P\zeta_0) \geq 1$. Since P is γ-admissible, $\gamma(\zeta_0, \zeta_1) \geq 1$ implies $\gamma(P\zeta_0, P\zeta_1) = \gamma(\zeta_1, \zeta_2) \geq 1$, and inductively $\gamma(\zeta_n, \zeta_{n+1}) \geq 1$ for every $n \geq 0$.

Step 2. Geometric decay of consecutive distances. Using $\gamma(\zeta_{n-1}, \zeta_n) \geq 1$ and the γ-φ contraction,

$$\bar{\Lambda}(\zeta_n, \zeta_{n+1}) \leq \gamma(\zeta_{n-1}, \zeta_n)\, \bar{\Lambda}(P\zeta_{n-1}, P\zeta_n) \leq \varphi\big(\bar{\Lambda}(\zeta_{n-1}, \zeta_n)\big).$$

As φ is nondecreasing, iterating gives $\bar{\Lambda}(\zeta_n, \zeta_{n+1}) \leq \varphi^n\big(\bar{\Lambda}(\zeta_0, \zeta_1)\big)$, exactly as in Equation (4); hence $\bar{\Lambda}(\zeta_n, \zeta_{n+1}) \to 0$.

Step 3. The sequence $\{\zeta_n\}$ is Cauchy. The telescoping estimate Equation (6) and the bound $\bar{\Lambda}(\zeta_k, \zeta_{k+1}) \leq \varphi^k(\bar{\Lambda}(\zeta_0, \zeta_1))$ give, exactly as in Step 3 of Theorem 1,

$$\bar{\Lambda}(\zeta_n, \zeta_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=1}^{k-1} \alpha(\zeta_j, \zeta_m, \zeta_{j+1}) \right) \theta(\zeta_k, \zeta_m, \zeta_{k+1})\, \varphi^k\big(\bar{\Lambda}(\zeta_0, \zeta_1)\big),$$

whose tail tends to 0 as $n, m \to \infty$ by the ratio test, using $(F\varphi 3)$ and (C_0). Thus $\{\zeta_n\}$ is Cauchy, and by completeness $\zeta_n \to \zeta$ for some $\zeta \in S$.

Step 4. Existence and uniqueness. Continuity of P gives $\zeta_{n+1} = P\zeta_n \to P\zeta$, while $\zeta_{n+1} \to \zeta$; by (C_0) and 2, $P\zeta = \zeta$. Finally, if a, b are fixed points with $\gamma(a, b) \geq 1$, then, since P is a γ-φ contraction,

$$\bar{\Lambda}(a,b) \le \gamma(a,b)\,\bar{\Lambda}(Pa,Pb) \le \varphi\big(\bar{\Lambda}(a,b)\big) < \bar{\Lambda}(a,b)$$

unless $\bar{\Lambda}(a,b) = 0$; hence $a = b$, and the fixed point is unique. □

4. Applications

4.1. A nonlinear algebraic equation

As a first application, we establish the existence and uniqueness of the solution of a nonlinear algebraic equation.

Theorem 4. *For every integer $m \ge 2$, the equation*

$$(\zeta+1)^m + 1 = (4^m+1)\,\zeta\,(\zeta+1)^m + 4^m\zeta \tag{12}$$

admits exactly one solution $\zeta' \in [0,+\infty)$.

Proof. Let $S = [0,+\infty)$ and $\bar{\Lambda}(\zeta,\mu) = |\zeta-\mu|$. Then $(S,\bar{\Lambda})$ is a complete metric space, hence a complete (θ,α)-metric space with the constant controls $\theta \equiv \alpha \equiv 1$ (Proposition 1(i) with $b = 1$). Define $P : S \to S$ by

$$P\zeta = \frac{(\zeta+1)^m + 1}{(4^m+1)(\zeta+1)^m + 4^m}. \tag{13}$$

Step 1. Fixed points of P are the solutions of Equation (12). For $\zeta \ge 0$, $P\zeta = \zeta$ holds if and only if $\zeta\big[(4^m+1)(\zeta+1)^m + 4^m\big] = (\zeta+1)^m + 1$, i.e., if and only if $(4^m+1)\zeta(\zeta+1)^m + 4^m\zeta = (\zeta+1)^m + 1$, which is exactly Equation (12). Thus the solutions of Equation (12) in $[0,\infty)$ are precisely the fixed points of P.

Step 2. P maps S into $(0,1)$. The numerator $(\zeta+1)^m + 1$ is positive, so $P\zeta > 0$. Moreover the denominator exceeds the numerator, since

$$\big[(4^m+1)(\zeta+1)^m+4^m\big]-\big[(\zeta+1)^m+1\big] = 4^m(\zeta+1)^m+4^m-1 = 4^m\big[(\zeta+1)^m+1\big]-1 > 0.$$

Hence $0 < P\zeta < 1$, and in particular $P(S) \subseteq S$ and P is continuous on S.

Step 3. Computation of $P\zeta - P\mu$. Write $u = (\zeta+1)^m$ and $v = (\mu+1)^m$, and abbreviate $A_\zeta = (4^m+1)u + 4^m$, $A_\mu = (4^m+1)v + 4^m$. Then

$$P\zeta - P\mu = \frac{u+1}{A_\zeta} - \frac{v+1}{A_\mu} = \frac{(u+1)A_\mu - (v+1)A_\zeta}{A_\zeta A_\mu}.$$

Expanding the numerator,

$$\begin{aligned}
(u+1)A_\mu - (v+1)A_\zeta &= (u+1)\big[(4^m+1)v + 4^m\big] - (v+1)\big[(4^m+1)u + 4^m\big] \\
&= (4^m+1)uv + 4^m u + (4^m+1)v + 4^m \\
&\quad - (4^m+1)uv - 4^m v - (4^m+1)u - 4^m \\
&= \big[4^m - (4^m+1)\big]u + \big[(4^m+1) - 4^m\big]v = v - u.
\end{aligned}$$

Therefore

$$P\zeta - P\mu = \frac{v-u}{A_\zeta A_\mu} = \frac{(\mu+1)^m - (\zeta+1)^m}{\big[(4^m+1)(\zeta+1)^m + 4^m\big]\big[(4^m+1)(\mu+1)^m + 4^m\big]}. \quad (14)$$

Step 4. The contraction estimate. Using the elementary factorisation $x^m - y^m = (x-y)\sum_{k=0}^{m-1} x^k y^{m-1-k}$ with $x = \zeta+1$, $y = \mu+1$,

$$\big|(\zeta+1)^m - (\mu+1)^m\big| = |\zeta-\mu| \sum_{k=0}^{m-1} (\zeta+1)^k(\mu+1)^{m-1-k}.$$

For the denominator, $A_\zeta = (4^m+1)(\zeta+1)^m + 4^m > 4^m(\zeta+1)^m$ and likewise, $A_\mu > 4^m(\mu+1)^m$, so $A_\zeta A_\mu > 16^m(\zeta+1)^m(\mu+1)^m$. Finally, since $\zeta+1 \geq 1$ and $\mu+1 \geq 1$, each of the m summands satisfies $(\zeta+1)^k(\mu+1)^{m-1-k} \leq (\zeta+1)^{m-1}(\mu+1)^{m-1}$, whence $\sum_{k=0}^{m-1}(\zeta+1)^k(\mu+1)^{m-1-k} \leq m\,(\zeta+1)^{m-1}(\mu+1)^{m-1}$. Substituting these three estimates into Equation (14),

$$|P\zeta - P\mu| \leq \tfrac{m\,(\zeta+1)^{m-1}(\mu+1)^{m-1}}{16^m(\zeta+1)^m(\mu+1)^m}\,|\zeta-\mu| =$$
$$\tfrac{m}{16^m(\zeta+1)(\mu+1)}\,|\zeta-\mu| \leq \tfrac{m}{16^m}\,|\zeta-\mu|,$$

where the last step uses $(\zeta+1)(\mu+1) \geq 1$.

Step 5. Conclusion. For every integer $m \geq 2$ we have $m/16^m < 1$ (indeed $16^m \geq 256 \gg m$). Hence P satisfies $\bar\Lambda(P\zeta, P\mu) \leq \varphi(\bar\Lambda(\zeta,\mu))$ with $\varphi(t) = \frac{m}{16^m}t$, which belongs to $\Phi_{\theta,\alpha}$ for the constant controls $\theta \equiv \alpha \equiv 1$: indeed φ is nondecreasing, $\varphi(t) < t$ for $t > 0$, and $\sum_n \varphi^n(t) = \sum_n (m/16^m)^n t < \infty$ with ratio $m/16^m < 1$. By Corollary 1, P has a unique fixed point $\zeta' \in S$, which by Step 1 is the unique solution of Equation (12). □

Example 3. *The equation* $(257\mu - 1)(\mu+1)^4 + 256\mu - 1 = 0$ *has a unique real solution* $\mu' \in [0, +\infty)$*. Indeed, expanding and regrouping*

$$(257\mu-1)(\mu+1)^4 + 256\mu - 1 = 0 \iff 257\mu(\mu+1)^4 - (\mu+1)^4 + 256\mu - 1 = 0,$$

and adding and subtracting suitably gives the equivalent form $(\mu+1)^4 + 1 = (4^4+1)\mu(\mu+1)^4 + 4^4\mu$*, since* $4^4 = 256$ *and* $4^4 + 1 = 257$*. This is Equation* (12) *with* $m = 4$*, so the claim follows from Theorem 4.*

4.2. A nonlinear Caputo fractional boundary value problem

To strengthen the connection of the theory with differential equations, we apply the main theorem to a fractional boundary value problem (BVP). Such fixed point treatments of fractional models are the subject of active current research [34–37].

Recall that, for $1 < q \leq 2$, the Caputo derivative of $x \in C^2[0,1]$ is ${}^cD^q x(t) = \frac{1}{\Gamma(2-q)}\int_0^t (t-s)^{1-q}x''(s)\,ds$, and the Riemann–Liouville integral is $I^q g(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)\,ds$, with the property $I^q\,{}^cD^q x(t) = x(t) - x(0) - x'(0)\,t$.

Consider the nonlinear Caputo fractional BVP

$$ {}^{c}D^{q}x(t) + f\big(t, x(t)\big) = 0, \quad t \in [0,1], \qquad x(0) = x(1) = 0, \tag{15}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Lemma 1. $x \in C[0,1]$ *solves Equation* (15) *if and only if*

$$x(t) = \int_0^1 G(t,s)\, f\big(s, x(s)\big)\, ds, \tag{16}$$

where the Green's function is

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t(1-s)^{q-1} - (t-s)^{q-1}, & 0 \le s \le t \le 1, \\ t(1-s)^{q-1}, & 0 \le t \le s \le 1. \end{cases}$$

Proof. Applying I^q to Equation (15) and using $I^q\, {}^{c}D^{q}x(t) = x(t) - x(0) - x'(0)t$,

$$x(t) = x(0) + x'(0)\, t - I^q f\big(\cdot, x(\cdot)\big)(t) = c_0 + c_1 t - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\big(s, x(s)\big)\, ds,$$

with $c_0 = x(0)$, $c_1 = x'(0)$. The condition $x(0) = 0$ gives $c_0 = 0$. The condition $x(1) = 0$ gives

$$0 = c_1 - \tfrac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} f\big(s, x(s)\big)\, ds, \quad \text{so} \quad c_1 = \tfrac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} f\big(s, x(s)\big)\, ds.$$

Substituting c_0, c_1 back,

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \left[\int_0^1 t(1-s)^{q-1} f\big(s, x(s)\big)\, ds - \int_0^t (t-s)^{q-1} f\big(s, x(s)\big)\, ds \right] \\ &= \int_0^1 G(t,s) f\big(s, x(s)\big)\, ds, \end{aligned}$$

where G is the stated Green's function (the term $(t-s)^{q-1}$ is present only for $s \le t$). The steps are reversible, which proves the equivalence. □

Theorem 5. *Assume there is a constant* $L > 0$ *with*

$$|f(t,x) - f(t,y)| \le L\, |x - y| \qquad \textit{for all } t \in [0,1],\ x, y \in \mathbb{R}, \tag{17}$$

and that $L < \frac{1}{2}\, \Gamma(q+1)$. *Then the BVP Equation* (15) *has a unique solution in* $C[0,1]$.

Proof. Let $S = C[0,1]$ and $\bar{\Lambda}(x,y) = \|x-y\|_\infty = \sup_{t \in [0,1]} |x(t) - y(t)|$. Then $(S, \bar{\Lambda})$ is a complete metric space, hence a complete (θ, α)-metric space with $\theta \equiv \alpha \equiv 1$. Define the operator $P : S \to S$ by

$$(Px)(t) = \int_0^1 G(t,s)\, f\big(s, x(s)\big)\, ds.$$

Because f is continuous and $G(t,\cdot)$ is integrable with $t \mapsto G(t,s)$ continuous, $Px \in C[0,1]$, so P is well defined; by Lemma 1, the fixed points of P are exactly the solutions of Equation (15).

Step 1. A bound for $\int_0^1 |G(t,s)|\,ds$. Fix $t \in [0,1]$. On $0 \le s \le t$ we have $|t(1-s)^{q-1} - (t-s)^{q-1}| \le t(1-s)^{q-1} + (t-s)^{q-1}$, while on $t \le s \le 1$ the kernel equals $t(1-s)^{q-1} \ge 0$. Hence

$$\int_0^1 |G(t,s)|\,ds \le \frac{1}{\Gamma(q)}\left[\int_0^1 t(1-s)^{q-1}\,ds + \int_0^t (t-s)^{q-1}\,ds\right].$$

Now $\int_0^1 (1-s)^{q-1}\,ds = \frac{1}{q}$ and $\int_0^t (t-s)^{q-1}\,ds = \frac{t^q}{q}$, so, using $\Gamma(q+1) = q\,\Gamma(q)$,

$$\int_0^1 |G(t,s)|\,ds \le \frac{1}{\Gamma(q)}\Big[\frac{t}{q} + \frac{t^q}{q}\Big] = \frac{t+t^q}{\Gamma(q+1)} \le \frac{2}{\Gamma(q+1)}, \tag{18}$$

because $t \le 1$ and $t^q \le 1$ on $[0,1]$.

Step 2. P is a contraction. For $x, y \in S$ and any $t \in [0,1]$, the Lipschitz conditions (17) and (18) give

$$\begin{aligned} |(Px)(t) - (Py)(t)| &\le \int_0^1 |G(t,s)|\,\big|f(s,x(s)) - f(s,y(s))\big|\,ds \\ &\le L\,\|x-y\|_\infty \int_0^1 |G(t,s)|\,ds \le \frac{2L}{\Gamma(q+1)}\,\|x-y\|_\infty. \end{aligned}$$

Taking the supremum over $t \in [0,1]$,

$$\bar{\Lambda}(Px,Py) = \|Px - Py\|_\infty \le \frac{2L}{\Gamma(q+1)}\,\bar{\Lambda}(x,y).$$

Step 3. Conclusion. By hypothesis $\kappa := \dfrac{2L}{\Gamma(q+1)} < 1$. Thus $\varphi(t) = \kappa t$ belongs to $\Phi_{\theta,\alpha}$ (it is nondecreasing, satisfies $\varphi(t) < t$, and $\sum_n \kappa^n t < \infty$), and P satisfies $\bar{\Lambda}(Px,Py) \le \varphi(\bar{\Lambda}(x,y))$. Since P is continuous, Corollary 1 yields a unique fixed point of P in $C[0,1]$, which by Lemma 1 is the unique solution of the BVP Equation (15). □

Example 4. *Take* $q = \frac{3}{2}$ *and* $f(t,x) = \dfrac{1}{4}\sin x + \dfrac{t}{5}$*. Then* $|f(t,x) - f(t,y)| = \frac{1}{4}|\sin x - \sin y| \le \frac{1}{4}|x-y|$*, so Equation* (17) *holds with* $L = \frac{1}{4}$*. Here* $\frac{1}{2}\Gamma(q+1) = \frac{1}{2}\Gamma(\frac{5}{2}) = \frac{1}{2}\cdot\frac{3\sqrt{\pi}}{4} = \frac{3\sqrt{\pi}}{8} \approx 0.6647$*, and* $L = 0.25 < 0.6647$*. By Theorem 5, the boundary value problem*

$${}^cD^{3/2}x(t) + \tfrac{1}{4}\sin x(t) + \tfrac{t}{5} = 0, \quad t \in [0,1], \qquad x(0) = x(1) = 0,$$

has a unique solution in $C[0,1]$.

5. Conclusion

In the present work we introduced the class of (θ,α)-metric spaces, which unifies b-metric spaces, new extended b-metric spaces, controlled metric type spaces and double controlled metric type spaces; the corresponding reductions were established explicitly in Proposition 1. Within this class we proved fixed point theorems for φ-contractive and Reich–Ćirić–Rus–type mappings, with the special cases presented as corollaries of a single general theorem, and we supplied non-trivial examples confirming that the framework is strictly broader than a b-metric space. As applications, we proved the existence and uniqueness of the solution of a nonlinear algebraic equation and of a nonlinear Caputo fractional boundary value problem, the latter reinforcing the relevance of the results to differential equations and control processes. A natural direction for future work is to study coupled and multivalued analogues in (θ, α)-metric spaces and their applications to fractional integro-differential systems.

Author contributions: Conceptualization, MA and UI; methodology, ZB and ILP; software, UI; validation, TK; formal analysis, ILP; investigation, MA; resources, MA; data curation, UI; original draft preparation, MA and ZB; review and editing, UI, TK and ILP; visualization, MA; supervision, TK and ILP; project administration, UI; funding acquisition, MA. All authors have read and agreed to the published version of the manuscript.

Funding: This research work was funded by the Deanship of Scientific Research, Islamic University of Madinah, Saudi Arabia.

Institutional review board statement: Not applicable.

Informed consent statement: Not applicable.

Data availability statement: This article has no associated data, since no datasets were produced or examined during the course of the study.

Acknowledgment: The authors extend their appreciation to the Deanship of Scientific Research, Islamic University of Madinah, Saudi Arabia, for funding this research work.

Conflict of interest: The authors declare that they have no competing interests.

AI use statement: During the preparation of this manuscript, the authors used Claude-AI to enhance the clarity, readability, and overall quality of the language. The authors critically reviewed, edited, and verified all generated content and accept full responsibility for the accuracy and integrity of the final published manuscript.

References

1. Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae. 1922; 3: 133–181.
2. Bakhtin IA. The contraction mapping principle in almost metric spaces. Functional Analysis. 1989; 30: 26–37.
3. Czerwik S. Contraction mappings in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis. 1993; 1: 5–11.
4. Huang H, Deng G, Radenović S. Fixed point theorems in b-metric spaces with applications to differential equations. Journal of Fixed Point Theory and Applications. 2018; 20(1): 52. doi: 10.1007/s11784-018-0491-z
5. Rezaei Roshan J, Parvaneh V, Sedghi S, et al. Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces. Fixed Point Theory and Applications. 2013; 2013(1): 159. doi: 10.1186/1687-1812-2013-159

6. Mukheimer A, Mlaiki N, Abodayeh K, et al. New theorems on extended b-metric spaces under new contractions. Nonlinear Analysis: Modelling and Control. 2019; 24(6). doi: 10.15388/NA.2019.6.2
7. Shatanawi W, Pitea A, Lazović R. Contraction conditions using comparison functions on b-metric spaces. Fixed Point Theory and Applications. 2014; 2014(1): 135. doi: 10.1186/1687-1812-2014-135
8. Younis M, Singh D, Shi L. Revisiting graphical rectangular b-metric spaces. Asian-European Journal of Mathematics. 2022; 15(4): 2250072. doi: 10.1142/S1793557122500723
9. Abdeljawad T, Abodayeh K, Mlaiki N. On fixed point generalizations to partial b-metric spaces. Journal of Computational Analysis and Applications. 2015; 19(5): 883–891.
10. Shatanawi W, Mustafa Z, Tahat N. Some Coincidence Point Theorems for Nonlinear Contraction in Ordered Metric Spaces. Fixed Point Theory and Applications. 2011; 2011(1): 68. doi: 10.1186/1687-1812-2011-68
11. Rasham T, Shabbir MS, Agarwal P, et al. On a pair of fuzzy dominated mappings on closed ball in the multiplicative metric space with applications. Fuzzy Sets and Systems. 2022; 437: 81–96. doi: 10.1016/j.fss.2021.09.002
12. Gupta V, Gondhi A. Fixed points of weakly compatible maps on modified intuitionistic fuzzy soft metric spaces. International Journal of System Assurance Engineering and Management. 2022; 13(3): 1232–1238. doi: 10.1007/s13198-021-01423-1
13. Gupta V, Mani N, Sharma R, et al. Some fixed point results and their applications on integral type contractive condition in fuzzy metric spaces. Boletim da Sociedade Paranaense de Matemática. 2022; 40: 1–9. doi: 10.5269/bspm.51777
14. Chauhan SS, Gupta V. Banach contraction theorem on fuzzy cone b-metric space. Journal of Applied Research and Technology. 2020; 18(4): 154–160. doi: 10.22201/icat.24486736e.2020.18.4.1188
15. Gamal M, Rasham T, Cholamjiak W, et al. New iterative scheme for fixed point results of weakly compatible maps in multiplicative G_M-metric space via various contractions with application. AIMS Mathematics. 2022; 7(8): 13681–13703. doi: 10.3934/math.2022754
16. Rasham T, Nazam M, Aydi H, et al. Hybrid pair of multivalued mappings in modular-like metric spaces and applications. AIMS Mathematics. 2022; 7(6): 10582–10595. doi: 10.3934/math.2022590
17. Younis M, Singh D, Chen L, et al. A study on the solutions of notable engineering models. Mathematical Modelling and Analysis. 2022; 27(3): 492–509. doi: 10.3846/mma.2022.15276
18. Kamran T, Samreen M, Ul Ain Q. A Generalization of b-Metric Space and Some Fixed Point Theorems. Mathematics. 2017; 5(2): 19. doi: 10.3390/math5020019
19. Mlaiki N, Aydi H, Souayah N, et al. Controlled Metric Type Spaces and the Related Contraction Principle. Mathematics. 2018; 6(10): 194. doi: 10.3390/math6100194
20. Aiadi SS, Othman WAM, Wong KB, et al. Fixed point theorems in controlled $J-$metric spaces. AIMS Mathematics. 2023; 8(2): 4753–4763. doi: 10.3934/math.2023235
21. Ahmad H, Younis M, Köksal ME. Double Controlled Partial Metric Type Spaces and Convergence Results. Journal of Mathematics. 2021; 2021: 1–11. doi: 10.1155/2021/7008737
22. Rasham T, Shoaib A, Alshoraify S, et al. Study of multivalued fixed point problems for generalized contractions in double controlled dislocated quasi metric type spaces. AIMS Mathematics. 2021; 7(1): 1058–1073. doi: 10.3934/math.2022063
23. Rezazgui AZ, Tallafha AA, Shatanawi W. Common fixed point results via $\mathcal{A}_\vartheta$-α-contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b-metric space. AIMS Mathematics. 2023; 8(3): 7225–7241. doi: 10.3934/math.2023363
24. Souayah N, Hidri L. New Fixed-Point Results in Controlled Metric Type Spaces with Applications. Axioms. 2025; 14(2): 85. doi: 10.3390/axioms14020085
25. Shammaky AE, Ahmad J. Common Fixed Point Theorems in Complex-Valued Controlled Metric Spaces with Application. Symmetry. 2024; 16(11): 1442. doi: 10.3390/sym16111442
26. Abdeljawad T, Mlaiki N, Aydi H, et al. Double Controlled Metric Type Spaces and Some Fixed Point Results. Mathematics. 2018; 6(12): 320. doi: 10.3390/math6120320
27. Bostan NU, Pazar Varol B. Fixed Point Results in Modular b-Metric-like Spaces with an Application. Axioms. 2024; 13(10): 726. doi: 10.3390/axioms13100726
28. Middlebrook C, Feng W. Integral Operators in b-Metric and Generalized b-Metric Spaces and Boundary Value Problems. Fractal and Fractional. 2024; 8(11): 674. doi: 10.3390/fractalfract8110674
29. Mebarki K, Boudaoui A, Belhenniche A, et al. Fixed-Point Theorems for Covariant and Contravariant Multivalued Mappings in Bipolar b-Metric Spaces. Mathematics. 2025; 13(18): 2983. doi: 10.3390/math13182983
30. Aydi H, Felhi A, Kamran T, et al. On nonlinear contractions in new extended b-metric spaces. Applications and Applied

Mathematics. 2019; 14(1): 537–547.

31. Rhoades BE. Some theorems on weakly contractive maps. Nonlinear Analysis: Theory, Methods & Applications. 2001; 47(4): 2683–2693. doi: 10.1016/S0362-546X(01)00388-1
32. Karapinar E, Petrusel A, Petrusel G. On admissible hybrid Geraghty contractions. Carpathian Journal of Mathematics. 2020; 36(3): 433–442. doi: 10.37193/CJM.2020.03.11
33. Shatanawi W, Abodayeh K, Mukheimer A. Some fixed point theorems in extended b-metric spaces. UPB Scientific Bulletin, Series A: Applied Mathematics and Physics. 2018; 80(4): 71–78.
34. Alamri B. Fixed Point Results in Extended -Metric Spaces with Applications to Caputo Fractional Differential Equations. Fractal and Fractional. 2026; 10(4): 261. doi: 10.3390/fractalfract10040261
35. Alfaqih WM, Sessa S, Saleh HN, et al. Solving Fractional Differential Equations via New Relation-Theoretic Fuzzy Fixed Point Theorems. Mathematics. 2025; 13(16): 2582. doi: 10.3390/math13162582
36. Erkan F, Hamal NA, Ntouyas SK, et al. Existence and Uniqueness Analysis for (k, ψ)-Hilfer and (k, ψ)-Caputo Sequential Fractional Differential Equations and Inclusions with Non-Separated Boundary Conditions. Fractal and Fractional. 2025; 9(7): 437. doi: 10.3390/fractalfract9070437
37. Alharbi N, Hussain N, Alsulami H. Efficient Fixed-Point Method with Application to a Fractional Blood Flow Model. Fractal and Fractional. 2025; 9(11): 752. doi: 10.3390/fractalfract9110752