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Article

Novel fixed point theorems for order-theoretic enriched contractions with applications to boundary value problems and matrix equations

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Abstract: This work puts forward a family of operators, which we term enriched ∇-preserving contractions, acting on a normed space carrying an arbitrary binary relation ∇. The family gathers several familiar classes under one roof: Banach contractions, order-theoretic contractions, and certain nonexpansive maps, among them averaged maps whose fixed points lie beyond the reach of ordinary contraction arguments; all arise through suitable choices of the constants and of the relation. Working in this relational framework, we show that any such operator admits a fixed point, even though its defining contractive estimate is demanded only on ∇-related pairs rather than across the whole space. A device used repeatedly is the symmetry of the norm: it forces ∇ and its symmetric closure to act compatibly with the operator, and this is what keeps the fixed-point conclusions meaningful. The fixed points themselves are located through the Krasnoselskii iteration, and a path criterion formulated in the symmetric closure of ∇ then yields their uniqueness; an almost contraction variant of the scheme is treated as well. By way of application, the theory settles the solvability, and under the stated hypotheses the uniqueness, of solutions to a Caputo fractional boundary value problem with an integral condition and to a nonlinear matrix equation. Numerical experiments over several groups of Lipschitz coefficients confirm the convergence threshold predicted by the theory, and the iteration is compared with Riemann-Hilbert analysis, Lie group integrators, physics-informed neural networks, and neural symbolic derivation tools, so that strengths and limits of the framework can be judged objectively.

Keywords: fixed point; enriched contraction; binary relation; order-theoretic contraction; almost contraction; Krasnoselskii iteration; matrix equation; fractional differential equation

1. Introduction and preliminaries

Few results in analysis are invoked as often as the Banach contraction principle [1], which underlies a great deal of both theoretical and applied work. The literature that grew out of it proceeds along two broad lines: widening the class of maps to which a contraction-type argument applies, and recasting the principle over more general ambient structures, together with the applications these advances make possible.

A decisive step in the first direction is due to Alam and Imdad [2]. Their idea was to weaken the contractive hypothesis itself: instead of insisting that the estimate hold for every pair of points, they require it only for those pairs linked by a prescribed binary relation. Spaces handled in this way are commonly referred to as relational metric spaces, and the viewpoint has been taken up widely. As one illustration, Din and coauthors [3] returned to the Perov-type theorems from this angle, working out applications in vector-valued metric spaces and supplying explicit examples. Sharper versions of the conclusions [2], valid under weaker assumptions, were subsequently obtained by Alam et al. [4]. The relational viewpoint has continued to widen since then: almost contractions were carried over to arbitrary binary relations by Khan [5], contractions governed by (c)-comparison functions were treated by Algehyne et al. [6], Geraghty-type maps with applications to fractional differential equations appear in the work of Alamer et al. [7], fixed-point algorithms on relational m-metric spaces were designed by Tariq et al. [8], and nonlinear almost contractions of Pant type under binary relations were examined by Alshaban et al. [9].

A second line of generalisation originates with Berinde and Păcurar [10], who broadened the Banach notion by isolating what are now called enriched contractions; the same authors developed Kannan-type versions of the idea [11]. This class properly contains the Banach contractions while also reaching particular nonexpansive maps that lie entirely outside the Banach setting, so that a wider variety of problems becomes accessible. The key observation of Berinde and Păcurar is that an enriched contraction, although not necessarily a contraction on its own, admits an averaged form that is; its unique fixed point can therefore still be reached by Krasnoselskii iteration in a Banach space. Enriched contractions thus furnish a versatile vehicle for fixed-point theory, one that extends to problems beyond the reach of ordinary contractions.

Let us recall the precise definition. In a normed space $(V, \|\cdot\|)$, a self-map $\Upsilon : V \to V$ qualifies as an enriched contraction provided one can find constants $\mathring{\mathfrak{m}} \geq 0$ and $\hbar \in [0, \mathring{\mathfrak{m}} + 1)$ with

$$\|\mathring{\mathfrak{m}}(\mathring{\mho} - \mathring{\mathfrak{v}}) + \Upsilon(\mathring{\mho}) - \Upsilon(\mathring{\mathfrak{v}})\| \leq \hbar \, \|\mathring{\mho} - \mathring{\mathfrak{v}}\|, \quad \text{for all } \mathring{\mho}, \mathring{\mathfrak{v}} \in V.$$

Because it interpolates between the contractive and the nonexpansive regimes yet still secures a unique fixed point, this concept has proved useful in the metric theory of fixed points. Suppose $\mathfrak{Z}$ is a convex subset of V, let $\Upsilon : \mathfrak{Z} \to \mathfrak{Z}$, and pick $\mathring{g} \in (0, 1]$. The associated averaged mapping is $\Upsilon_{\mathring{g}} : \mathfrak{Z} \to \mathfrak{Z}$, given by $\Upsilon_{\mathring{g}}(\mathring{\mathfrak{v}}) = (1 - \mathring{g})\mathring{\mathfrak{v}} + \mathring{g}\Upsilon(\mathring{\mathfrak{v}})$. One verifies at once that Υ and $\Upsilon_{\mathring{g}}$ share the same set of fixed points. Enriched Ćirić-Reich-Rus contractions were investigated by Berinde and Păcurar [12], enriched φ-contractions [13], and enriched weak contractions combined with order-theoretic structure [14]; for an extensive treatment of these themes we refer to the literature [10, 12, 14] and the works cited there.

Alongside these analytic developments, the computational treatment of nonlinear differential models has been reshaped by several families of tools, and it is useful to record where fixed-point iteration stands among them. For integrable models, Riemann-Hilbert analysis produces exact soliton profiles together with their long-time behaviour, as in the improved treatment of the derivative nonlinear Schrödinger equation by Kuang and Tian [15]. Lie group and geometry-preserving integrators approximate flows while respecting the invariants of the underlying system, and so retain its qualitative features over long horizons [16]. Physics-informed neural networks embed the residual of a differential equation in a training loss [17]; fractional variants extend them to Caputo-type operators [18], and a refined version built on a quadratic residual term with reinforcement-learning-based sampling has recently been used to predict PT-symmetric solitons in competing cubic-quintic saturable media [19]. In parallel, neural symbolic architectures such as Kolmogorov-Arnold networks recover closed-form structure from data [20], and derivation methods enhanced by artificial intelligence aim at automating parts of the symbolic work that analysis requires [21]. These techniques reach high-dimensional and strongly nonlinear problems that closed-form analysis cannot touch, but they generally arrive without existence or uniqueness certificates for the underlying problem. Fixed-point theorems of the kind developed here supply that missing certificate, and they do so constructively, because the Krasnoselskii scheme that carries the proofs is at the same time a convergent algorithm. Section 3.1 quantifies this behaviour on a fractional boundary value problem, and Section 3.2 places the approaches side by side.

The order-theoretic terminology used below is collected next.

Definition 1 ([22])**.** *Fix a binary relation $\nabla \subseteq V^2$ and two points $\mathring{\mathfrak{v}}, \mathring{\partial} \in V$. These points are said to be ∇-comparable when at least one of $(\mathring{\mathfrak{v}}, \mathring{\partial}) \in \nabla$ and $(\mathring{\partial}, \mathring{\mathfrak{v}}) \in \nabla$ holds; we then write $[\mathring{\mathfrak{v}}, \mathring{\partial}] \in \nabla$.*

Proposition 1 ([2])**.** *Let $(V, \mathring{\Omega})$ be a metric space carrying a relation $\nabla \subseteq V^2$, let $\Upsilon : V \to V$, and let $\digamma \in [0, 1)$. Then the two contractive requirements below coincide:*

(I) $(\mathring{\mho}, \mathring{\partial}) \in \nabla \implies \mathring{\Omega}(\Upsilon\mathring{\mho}, \Upsilon\mathring{\partial}) \leq \digamma\mathring{\Omega}(\mathring{\mho}, \mathring{\partial})$,

(II) $[\mathring{\mho}, \mathring{\partial}] \in \nabla \implies \mathring{\Omega}(\Upsilon\mathring{\mho}, \Upsilon\mathring{\partial}) \leq \digamma\mathring{\Omega}(\mathring{\mho}, \mathring{\partial})$.

Definition 2 ([22])**.** *Suppose $V \neq \varnothing$ and $\nabla \subseteq V^2$. The inverse of ∇ is $\nabla^{-1} = \{(\mathring{\mho}, \mathring{\partial}) \in V^2 : (\mathring{\partial}, \mathring{\mho}) \in \nabla\}$; its reflexive closure is $\nabla^{\#} = \nabla \cup \Delta_V$, where Δ_V denotes the diagonal of V; and its symmetric closure is $\nabla^s = \nabla \cup \nabla^{-1}$.*

Proposition 2 ([2,22])**.** *For any $\nabla \subseteq V^2$, $[\mathring{\mathfrak{v}}, \mathring{\partial}] \in \nabla \iff (\mathring{\mathfrak{v}}, \mathring{\partial}) \in \nabla^s$.*

Definition 3 ([2])**.** *Take $\nabla \subseteq V^2$. A sequence $\{\mathring{\mathfrak{v}}_j\} \subset V$ is termed ∇-preserving when $(\mathring{\mathfrak{v}}_j, \mathring{\mathfrak{v}}_{j+1}) \in \nabla, \quad \forall j \in \mathbb{N}_0$.*

Definition 4 ([2])**.** *Consider a metric space $(V, \mathring{\Omega})$ together with $\nabla \subseteq V^2$. We call ∇ $\mathring{\Omega}$-self-closed if, whenever a ∇-preserving sequence $\{\mathring{\mathfrak{v}}_j\}$ converges to a point $\mathring{\mathfrak{v}}$ in $(V, \mathring{\Omega})$, some subsequence $\{\mathring{\mathfrak{v}}_{j_l}\}$ satisfies $(\mathring{\mathfrak{v}}_{j_l}, \mathring{\mathfrak{v}}) \in \nabla$ for every $l \in \mathbb{N}_0$.*

Definition 5 ([2])**.** *Let $\Upsilon : V \to V$ and $\nabla \subseteq V^2$. We say ∇ is Υ-closed provided that, for every $\mathring{\mho}, \mathring{\partial} \in V$,*

$$(\mathring{\mho}, \mathring{\partial}) \in \nabla \implies (\Upsilon\mathring{\mho}, \Upsilon\mathring{\partial}) \in \nabla.$$

Proposition 3 ([23])**.** *Whenever ∇ is Υ-closed, it is Υ^j-closed for each $j \in \mathbb{N}_0$.*

Proposition 4 ([2])**.** *With V, Υ, and ∇ as in Definition 5, the Υ-closedness of ∇ implies that of ∇^s.*

Definition 6 ([24])**.** *A metric space $(V, \mathring{\Omega})$ is ∇-complete when each ∇-preserving Cauchy sequence in V has a limit in V.*

Every complete metric space is automatically ∇-complete, whatever ∇ may be; the universal relation recovers ordinary completeness as the special case of ∇-completeness.

Definition 7 ([24])**.** *A map Υ is ∇-continuous at a point $\mathring{\mathfrak{v}} \in V$ if $\Upsilon(\mathring{\mathfrak{v}}_j) \to \Upsilon(\mathring{\mathfrak{v}})$ for every ∇-preserving sequence $\{\mathring{\mathfrak{v}}_j\}$ with limit $\mathring{\mathfrak{v}}$; it is ∇-continuous on V when this holds at each point of V.*

Ordinary continuity always entails ∇-continuity for any ∇; for the universal relation the converse holds too, so the two notions agree.

Definition 8 ([25])**.** *Let $V \neq \varnothing$ and $\nabla \subseteq V^2$. A subset $\mathcal{E} \subseteq V$ is ∇-directed when every pair $\mathring{\mathfrak{v}}, \mathring{\partial} \in \mathcal{E}$ admits a common $\mathring{\pi} \in V$ with $(\mathring{\mathfrak{v}}, \mathring{\pi}) \in \nabla \quad$ and $\quad (\mathring{\partial}, \mathring{\pi}) \in \nabla$.*

Definition 9 ([26])**.** *Let $V \neq \varnothing$ and $\nabla \subseteq V^2$. For $\mathring{\mathfrak{v}}, \mathring{\partial} \in V$ and $k \in \mathbb{N}$, by a path of length k in ∇ joining $\mathring{\mathfrak{v}}$ to $\mathring{\partial}$ we mean a finite list $\{\mathring{\pi}_0, \mathring{\pi}_1, \mathring{\pi}_2, \ldots, \mathring{\pi}_k\} \subseteq V$ for which $\mathring{\pi}_0 = \mathring{\mathfrak{v}}$, $\mathring{\pi}_k = \mathring{\partial}$, and consecutive terms satisfy $(\mathring{\pi}_i, \mathring{\pi}_{i+1}) \in \nabla$ for $0 \leq i \leq k-1$.*

The following notation is fixed for the remainder of the paper: $\mathrm{Fix}(\Upsilon)$ stands for the fixed-point set of Υ; ∇_Υ abbreviates $\{\mathring{\mathfrak{v}} \in V : (\mathring{\mathfrak{v}}, \Upsilon(\mathring{\mathfrak{v}})) \in \nabla\}$; $\mathrm{Paths}(\mathring{\mathfrak{v}}, \mathring{\mho}; \nabla)$ is the totality of paths in ∇ running from $\mathring{\mathfrak{v}}$ to $\mathring{\mho}$; and to each $\mathring{\mathfrak{m}} \geq 0$ we attach $\mathring{g} = \frac{1}{\mathring{\mathfrak{m}}+1} \in (0, 1]$.

Drawing on the enriched-contraction programme of Berinde and Păcurar [10, 12] and the relation-theoretic programme of Alam, Imdad, and their collaborators [2, 4], we put forward the notion of an enriched ∇-preserving contraction and prove the attendant fixed-point theorems over a normed space equipped with an arbitrary relation. Several worked examples accompany the theory, and two applications follow: existence and, where warranted, uniqueness of solutions for a fractional-differential-equation boundary value problem and for a nonlinear matrix equation. A numerical study then verifies the contraction threshold of the fractional application over batches of Lipschitz coefficients, reports the residual statistics of the resulting iterations, and compares the scheme with the computational tools recalled above.

2. Main results

To begin, we recast ∇-completeness, ∇-continuity, and $\|\cdot\|$-self-closedness for normed spaces.

Definition 10 ([14])**.** *A normed space $(V, \|\cdot\|)$ is ∇-complete if each ∇-preserving Cauchy sequence has a norm limit in V.*

Definition 11 ([14])**.** *A map Υ is ∇-continuous at $\mathring{\mathfrak{v}}$ when $\|\Upsilon(\mathring{\mathfrak{v}}_j) - \Upsilon(\mathring{\mathfrak{v}})\| \to 0$ for every ∇-preserving sequence $\{\mathring{\mathfrak{v}}_j\}$ satisfying $\|\mathring{\mathfrak{v}}_j - \mathring{\mathfrak{v}}\| \to 0$; it is ∇-continuous on V when the same is true at each $\mathring{\mathfrak{v}} \in V$.*

Definition 12 ([14])**.** *Let $(V, \|\cdot\|)$ be a normed space. We call ∇ $\|\cdot\|$-self-closed if, for any ∇-preserving sequence $\{\mathring{\mathfrak{v}}_j\}$ converging in norm to $\mathring{\mathfrak{v}}$, there is a subsequence $\{\mathring{\mathfrak{v}}_{j_l}\}$ with $[\mathring{\mathfrak{v}}_{j_l}, \mathring{\mathfrak{v}}] \in \nabla$ for every $l \in \mathbb{N}_0$.*

With these in hand, we introduce the central definition of the paper.

Definition 13. *Let $(V, \|\cdot\|)$ be a normed space carrying a binary relation ∇. A self-map $\Upsilon : V \to V$ is an enriched ∇-preserving contraction if there are constants $\mathring{\mathfrak{m}} \geq 0$ and $\hbar \in [0, \mathring{\mathfrak{m}} + 1)$ for which*

$$\|\mathring{\mathfrak{m}}(\mathring{\mho} - \mathring{\partial}) + \Upsilon\mathring{\mho} - \Upsilon\mathring{\partial}\| \leq \hbar\|\mathring{\mho} - \mathring{\partial}\| \tag{1}$$

holds for every $\mathring{\mho}, \mathring{\partial} \in V$ with $(\mathring{\mho}, \mathring{\partial}) \in \nabla$.

To make the constants explicit, we shall speak of an $(\mathring{\mathfrak{m}},\hbar)$-enriched ∇-preserving contraction.

Since the norm does not distinguish $\mathring{\mho} - \mathring{\partial}$ from $\mathring{\partial} - \mathring{\mho}$, the contractive estimate is insensitive to swapping $\mathring{\mho}$ and $\mathring{\partial}$. The following is immediate.

Proposition 5. *A map Υ is an enriched ∇-preserving contraction precisely when it is an enriched ∇^s-preserving contraction.*

Remark 1. *The name enriched ∇-preserving contraction records two facts: the contractive estimate* (1) *is imposed only along ∇-related pairs, and the Krasnoselskii sequences generated in the proofs below are ∇-preserving in the sense of Definition 3. The requirement that the averaged operator $\Upsilon_{\mathring{g}}$ carry ∇-related pairs to ∇-related pairs is not part of Definition 13 itself; it enters separately, through condition (2) of Property 1.*

The structural assumptions linking the space $(V, \|\cdot\|)$, the relation ∇, and the map Υ are gathered into the following property, which precedes our fixed-point theorem for $(\mathring{\mathfrak{m}},\hbar)$-enriched ∇-preserving contractions.

Property 1. *Let $(V, \|\cdot\|)$ be a normed space, ∇ a binary relation on V, and $\Upsilon : V \to V$. The triple (V, ∇, Υ) is said to satisfy Property 1 if:*

1. *$(V, \|\cdot\|)$ is ∇-complete;*
2. *∇ is $\Upsilon_{\mathring{g}}$-closed, where $\mathring{g} = \frac{1}{\mathring{\mathfrak{m}}+1}$;*
3. *either ∇ is $\|\cdot\|$-self-closed or $\Upsilon_{\mathring{g}}$ is ∇-continuous;*
4. *$\nabla_{\Upsilon_{\mathring{g}}} \neq \varnothing$.*

Theorem 1. *Suppose $(V, \|\cdot\|)$ is a normed space, $\nabla \subseteq V \times V$, and Υ is an $(\mathring{\mathfrak{m}},\hbar)$-enriched ∇-preserving contraction for which Property 1 is satisfied. Then Υ*

possesses a fixed point in V.

Proof. The argument is split into four steps.

Step 1. Unravelling the definition of an enriched ∇-preserving contraction, we may recast Equation (1) as

$$\|(\frac{1}{\mathring{g}}-1)(\mathring{\mho}-\mathring{\partial})+\Upsilon\mathring{\mho}-\Upsilon\mathring{\partial}\| \leq \hbar\|\mathring{\mho}-\mathring{\partial}\|, \quad \forall\mathring{\mho},\mathring{\partial}\in V \text{ with } (\mathring{\mho},\mathring{\partial})\in\nabla,$$

equivalently,

$$\|(1-\mathring{g})(\mathring{\mho}-\mathring{\partial})+\mathring{g}(\Upsilon\mathring{\mho}-\Upsilon\mathring{\partial})\| \leq \mathring{g}\hbar\|\mathring{\mho}-\mathring{\partial}\|, \quad \forall\mathring{\mho},\mathring{\partial}\in V \text{ with } (\mathring{\mho},\mathring{\partial})\in\nabla.$$

Putting $\digamma := \mathring{g}\hbar \in [0,1)$, the estimate reads

$$\|\Upsilon_{\mathring{g}}\mathring{\mho}-\Upsilon_{\mathring{g}}\mathring{\partial}\| \leq \digamma\|\mathring{\mho}-\mathring{\partial}\|, \quad \forall\mathring{\mho},\mathring{\partial}\in V \text{ with } (\mathring{\mho},\mathring{\partial})\in\nabla. \tag{2}$$

Note that $\digamma < 1$ holds exactly because $\hbar < \mathring{\mathfrak{m}}+1 = 1/\mathring{g}$.

Step 2. Part 4 in Property 1 supplies a point $\mathring{\mho}_0 \in \nabla_{\Upsilon_{\mathring{g}}}$. Beginning at $\mathring{\mho}_0$, form the Krasnoselskii iterates $\{\mathring{\mho}_j\}$ via $\mathring{\mho}_{j+1} = (1-\mathring{g})\mathring{\mho}_j + \mathring{g}\Upsilon\mathring{\mho}_j = \Upsilon_{\mathring{g}}\mathring{\mho}_j, \quad \forall j\in\mathbb{N}$. As $\left(\mathring{\mho}_0, \Upsilon_{\mathring{g}}\mathring{\mho}_0\right)\in\nabla$, combining the $\Upsilon_{\mathring{g}}$-closedness of ∇ with Proposition 3 gives

$$(\Upsilon_{\mathring{g}}^{j}\mathring{\mho}_0, \Upsilon_{\mathring{g}}^{j+1}\mathring{\mho}_0)\in\nabla, \quad \forall j\in\mathbb{N}. \tag{3}$$

and consequently

$$(\mathring{\mho}_j, \mathring{\mho}_{j+1})\in\nabla, \quad \forall j\in\mathbb{N}. \tag{4}$$

Thus the iterates $\{\mathring{\mho}_j\}$ form a ∇-preserving sequence.

Step 3. Invoking Equation (2) alongside Equations (3) and (4), we find for each $j\in\mathbb{N}$,

$$\|\mathring{\mho}_{j+1}-\mathring{\mho}_{j+2}\| = \|\Upsilon_{\mathring{g}}\mathring{\mho}_j-\Upsilon_{\mathring{g}}\mathring{\mho}_{j+1}\| \leq \digamma\|\mathring{\mho}_j-\mathring{\mho}_{j+1}\|$$

which, by induction, yields

$$\|\mathring{\mho}_{j+1}-\mathring{\mho}_{j+2}\| \leq \digamma^{j+1}\|\mathring{\mho}_0-\Upsilon_{\mathring{g}}\mathring{\mho}_0\| \text{ for all } j\in\mathbb{N}. \tag{5}$$

For any $r, e\in\mathbb{N}$ with $r<e$, the triangle inequality and Equation (5) produce

$$\begin{aligned}
\|\mathring{\mho}_r-\mathring{\mho}_e\| &\leq \|\mathring{\mho}_r-\mathring{\mho}_{r+1}\| + \|\mathring{\mho}_{r+1}-\mathring{\mho}_{r+2}\| + \cdots + \|\mathring{\mho}_{e-1}-\mathring{\mho}_e\| \\
&\leq \left(\digamma^r+\digamma^{r+1}+\cdots+\digamma^{e-1}\right)\|\mathring{\mho}_0-\Upsilon_{\mathring{g}}\mathring{\mho}_0\| \\
&= \digamma^r\left(1+\digamma+\digamma^2+\cdots+\digamma^{e-r-1}\right)\|\mathring{\mho}_0-\Upsilon_{\mathring{g}}\mathring{\mho}_0\| \\
&\leq \frac{\digamma^r}{1-\digamma}\|\mathring{\mho}_0-\Upsilon_{\mathring{g}}\mathring{\mho}_0\|.
\end{aligned}$$

As $r, e\to\infty$ the right-hand side tends to 0, so $\|\mathring{\mho}_r-\mathring{\mho}_e\|\to 0$; the sequence $\{\mathring{\mho}_j\}$ is therefore ∇-preserving and Cauchy in V. Completeness with respect to ∇ furnishes a limit $\mathring{\pi}\in V$, with $\mathring{\mho}_j\to\mathring{\pi}$ in norm.

Step 4. We verify that $\mathring{\pi}$ is fixed by Υ, treating separately the two options offered by

Part 3 in Property 1.

Case 1: $\Upsilon_{\mathring{g}}$ is ∇-continuous. Because $\{\mathring{\mho}_j\}$ is ∇-preserving and converges to $\mathring{\pi}$ in norm, $\mathring{\mho}_{j+1} = \Upsilon_{\mathring{g}}(\mathring{\mho}_j) \to \Upsilon_{\mathring{g}}(\mathring{\pi})$. Limits being unique, $\Upsilon_{\mathring{g}}(\mathring{\pi}) = \mathring{\pi}$, so $\mathring{\pi}$ is fixed by $\Upsilon_{\mathring{g}}$, and hence by Υ.

Case 2: ∇ is $\|\cdot\|$-self-closed. The sequence being ∇-preserving with $\mathring{\mho}_j \to \mathring{\pi}$, some subsequence $\{\mathring{\mho}_{j_l}\}$ satisfies $[\mathring{\mho}_{j_l}, \mathring{\pi}] \in \nabla$ for all $l \in \mathbb{N}$. From Equation (2), Proposition 5, and $\mathring{\mho}_{j_l} \to \mathring{\pi}$,

$$\|\mathring{\mho}_{j_l+1} - \Upsilon_{\mathring{g}}\mathring{\pi}\| = \|\Upsilon_{\mathring{g}}\mathring{\mho}_{j_l} - \Upsilon_{\mathring{g}}\mathring{\pi}\| \leq \digamma\|\mathring{\mho}_{j_l} - \mathring{\pi}\| \to 0 \text{ as } l \to \infty.$$

Consequently $\mathring{\mho}_{j_l+1} \xrightarrow{\|\cdot\|} \Upsilon_{\mathring{g}}(\mathring{\pi})$, and uniqueness of limits forces $\Upsilon_{\mathring{g}}(\mathring{\pi}) = \mathring{\pi}$. Thus $\mathring{\pi}$ is fixed by $\Upsilon_{\mathring{g}}$, and therefore by Υ. $\square$

Uniqueness requires one further hypothesis, isolated in the next theorem. Since the contractive estimate is insensitive to the order of its arguments (Proposition 5), the natural setting for this hypothesis is the symmetric closure ∇^s: paths are understood in the sense of Definition 9 applied to ∇^s, so that consecutive terms need only be ∇-comparable. This convention is used throughout the remainder of the paper.

Theorem 2. *Retain the hypotheses of Theorem 1, and assume in addition that Paths*$(\mathring{\mathfrak{v}}, \mathring{\partial}; \nabla^s) \neq \varnothing$ *for every* $\mathring{\mathfrak{v}}, \mathring{\partial} \in V$*. Then the fixed point of* Υ *is unique.*

Proof. Existence is already guaranteed by Theorem 1. To see uniqueness, let $\mathring{\mathfrak{v}}^*$ and $\mathring{\partial}^*$ both be fixed by Υ; both are then fixed by $\Upsilon_{\mathring{g}}$ as well. The hypothesis provides a path $\{\mathring{\pi}_0, \mathring{\pi}_1, \mathring{\pi}_2, \ldots, \mathring{\pi}_j\}$ of finite length j in ∇^s from $\mathring{\mathfrak{v}}^*$ to $\mathring{\partial}^*$; that is,

$$\mathring{\pi}_0 = \mathring{\mathfrak{v}}^*, \quad \mathring{\pi}_j = \mathring{\partial}^*, \quad (\mathring{\pi}_i, \mathring{\pi}_{i+1}) \in \nabla^s, \text{ i.e., } [\mathring{\pi}_i, \mathring{\pi}_{i+1}] \in \nabla, \quad \forall i\,(0 \leq i \leq j-1). \tag{6}$$

By Proposition 4, the $\Upsilon_{\mathring{g}}$-closedness of ∇ carries over to ∇^s, and Proposition 3 extends it to all iterates $\Upsilon_{\mathring{g}}^k$; hence

$$[\Upsilon_{\mathring{g}}^k\mathring{\pi}_i, \Upsilon_{\mathring{g}}^k\mathring{\pi}_{i+1}] \in \nabla \quad \forall i\,(0 \leq i \leq j-1) \quad \text{and for each } k \in \mathbb{N}_0. \tag{7}$$

Proposition 5 makes the averaged estimate (2) available along ∇^s-related pairs, so it applies to each pair in Equation (7). Combining Equations (6) and (7), the contractive estimate (2), and the triangle inequality, we get

$$\|\mathring{\mathfrak{v}}^* - \mathring{\partial}^*\| = \|\Upsilon_{\mathring{g}}^k\mathring{\pi}_0 - \Upsilon_{\mathring{g}}^k\mathring{\pi}_j\| \leq \sum_{i=0}^{j-1}\|\Upsilon_{\mathring{g}}^k\mathring{\pi}_i - \Upsilon_{\mathring{g}}^k\mathring{\pi}_{i+1}\| \leq \digamma\sum_{i=0}^{j-1}\|\Upsilon_{\mathring{g}}^{k-1}\mathring{\pi}_i - \Upsilon_{\mathring{g}}^{k-1}\mathring{\pi}_{i+1}\|$$

$$\leq \digamma^2\sum_{i=0}^{j-1}\|\Upsilon_{\mathring{g}}^{k-2}\mathring{\pi}_i - \Upsilon_{\mathring{g}}^{k-2}\mathring{\pi}_{i+1}\| \leq \cdots \leq \digamma^k\sum_{i=0}^{j-1}\|\mathring{\pi}_i - \mathring{\pi}_{i+1}\| \to 0 \quad \text{as } k \to \infty,$$

whence $\mathring{\mathfrak{v}}^* = \mathring{\partial}^*$. Therefore $\Upsilon_{\mathring{g}}$, and with it Υ, has exactly one fixed point. $\square$

Completeness of ∇, or ∇^s-directedness of V, suffices for uniqueness, as we now record.

Corollary 1. *Assume Υ meets the hypotheses of Theorem 1. Then Υ has a unique fixed point if either of the following holds:*

$$(e') : \nabla \text{ is complete,}$$

$$(e'') : V \text{ is } \nabla^s\text{-directed.}$$

Proof. Under (e′), every pair satisfies $[\mathring{\mathfrak{v}}, \mathring{\partial}] \in \nabla$, so $\{\mathring{\mathfrak{v}}, \mathring{\partial}\}$ is already a length-1 path in ∇^s; hence $\mathrm{Paths}(\mathring{\mathfrak{v}}, \mathring{\partial}; \nabla^s) \neq \varnothing$ and Theorem 2 applies.

Under (e″), each pair $\mathring{\mathfrak{v}}, \mathring{\partial}$ admits some $\mathring{\pi}$ with $[\mathring{\mathfrak{v}}, \mathring{\pi}], [\mathring{\partial}, \mathring{\pi}] \in \nabla$, so $\{\mathring{\mathfrak{v}}, \mathring{\pi}, \mathring{\partial}\}$ is a length-2 path in ∇^s; again $\mathrm{Paths}(\mathring{\mathfrak{v}}, \mathring{\partial}; \nabla^s) \neq \varnothing$, and Theorem 2 gives the claim. $\square$

The choice $\mathring{\mathfrak{m}} = 0$ in Theorem 1 specialises to an order-theoretic fixed-point statement for normed spaces.

Corollary 2. *Let $(V, \|\cdot\|)$ be a normed space, ∇ a binary relation on V, and Υ a ∇-preserving contraction with Property 1. Then Υ has a fixed point in V.*

Taking instead $\nabla = V \times V$ together with $\mathring{\mathfrak{m}} = 0$ returns the classical Banach principle in the normed-space setting.

Corollary 3. *Let $(V, \|\cdot\|)$ be a Banach space and Υ a Banach contraction. Then Υ has a fixed point in V.*

Example 1. *Take $V = \mathbb{R}$ under its usual norm; then $(V, \|\cdot\|)$ is complete and, in particular, ∇-complete whatever ∇ may be. On V put $\nabla = \{(\mathring{\pi}, \mathring{\partial}) \in \mathbb{R}^2 : \mathring{\pi} - \mathring{\partial} \geq 0,\ \mathring{\pi} \in \mathbb{Q}\}$, and let $\Upsilon : V \to V$ be*

$$\Upsilon(\mathring{\pi}) = 8 - \frac{\mathring{\pi}}{3}.$$

With $\mathring{\mathfrak{m}} = 1$, so that $\mathring{g} = \frac{1}{2}$,

$$\Upsilon_{\frac{1}{2}}(\mathring{\pi}) = 4 + \frac{\mathring{\pi}}{3}.$$

Evidently ∇ is $\Upsilon_{1/2}$-closed, and the continuity of $\Upsilon_{1/2}$ makes it ∇-continuous. Whenever $\mathring{\pi}, \mathring{\partial} \in V$ satisfy $(\mathring{\pi}, \mathring{\partial}) \in \nabla$,

$$\|1 \cdot (\mathring{\pi} - \mathring{\partial}) + \Upsilon(\mathring{\pi}) - \Upsilon(\mathring{\partial})\| = \left|(\mathring{\pi} - \mathring{\partial}) - \frac{\mathring{\pi} - \mathring{\partial}}{3}\right| = \frac{2}{3}|\mathring{\pi} - \mathring{\partial}| < \frac{5}{6}\|\mathring{\pi} - \mathring{\partial}\|.$$

Hence Υ is an $(\mathring{\mathfrak{m}},\hbar)$-enriched ∇-preserving contraction with $\mathring{\mathfrak{m}} = 1$, $\hbar = \frac{5}{6}$, and $\nabla_{\Upsilon_{1/2}} \neq \varnothing$. Theorem 1 therefore applies and produces a fixed point in V. The path hypothesis of Theorem 2 is met as well: given any $\mathring{\pi}, \mathring{\partial} \in V$, pick a rational $\mathring{w} \geq \max\{\mathring{\pi}, \mathring{\partial}\}$; then $(\mathring{w}, \mathring{\pi}) \in \nabla$ and $(\mathring{w}, \mathring{\partial}) \in \nabla$, so $\{\mathring{\pi}, \mathring{w}, \mathring{\partial}\}$ is a path of length 2 in ∇^s joining $\mathring{\pi}$ to $\mathring{\partial}$, and $\mathrm{Paths}(\mathring{\pi}, \mathring{\partial}; \nabla^s) \neq \varnothing$. By Theorem 2 the fixed point, $\mathring{\pi} = 6$, is the only one.

Note that the relation ∇ of Example 1 is a near-order, lacking reflexivity, irreflexivity, and symmetry alike; consequently it is none of the following: a preorder, a partial order, a strict order, a tolerance, or the symmetric closure of any relation.

Example 2. *Equip $V = \mathbb{R}$ with the usual norm $\|\cdot\|$; the space is then complete, hence ∇-complete. Take $\nabla = \{(0,0),(0,1),(1,0),(1,1),(0,2)\}$ on V and define $\Upsilon : V \to V$ by*

$$\Upsilon(\mathring{\pi}) = \begin{cases} -\mathring{\pi}, & \textit{if } \mathring{\pi} \le 1, \\ 2-\mathring{\pi}, & \textit{if } \mathring{\pi} > 1. \end{cases}$$

For $\mathring{\mathfrak{m}} = 1$ we have $\mathring{g} = \frac{1}{2}$, giving

$$\Upsilon_{\frac{1}{2}}(\mathring{\pi}) = \begin{cases} 0, & \textit{if } \mathring{\pi} \le 1, \\ 1, & \textit{if } \mathring{\pi} > 1. \end{cases}$$

Here ∇ is $\Upsilon_{1/2}$-closed, yet $\Upsilon_{1/2}$ fails to be continuous. Suppose $\{\mathring{\pi}_n\}$ is ∇-preserving with

$$\mathring{\pi}_n \xrightarrow{\|\cdot\|} \mathring{\pi},$$

hence $(\mathring{\pi}_n, \mathring{\pi}_{n+1}) \in \nabla$ for every $n \in \mathbb{N}$. Because $(\mathring{\pi}_n, \mathring{\pi}_{n+1}) \neq (0,2)$, each term satisfies $\mathring{\pi}_n \in \{0,1\}$ for all $n \in \mathbb{N}$, i.e., $\{\mathring{\pi}_n\} \subseteq \{0,1\}$. Since $\{0,1\}$ is closed, the limit $\mathring{\pi}$ also lies in $\{0,1\}$, and so $[\mathring{\pi}_n, \mathring{\pi}] \in \nabla$ for all n. Therefore ∇ is $\|\cdot\|$-self-closed.

A short computation confirms that Υ is an $(\mathring{\mathfrak{m}},\hbar)$-enriched ∇-preserving contraction with $\mathring{\mathfrak{m}} = 1$ and $\hbar = \frac{1}{2}$. Theorem 1 then applies, and Υ has the fixed point $\mathring{\pi} = 0$.

By contrast, the results of the studies by Alam and Imdad [2], Alam et al. [4] are inapplicable, since ∇ is not Υ-closed: although $(0,1) \in \nabla$, one has $(\Upsilon 0, \Upsilon 1) = (0,-1) \notin \nabla$.

The relation ∇ of Example 2 evades every usual label: it is not reflexive, irreflexive, symmetric, antisymmetric, transitive, complete, or weakly complete.

The example below is especially telling, exhibiting the role played by the weaker forms of completeness and continuity.

Example 3. *On $V = [0,2)$, taken with the usual norm $\|\cdot\|$, introduce $\nabla = \{(\mathring{\mathfrak{v}}, \mathring{\partial}) : \frac{1}{4} \le \mathring{\mathfrak{v}} \le \mathring{\partial} \le \frac{1}{3} \text{ or } \frac{1}{2} \le \mathring{\mathfrak{v}} \le \mathring{\partial} \le 1\}$. The space $(V, \|\cdot\|)$ is not complete in the ordinary sense, but it is ∇-complete.*

Let $\Upsilon : V \to V$ be given by

$$\Upsilon(\mathring{\mathfrak{v}}) = \begin{cases} \frac{1}{2} - \mathring{\mathfrak{v}}, & \textit{if } 0 \le \mathring{\mathfrak{v}} < \frac{1}{2}, \\ 2-\mathring{\mathfrak{v}}, & \textit{if } \frac{1}{2} \le \mathring{\mathfrak{v}} < 2. \end{cases}$$

Setting $\mathring{\mathfrak{m}} = 1$, so $\mathring{g} = \frac{1}{2}$,

$$\Upsilon_{\frac{1}{2}}(\mathring{\mathfrak{v}}) = \begin{cases} \frac{1}{4}, & \textit{if } 0 \le \mathring{\mathfrak{v}} < \frac{1}{2}, \\ 1, & \textit{if } \frac{1}{2} \le \mathring{\mathfrak{v}} < 2. \end{cases}$$

Then $\Upsilon_{1/2}$ is ∇-continuous but not continuous in the usual sense, and ∇ is $\Upsilon_{1/2}$-closed. One checks directly that Υ is an $(\mathring{\mathfrak{m}},\hbar)$-enriched ∇-preserving contraction with $\mathring{\mathfrak{m}} = 1$ and some $\hbar \in [0,1)$. Hence all hypotheses of Theorem 1 hold, and Υ has at least one fixed point in V.

However, V is not ∇^s-connected; in particular, there is no path in ∇^s joining $\frac{1}{4}$ and 1, so Paths$(\frac{1}{4}, 1; \nabla^s) = \varnothing$. Consequently, the uniqueness criterion of Theorem 2 does not apply, and indeed Υ has two distinct fixed points, $\mathring{\mathfrak{v}} = \frac{1}{4}$ and $\mathring{\mathfrak{v}} = 1$. Moreover, ∇ is not Υ-closed: $(\frac{1}{2}, 1) \in \nabla$, but $(\Upsilon(\frac{1}{2}), \Upsilon(1)) = (\frac{3}{2}, 1) \notin \nabla$. Thus the main results of studies by Alam and Imdad [2], Alam et al. [4] again cannot be applied here. The numerical experiment and the convergence behavior of Υ are illustrated in ***Figure 1****, whose two panels share identical simulation conditions.* ***Figure 1a*** *runs the Krasnoselskii iteration $\mathring{\mho}_{j+1} = \Upsilon_{1/2}(\mathring{\mho}_j)$ generated by the averaged operator; every trajectory lands on a fixed point after a single step and remains there, the starting values below $\frac{1}{2}$ reaching $\frac{1}{4}$ and the remaining ones reaching 1.* ***Figure 1b*** *repeats the run without averaging, that is with $\mathring{g} = 1$; the starting values 0.5, 0.75, and 1.8 then lock into period-two cycles and never settle, so the averaging step is exactly what produces convergence for this operator.*

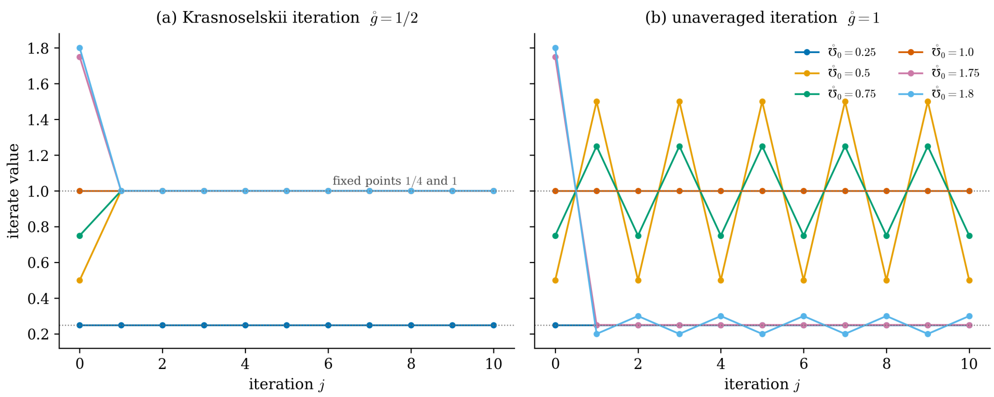

Figure 1. Iteration history for Example 3. Simulation conditions: enrichment parameter $\mathring{\mathfrak{m}} = 1$, averaging parameter $\mathring{g} = \frac{1}{2}$ **(a)** and $\mathring{g} = 1$ **(b)**, contraction bound $F = \mathring{g}\hbar$ with $\hbar \in [0,1)$, initial points $\mathring{\mho}_0 = 0.25, 0.5, 0.75, 1.0, 1.75$, and 1.8, maximum number of iterations 10, stopping threshold 10^{-10} on $|\mathring{\mho}_{j+1} - \mathring{\mho}_j|$; dotted horizontal lines mark the fixed points $\frac{1}{4}$ and 1.

Next we put forward the enriched ∇-preserving almost contraction.

Definition 14. *Let $(V, \|\cdot\|)$ be a normed space and $\nabla \subseteq V^2$. A self-map $\Upsilon : V \to V$ is an enriched ∇-preserving almost contraction provided there are constants $\mathring{\mathfrak{m}} \geq 0$, $\mathfrak{L} \geq 0$, and $\hbar \in [0, \mathring{\mathfrak{m}} + 1)$ for which*

$$\|\mathring{\mathfrak{m}}(\mathring{\mathfrak{v}} - \mathring{\partial}) + \Upsilon\mathring{\mathfrak{v}} - \Upsilon\mathring{\partial}\| \leq \hbar\|\mathring{\mathfrak{v}} - \mathring{\partial}\| + \mathfrak{L}\,\|(\mathring{\mathfrak{m}} + 1)(\mathring{\partial} - \mathring{\mathfrak{v}}) + \mathring{\mathfrak{v}} - \Upsilon\mathring{\mathfrak{v}}\|, \quad (8)$$

holds whenever $\mathring{\mathfrak{v}}, \mathring{\partial} \in V$ and $(\mathring{\mathfrak{v}}, \mathring{\partial}) \in \nabla$.

To exhibit the constants, we speak of a $(\mathring{\mathfrak{m}},\hbar,\mathfrak{L})$-enriched ∇-preserving almost contraction. For $\mathring{\mathfrak{m}} = 0$ the second term on the right of estimate (8) collapses to $\mathfrak{L}\|\mathring{\partial} - \Upsilon\mathring{\mathfrak{v}}\|$, and estimate (8) becomes the classical almost contraction condition of Berinde, imposed along ∇-related pairs; also see the relational versions in previous studies [5,9].

Unlike Equation (1), the estimate (8) is not insensitive to swapping $\mathring{\mathfrak{v}}$ and $\mathring{\partial}$: the term carrying $\mathfrak{L}$ singles out the first argument. No analogue of Proposition 5 is therefore available for almost contractions, and the direction of comparability has to be controlled explicitly in the limit step of the existence proof. The following strengthening of Definition 12 does exactly that.

Definition 15. *Let $(V, \|\cdot\|)$ be a normed space and $\nabla \subseteq V^2$. We call ∇ directionally $\|\cdot\|$-self-closed if, for any ∇-preserving sequence $\{\mathring{\mathfrak{v}}_j\}$ converging in norm to a point $\mathring{\mathfrak{v}} \in V$, there is a subsequence $\{\mathring{\mathfrak{v}}_{j_l}\}$ with $(\mathring{\mathfrak{v}}_{j_l}, \mathring{\mathfrak{v}}) \in \nabla$ for every $l \in \mathbb{N}_0$.*

When ∇ is symmetric the notion agrees with $\|\cdot\|$-self-closedness; in general it is stronger.

Property 2. *Let $(V, \|\cdot\|)$ be a normed space, ∇ a binary relation on V, and $\Upsilon : V \to V$. The triple (V, ∇, Υ) is said to satisfy Property 2 if Parts 1,2, and 4 of Property 1 hold and in place of Part 3, either ∇ is directionally $\|\cdot\|$-self-closed or $\Upsilon_{\mathring{g}}$ is ∇-continuous.*

Theorem 3. *Suppose $(V, \|\cdot\|)$ is a normed space carrying a relation ∇, and Υ is a $(\mathring{\mathfrak{m}}, \hbar, \mathfrak{L})$-enriched ∇-preserving almost contraction obeying Property 2. Then Υ possesses a fixed point in V.*

Proof. As before, the argument is split into 4 steps. **Step 1.** Since $\mathring{\mathfrak{m}} = \frac{1}{\mathring{g}} - 1$, the definition of an enriched ∇-preserving almost contraction turns estimate (8) into

$$\left\|\Big(\frac{1}{\mathring{g}} - 1\Big)(\mathring{\mho} - \mathring{\partial}) + \Upsilon\mathring{\mho} - \Upsilon\mathring{\partial}\right\| \leq \hbar\|\mathring{\mho} - \mathring{\partial}\| + \mathfrak{L}\left\|\frac{1}{\mathring{g}}(\mathring{\partial} - \mathring{\mho}) + \mathring{\mho} - \Upsilon\mathring{\mho}\right\|$$

for all $\mathring{\mho}, \mathring{\partial} \in V$ with $(\mathring{\mho}, \mathring{\partial}) \in \nabla$. Multiplying both sides by $\mathring{g} > 0$ and using the homogeneity of the norm gives

$$\|(1 - \mathring{g})(\mathring{\mho} - \mathring{\partial}) + \mathring{g}(\Upsilon\mathring{\mho} - \Upsilon\mathring{\partial})\| \leq \mathring{g}\hbar\|\mathring{\mho} - \mathring{\partial}\| + \mathfrak{L}\,\|(\mathring{\partial} - \mathring{\mho}) + \mathring{g}(\mathring{\mho} - \Upsilon\mathring{\mho})\|.$$

On the left, $(1-\mathring{g})(\mathring{\mho} - \mathring{\partial}) + \mathring{g}(\Upsilon\mathring{\mho} - \Upsilon\mathring{\partial}) = \Upsilon_{\mathring{g}}\mathring{\mho} - \Upsilon_{\mathring{g}}\mathring{\partial}$. On the right, $\mathring{g}(\mathring{\mho} - \Upsilon\mathring{\mho}) = \mathring{\mho} - \Upsilon_{\mathring{g}}\mathring{\mho}$, so the final norm equals $\|\mathring{\partial} - \Upsilon_{\mathring{g}}\mathring{\mho}\|$. Writing $F := \mathring{g}\hbar$, and noting that $F < 1$ holds precisely because $\hbar < \mathring{\mathfrak{m}} + 1 = 1/\mathring{g}$, we arrive at

$$\|\Upsilon_{\mathring{g}}\mathring{\mho} - \Upsilon_{\mathring{g}}\mathring{\partial}\| \leq F\,\|\mathring{\mho} - \mathring{\partial}\| + \mathfrak{L}\,\|\mathring{\partial} - \Upsilon_{\mathring{g}}\mathring{\mho}\|, \quad \forall \mathring{\mho}, \mathring{\partial} \in V \text{ with } (\mathring{\mho}, \mathring{\partial}) \in \nabla. \tag{9}$$

Both the coefficient of the first term and the expression multiplying $\mathfrak{L}$ transform exactly as stated, each step above being reversible.

Step 2. Property 2 yields some $\mathring{\mho}_0 \in \nabla_{\Upsilon_{\mathring{g}}}$. From $\mathring{\mho}_0$ build the Krasnoselskii iterates $\{\mathring{\mho}_j\}$ through

$$\mathring{\mho}_{j+1} = (1 - \mathring{g})\mathring{\mho}_j + \mathring{g}\Upsilon\mathring{\mho}_j = \Upsilon_{\mathring{g}}\mathring{\mho}_j \quad \text{for } j \in \mathbb{N}.$$

As $\left(\mathring{\mho}_0, \Upsilon_{\mathring{g}}\mathring{\mho}_0\right) \in \nabla$, the $\Upsilon_{\mathring{g}}$-closedness of ∇ with Proposition 3 gives

$$\left(\Upsilon_{\mathring{g}}^{j}\mathring{\mho}_0, \Upsilon_{\mathring{g}}^{j+1}\mathring{\mho}_0\right) \in \nabla, \quad \forall j \in \mathbb{N}, \tag{10}$$

and hence

$$\left(\mathring{\mho}_j, \mathring{\mho}_{j+1}\right) \in \nabla, \quad \forall j \in \mathbb{N}. \tag{11}$$

So $\{\mathring{\mho}_j\}$ is ∇-preserving. Applying estimate (9) to the pair $(\mathring{\mho}_j, \mathring{\mho}_{j+1}) \in \nabla$ and using the identity $\mathring{\mho}_{j+1} = \Upsilon_{\mathring{g}}\mathring{\mho}_j$, the term carrying $\mathfrak{L}$ vanishes:

$$\begin{aligned}\|\mathring{\mho}_{j+1} - \mathring{\mho}_{j+2}\| = \|\Upsilon_{\mathring{g}}\mathring{\mho}_j - \Upsilon_{\mathring{g}}\mathring{\mho}_{j+1}\| &\leq \digamma\,\|\mathring{\mho}_j - \mathring{\mho}_{j+1}\| + \mathfrak{L}\,\|\mathring{\mho}_{j+1} - \Upsilon_{\mathring{g}}\mathring{\mho}_j\| \\ &= \digamma\,\|\mathring{\mho}_j - \mathring{\mho}_{j+1}\|,\end{aligned}$$

which, by induction, leads to

$$\|\mathring{\mho}_{j+1} - \mathring{\mho}_{j+2}\| \leq \digamma^{j+1}\|\mathring{\mho}_0 - \Upsilon_{\mathring{g}}\mathring{\mho}_0\| \text{ for all } j \in \mathbb{N}. \tag{12}$$

Step 3. Just as in Theorem 1, $\{\mathring{\mho}_j\}$ is Cauchy in V, and ∇-completeness produces a limit $\mathring{\pi} \in V$ with $\mathring{\mho}_j \xrightarrow{\|\cdot\|} \mathring{\pi}$.

Step 4. We appeal to Property 2. Should $\Upsilon_{\mathring{g}}$ be ∇-continuous, the argument of Theorem 1 shows verbatim that $\mathring{\pi}$ is fixed by Υ. If instead ∇ is directionally $\|\cdot\|$-self-closed, then, the sequence being ∇-preserving with limit $\mathring{\pi}$, a subsequence $\{\mathring{\mho}_{j_l}\}$ obeys $(\mathring{\mho}_{j_l}, \mathring{\pi}) \in \nabla$ for every $l \in \mathbb{N}_0$. The estimate (9) may then be applied directly to each pair $(\mathring{\mho}_{j_l}, \mathring{\pi})$, with $\mathring{\mho}_{j_l}$ in the first slot, and no appeal to Proposition 5 is needed:

$$\begin{aligned}\|\mathring{\mho}_{j_l+1} - \Upsilon_{\mathring{g}}\mathring{\pi}\| = \|\Upsilon_{\mathring{g}}\mathring{\mho}_{j_l} - \Upsilon_{\mathring{g}}\mathring{\pi}\| &\leq \digamma\,\|\mathring{\mho}_{j_l} - \mathring{\pi}\| + \mathfrak{L}\,\|\mathring{\pi} - \Upsilon_{\mathring{g}}\mathring{\mho}_{j_l}\| \\ &= \digamma\,\|\mathring{\mho}_{j_l} - \mathring{\pi}\| + \mathfrak{L}\,\|\mathring{\pi} - \mathring{\mho}_{j_l+1}\| \to 0\end{aligned}$$

as $l \to \infty$, both terms on the right vanishing because $\mathring{\mho}_j \xrightarrow{\|\cdot\|} \mathring{\pi}$. Thus $\mathring{\mho}_{j_l+1} \xrightarrow{\|\cdot\|} \Upsilon_{\mathring{g}}(\mathring{\pi})$, and uniqueness of limits gives $\Upsilon_{\mathring{g}}(\mathring{\pi}) = \mathring{\pi}$. Therefore $\mathring{\pi}$ is fixed by $\Upsilon_{\mathring{g}}$, hence by Υ. $\square$

Putting $\mathring{\mathfrak{m}} = 0$ in Theorem 3 specialises to an order-theoretic statement for almost contractions on normed spaces.

Corollary 4. *Let $(V, \|\cdot\|)$ be a normed space, $\nabla \subseteq V^2$, and Υ a ∇-preserving almost contraction satisfying Property 2. Then Υ has a fixed point in V.*

With $\nabla = V \times V$ and $\mathring{\mathfrak{m}} = 0$, Theorem 3 delivers a fixed-point result for almost contractions on normed spaces.

Corollary 5. *Let $(V, \|\cdot\|)$ be a Banach space and Υ an almost contraction. Then Υ admits a fixed point.*

3. Applications to fractional differential equations

As an application in the spirit of the study by Baleanu and Rezapour [27], we use our theorems to address existence and uniqueness for a nonlinear fractional differential equation. Throughout this section the fractional order is written ϱ and $\eta \in (0,1)$ is a fixed number attached to the boundary condition. Recall that $\mathfrak{D}^{\varrho}$ denotes the Caputo fractional derivative of order ϱ:

$$\mathfrak{D}^{\varrho}\Upsilon(\chi) = \frac{1}{\Gamma(\mathtt{w}-\varrho)}\int_0^{\chi}(\chi-\mathtt{a})^{\mathtt{w}-\varrho-1}\Upsilon^{(\mathtt{w})}(\mathtt{a})\,d\mathtt{a}, \quad \mathtt{w}-1<\varrho<\mathtt{w},\, \mathtt{w}=\lfloor\varrho\rfloor+1,$$

and that the Riemann-Liouville fractional integral of order ϱ is given by Sudsutad and Tariboon [28]

$$\mathcal{I}^{\varrho}\Upsilon(\chi) = \frac{1}{\Gamma(\varrho)}\int_0^{\chi}(\chi-\mathtt{a})^{\varrho-1}\Upsilon(\mathtt{a})\,d\mathtt{a}, \quad \varrho>0.$$

Our ambient space is $V = \mathcal{C}([0,1],\mathbb{R})$, the continuous real-valued functions on $[0,1]$, taken under the supremum norm

$$\|\mathring{\partial}\| = \sup_{\chi\in[0,1]}|\mathring{\partial}(\chi)|.$$

The boundary value problem under study couples the fractional equation with an integral condition on the initial slope:

$$\begin{gathered}\mathfrak{D}^{\varrho}\mathring{\partial}(\chi) = \mathring{\mathtt{e}}(\chi,\mathring{\partial}(\chi)), \quad \chi\in(0,1), \quad \varrho\in(1,2], \\ \mathring{\partial}(0)=0, \qquad \mathring{\partial}'(0) = \int_0^{\eta}\mathring{\partial}(\mathtt{a})\,d\mathtt{a},\end{gathered} \tag{13}$$

where $\mathring{\mathtt{e}} : [0,1]\times\mathbb{R}\to\mathbb{R}^+$ is continuous and $\eta\in(0,1)$. Conditions of this kind arise when the initial growth rate of a process is tied to an average of its early history, and three-point and integral variants of them are classical in the fractional setting [27,28]; relational fixed-point methods have been applied to fractional models of a similar flavour [7].

The first step is to identify the solutions of Equation (13) with the fixed points of an integral operator; the operator and the boundary conditions are matched to one another in the following lemma.

Lemma 1. *Define $\Upsilon : V\to V$ by*

$$\begin{aligned}\Upsilon\mathring{\partial}(\chi) &= \frac{1}{\Gamma(\varrho)}\int_0^{\chi}(\chi-\mathtt{a})^{\varrho-1}\mathring{\mathtt{e}}(\mathtt{a},\mathring{\partial}(\mathtt{a}))\,d\mathtt{a} \\ &\quad + \frac{2\chi}{(2-\eta^2)\Gamma(\varrho)}\int_0^{\eta}\int_0^{\mathtt{a}}(\mathtt{a}-\mathtt{r})^{\varrho-1}\mathring{\mathtt{e}}(\mathtt{r},\mathring{\partial}(\mathtt{r}))\,d\mathtt{r}\,d\mathtt{a}\end{aligned} \tag{14}$$

for $\chi\in[0,1]$. Then $\mathring{\partial}\in V$ solves Equation (13) *if and only if $\mathring{\partial}=\Upsilon\mathring{\partial}$.*

Proof. For $\varrho\in(1,2]$, applying $\mathcal{I}^{\varrho}$ to the fractional equation shows that its solutions are exactly the functions $\mathring{\partial}(\chi) = \mathcal{I}^{\varrho}\mathring{\mathtt{e}}(\cdot,\mathring{\partial}(\cdot))(\chi)+c_0+c_1\chi$ with real constants c_0, c_1.

The condition $\mathring{\partial}(0) = 0$ forces $c_0 = 0$. Since $\varrho > 1$, the derivative of $\mathcal{I}^{\varrho}\mathring{\mathtt{e}}$ is $\mathcal{I}^{\varrho-1}\mathring{\mathtt{e}}$, which vanishes at $\chi = 0$; hence $\mathring{\partial}'(0) = c_1$. The integral condition then reads

$$c_1 = \int_0^{\eta} \mathring{\partial}(\mathtt{a})\, d\mathtt{a} = \int_0^{\eta} \mathcal{I}^{\varrho}\mathring{\mathtt{e}}(\mathtt{a})\, d\mathtt{a} + c_1 \frac{\eta^2}{2},$$

so that

$$c_1\Big(1 - \frac{\eta^2}{2}\Big) = \int_0^{\eta} \mathcal{I}^{\varrho}\mathring{\mathtt{e}}(\mathtt{a})\, d\mathtt{a},$$

that is,

$$c_1 = \frac{2}{2-\eta^2} \cdot \frac{1}{\Gamma(\varrho)} \int_0^{\eta}\int_0^{\mathtt{a}} (\mathtt{a}-\mathtt{r})^{\varrho-1}\, \mathring{\mathtt{e}}(\mathtt{r}, \mathring{\partial}(\mathtt{r}))\, d\mathtt{r}\, d\mathtt{a},$$

where $\eta \in (0,1)$ guarantees $2-\eta^2 > 0$. Substituting c_1 back produces exactly Equation (14), and every step is reversible. $\square$

We equip V with the relation ∇ defined by

$$(\mathring{\partial}, \mathring{\varsigma}) \in \nabla \iff \mathring{\partial}(\chi) + \mathring{\varsigma}(\chi) \geq 0 \;\text{ for all } \chi \in [0,1].$$

Assumption 1. *There is $\Lambda > 0$ such that*

$$|\mathring{\mathtt{e}}(\chi, \mathtt{u}) - \mathring{\mathtt{e}}(\chi, \mathtt{v})| \leq \Lambda\, |\mathtt{u} - \mathtt{v}|$$

for every $\chi \in [0,1]$ and all $\mathtt{u}, \mathtt{v} \in \mathbb{R}$ with $\mathtt{u} + \mathtt{v} \geq 0$, so that the bound is demanded exactly on the pairs of values produced by ∇-related functions.

Assumption 2. $\Lambda\delta < 1$.

Theorem 4. *Let $\varrho \in (1,2]$ and $\eta \in (0,1)$, let $\mathring{\mathtt{e}} : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ be continuous, and set*

$$\delta := \frac{1}{\Gamma(\varrho+1)} + \frac{2\,\eta^{\varrho+1}}{(2-\eta^2)\,\Gamma(\varrho+2)}. \tag{15}$$

If Assumptions 1 and 2 hold, then Equation (13) admits one and only one solution $\mathring{\partial}^* \in V$, and for every $\mathring{\partial}_0$ with $(\mathring{\partial}_0, \Upsilon\mathring{\partial}_0) \in \nabla$ (the zero function is one such choice) the iterates $\Upsilon^k\mathring{\partial}_0$ converge to $\mathring{\partial}^*$ in the supremum norm.

Proof. By Lemma 1, it is enough to produce a unique fixed point of Υ. We check the hypotheses of Theorem 1 with $\mathring{\mathfrak{m}} = 0$, so that $\mathring{g} = 1$ and $\Upsilon_{\mathring{g}} = \Upsilon$, and then the path hypothesis of Theorem 2.

Step 1. The space $(V, \|\cdot\|)$ is a Banach space, hence ∇-complete. Let $(\mathring{\partial}, \mathring{\varsigma}) \in \nabla$. Since $\mathring{\mathtt{e}} \geq 0$, both integrals in Equation (14) are nonnegative, so $\Upsilon\mathring{\partial}(\chi) \geq 0$ and $\Upsilon\mathring{\varsigma}(\chi) \geq 0$ for every χ, whence

$$\Upsilon\mathring{\partial}(\chi) + \Upsilon\mathring{\varsigma}(\chi) \geq 0, \qquad (\Upsilon\mathring{\partial}, \Upsilon\mathring{\varsigma}) \in \nabla.$$

Thus ∇ is Υ-closed. Moreover $\Upsilon\mathring{\partial}_0 \geq 0$ holds for every $\mathring{\partial}_0$, so the zero function satisfies $(0, \Upsilon 0) \in \nabla$ and $\nabla_{\Upsilon} \neq \varnothing$.

Step 2. Let $(\mathring{\partial}, \mathring{\varsigma}) \in \nabla$ and $\chi \in [0,1]$. Then $\mathring{\partial}(\mathtt{a}) + \mathring{\varsigma}(\mathtt{a}) \geq 0$ for every $\mathtt{a}$, so Assumption applies to the pairs of values $\big(\mathring{\partial}(\mathtt{a}), \mathring{\varsigma}(\mathtt{a})\big)$ that actually enter the integrals; this is the point at which the Lipschitz condition and the relation are matched to each other. Write $\omega(\mathtt{a}) := |\mathring{\mathtt{e}}(\mathtt{a}, \mathring{\partial}(\mathtt{a})) - \mathring{\mathtt{e}}(\mathtt{a}, \mathring{\varsigma}(\mathtt{a}))|$; Assumption 1 gives the pointwise bound $\omega \leq \Lambda \|\mathring{\partial} - \mathring{\varsigma}\|$. Subtracting the two copies of Equation (14) and using the triangle inequality,

$$
\begin{aligned}
|\Upsilon\mathring{\partial}(\chi) - \Upsilon\mathring{\varsigma}(\chi)| &\leq \frac{1}{\Gamma(\varrho)} \int_0^{\chi} (\chi - \mathtt{a})^{\varrho-1}\, \omega(\mathtt{a})\, d\mathtt{a} \\
&\quad + \frac{2\chi}{(2-\eta^2)\Gamma(\varrho)} \int_0^{\eta} \int_0^{\mathtt{a}} (\mathtt{a} - \mathtt{r})^{\varrho-1}\, \omega(\mathtt{r})\, d\mathtt{r}\, d\mathtt{a} \\
&\leq \Lambda\, \|\mathring{\partial} - \mathring{\varsigma}\| \Bigg(\frac{1}{\Gamma(\varrho)} \int_0^{\chi} (\chi - \mathtt{a})^{\varrho-1}\, d\mathtt{a} \\
&\qquad + \frac{2}{(2-\eta^2)\Gamma(\varrho)} \int_0^{\eta} \int_0^{\mathtt{a}} (\mathtt{a} - \mathtt{r})^{\varrho-1}\, d\mathtt{r}\, d\mathtt{a} \Bigg).
\end{aligned}
$$

The two kernel integrals are evaluated in closed form. First,

$$
\frac{1}{\Gamma(\varrho)} \int_0^{\chi} (\chi - \mathtt{a})^{\varrho-1}\, d\mathtt{a} = \frac{\chi^{\varrho}}{\Gamma(\varrho+1)} \leq \frac{1}{\Gamma(\varrho+1)},
$$

since $\chi \leq 1$. Second, the inner integral equals $\mathtt{a}^{\varrho}/\varrho$, so

$$
\frac{2}{(2-\eta^2)\Gamma(\varrho)} \int_0^{\eta} \frac{\mathtt{a}^{\varrho}}{\varrho}\, d\mathtt{a} = \frac{2}{(2-\eta^2)\Gamma(\varrho)} \cdot \frac{\eta^{\varrho+1}}{\varrho(\varrho+1)} = \frac{2\,\eta^{\varrho+1}}{(2-\eta^2)\,\Gamma(\varrho+2)},
$$

because $\Gamma(\varrho+2) = (\varrho+1)\,\varrho\,\Gamma(\varrho)$. Taking the supremum over $\chi \in [0,1]$ and recalling Equation (15),

$$
\|\Upsilon\mathring{\partial} - \Upsilon\mathring{\varsigma}\| \leq \Lambda\delta\, \|\mathring{\partial} - \mathring{\varsigma}\|,
$$

and $\Lambda\delta < 1$ by Assumption 2. Thus Υ satisfies Equation (1) with $\mathring{\mathtt{m}} = 0$ and $\hbar = \Lambda\delta$ on ∇-related pairs.

Step 3. We show that Υ is continuous on all of V, which implies ∇-continuity for any relation. Let $\mathring{\partial}_j \to \mathring{\partial}$ uniformly on $[0,1]$. All the functions $\mathring{\partial}_j, \mathring{\partial}$ take values in a common compact interval $\mathtt{J} \subset \mathbb{R}$, and $\mathring{\mathtt{e}}$ is uniformly continuous on the compact set $[0,1] \times \mathtt{J}$; hence $\mathring{\mathtt{e}}(\cdot, \mathring{\partial}_j(\cdot)) \to \mathring{\mathtt{e}}(\cdot, \mathring{\partial}(\cdot))$ uniformly. The kernel computation of Step 2 shows that a uniform bound $\varepsilon_j := \sup_{\mathtt{a}} |\mathring{\mathtt{e}}(\mathtt{a}, \mathring{\partial}_j(\mathtt{a})) - \mathring{\mathtt{e}}(\mathtt{a}, \mathring{\partial}(\mathtt{a}))|$ on the integrand propagates to the bound $\|\Upsilon\mathring{\partial}_j - \Upsilon\mathring{\partial}\| \leq \varepsilon_j\, \delta$, so $\Upsilon\mathring{\partial}_j \to \Upsilon\mathring{\partial}$ uniformly. This argument does not use the Lipschitz bound, and so it is not restricted to ∇-related pairs.

Step 4. Steps 1–3 verify Property 1 for the triple (V, ∇, Υ) with $\mathring{\mathtt{m}} = 0$, so Theorem 1 yields a fixed point of Υ, that is, a solution of Equation (13). For uniqueness we check the path hypothesis of Theorem 2. Given $\mathring{\partial}, \mathring{\varsigma} \in V$, let $\mathtt{w}$ denote the constant function with value $\max\{\|\mathring{\partial}\|, \|\mathring{\varsigma}\|\}$. Then $\mathring{\partial}(\chi) + \mathtt{w} \geq 0$ and $\mathtt{w} + \mathring{\varsigma}(\chi) \geq 0$ for every χ, so $(\mathring{\partial}, \mathtt{w}) \in \nabla$ and $(\mathtt{w}, \mathring{\varsigma}) \in \nabla$; the list $\{\mathring{\partial}, \mathtt{w}, \mathring{\varsigma}\}$ is a path of length 2, and $\mathrm{Paths}(\mathring{\partial}, \mathring{\varsigma}; \nabla^s) \neq \varnothing$. Theorem 2 therefore supplies uniqueness, and its proof, run with $\mathring{g} = 1$, is precisely the statement that the Picard-Krasnoselskii iterates $\Upsilon^k \mathring{\partial}_0$ converge to the solution from any admissible starting point. $\square$

3.1. Numerical verification over batches of Lipschitz coefficients

To probe the sharpness of the threshold $\Lambda\delta < 1$ we ran the iteration of Theorem 4 on the test nonlinearity

$$\mathring{\mathtt{e}}(\chi, \mathtt{u}) = \Lambda\big(|\mathtt{u}| + 1 - \chi\big),$$

which is continuous, nonnegative on $[0,1] \times \mathbb{R}$, and satisfies Assumption 1 with the stated Λ, because $\big||\mathtt{u}| - |\mathtt{v}|\big| \leq |\mathtt{u} - \mathtt{v}|$. Seven parameter groups G1–G7 were formed by varying the order ϱ, the boundary parameter η, and the Lipschitz coefficient Λ; four groups satisfy the threshold and three deliberately violate it. The simulation conditions, identical across the groups, are as follows: the interval $[0,1]$ is discretized by $1,201$ uniform nodes; the Riemann-Liouville integral is evaluated by a product-trapezoidal rule that interpolates the integrand piecewise linearly and integrates the weakly singular kernel exactly; the outer integral over $[0,\eta]$ uses the composite trapezoidal rule; the iteration $\mathring{\partial}_{k+1} = \Upsilon\mathring{\partial}_k$ starts from $\mathring{\partial}_0 \equiv 0$, which is admissible by Theorem 4; the stopping threshold is $\|\mathring{\partial}_k - \mathring{\partial}_{k-1}\| \leq 10^{-10}$.

Table 1 and **Figure 2** summarise the outcome. Three observations matter. First, in every group that satisfies the hypothesis of Theorem 4 (G1–G4) the iteration converges within at most ten steps, the terminal residuals sit at the level of 10^{-13} to 10^{-12}, and the observed contraction ratios remain far below the guaranteed bound $\Lambda\delta$; the threshold predicted by the theorem is therefore valid wherever it applies. Second, groups G5 and G6 violate the threshold yet still converge, which is consistent with the nature of the hypothesis: $\Lambda\delta < 1$ is sufficient, not necessary. The reason is structural. The first term of Equation (14) is a Volterra operator, whose spectral radius is zero, however large its norm may be, so the asymptotic rate of the iteration is governed by the rank-one boundary coupling alone. A discretized eigenvalue computation for $(\varrho, \eta) = (1.5, 0.5)$ puts the spectral radius of the linearised iteration at $0.031077\,\Lambda$, which predicts divergence only beyond $\Lambda \approx 32.18$. Third, group G7 with $\Lambda = 40$ does diverge, and its measured residual growth ratio 1.243 coincides with the predicted value $40 \times 0.031077 = 1.243$ to three decimal places. The theorem's certificate is thus confirmed on its own terms, and the gap between the certified region and the actual convergence region is explained and quantified by the spectral structure of the operator.

Table 1. Batch statistics of the Krasnoselskii iteration for problem (13) with $\mathring{\mathtt{e}}(\chi, \mathtt{u}) = \Lambda(|\mathtt{u}| + 1 - \chi)$.

Group	ϱ	η	Λ	δ	$\Lambda\delta$	Iterations	Residual	Observed ratio
G1	1.2	0.3	0.50	0.9382	0.4691	9	2.2×10^{-12}	0.083
G2	1.5	0.5	0.75	0.8130	0.6098	9	3.3×10^{-13}	0.067
G3	1.5	0.5	1.00	0.8130	0.8130	10	1.8×10^{-13}	0.057
G4	1.8	0.7	1.20	0.7004	0.8405	10	3.7×10^{-13}	0.055
G5	1.9	0.9	1.30	0.7809	1.0152	12	4.2×10^{-12}	0.107
G6	1.5	0.5	3.00	0.8130	2.4391	14	2.4×10^{-12}	0.098
G7	1.5	0.5	40.0	0.8130	32.522	diverges	not attained	1.243

Note: Columns: contraction constant δ from Equation (15), theoretical bound $\Lambda\delta$, iterations to the threshold 10^{-10}, terminal fixed-point residual $\|\Upsilon\mathring{\partial} - \mathring{\partial}\|$, and mean contraction ratio observed over the final iterations.

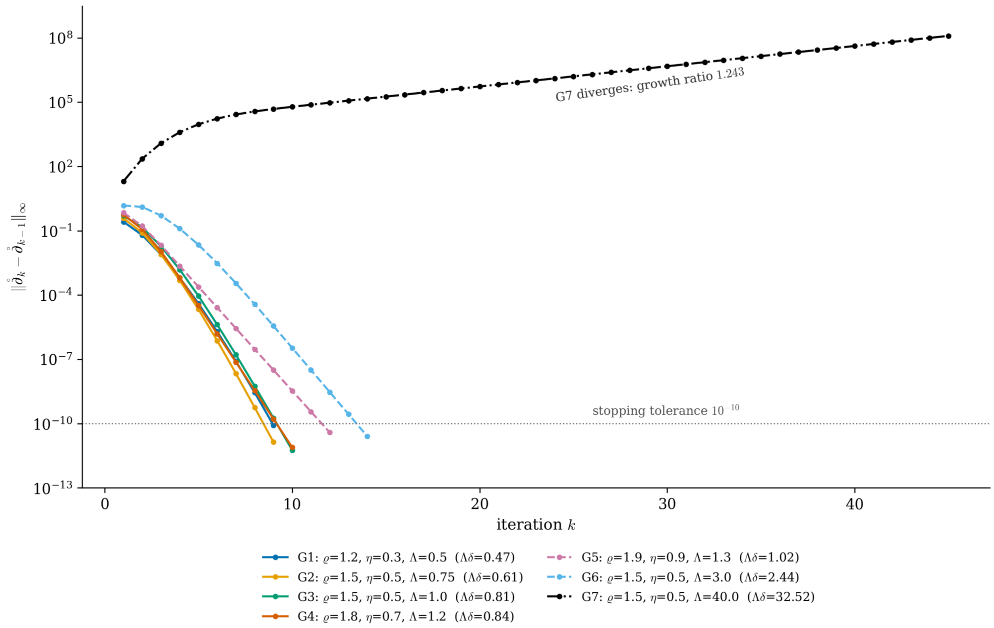

Figure 2. Residual histories $\|\mathring{\partial}_k - \mathring{\partial}_{k-1}\|$ for the seven groups of **Table 1**.

Note: Simulation conditions: initial iterate $\mathring{\partial}_0 \equiv 0$, grid of $1,201$ nodes, product-trapezoidal quadrature for the fractional kernel, stopping threshold 10^{-10} (dotted line), iteration cap 400. Groups G1–G4 satisfy $\Lambda\delta < 1$; G5 and G6 violate the bound but stay below the spectral threshold; G7 lies beyond it and diverges at the predicted ratio.

3.2. Position of the method among current solution frameworks

The literature recalled in Section 1 offers several routes to nonlinear differential and matrix models, and it is fair to ask where the present framework sits among them. **Table 2** organises the comparison along four axes: whether the method comes with a rigorous existence and uniqueness certificate, how readily it adapts to high-dimensional matrix problems and to fractional operators, how stable the delivered object is over long simulation horizons, and how much manual derivation the method demands.

Table 2. Qualitative comparison of solution frameworks along the four axes discussed in the text.

Framework	Existence and uniqueness certificate	High-dimensional matrix and fractional reach	Long-horizon stability	Manual derivation workload
Krasnoselskii iteration for enriched ∇-preserving contractions (this work)	yes, constructive, under $\Lambda\delta < 1$ and Property 1	natural: Sections 3 and 4 treat Caputo and matrix models directly	geometric error decay; iterates stay in the admissible cone	moderate: one verifies the hypotheses once per model
Riemann-Hilbert analysis [15]	yes, via inverse scattering, for integrable models	restricted to integrable hierarchies; dimension raises the algebra sharply	excellent: solutions are exact	high: contour and pole analysis by hand
Lie group and geometry-preserving integrators [16]	no; well-posedness assumed	good on matrix manifolds; fractional structure not intrinsic	very good: invariants preserved by design	moderate: algebra of the symmetry group
Improved physics-informed neural networks [17–19]	no general certificate; residual small does not imply well-posedness	strong in high dimension; fractional variants available	drift possible outside the training window	low derivation, high tuning and training cost
Neural symbolic and AI-assisted derivation [20,21]	certificates only after separate verification	promising; matrix and fractional cases largely open	depends on the recovered expression	low by design; the method aims to automate derivation

Read along the first axis, the fixed-point route is the only entry of **Table 2** that produces the existence and uniqueness statement as an output rather than assuming it, and the certificate is constructive because the proof and the algorithm coincide.

Read along the remaining axes, its limits appear just as plainly. The method needs a Lipschitz-type structure and a verified threshold before it says anything; its convergence degrades as $\Lambda\delta$ approaches 1, and it produces a numerical fixed point rather than a closed-form expression, so questions that demand explicit solution profiles are better served by Riemann-Hilbert analysis when the model is integrable. On problems whose dimension defeats quadrature, physics-informed networks reach further than the discretization used in Section 3.1, at the price of the missing certificate, while geometry-preserving integrators are preferable when invariant conservation over long horizons is the dominant requirement. The frameworks are therefore complementary rather than competing: a practical pipeline can let a network or a symbolic engine propose a candidate solution and let a fixed-point theorem of the present kind certify and refine it, and the closing section lists concrete steps in that direction.

4. Applications to a nonlinear matrix equation

Fixed-point theorems on ordered metric spaces have been studied extensively and applied widely in mathematics and the sciences, particularly to differential, integral, and matrix equations [29–31].

Write $\mathring{\mathscr{J}}_s(j)$ for the set of $j \times j$ complex matrices, and inside it let $\mathtt{Z}(j)$, $\mathtt{S}(j)$, and $\mathtt{Z}^+(j)$ denote, respectively, the Hermitian, the positive definite, and the positive semidefinite matrices. By $\mathfrak{Q} \succ 0$ we mean that $\mathfrak{Q}$ is positive definite, and by $\mathfrak{Q} \succeq 0$ that it is positive semidefinite. We write $\mathtt{C}_1 \succ \mathtt{C}_2$ (or $\mathtt{C}_1 \succeq \mathtt{C}_2$) when $\mathtt{C}_1 - \mathtt{C}_2$ is positive definite (or positive semidefinite). The spectral norm of a matrix $\mathfrak{Q}$ is

$$\|\mathfrak{Q}\| = \sqrt{\lambda^+ (\mathfrak{Q}^*\mathfrak{Q})}$$

with $\lambda^+ (\mathfrak{Q}^*\mathfrak{Q})$ the largest eigenvalue of $\mathfrak{Q}^*\mathfrak{Q}$ and $\mathfrak{Q}^*$ the conjugate transpose. Its trace norm is

$$\|\mathfrak{Q}\|_{tr} = \sum_{k=1}^{j} s_k(\mathfrak{Q})$$

the $s_k(\mathfrak{Q})$, $1 \le k \le j$, being the singular values of $\mathfrak{Q} \in \mathring{\mathscr{J}}_s(j)$. Under $\|\cdot\|_{tr}$, the space $\mathtt{Z}(j)$ is a complete normed space [31–33]. On $\mathtt{Z}(j)$ we adopt the order $\preceq$ defined by $\mathfrak{Q}_1 \preceq \mathfrak{Q}_2 \iff \mathfrak{Q}_2 \succeq \mathfrak{Q}_1$.

Our aim here is to solve the nonlinear matrix equation

$$\mathtt{K} = \mathtt{S} + \sum_{i=1}^{m} \mathtt{C}_i^* \mathtt{Y}(\mathtt{K}) \mathtt{C}_i. \tag{16}$$

In it, $\mathtt{Y}$ is continuous and order-preserving with $\mathtt{Y}(0) = 0$, $\mathtt{S}$ is Hermitian positive definite, and $\mathtt{C}_1, \ldots, \mathtt{C}_m$ are arbitrary $j \times j$ matrices (with adjoints $\mathtt{C}_i^*$).

Two auxiliary lemmas will be needed.

Lemma 2 ([31]). *For $\mathfrak{Q}_1, \mathfrak{Q}_2 \in \mathtt{Z}(j)$ with $\mathfrak{Q}_1 \succeq 0$ and $\mathfrak{Q}_2 \succeq 0$,*

$$0 \le \operatorname{tr}(\mathfrak{Q}_1\mathfrak{Q}_2) \le \|\mathfrak{Q}_1\| \operatorname{tr}(\mathfrak{Q}_2)$$

Lemma 3 ([34]). *If $\mathfrak{Q}_1 \in \mathtt{Z}(j)$ satisfies $\mathfrak{Q}_1 \prec I_j$, then $\|\mathfrak{Q}_1\| < 1$.*

Theorem 5. *Fix $L > 0$ and $\varkappa \in (0,1)$, and suppose Equation* (16) *obeys the following:*

1. *whenever $\mathfrak{Q}_1, \mathfrak{Q}_2 \in \mathtt{Z}(j)$ satisfy $\mathfrak{Q}_1 \preceq \mathfrak{Q}_2$ and $\sum_{i=1}^{m} \mathtt{C}_i^* \mathtt{Y}(\mathfrak{Q}_1)\,\mathtt{C}_i \neq \sum_{i=1}^{m} \mathtt{C}_i^* \mathtt{Y}(\mathfrak{Q}_2)\,\mathtt{C}_i$,*

$$|\mathrm{tr}\,(\mathtt{Y}(\mathfrak{Q}_2) - \mathtt{Y}(\mathfrak{Q}_1))| \leq \frac{\varkappa^2\,|\mathrm{tr}\,(\mathfrak{Q}_2 - \mathfrak{Q}_1)|}{L}; \tag{17}$$

2. *$\sum_{i=1}^{m} \mathtt{C}_i \mathtt{C}_i^* \prec L I_j$.*

Then Equation (16) *has one and only one solution in $\mathtt{Z}(j)$.*

Proof. Let $\Upsilon : \mathtt{Z}(j) \to \mathtt{Z}(j)$ be

$$\Upsilon(\mathtt{K}) = \mathtt{S} + \sum_{i=1}^{m} \mathtt{C}_i^* \mathtt{Y}(\mathtt{K}) \mathtt{C}_i, \quad \forall \mathtt{K} \in \mathtt{Z}(j). \tag{18}$$

Solving Equation (16) then amounts to solving the fixed-point equation

$$\Upsilon(\mathtt{K}) = \mathtt{K}. \tag{19}$$

On $\mathtt{Z}(j)$ we work with the relation $\nabla := \{(\mathfrak{Q}_1, \mathfrak{Q}_2) \in \mathtt{Z}(j)^2 : \mathfrak{Q}_1 \preceq \mathfrak{Q}_2\}$, and we verify the hypotheses of Corollary 2, the contraction corollary of Theorem 1 with $\mathring{\mathfrak{m}} = 0$, one at a time.

Step 1. Under the trace norm, $\mathtt{Z}(j)$ is a Banach space [31–33], hence ∇-complete. Let $\mathfrak{Q}_1 \preceq \mathfrak{Q}_2$. Since $\mathtt{Y}$ is order-preserving, $\mathtt{Y}(\mathfrak{Q}_1) \preceq \mathtt{Y}(\mathfrak{Q}_2)$; note that only this weak inequality is available, an order-preserving map need not separate distinct arguments strictly. Congruence preserves positive semidefiniteness, so

$$\Upsilon(\mathfrak{Q}_2) - \Upsilon(\mathfrak{Q}_1) = \sum_{i=1}^{m} \mathtt{C}_i^* \big(\mathtt{Y}(\mathfrak{Q}_2) - \mathtt{Y}(\mathfrak{Q}_1)\big) \mathtt{C}_i \succeq 0,$$

that is, $\Upsilon(\mathfrak{Q}_1) \preceq \Upsilon(\mathfrak{Q}_2)$, and ∇ is Υ-closed.

Step 2. Since $\mathtt{S} \succ 0$ we have $0 \preceq \mathtt{S}$, so the order-preservation of $\mathtt{Y}$ together with $\mathtt{Y}(0) = 0$ gives $\mathtt{Y}(\mathtt{S}) \succeq 0$, hence

$$\Upsilon(\mathtt{S}) - \mathtt{S} = \sum_{i=1}^{m} \mathtt{C}_i^*\, \mathtt{Y}(\mathtt{S})\, \mathtt{C}_i \succeq 0.$$

Thus $(\mathtt{S}, \Upsilon(\mathtt{S})) \in \nabla$ and $\nabla_\Upsilon \neq \varnothing$; no separate hypothesis is needed for this.

Step 3. The map $\mathtt{Y}$ is continuous by assumption and the congruence sums are linear, so Υ is continuous on the finite-dimensional space $\mathtt{Z}(j)$; continuity implies ∇-continuity for any relation.

Step 4. Take $\mathfrak{Q}_1 \preceq \mathfrak{Q}_2$. If $\sum_i \mathtt{C}_i^* \mathtt{Y}(\mathfrak{Q}_1) \mathtt{C}_i = \sum_i \mathtt{C}_i^* \mathtt{Y}(\mathfrak{Q}_2) \mathtt{C}_i$, then $\Upsilon(\mathfrak{Q}_1) = \Upsilon(\mathfrak{Q}_2)$ and the contractive inequality holds trivially. Otherwise estimate (17) is in force. By Step

1, $\Upsilon(\mathfrak{Q}_2) - \Upsilon(\mathfrak{Q}_1) \succeq 0$, and for a positive semidefinite matrix the trace norm is the trace itself; likewise $\mathtt{Y}(\mathfrak{Q}_2) - \mathtt{Y}(\mathfrak{Q}_1) \succeq 0$ and $\mathfrak{Q}_2 - \mathfrak{Q}_1 \succeq 0$. Hence

$$\begin{aligned}
\left\|\Upsilon\left(\mathfrak{Q}_2\right)-\Upsilon\left(\mathfrak{Q}_1\right)\right\|_{tr} &= \operatorname{tr}\left(\Upsilon\left(\mathfrak{Q}_2\right)-\Upsilon\left(\mathfrak{Q}_1\right)\right) \\
&= \operatorname{tr}\left(\sum_{i=1}^{m} \mathtt{C}_i^*\left(\mathtt{Y}\left(\mathfrak{Q}_2\right)-\mathtt{Y}\left(\mathfrak{Q}_1\right)\right)\mathtt{C}_i\right) \\
&= \operatorname{tr}\left(\left(\sum_{i=1}^{m} \mathtt{C}_i\mathtt{C}_i^*\right)\left(\mathtt{Y}\left(\mathfrak{Q}_2\right)-\mathtt{Y}\left(\mathfrak{Q}_1\right)\right)\right) \\
&\leq \left\|\sum_{i=1}^{m} \mathtt{C}_i\mathtt{C}_i^*\right\| \left\|\mathtt{Y}\left(\mathfrak{Q}_2\right)-\mathtt{Y}\left(\mathfrak{Q}_1\right)\right\|_{tr} \\
&\leq \frac{\left\|\sum\limits_{i=1}^{m} \mathtt{C}_i\mathtt{C}_i^*\right\|}{L}\left(\varkappa^2\left\|\mathfrak{Q}_2-\mathfrak{Q}_1\right\|_{tr}\right) \\
&\leq \varkappa^2\left\|\mathfrak{Q}_2-\mathfrak{Q}_1\right\|_{tr},
\end{aligned}$$

where the third equality uses the linearity and the cyclic property of the trace, the first inequality is Lemma 2, applicable because both factors are positive semidefinite, the second is Equation (17) combined with $\operatorname{tr}(\mathtt{Y}(\mathfrak{Q}_2) - \mathtt{Y}(\mathfrak{Q}_1)) = \|\mathtt{Y}(\mathfrak{Q}_2) - \mathtt{Y}(\mathfrak{Q}_1)\|_{tr}$ and $|\operatorname{tr}(\mathfrak{Q}_2 - \mathfrak{Q}_1)| = \|\mathfrak{Q}_2 - \mathfrak{Q}_1\|_{tr}$, and the third follows from Lemma 3, which turns Assumption 2 into $\|\sum\limits_{i} \mathtt{C}_i\mathtt{C}_i^*\| < L$. Since $\varkappa^2 < 1$, the operator Υ is a ∇-preserving contraction.

Step 5. Steps 1-4 verify every hypothesis of Corollary 2, so Υ has a fixed point $\mathtt{K}^* \in \mathtt{Z}(j)$, which solves Equation (16). For uniqueness, observe that any two Hermitian matrices admit a common upper bound: with $\mu := \max\{\lambda^+(\mathfrak{Q}_1), \lambda^+(\mathfrak{Q}_2)\}$ one has $\mathfrak{Q}_1 \preceq \mu I_j$ and $\mathfrak{Q}_2 \preceq \mu I_j$, so $\{\mathfrak{Q}_1, \mu I_j, \mathfrak{Q}_2\}$ is a path of length 2 in ∇^s and $\operatorname{Paths}(\mathfrak{Q}_1, \mathfrak{Q}_2; \nabla^s) \neq \varnothing$ for every pair. Theorem 2 then shows that $\mathtt{K}^*$ is the only solution. $\square$

5. Conclusion and future directions

We have introduced enriched ∇-preserving contractions, a class that brings order-theoretic contractions, Banach contractions, and particular nonexpansive maps under a single heading. Which maps are covered is governed by the enrichment constant and by the relation, and Examples 2 and 3 delimit the reach of the notion concretely: they exhibit operators that the class captures although the earlier relational theorems of previous studies [2, 4] do not apply to them, without any suggestion that every nonexpansive map is included. For maps of this kind on a normed space carrying an arbitrary relation, the Krasnoselskii iteration was used to locate the fixed points, and suitable choices of the constants or of the relation return a number of classical fixed-point theorems. The worked examples, together with the two applications (solvability of a fractional boundary value problem and of a nonlinear matrix equation), indicate the scope of the framework. The numerical study of Section 3.1 adds a quantitative check: wherever the threshold $\Lambda\delta < 1$ holds, the iteration converges at least at the certified rate, and the gap between the certified region and the observed convergence region is accounted for, indeed predicted to three decimal places, by the spectral structure of the underlying operator.

Several avenues remain open. One is to weaken the present hypotheses without sacrificing the conclusions; another is to probe the fractal behaviour of enriched contractions from the relational standpoint. Two further directions are suggested by the comparison of Section 3.2. On the analytical side, exact solutions delivered by Riemann-Hilbert analysis [15] can serve as reference benchmarks on integrable test models, so that the convergence accuracy of the Krasnoselskii iterates is measured against a closed form rather than against a finer discretization, while structure-preserving Lie group schemes [16] offer a disciplined way of carrying the iteration onto matrix manifolds. On the symbolic side, derivation methods enhanced by artificial intelligence merit systematic study: multimodal neural symbolic architectures [20,21] could propose candidate relations ∇, enrichment constants, and averaged operators for a given equation, and could assemble draft existence and uniqueness arguments for enriched ∇-preserving maps, with refined physics-informed networks [19] supplying informed initial iterates. Any such proposal would still pass through the theorems of Section 2 for certification, so the rigour of the conclusions is retained while the manual derivation work is reduced. Either line could sharpen our grasp of the dynamics and structure of such maps, with repercussions for both theory and applications.

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