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Article

Stability and controllability for fractional damped differential equations with ψ-Hilfer derivatives

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Abstract: This study investigates the existence, uniqueness, finite-time stability and controllability for a class of nonlinear ψ-Hilfer fractional differential equations with damping terms, and the main mathematical contributions are achieved through several techniques including parameter selection, reasonable application of weighting methods and transformations. Specifically, by leveraging the generalized Laplace transform method and weighted norm technique, the explicit representation of solutions is derived. The existence and uniqueness of solutions are established via the generalized Banach contraction principle and Schauder fixed point theorem. Then finite-time stability criteria are obtained via inequality estimates and fractional Gronwall inequalities, each providing complementary insights into the behavior of systems within a bounded time interval. Furthermore, we establish controllability criteria for linear and nonlinear systems, the latter relies on the successive approximation method. Finally, several numerical examples are presented to validate our findings. These examples illustrate the necessity of the two existence theorems, since their respective assumptions are not always simultaneously satisfied in practical applications. In particular, we examined the magnitude of the error generated when the controllability conditions for the nonlinear system are not satisfied. These results contribute to the broader understanding of fractional damped systems governed by ψ-Hilfer derivatives and offer practical tools for analyzing stability and controllability in complex damped dynamical systems.

Keywords: ψ-Hilfer fractional derivative; finite-time stability; controllability; generalized Laplace transform; fixed-point theorem

1. Introduction

Fractional calculus traces its origins to the late 17th century, emerging from far-reaching discussions between mathematicians regarding the extension of integer-order differentiation [1]. Building upon these early insights, the theoretical framework underwent progressive refinement over subsequent centuries. Currently, fractional calculus attracts growing research interest due to its applications across various fields, particularly in automation systems [2], computational epidemiology [3], power engineering [4], quantitative risk analysis [5], viscoelastic materials [6], etc.

Recently, research on Caputo and Riemann-Liouville fractional differential equations has surged, encompassing diverse areas such as process controllability investigations, studies on boundary value problems and so on [7–9]. A variety of analytical methods, including the nonlinear Henry–Gronwall type inequality and extended semigroup theory [10], have been developed and applied to address the challenges inherent in these problems. Furthermore, the Hilfer fractional derivative emerged as a sustained research focus, serving as a generalized framework that unifies both the Caputo and Riemann-Liouville fractional derivatives [11]. The establishment of Hilfer derivative expands the theoretical aspects of fractional calculus, also assists researchers and application engineers in selecting the most appropriate models for varied physical or engineering problems [12–15]. Moreover, as fractional integrals and derivatives with different kernels lead to a wide range of definitions, Sousa and Oliveira [16] introduced the ψ-Hilfer fractional operator, which includes Hilfer, Hadamard, Marchaud, and others. Then Jarad and Abdeljawad [17] and Fahad et al. [18] proposed and investigated various properties of the ψ-Laplace transform (also called generalized Laplace transform) through different yet complementary approaches. Their work provides powerful mathematical tools for solution representations of ψ-Hilfer fractional differential equations even when parameter k is considered [19]. In recent years, Tunç et al. [20] investigated existence and stability for a nonlinear ψ-Hilfer fractional delay integro-differential equation based on them. Furthermore, scholars have investigated the existence and controllability of solutions for various ψ-Hilfer fractional differential equations [21–24].

On the other hand, fractional differential equations with damping terms arise from numerous practical physical problems and possess significant application value [25, 26]. They are typically used to describe damped oscillations in oscillatory systems, where the energy of systems is gradually dissipated. At the theoretical level, such equations regulate dissipation intensity through damping terms while resolving anomalous diffusion and non-exponential decay in combination with fractional derivative operators, revealing damping-dependent power-law attenuation and dissipation-induced hysteretic responses that are beyond the explanatory capacity of classical models. Consequently, a number of scholars have engaged in research within this domain.

In the literature [27], Balachandran et al. researched the following Caputo fractional damped dynamical systems:

$$\begin{cases} {}^C D^{\alpha} x(t) - A\, {}^C D^{\beta} x(t) = f(t, x(t)), t \in [0,\ T], \\ x(0) = x_0,\ \ x'(0) = x'_0, \end{cases}$$

where $0 < \beta \leqslant 1 < \alpha \leqslant 2$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. They got the solutions of systems by Laplace transform, and used matrix Mittag-Leffler functions to represent them. Subsequently, they presented theorems about the controllability of both linear and nonlinear systems. Naveen et al. [28] investigated Hilfer fractional damped dynamical systems containing three progressively more complex forms of nonlinear terms by the same method. The authors applied fixed point theorems and the successive approximation technique to systematically analyze their controllability. However, they provided three initial conditions, which cannot guarantee well-posedness. Arthi et al. [29] expanded the system [27] by incorporating distributed delays, enhancing the system's universality in another way.

Additionally, finite-time stability (FTS) represents a pivotal concept in dynamical systems theory, focusing on the boundedness of system states over a specified finite time interval, in contrast to classical stability notions such as Lyapunov, asymptotic, or exponential stability, which concern system behavior over an infinite time horizon. Within automatic control, FTS has emerged as a prominent subfield, attracting considerable research attention due to its practical relevance in ensuring prescribed performance within operational time windows. The study of FTS for fractional-order systems, particularly those involving damping effects, holds substantial significance, as it addresses realistic constraints where states must remain within acceptable bounds during finite durations. Methodologically, establishing sufficient conditions for FTS often relies on inequality techniques, including Gronwall-type inequalities and linear matrix inequalities, as demonstrated in a series of recent works [30–32] and the references therein.

Not long ago, Jeelani et al. [33] studied the solutions of ψ-Hilfer fractional differential equations, as given by

$$\begin{cases} {}^{H}D_{t_0}^{\beta,\zeta;\psi} w(t) = f(t, w(t)), \quad t_0 \leq t, \\ I_0^{1-p;\psi} w(t_0) = w_0, \end{cases}$$

with $0 < \beta < 1$, $0 \leq \zeta \leq 1$ and $p = \beta + \zeta(1-\beta)$, ${}^{H}D_{t_0}^{\beta,\zeta;\psi}(\cdot)$ represents the ψ-Hilfer fractional derivative, and $I_{t_0}^{1-p;\Psi}(\cdot)$ is the ψ-fractional integral. Authors employed fixed point theorem primarily and developed a methodology for examining the stability and attractivity of solutions. And Chalishajar et al. [34] investigated the controllability and stability of a fractional-order stochastic equation.

Inspired by the papers mentioned above, in this article, we direct our attention to the stability and controllability for the following nonlinear damped fractional differential equations

$$\begin{cases} {}^{H}D_{a^+}^{\alpha_1,\beta_1;\psi} y(t) - P\, {}^{H}D_{a^+}^{\alpha_2,\beta_2;\psi} y(t) = f(t, y(t)), \quad a < t \leq b, \\ D_{a^+}^{\zeta-1;\psi} y(a) = c_1, \;\; I_{a^+}^{2-\zeta;\psi} y(a) = c_2, \end{cases} \tag{1}$$

where ${}^{H}D_{a^+}^{\alpha_1,\beta_1;\psi}(\cdot)$, ${}^{H}D_{a^+}^{\alpha_2,\beta_2;\psi}(\cdot)$ are ψ-Hilfer fractional derivatives with $0 < \alpha_2 \leq 1 < \alpha_1 \leq 2$, $\beta_1, \beta_2 \in [0,1]$, $\zeta = \alpha_1 + \beta_1(2-\alpha_1)$, $\nu = \alpha_2 + \beta_2(1-\alpha_2)$ such that $\zeta - \nu > 1$. $D_{a^+}^{\zeta-1;\psi}(\cdot)$, $I_{a^+}^{2-\zeta;\psi}(\cdot)$ are ψ-Riemann-Liouville (ψ-R-L) fractional derivative and ψ-fractional integral, respectively. Let $J = [a,b]$, $P \in \mathbb{R}^{n\times n}$, c_1,$c_2 \in \mathbb{R}^n$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $\psi : J \to \mathbb{R}$ are prescribed functions, whose form will be given later.

For this system, we have the following contributions and novelties.

- The well-posedness of the equations hinges on parameter selection. We choose suitable values for ζ and ν to progress the problem.
- The added damping term increases the complexity of the system, and attempting to solve it via direct integration on both sides would render the process more complex. Thus, we introduce a generalized Laplace transform to facilitate a more effective method and find an explicit solution.

- Due to the singularity of solutions at zero caused by the presence of Hilfer derivatives, an appropriate weighted space needs to be chosen to investigate the existence of solutions. And we establish criteria for stability and controllability based on this space.
- By adjusting the parameters α_i, β_i $(i = 1, 2)$ and kernel function ψ, our model can cover the problems and equations from many previous papers.

It should be specifically noted that the synergy between the ψ-Hilfer fractional operator and the damping term is not merely a mathematical generalization; it provides a unified framework for modeling real-world phenomena characterized by history-dependent dissipation evolving under a tunable internal time scale.

We take epidemiological modeling with non-exponential decay as an example. The ψ-Hilfer fractional derivative has been successfully applied to model the dynamics of infectious diseases such as monkeypox, where the integer-order formulation is extended to a fractional-order system to capture memory effects in disease transmission. The damping term in our framework naturally corresponds to the attenuation of infection rates due to public health interventions or acquired immunity, while the ψ-kernel allows the model to adapt to time-varying intervention strategies (e.g., changes in social distancing policies over the course of an outbreak). Our finite-time stability criteria provide quantitative thresholds for containment within a predefined time window, directly informing public health decision-making.

The paper is structured as follows. Section 2 introduces essential definitions and lemmas, establishes the representation of solutions for the linear form of Equation (1) via Mittag-Leffler matrix-valued functions, and provides rigorous proof. Section 3 derives two sufficient conditions for the existence of solutions to Equation (1). In Section 4, some criteria for Equation (1) to have finite-time stability are given. Section 5 mentions controllability for the systems. In Section 6, three examples are presented to exemplify our results. Section 7 concludes the article.

2. Preliminaries

This section recalls several definitions and lemmas used throughout the work. Here $\|x\|$ and $\|P\|$ stand for ordinary norms of a vector and matrix, respectively. The space $C(J, \mathbb{R}^n)$, consisting of all continuous functions on J with values in $\mathbb{R}^n$, is a Banach space under the norm $\|y\|_C = \max_{t\in J} \|y(t)\|$.

Definition 1 ([1]). *Let $\alpha > 0, f \in L^1(J)$, and $\psi(t)$ be an positive increasing function on $(a, b]$, having a continuous derivative on (a, b). The fractional integral of a function f with respect to ψ of α order on J is defined by*

$$I_{a^+}^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s) \left(\psi(t) - \psi(s)\right)^{\alpha-1} f(s) ds, \ t > a.$$

If $\psi'(t) \neq 0$ $(a < t < b)$, the R-L fractional derivative of a function f with respect *to ψ of α order on J is defined as follows*

$$
\begin{aligned}
D_{a^+}^{\alpha;\psi} f(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha;\psi} f(t) \\
&= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1} f(s)ds,\ t>a.
\end{aligned}
$$

Lemma 1 ([1])**.** *Let $\alpha_1, \alpha_2 \in (0, +\infty)$ and $y \in L^1(J)$. Then, the ψ-R-L integral has the following property on J a.e.,*

$$I_{a^+}^{\alpha_1;\psi} I_{a^+}^{\alpha_2;\psi} y(t) = I_{a^+}^{\alpha_1+\alpha_2;\psi} y(t).$$

Definition 2 ([16])**.** *Let $n-1 \leq \alpha \leq n$ with $n \in \mathbb{N}$ and $f, \psi \in C^n(J)$ be two function such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The ψ-Hilfer fractional derivative of f of order α and type $\beta \in [0,1]$ is given by*

$${}^H D_{a^+}^{\alpha,\beta;\psi} f(t) = I_{a^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(t).$$

Remark 1. *All functions ψ mentioned in following lemmas or definitions conform to Definition 2.*

Definition 3. *The weighted space $C_{2-\zeta;\psi}(J, \mathbb{R}^n)$ is defined by*

$$C_{2-\zeta;\psi}(J, \mathbb{R}^n) = \left\{ f : J \to \mathbb{R}^n : (\psi(\cdot)-\psi(a))^{2-\zeta} f \in C[a,b] \right\},$$

it is clearly a Banach space with the norm $\|\cdot\|_{C_{2-\zeta;\psi}}$ defined by

$$\|f\|_{C_{2-\zeta;\psi}} = \|\left(\psi(\cdot)-\psi(a)\right)^{2-\zeta} f\|_C = \max_{t\in J}\|\left(\psi(t)-\psi(a)\right)^{2-\zeta} f(t)\|.$$

Lemma 2 ([17])**.** *Let $f : [a,\infty) \to \mathbb{R}^n$ be a piecewise continuous function and of ψ-exponential order which means there exist non-negative constants M,c,T such that $\|f(t)\| \leq Me^{c\psi(t)}$ for $t \geq T$. Then Equation* (2) *exists for $s > c$*

$$\mathcal{L}_\psi\{f(t)\}(s) = F_\psi(s) = \int_a^\infty e^{-s(\psi(t)-\psi(a))} f(t)\psi'(t)dt. \tag{2}$$

The formula above is known as the generalized Laplace transform.

Definition 4 ([17])**.** *Let f and h be two functions which are piecewise continuous at each interval $[a,b]$ and of ψ-exponential order. The generalized convolution of f and h is defined by*

$$(f *_\psi h)(t) = \int_a^t f(s)h\left(\psi^{-1}(\psi(t)+\psi(a)-\psi(s))\right)\psi'(s)ds. \tag{3}$$

Lemma 3 ([17])**.** *If $f_1, f_2 : J \to \mathbb{R}^n$ are piecewise continuous functions and of*

ψ-exponential order, then for any constant a_1 and a_2
(i) $\mathcal{L}_\psi\{a_1 f_1(t) + a_2 f_2(t)\} = a_1\mathcal{L}_\psi\{f_1(t)\} + a_2\mathcal{L}_\psi\{f_2(t)\}$.
(ii) $\mathcal{L}_\psi\{f_1 *_\psi f_2\} = \mathcal{L}_\psi\{f_1\}\cdot\mathcal{L}_\psi\{f_2\}$.

Definition 5 ([1])**.** *Two parameters Mittag-Leffler function for a matrix $P_{n\times n}$ is defined as*

$$E_{\alpha,\beta}(P) = \sum_{k=0}^{\infty}\frac{P^k}{\Gamma(k\alpha+\beta)},$$

for all α, $\beta > 0$.

Lemma 4. *Let α, $\beta > 0$, and I be identity matrix. Then*

$$\mathcal{L}_\psi\left\{(\psi(t)-\psi(a))^{\beta-1}E_{\alpha,\beta}\left(P(\psi(t)-\psi(a))^{\alpha}\right)\right\} = s^{\alpha-\beta}(s^\alpha I - P)^{-1}.$$

Proof.

$$\begin{aligned}
&\mathcal{L}_\psi\left\{(\psi(t)-\psi(a))^{\beta-1}E_{\alpha,\beta}\left(P(\psi(t)-\psi(a))^{\alpha}\right)\right\}\\
&=\sum_{k=0}^{\infty}\frac{P^k}{\Gamma(k\alpha+\beta)}\mathcal{L}_\psi\left\{(\psi(t)-\psi(a))^{k\alpha+\beta-1}\right\}\\
&=\sum_{k=0}^{\infty}\frac{P^k\int_a^\infty e^{-s(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{k\alpha+\beta-1}\psi'(t)dt}{\Gamma(k\alpha+\beta)}\\
&=\sum_{k=0}^{\infty}\frac{P^k}{\Gamma(k\alpha+\beta)}\int_0^\infty e^{-su}u^{k\alpha+\beta-1}du\\
&=\sum_{k=0}^{\infty}\frac{P^k}{\Gamma(k\alpha+\beta)}\frac{\Gamma(k\alpha+\beta)}{s^{k\alpha+\beta}} = \frac{1}{s^\beta}\sum_{k=0}^{\infty}\left(\frac{P}{s^\alpha}\right)^k.
\end{aligned}$$

Taking s such that $\|P\| \le |s|^\alpha$, it has

$$\begin{aligned}
&\mathcal{L}_\psi\left\{(\psi(t)-\psi(a))^{\beta-1}E_{\alpha,\beta}\left(P(\psi(t)-\psi(a))^{\alpha}\right)\right\} =\\
&\tfrac{1}{s^\beta}\left(I-\tfrac{P}{s^\alpha}\right)^{-1} = s^{\alpha-\beta}(s^\alpha I - P)^{-1}.
\end{aligned}$$

□

Lemma 5 ([18])**.** *Assume that $\alpha > 0$, $n = \lfloor\alpha\rfloor + 1$, $0 \le \beta \le 1$, and $f \in C^n_{2-\zeta;\psi}(J,\mathbb{R}^n)$ be a piecewise continuous function of ψ-exponential order, then*

$$\mathcal{L}_\psi\left\{{}^H D_{a^+}^{\alpha,\beta;\psi}f(t)\right\} = s^\alpha\mathcal{L}_\psi\{f(t)\} - \sum_{i=0}^{n-1}s^{n(1-\beta)+\alpha\beta-i-1}\left(I_{a^+}^{(1-\beta)(n-\alpha)-i}f\right)(a). \quad (4)$$

Lemma 6. *Assume $\zeta - \nu > 1$, and $h(t)$ be a continuous function on J. Then the linear form of Equation* (1)*.*

$$\begin{cases} {}^H D_{a^+}^{\alpha_1,\beta_1;\psi}y(t) - P\,{}^H D_{a^+}^{\alpha_2,\beta_2;\psi}y(t) = h(t), \quad a < t \le b,\\ D_{a^+}^{\zeta-1;\psi}y(a) = c_1,\ \ I_{a^+}^{2-\zeta;\psi}y(a) = c_2, \end{cases} \quad (5)$$

has a solution in space $C_{2-\zeta;\psi}(J,\mathbb{R}^n)$ *as follows:*

$$\begin{aligned}y(t)=&\left(\psi(t)-\psi(a)\right)^{\zeta-1}E_{\alpha_1-\alpha_2,\zeta}\left(P\left(\psi(t)-\psi(a)\right)^{\alpha_1-\alpha_2}\right)c_1\\&+\left(\psi(t)-\psi(a)\right)^{\zeta-2}E_{\alpha_1-\alpha_2,\zeta-1}\left(P\left(\psi(t)-\psi(a)\right)^{\alpha_1-\alpha_2}\right)c_2\\&+\int_a^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\left(P\left(\psi(t)-\psi(s)\right)^{\alpha_1-\alpha_2}\right)h(s)ds.\end{aligned}$$

Proof. Applying generalized Laplace transform to both sides of Equation (5) and using Lemma 5, we have

$$\begin{aligned}s^{\alpha_1}Y_\psi(s)-s^{\beta_1(\alpha_1-2)+1}I_{a+}^{2-\zeta;\psi}y(a)-s^{\beta_1(\alpha_1-2)}D_{a+}^{\zeta-1;\psi}y(a)&\\-Ps^{\alpha_2}Y_\psi(s)+Ps^{\beta_2(\alpha_2-1)}I_{a+}^{1-\nu;\psi}y(a)&=H_\psi(s).\end{aligned}$$

Then we show $I_{a+}^{1-\nu;\psi}y(a)=0$ with initial conditions and $\zeta-\nu>1$.

As $I_{a^+}^{2-\zeta;\psi}y(a)=c_2$, if we let $q(t)=I_{a^+}^{2-\zeta;\psi}y(t)-c_2$, then $q(t)\to 0$ when $t\to a^+$. Furthermore, due to the continuity of ψ, $\left(\psi(t)-\psi(a)\right)^{\zeta-\nu-1}\to 0$ when $t\to a^+$. Then for arbitrary $\varepsilon>0$, there exists $\delta_1>0$, when $t\in(a,a+\delta_1)$

$$\|q(t)\|<1\ \ and\ \ \left(\psi(t)-\psi(a)\right)^{\zeta-\nu-1}<\Gamma(\zeta-\nu)\varepsilon.$$

By Lemma 1, when $a<t<a+\delta_1$

$$\begin{aligned}\|I_{a^+}^{1-\nu;\psi}y(t)\|&=\|I_{a^+}^{\zeta-\nu-1;\psi}I_{a^+}^{2-\zeta;\psi}y(t)\|\\&=\|I_{a^+}^{\zeta-\nu-1;\psi}c_2+I_{a^+}^{\zeta-\nu-1;\psi}q(t)\|\\&\le\|I_{a^+}^{\zeta-\nu-1;\psi}c_2\|+\|I_{a^+}^{\zeta-\nu-1;\psi}q(t)\|\\&\le\frac{\|c_2\|}{\Gamma(\zeta-\nu-1)}\int_a^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\zeta-\nu-2}ds\\&+\frac{1}{\Gamma(\zeta-\nu-1)}\int_a^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\zeta-\nu-2}\|q(s)\|ds\\&\le\frac{\|c_2\|+1}{\Gamma(\zeta-\nu-1)}\int_a^t\psi'(s)\left(\psi(t)-\psi(s)\right)^{\zeta-\nu-2}ds\\&=(\|c_2\|+1)\frac{\left(\psi(t)-\psi(a)\right)^{\zeta-\nu-1}}{\Gamma(\zeta-\nu)}<(\|c_2\|+1)\varepsilon,\end{aligned}$$

which means $I_{a+}^{1-\nu;\psi}y(a)=0$. Then

$$\begin{aligned}Y_\psi(s)=&s^{\beta_1(\alpha_1-2)}\left(s^{\alpha_1}I-Ps^{\alpha_2}\right)^{-1}c_1+s^{\beta_1(\alpha_1-2)+1}\left(s^{\alpha_1}I-Ps^{\alpha_2}\right)^{-1}c_2\\&+H_\psi(s)\left(s^{\alpha_1}I-Ps^{\alpha_2}\right)^{-1}\\=&s^{\alpha_1-\alpha_2-\zeta}\left(s^{\alpha_1-\alpha_2}I-P\right)^{-1}c_1+s^{\alpha_1-\alpha_2-\zeta+1}\left(s^{\alpha_1-\alpha_2}I-P\right)^{-1}c_2\\&+s^{-\alpha_2}\left(s^{\alpha_1-\alpha_2}I-P\right)^{-1}H_\psi(s).\end{aligned}$$

Taking the generalized Laplace inverse transform of the equation above and using

Lemma 4, we obtain

$$\begin{aligned} y(t) =& (\psi(t)-\psi(a))^{\zeta-1} E_{\alpha_1-\alpha_2,\zeta}\left(P(\psi(t)-\psi(a))^{\alpha_1-\alpha_2}\right) c_1 \\ &+ (\psi(t)-\psi(a))^{\zeta-2} E_{\alpha_1-\alpha_2,\zeta-1}\left(P(\psi(t)-\psi(a))^{\alpha_1-\alpha_2}\right) c_2 \\ &+ h(t) *_\psi \left[(\psi(t)-\psi(a))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\psi(t)-\psi(a))^{\alpha_1-\alpha_2}\right)\right] \\ =& (\psi(t)-\psi(a))^{\zeta-1} E_{\alpha_1-\alpha_2,\zeta}\left(P(\psi(t)-\psi(a))^{\alpha_1-\alpha_2}\right) c_1 \\ &+ (\psi(t)-\psi(a))^{\zeta-2} E_{\alpha_1-\alpha_2,\zeta-1}\left(P(\psi(t)-\psi(a))^{\alpha_1-\alpha_2}\right) c_2 \\ &+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\psi(t)-\psi(s))^{\alpha_1-\alpha_2}\right) h(s)ds. \end{aligned}$$

□

Remark 2. *The uniqueness of the solution is guaranteed by the uniqueness of the inverse generalized Laplace transform, as shown in Theorem 3.10 of the literature [18].*

Theorem 1 (Schauder's FPT). *Suppose that X is a Banach space. If the following two conditions are satisfied, then there exists at least one fixed point $y \in O$ of X such that $Ty = y$.*

(i) *The operator $T : X \to X$ is continuous and compact;*

(ii) *There exists a nonempty bounded convex closed set $O \subseteq X$, such that $TO \subseteq O$.*

3. Existence results

In this section, we study the existence of solutions to Equation (1) and begin with the definition of solutions to this problem.

Definition 6. *A function $y \in C_{2-\zeta;\psi}(J, \mathbb{R}^n)$ is said to be a solution of Equation* (1)*, if y satisfies*

$$\begin{aligned} y(t) =& (\psi(t)-\psi(a))^{\zeta-1} E_{\alpha_1-\alpha_2,\zeta}\left(P(\psi(t)-\psi(a))^{\alpha_1-\alpha_2}\right) c_1 \\ &+ (\psi(t)-\psi(a))^{\zeta-2} E_{\alpha_1-\alpha_2,\zeta-1}\left(P(\psi(t)-\psi(a))^{\alpha_1-\alpha_2}\right) c_2 \\ &+ \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\psi(t)-\psi(s))^{\alpha_1-\alpha_2}\right) f(s,y(s))ds. \end{aligned}$$

For the forthcoming analysis, we consider the following assumptions.

H1. *$f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.*

H2. *There exists a function $\Phi(\cdot) \in L^1(J, \mathbb{R}^+)$ and a constant $L > 0$, such that*

$$\|f(t,y)\| \leq \Phi(t) + L(\psi(t)-\psi(a))^{2-\zeta}\|y\|,$$

for all $y \in \mathbb{R}^n$ and a.e. $t \in J$.

H3. *There exists a constant $k > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leqslant k\|x-y\|,$$

for all $x, y \in \mathbb{R}^n$ *and* $t \in J$.

For notational simplicity, we denote

$$\Psi(t) = \psi(t) - \psi(a), \quad D = \Psi(b) - \Psi(a), \quad M = E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\| D^{\alpha_1-\alpha_2}\right),$$
$$M_1 = \max_{t\in J} \Psi'(t).$$

Here, $\Psi(t)$ is the function obtained by shifting the initial point of $\psi(t)$ to zero.

Before presenting the existence theorems, we emphasize that the solution space $C_{2-\zeta;\psi}(J, \mathbb{R}^n)$ is the natural one for the problem: we must impose restrictions on the growth or singularity of the solutions. The space $C^{2-\zeta;\psi}(J, \mathbb{R}^n)$ is precisely the one that absorbs such singularities via the weight function ψ, so that although the solution may be unbounded near the origin, its weighted norm remains finite, and the integral operator maps this space into itself under the growth condition H2. The regularity assumptions on ψ (strictly increasing and belongs to $C^1(J)$) and on f (continuous, as specified in H1) are standard and guarantee that all integrals and derivatives are well defined in the sense of Lebesgue. And the growth or Lipschitz conditions H2–H6 are imposed in the weighted norm to ensure well-posedness.

Theorem 2. *Suppose that the conditions H1–H3 are valid, then Equation* (1) *has a unique solution on* $C_{2-\zeta;\psi}(J, \mathbb{R}^n)$.

Proof. Let operator $T : C_{2-\zeta;\psi}(J, \mathbb{R}^n) \to C_{2-\zeta;\psi}(J, \mathbb{R}^n)$ be defined by

$$\begin{aligned}(Ty)(t) =& \Psi(t)^{\zeta-1} E_{\alpha_1-\alpha_2,\zeta}\left(P\Psi(t)^{\alpha_1-\alpha_2}\right) c_1 + \Psi(t)^{\zeta-2} E_{\alpha_1-\alpha_2,\zeta-1}\left(P\Psi(t)^{\alpha_1-\alpha_2}\right) c_2 \\ &+ \int_a^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\Psi(t)-\Psi(s))^{\alpha_1-\alpha_2}\right) f(s, y(s)) ds \\ :=& A(t) + B(t) + C(t),\end{aligned}$$

First, we verify T maps $C_{2-\zeta;\psi}(J, \mathbb{R}^n)$ into itself. It is readily seen that $\Psi(t)^{2-\zeta} A(t)$, $\Psi(t)^{2-\zeta} B(t) \in C(a, b]$, and their limits exist at the endpoint a. And for $t \in (a, b]$, we define

$$H(t, s) = \begin{cases} \kappa(t, s) f(s, y(s)), & a < s \le t, \\ 0, & t < s \le b, \end{cases}$$

where $\kappa(t, s) = \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\Psi(t) - \Psi(s))^{\alpha_1-\alpha_2}\right)$.

As H2 is valid, we have

$$\begin{aligned}\|H(t, s)\| &\le \|\kappa(t, s) f(s, y(s))\| \\ &\le \|\kappa(t, s)\| \left(\Phi(s) + L\Psi(s)^{2-\zeta}\|y(s)\|\right) \\ &\le M D^{\alpha_1-1} \psi'(s)\left(\Phi(s) + L\|y\|_{C_{2-\zeta;\psi}}\right) := G(s).\end{aligned}$$

By dominated convergence theorem, for any $t_0 \in (a, b]$, it holds that

$$\lim_{t\to t_0} \int_a^b H(t, s) ds = \int_a^b H(t_0, s) ds,$$

thus, we get

$$\lim_{t\to t_0}\Psi(t)^{2-\zeta}C(t)=\Psi(t_0)^{2-\zeta}C(t_0).$$

Meanwhile, it has

$$\Psi(t)^{2-\zeta}\|C(t)\|\leq\Psi(t)^{2-\zeta}\int_a^t\|\kappa(t,s)f(s,x(s))\|ds\leq\Psi(t)^{2-\zeta}\int_a^t G(s)ds. \quad (6)$$

Obviously, the final part of the above inequality tends to 0 when $t\to a^+$, which means the limit of $\Psi(t)^{2-\zeta}C(t)$ exist at the endpoint a. Thus, we can infer that T maps $C_{2-\zeta;\psi}(J,\mathbb{R}^n)$ into itself.

Next, we turn to proving that T^n is a contraction operator on $C_{2-\zeta;\psi}(J,\mathbb{R}^n)$. Taking $x,y\in C_{2-\zeta;\psi}(J,\mathbb{R}^n)$ arbitrary, due to H3, we derive

$$\begin{aligned}
&\Psi(t)^{2-\zeta}\|Tx(t)-Ty(t)\|\\
&\leq\Psi(t)^{2-\zeta}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\left(\|P\|(\Psi(t)-\Psi(s))^{\alpha_1-\alpha_2}\right)\|f(s,x(s))\\
&-f(s,y(s)\|ds\\
&\leq M\Psi(t)^{2-\zeta}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}\Psi(s)^{\zeta-2}\Psi(s)^{2-\zeta}\left\|f(s,x(s))-f(s,y(s)\right\|ds\\
&\leq kM\Psi(t)^{2-\zeta}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}\Psi(s)^{\zeta-2}\Psi(s)^{2-\zeta}\left\|x(s)-y(s)\right\|ds\\
&\leq kM\Psi(t)^{2-\zeta}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}\Psi(s)^{\zeta-2}ds\|x-y\|_{C_{2-\zeta;\psi}}\\
&\leq\frac{kM\Psi(t)^{\alpha_1}\Gamma(\alpha_1)\Gamma(\zeta-1)}{\Gamma(\alpha_1+\zeta-1)}\|x-y\|_{C_{2-\zeta;\psi}}.
\end{aligned}$$

Also, it has

$$\begin{aligned}
&\Psi(t)^{2-\zeta}\|(T^2x)(t)-(T^2y)(t)\|\\
&\leq\Psi(t)^{2-\zeta}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\left(\|P\|(\Psi(t)-\Psi(s))^{\alpha_1-\alpha_2}\right)\|f(s,Tx(s))\\
&-f(s,Ty(s)\|ds\\
&\leq M\Psi(t)^{2-\zeta}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}\Psi(s)^{\zeta-2}\Psi(s)^{2-\zeta}\left\|f(s,Tx(s))-f(s,Ty(s)\right\|ds\\
&\leq kM\Psi(t)^{2-\zeta}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}\Psi(s)^{\zeta-2}\Psi(s)^{2-\zeta}\left\|Tx(s)-Ty(s)\right\|ds\\
&\leq\frac{(kM)^2\Psi(t)^{2-\zeta}\Gamma(\alpha_1)\Gamma(\zeta-1)}{\Gamma(\alpha_1+\zeta-1)}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}\Psi(s)^{\alpha_1+\zeta-2}ds\|x-y\|_{C_{2-\zeta;\psi}}\\
&\leq\frac{[kM\Psi(t)^{\alpha_1}\Gamma(\alpha_1)]^2\Gamma(\zeta-1)}{\Gamma(2\alpha_1+\zeta-1)}\|x-y\|_{C_{2-\zeta;\psi}}.
\end{aligned}$$

By induction on n, it follows easily that

$$\Psi(t)^{2-\zeta}\|(T^nx)(t)-(T^ny)(t)\|\leq\frac{[kM\Psi(t)^{\alpha_1}\Gamma(\alpha_1)]^n\,\Gamma(\zeta-1)}{\Gamma(n\alpha_1+\zeta-1)}\|x-y\|_{C_{2-\zeta;\psi}},$$

which implies

$$\|T^nx-T^ny\|_{C_{2-\zeta;\psi}}\leq\frac{[kMD^{\alpha_1}\Gamma(\alpha_1)]^n\,\Gamma(\zeta-1)}{\Gamma(n\alpha_1+\zeta-1)}\|x-y\|_{C_{2-\zeta;\psi}}.$$

Since $\frac{[kMD^{\alpha_1}\Gamma(\alpha_1)]^n}{\Gamma(n\alpha_1+\zeta-1)}$ is the general term of $E_{\alpha_1,\zeta-1}(kMD^{\alpha_1}\Gamma(\alpha_1))$ which is uniformly convergent on J, for n large enough, such as $log_{kMD^{\alpha_1}}$ we obtain that

$$\frac{[kMD^{\alpha_1}\Gamma(\alpha_1)]^n\,\Gamma(\zeta-1)}{\Gamma(n\alpha_1+\zeta-1)}<1.$$

Hence, T^n is a contraction operator on $C_{2-\zeta;\psi}\left(J,\mathbb{R}^n\right)$. Applying the generalized Banach contraction principle, T has a unique fixed point, which is the solution of Equation (1) on $C_{2-\zeta;\psi}\left(J,\mathbb{R}^n\right)$. $\square$

Theorem 3. *Suppose that H1 and H2 are valid, if*

$$MD^{2-\zeta+\alpha_1}L<\alpha_1, \tag{7}$$

then Equation (1) *has a solution on* $C_{2-\zeta;\psi}\left(J,\mathbb{R}^n\right)$.

Proof. Set $B_\lambda:=\{y\in C_{2-\zeta;\psi}(J,\mathbb{R}^n):\|y\|_{2-\zeta;\psi}\le\lambda\}$, then for arbitrary $y\in B_\lambda$

$$\begin{aligned}\Psi(t)^{2-\zeta}\|Ty(t)\|\le&\Psi(t)E_{\alpha_1-\alpha_2,\zeta}\left(\|P\Psi(t))^{\alpha_1-\alpha_2}\|\right)\|c_1\|\\&+E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\Psi(t))^{\alpha_1-\alpha_2}\|\right)\|c_2\|\\&+\Psi(t)^{2-\zeta}\int_a^t\|\kappa(t,s)f(s,x(s))\|ds\\\le&DE_{\alpha_1-\alpha_2,\zeta}\left(\|P\|(\Psi(b)-\Psi(a))^{\alpha_1-\alpha_2}\right)\|c_1\|\\&+E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\|(\psi(b)-\psi(a))^{\alpha_1-\alpha_2}\right)\|c_2\|\\&+M_1D^{1-\zeta+\alpha_1}E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\|(\Psi(b)-\Psi(a))^{\alpha_1-\alpha_2}\right)\|\Phi\|_{L_1}\\&+ML\lambda\Psi(t)^{2-\zeta}\int_a^t\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}ds\\\le&MD\|c_1\|+M\|c_2\|+MM_1D^{1-\zeta+\alpha_1}\|\Phi\|_{L^1}+\frac{MD^{2-\zeta+\alpha_1}L\lambda}{\alpha_1}.\end{aligned}$$

Since $MD^{2-\zeta+\alpha_1}L<\alpha_1$, we can take $\lambda>0$ large enough such that

$$MD\|c_1\|+M\|c_2\|+MM_1D^{1-\zeta+\alpha_1}\|\Phi\|_{L^1}+\frac{MD^{2-\zeta+\alpha_1}L\lambda}{\alpha_1}\le\lambda, \tag{8}$$

which implies $Ty\in B_\lambda$.

Then we verify the continuity of T on B_λ. Let $\{y_m\}_{m\ge1}\subseteq B_\lambda$ with $\lim\limits_{m\to\infty}y_m=y$ in B_λ. According to H1 and H2, for a.e. $t\in J$, we derive

$$\|f\left(t,y_m(t)\right)-f\left(t,y(t)\right)\|\le2\left(\Phi(t)+L\lambda\right)$$

and

$$\|f\left(t,y_m(t)\right)-f\left(t,y(t)\right)\|\to0,\quad m\to+\infty.$$

By dominated convergence theorem, we obtain

$$\Psi(t)^{2-\zeta}\|Ty_m(t)-Ty(t)\|\le MM_1D^{1-\zeta+\alpha_1}\int_a^b\|f(s,y_m(s))-f(s,y(s)\|\,ds\to0$$

as $m\to+\infty$. Thus, we can infer that T is continuous on B_λ.

Now we prove TB_λ is a relatively compact set in $C_{2-\zeta;\psi}(J,\mathbb{R}^n)$.

Let operator $F : C_{2-\zeta;\psi}(J,\mathbb{R}^n) \to C(J,\mathbb{R}^n)$ be defined by

$$(Fy)(t) = \Psi(t)^{2-\zeta}y(t),$$

F is obviously an isometric isomorphism. It equals to prove that $F(TB_\lambda)$ is relatively compact in $C(J,\mathbb{R}^n)$. First, for any $y \in B_\lambda$, $F(TB_\lambda)$ is a uniformly bounded set according to Equation (8). Then we show $F(TB_\lambda)$ is a equicontinuous set in $C(J,\mathbb{R}^n)$.

In fact, for any $a \le t_1 < t_2 \le b$, $y \in B_\lambda$, it is apparent that

$$\|\Psi(t)^{2-\zeta}A(t_2) - \Psi(t)^{2-\zeta}A(t_1)\| \to 0,\ \|\Psi(t)^{2-\zeta}B(t_2) - \Psi(t)^{2-\zeta}B(t_1)\| \to 0,$$

as $t_1 \to t_2$.

Moreover, we have

$$\begin{aligned}
&\|\Psi(t)^{2-\zeta}C(t_2) - \Psi(t)^{2-\zeta}C(t_1)\| \\
\le& \left[\Psi(t_2)^{2-\zeta} - \Psi(t_1)^{2-\zeta}\right] \int_a^{t_1} \|\kappa(t_2,s)f(s,y(s))\|ds \\
&+ \Psi(t_2)^{2-\zeta} \int_{t_1}^{t_2} \|\kappa(t_2,s)f(s,y(s))\|ds \\
&+ \Psi(t_1)2-\zeta \int_a^{t_1} [\kappa(t_2,s) - \kappa(t_1,s)]\,\|f(s,y(s))\|ds \\
\le& \left[\Psi(t_2)^{2-\zeta} - \Psi(t_1)^{2-\zeta}\right] \int_a^{t_1} MM_1D^{\alpha_1-1}\left(\Phi(s) + L\lambda\right)ds \\
&+ \Psi(t_2)^{2-\zeta} \int_{t_1}^{t_2} MM_1D^{\alpha_1-1}\left(\Phi(s) + L\lambda\right)ds \\
&+ \Psi(t_1)^{2-\zeta} \int_a^{t_1} \|\kappa(t_1,s) - \kappa(t_2,s)\|\left(\Phi(s) + L\lambda\right)ds.
\end{aligned}$$

As $\kappa(t,s) = \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\Psi(t)-\Psi(s))^{\alpha_1-\alpha_2}\right)$, it has

$$\|\kappa(t_1,s) - \kappa(t_2,s)\| \le 2MD^{\alpha_1-1}\psi'(s) \le 2MD^{\alpha_1-1}M_1.$$

And when we fix t_2, for the previous t_1, $\|\kappa(t_1,s) - \kappa(t_2,s)\|$ is pointwise convergence. By dominated convergence theorem, we obtain

$$\|\Psi(t_2)^{2-\zeta}C(t_2) - \Psi(t_1)^{2-\zeta}C(t_1)\| \to 0, \quad t_1 \to t_2,$$

which implies $\|F(Ty(t_1)) - F(Ty(t_1))\| \to 0$, for all y$\in B_\lambda$. According to Arzelà-Ascoli Theorem, $F(TB_\lambda)$ is relatively compact in $C(J,\mathbb{R}^n)$. By the Schauder fixed point theorem, we deduce that T has at least a fixed point $y \in C_{2-\zeta;\psi}(J,\mathbb{R}^n)$, which is a solution of Equation (1), and the proof is completed. □

Next, we give several corollaries that can be proved by the above methods.

Corollary 1. *Taking $\psi(t) = t$, $\beta_1 = \beta_2 = 1$, if the conditions H1, H2 and Equation* (7) *are valid, then Equation* (1) *turns to the Caputo fractional damped dynamical system [27] and has a solution on $C\left(J,\mathbb{R}^n\right)$.*

Proof. Although $\zeta - \nu = 1$, which means the solution now contains an extra bounded term:

$$-\Psi(t)^{\alpha_1-\alpha_2+\nu-1}E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+\nu}\left(P\Psi(t)^{\alpha_1-\alpha_2}\right)Pc_2.$$

It has no impact on the proof derived by the method in Theorem 3 and is straightforward. □

Corollary 2. *Taking $\psi(t) = t$, if the conditions H1, H2 and Equation* (7) *are valid, then Equation* (1) *turns to the Hilfer fractional damped dynamical system [28] and has a solution on $C_{2-\zeta;\psi}(J,\mathbb{R}^n)$.*

Corollary 3. *Taking $\psi(t) = t$, $\beta_1 = \beta_2 = 0$, if the conditions H1, H2 and Equation* (7) *are valid, then Equation* (1) *turns to the Riemann-Liouville fractional damped dynamical system and has a solution on $C_{2-\zeta;\psi}(J,\mathbb{R}^n)$.*

Remark 3. *The above corollaries all follow from Theorem 3. Furthermore, based on Theorem 2, the system in Corollary 1 has a unique solution. If conditions H1 and H3 are satisfied. And when conditions H1, H2, and H3 are satisfied, the system described in Corollaries 2 and 3 has a unique solution too.*

Beyond these cases, by adjusting the parameters α_1, α_2, β_1, β_2 and matrix P while fixing the function $\psi(t)$, many other conclusions that may be applied to various fields were derived, which will not be exhaustively enumerated here.

4. Stability results

In this section, we will present the finite-time stability of solutions to Equation (1). We present the definition of finite-time stability and a Gronwall inequality first.

Definition 7 ([35])**.** *Equation* (1) *is said to be finite time stable with respect to $\{J,\delta,\eta\}$ if and only if $\|c_1\| < \delta$ and $\|c_2\| < \delta$, implies the solution y of Equation* (1) *satisfying $\|y\|_{C_{2-\zeta;\psi}} < \eta$ and $0 < \delta < \eta$.*

Lemma 7 ([36])**.** *Let $y(t)$ be a nonnegative and locally integrable function on $[a,b)$ $(b \le \infty)$, also let $g(t)$ and $\theta(t)$ be nonnegative, monotonically increasing continuous functions on $[a,b)$ with $\theta(t) \le M$ for some constant $M > 0$. Suppose $\alpha > 0$ and the following inequality holds for a.e. $t \in [a,b)$*

$$y(t) \le \theta(t) + g(t)\int_a^t (t-s)^{\alpha-1}y(s)\,ds, \quad \forall t \in [a,b).$$

Then, for a.e. $t \in [a,b)$, the solution satisfies

$$y(t) \le \theta(t)\,E_\alpha\Big(g(t)\zeta(\alpha)t^\alpha\Big),$$

where $E_\alpha(z)$ is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(j\alpha+1)}, \quad \alpha > 0,\ z \in \mathbb{R}.$$

Then we impose the following special growth conditions on function f.

H4. *There exists a function $\omega(\cdot) \in C(J, \mathbb{R}^+)$ such that*

$$\|f(t, y)\| \leq \omega(t),$$

for all $y \in \mathbb{R}^n$ and a.e. $t \in J$.

H5. *There exists a function $\varphi(\cdot) \in L^q(J, \mathbb{R}^+)$, $q > 1$, such that*

$$\|f(t, y)\| \leq \varphi(t),$$

for all $y \in \mathbb{R}^n$ and a.e. $t \in J$.

Remark 4. *It is worth noting that conditions H4 and H5 are stronger than H2. When H1, H4 or H1, H5 hold, TB_λ is uniformly bounded, which guarantees the existence of solutions to the Equation* (1) *according to Theorem 3.*

Theorem 4. *Assume that conditions H1, H4, and Equation* (7) *are valid. If the inequality*

$$MD\delta + M\delta + \frac{MD^{2-\zeta+\alpha_1}\|\omega\|_C}{\alpha_1} < \eta$$

holds, then Equation (1) *is finite time stable.*

Proof. For $t \in J$, it has

$$\begin{aligned}
\Psi(t)^{2-\zeta}\|y(t)\| \leq & \Psi(t)E_{\alpha_1-\alpha_2,\zeta}\left(\|P\Psi(t)^{\alpha_1-\alpha_2}\|\right)\|c_1\| + E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\Psi(t)^{\alpha_1-\alpha_2}\|\right)\|c_2\| \\
& + \Psi(t)^{2-\zeta}\int_a^t \|\kappa(t,s)f(s,x(s))\|ds \\
\leq & DE_{\alpha_1-\alpha_2,\zeta}\left(\|P\|D^{\alpha_1-\alpha_2}\right)\|c_1\| + E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\|D^{\alpha_1-\alpha_2}\right)\|c_2\| \\
& + MD^{2-\zeta}\|\omega\|_C \int_a^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}ds \\
\leq & MD\delta + M\delta + \frac{MD^{2-\zeta+\alpha_1}\|\omega\|_C}{\alpha_1},
\end{aligned}$$

also

$$\|y\|_{C_{2-\zeta;\psi}} = \max_{t\in J}\|\Psi(t)^{2-\zeta}y(t)\| \leq MD\delta + M\delta + \frac{MD^{2-\zeta+\alpha_1}\|\omega\|_C}{\alpha_1} < \eta,$$

which means Equation (1) is finite time stable. □

Theorem 5. *Assume that conditions H1, H5, and Equation* (7) *are valid. If the inequality*

$$MD\delta + M\delta + \frac{MM_1^{\frac{p-1}{p}}D^{\alpha_1-\zeta+\frac{1}{p}+1}\|\varphi\|_{L^q}}{(p\alpha_1-p+1)^{\frac{1}{p}}} < \eta$$

holds, where $\frac{1}{q} = 1 - \frac{1}{p}$, then Equation (1) *is finite time stable.*

Proof. For $t \in J$, by Hölder inequality, it has

$$
\begin{aligned}
\Psi(t)^{2-\zeta}\|y(t)\| \leq & \Psi(t)E_{\alpha_1-\alpha_2,\zeta}\left(\|P(\Psi(t)^{\alpha_1-\alpha_2}\|\right)\|c_1\| + E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\Psi(t)^{\alpha_1-\alpha_2}\|\right)\|c_2\| \\
& + \Psi(t)^{2-\zeta}\int_a^t \|\kappa(t,s)f(s,x(s))\|ds \\
\leq & DE_{\alpha_1-\alpha_2,\zeta}\left(\|P\|D^{\alpha_1-\alpha_2}\right)\|c_1\| + E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\|D^{\alpha_1-\alpha_2}\right)\|c_2\| \\
& + MD^{2-\zeta}\|\varphi\|_{L^q}\left(\int_a^t \left[\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}\right]^p ds\right)^{\frac{1}{p}} \\
\leq & DE_{\alpha_1-\alpha_2,\zeta}\left(\|P\|D^{\alpha_1-\alpha_2}\right)\delta + E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\|D^{\alpha_1-\alpha_2}\right)\delta \\
& + MM_1^{\frac{p-1}{p}}D^{2-\zeta}\|\varphi\|_{L^q}\left(\int_a^t \Psi'(s)(\Psi(t)-\Psi(s))^{p(\alpha_1-1)}ds\right)^{\frac{1}{p}} \\
\leq & MD\delta + M\delta + \frac{MM_1^{\frac{p-1}{p}}D^{\alpha_1-\zeta+\frac{1}{p}+1}\|\varphi\|_{L^q}}{(p\alpha_1-p+1)^{\frac{1}{p}}},
\end{aligned}
$$

also

$$
\begin{aligned}
\|y\|_{C_{2-\zeta;\psi}} &= \max_{t\in J}\|\Psi(t)^{2-\zeta} \\
y(t)\| &\leq MD\delta + M\delta + \frac{MM_1^{\frac{p-1}{p}}D^{\alpha_1-\zeta+\frac{1}{p}+1}\|\varphi\|_{L^q}}{(p\alpha_1-p+1)^{\frac{1}{p}}} < \eta,
\end{aligned}
$$

which means Equation (1) is finite time stable. □

In Theorems 4 and 5, finite-time stability was analyzed via elementary norm estimates. Subsequently, we employ the fractional Gronwall inequality to establish a stability criterion.

Theorem 6. *Assume that conditions H1, H2 and Equation* (7) *are valid. If the inequality*

$$
\left(MD\delta + M\delta + MM_1D^{\alpha_1-\zeta+1}\|\Phi\|_{L^1}\right)E_{\alpha_1}\left(\zeta(\alpha_1)MM_1^{\alpha_1}D^{2-\zeta}Lb^{\alpha_1}\right) < \eta
$$

holds, then Equation (1) *is finite time stable.*

Proof. For $t \in J$, it has

$$
\begin{aligned}
\Psi(t)^{2-\zeta}\|y(t)\| \leq & \Psi(t)E_{\alpha_1-\alpha_2,\zeta}\left(\|P\Psi(t)^{\alpha_1-\alpha_2}\|\right)\|c_1\| + E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\Psi(t)^{\alpha_1-\alpha_2}\|\right)\|c_2\| \\
& + \Psi(t)^{2-\zeta}\int_a^t \|\kappa(t,s)f(s,x(s))\|ds \\
\leq & DE_{\alpha_1-\alpha_2,\zeta}\left(\|P\|D^{\alpha_1-\alpha_2}\right)\delta + E_{\alpha_1-\alpha_2,\zeta-1}\left(\|P\|D^{\alpha_1-\alpha_2}\right)\delta \\
& + MD^{2-\zeta}\int_a^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}\left(\Phi(t) + L\Psi(s)^{2-\zeta}\|y(s)\|\right)ds \\
\leq & MD\delta + M\delta + MM_1D^{\alpha_1-\zeta+1}\|\Phi\|_{L^1} \\
& + MM_1^{\alpha_1}D^{2-\zeta}L\int_a^t (t-s)^{\alpha_1-1}\Psi(s)^{2-\zeta}\|y(s)\|ds.
\end{aligned}
$$

By Lemma 7, applying the Gronwall inequality to the right-hand side of the above inequality yields

$$
\begin{aligned}
\|y\|_{C_{2-\zeta;\psi}} &= \max_{t\in J}\|\Psi(t)^{2-\zeta}y(t)\| \\
&\leq \left(MD\delta + M\delta + MM_1D^{\alpha_1-\zeta+1}\|\Phi\|_{L^1}\right)E_{\alpha_1}\left(\zeta(\alpha_1)MM_1^{\alpha_1}D^{2-\zeta}Lb^{\alpha_1}\right) < \eta,
\end{aligned}
$$

which means Equation (1) is finite time stable. □

5. Controllability results

This part addresses the controllability of Equation (1). We now give a definition and begin with the linear system.

Definition 8. *The Equation* (1) *is considered to be controllable on J if for each vector* $c_1, c_2, x_1 \in \mathbb{R}^n$*, there exists a control input* $u \in L^2(J, \mathbb{R}^m)$ *such that the solution of Equation* (1) *starting from initial conditions satisfies* $y(b) = x_1$.

Theorem 7. *The system*

$$\begin{cases} {}^H D_{a^+}^{\alpha_1,\beta_1;\psi} y(t) - P\, {}^H D_{a^+}^{\alpha_2,\beta_2;\psi} y(t) = Qu(t), \quad a < t \leq b, \\ D_{a^+}^{\zeta-1;\psi} y(a) = c_1, \;\; I_{a^+}^{2-\zeta;\psi} y(a) = c_2, \end{cases} \tag{9}$$

is controllable on J *if and only if the* $n \times n$ *Gramian matrix*

$$G = \int_a^b \Psi'(s)(\Psi(b) - \Psi(s))^{2\alpha_1-2} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\Psi(b) - \Psi(s))^{\alpha_1-\alpha_2}\right)$$
$$QQ^T E_{\alpha_1-\alpha_2,\alpha_1}\left(P^T(\Psi(b) - \Psi(s))^{\alpha_1-\alpha_2}\right) ds$$

is invertible, where $Q \in R^{n\times m}$.

Proof. If G is invertible, then for given initial conditions c_1, c_2 and final state x_1, we introduce the control function

$$u(t) = \Psi'(t)(\Psi(b) - \Psi(t))^{\alpha_1-1} Q^T E_{\alpha_1-\alpha_2,\alpha_1}\left(P^T(\Psi(b) - \Psi(t))^{\alpha_1-\alpha_2}\right) G^{-1} y_1,$$

where

$$y_1 = x_1 - D^{\zeta-1} E_{\alpha_1-\alpha_2,\zeta}\left(PD^{\alpha_1-\alpha_2}\right) c_1 - D^{\zeta-2} E_{\alpha_1-\alpha_2,\zeta-1}\left(PD^{\alpha_1-\alpha_2}\right) c_2.$$

Substitute $u(t)$ and evaluate at $t = b$, we derive

$$\begin{aligned} y(b) =& D^{\zeta-1} E_{\alpha_1-\alpha_2,\zeta}\left(PD^{\alpha_1-\alpha_2}\right) c_1 + D^{\zeta-2} E_{\alpha_1-\alpha_2,\zeta-1}\left(PD^{\alpha_1-\alpha_2}\right) c_2 \\ &+ \int_a^b \Psi'(s)(\Psi(b) - \Psi(s))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\Psi(b) - \Psi(s))^{\alpha_1-\alpha_2}\right) Qu(s) ds. \end{aligned}$$

The integral term becomes

$$\begin{aligned} &\int_a^b \Psi'(s)(\Psi(b) - \Psi(s))^{2\alpha_1-2} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\Psi(b) - \Psi(s))^{\alpha_1-\alpha_2}\right) QQ^T \\ &\times E_{\alpha_1-\alpha_2,\alpha_1}\left(P^T(\Psi(b) - \Psi(s))^{\alpha_1-\alpha_2}\right) ds\, G^{-1} y_1 = GG^{-1} y_1 = y_1. \end{aligned}$$

Thus, $y(b) = x_1$, proving controllability.

On the other hand, suppose controllability holds while G lacks invertibility. Then one can identify a nonzero $z \in \mathbb{R}^n$ causing $z^T N z = 0$, which implies

$$z^T \Psi'(s)(\Psi(b) - \Psi(s))^{\alpha_1-1} E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\Psi(b) - \Psi(s))^{\alpha_1-\alpha_2}\right) Q = 0 \quad \text{for a.e. } s \in J.$$

By controllability, there exists a control $u(t)$ steering the system from zero initial conditions to $x_1 = z$ at $t = b$. The solution at $t = b$ gives

$$z = \int_a^b \Psi'(s)(\Psi(b) - \Psi(s))^{\alpha_1 - 1} E_{\alpha_1 - \alpha_2, \alpha_1}\left(P(\Psi(b) - \Psi(s))^{\alpha_1 - \alpha_2}\right) Qu(s)ds.$$

Multiplying both sides by z^T yields $z^T z = 0$, contradicting $z \neq 0$. Hence, G must be invertible. □

Then we consider the nonlinear Hilfer fractional damped dynamical system

$$\begin{cases} {}^H D_{a^+}^{\alpha_1,\beta_1;\psi} y(t) - P\,{}^H D_{a^+}^{\alpha_2,\beta_2;\psi} y(t) = Qu(t) + f(t, y(t)), \quad t \in (a, b], \\ D_{a^+}^{\zeta-1;\psi} y(a) = c_1, \;\; I_{a^+}^{2-\zeta;\psi} y(a) = c_2, \end{cases} \tag{10}$$

and a new condition.

H6. $\|f(t, y(t))\| \leq b_1$, *for all* $t \in J$, $y(t) \in \mathbb{R}^n$.

Define the following constants and functions:

$$\begin{aligned} &\eta_i(t) = \Psi(t)^{\zeta - i} E_{\alpha_1-\alpha_2,\zeta-i+1}\left(P\Psi(t)^{\alpha_1-\alpha_2}\right), \quad a_i = \|\eta_i\|_{C_{2-\zeta;\psi}}, \;\; i = 1, 2 \\ &L = \|c_1\|a_1 + \|c_2\|a_2, \quad M_2 = M_1 \|\Psi(t)^{\alpha_1 - 1} E_{\alpha_1-\alpha_2,\alpha_1}\left(P\Psi(t)\right)^{\alpha_1-\alpha_2}\right)\|, \quad M_3 = \|Q\|, \\ &M_4 = \|Q^T \kappa^T(b, s) G^{-1}\|, \quad r = M_4\left[\|x_1\|_{C_{2-\zeta;\psi}} + M_2 D^{2-\zeta} b_1 (b-a) + L\right]. \end{aligned}$$

Theorem 8. *If Equation* (9) *is controllable, then Equation* (10) *is also controllable on* J*, provided that the function* f *satisfies conditions H3 and H6.*

Proof. We use the successive approximation technique. Define the sequence of functions $\{y_n(t)\}$ by

$$\begin{aligned} y_0(t) &= y_0, \\ y_{n+1}(t) &= \eta_1(t)c_1 + \eta_2(t)c_2 + \int_a^t \kappa(t, s)Qu_n(s)ds \\ &\quad + \int_a^t \kappa(t, s) f(s, y_n(s))ds, \quad n = 0, 1, 2, \ldots, \end{aligned}$$

the control u_n is given by

$$\begin{aligned} u_n(t) &= Q^T \kappa^T(b, t) G^{-1} \\ &\quad \times \left\{ x_1 - \eta_1(t)c_1 - \eta_2(t)c_2 - \int_a^b \kappa(b, s) f(s, y_n(s))ds \right\}. \end{aligned}$$

Then we proved that $\{y_n(t)\}$ is a Cauchy sequence in $C_{2-\zeta;\psi}$. First, note that

$$\begin{aligned} \Psi(t)^{2-\zeta}\|u_n(t)\| &\leq \|Q^T \kappa^T(b, s) G^{-1}\| \\ &\quad \times \left[\|x_1\|_{C_{2-\zeta;\psi}} + \|c_1\|a_1 + \|c_2\|a_2 + D^{2-\zeta} \int_a^b \|\kappa(b, s)\| \|f(s, y_n(s))\| ds\right] \\ &\leq M_4\left[\|x_1\|_{C_{2-\zeta;\psi}} + L + D^{2-\zeta} M_2 (b-a) b_1\right] = r. \end{aligned}$$

Also

$$\begin{aligned}\Psi(t)^{2-\zeta}\|u_n(t)-u_{n-1}(t)\| \leq & \|Q^T\kappa^T(b,s)G^{-1}\| \\ & \times \Psi(t)^{2-\zeta}\int_a^b \|\kappa(b,s)\|\|f(s,y_{n-1}(s))-f(s,y_n(s))\|ds \\ \leq & kM_2M_4(b-a)\|y_{n-1}-y_n\|_{C_{2-\zeta;\psi}}.\end{aligned}$$

Then

$$\begin{aligned}\Psi(t)^{2-\zeta}\|y_{n+1}(t)-y_n(t)\| \leq & \Psi(t)^{2-\zeta}\int_a^t \|\kappa(t,s)\|\|Q\|\|u_n(s)-u_{n-1}(s)\|ds \\ & + \Psi(t)^{2-\zeta}\int_a^t \|\kappa(t,s)\|\|f(s,y_n(s))-f(s,y_{n-1}(s))\|ds \\ \leq & M_2M_3(kM_2M_4(b-a))\int_a^t \|y_{n-1}-y_n\|_{C_{2-\zeta;\psi}}ds \\ & + kM_2\int_a^t \|y_{n-1}-y_n\|_{C_{2-\zeta;\psi}}ds \\ \leq & (kM_2^2M_3M_4(b-a)+kM_2)\int_a^t \|y_{n-1}-y_n\|_{C_{2-\zeta;\psi}}ds.\end{aligned}$$

Furthermore, there exists $\mathcal{P} = \frac{L+\|y_0\|+1}{b-a} + (M_2M_3r + M_2b_1)$

$$\begin{aligned}\Psi(t)^{2-\zeta}\|y_1(q)-y_0(q)\| \leq & \|c_1\|a_1 + \|c_2\|a_2 + \|y_0\| \\ & + D^{2-\zeta}\int_a^t \|Q\|\|\kappa(t,s)\|\|u_0(s)\| + \|\kappa(t,s)\|\|f(s,y_0(s))\|ds \\ \leq & L + \|y_0\| + (b-a)D^{2-\zeta}(M_2M_3r + M_2b_1) < (b-a)\mathcal{P}.\end{aligned}$$

Using induction, we obtain

$$\|y_{n+1}(t)-y_n(t)\|_{C_{2-\zeta;\psi}} \leq \mathcal{P}(kM_2^2M_3M_4(b-a)+kM_2)\frac{(b-a)^{n+1}}{(n+1)!}.$$

As the right-hand side diminishes for large n, $\{y_n(t)\}$ forms a Cauchy sequence in $C_{2-\zeta;\psi}(J,\mathbb{R}^n)$. By completeness, we make $y(q)$ be the function $y_n(t)$ converges uniformly to, on J. When $n\to\infty$, we have

$$\begin{aligned}y(t) =& \Psi(t)^{\zeta-1}E_{\alpha_1-\alpha_2,\zeta}\left(P\Psi(t)^{\alpha_1-\alpha_2}\right)c_1 + \Psi(t)^{\zeta-2}E_{\alpha_1-\alpha_2,\zeta-1}\left(P\Psi(t)^{\alpha_1-\alpha_2}\right)c_2 \\ & + \int_a^t \Psi'(s)(\Psi(t)-\Psi(s))^{\alpha_1-1}E_{\alpha_1-\alpha_2,\alpha_1}\left(P(\Psi(t)-\Psi(s))^{\alpha_1-\alpha_2}\right) \\ & (Qu(s)+f(s,y(s)))\,ds,\end{aligned}$$

where

$$u(t) = Q^T\kappa^T(b,t)G^{-1}\left(y_1 - \int_a^t \kappa(b,s)f(s,y_n(s))ds\right).$$

we obtain $y(b) = x_1$, proving controllability. □

Controllability only guarantees that the endpoint can be reached, but does not ensure that no explosion occurs midway. The a priori bound $\|y\|_{C_{2-\zeta;\psi}} < \eta$ provided by stability can be used to infer backward an upper bound for the norm of the control input $u(t)$. Specifically, in Theorem 8, when estimating the norm of $u_n(t)$, one need not rely solely on the crude constant b_1. The result from Theorem 4 can be employed to obtain $\|y_n(s)\| \leq \eta$, and consequently redefine the bounded constant r for u_n as:

$$r = M_4 \left[\|x_1\|_C + L + D^{2-\zeta} M_2(b-a) \cdot \min\{\eta, b_1\} \right].$$

This implies that the control law in the controllability proof is bounded, and this bound arises precisely from the stability analysis, thereby ensuring that the actuator does not exceed its physical limits.

6. Examples

In this part, three examples will be presented. The first and second examples underscores the significance of both existence theorems, with the second example serving as an application of the stability criterion. For a specific case, we apply three theorems simultaneously to derive the optimal relative threshold. In the third example, we verify the controllability of a system and visually demonstrate the method of successive approximations. First, we construct an example that satisfies the conditions mentioned in Theorem 2 rather than Theorem 3.

Example 1. *Supposed that* $\alpha_1 = 1.8$, $\alpha_2 = 0.7$, $\beta_1 = 0.5$, $\beta_2 = 0.6$. *Considering a nonlinear system as follows*

$$\begin{cases} {}^H D_{1^+}^{1.8,0.5;\psi} y(t) - P \, {}^H D_{1^+}^{0.7,0.6;\psi} y(t) = f(t, y(t)), \quad 1 < t \leq 2, \\ D_{1^+}^{0.9;\psi} y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ I_{1^+}^{0.1;\psi} y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{cases}$$

where $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, $P = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $\psi(t) = \ln t$, $f(t, y(t)) = \begin{bmatrix} 0.05 \\ 0.1(\ln t)^{0.1} y_2(t) + 0.1\sin(y_2(t)) \end{bmatrix}$, $\zeta - \nu = 1.9 - 0.88 = 1.02 > 1$.

It is obvious that H1 stands. Then we verify H2, taking $\Phi(t) = 0.2$, $L = 0.1$, *it has*

$$\|f(t, y(t))\| \leq 0.05 + 0.1 + 0.1(\ln t)^{0.1} \|y\| = \Phi(t) + L\left(\psi(t) - \psi(a)\right)^{2-\zeta} \|y\|.$$

Meanwhile, take $k = 0.2$, *we derive*

$$\begin{aligned} \|f(t,x) - f(t,y)\| &= \begin{bmatrix} 0 \\ 0.1(\ln t)^{0.1} |x_2(t) - y_2(t)\| + 0.1\left|\sin(x_2(t)) - \sin(y_2(t))\right| \end{bmatrix} \\ &\leq 0.2 \begin{bmatrix} 0 \\ |x_2(t) - y_2(t)| \end{bmatrix} = k\|x - y\|, \end{aligned}$$

then Theorem 2 guarantees a unique solution for the system.

However, we obtain $D = \ln 2 \approx 0.6931$, $M = E_{1.1,0.9}(8 \times 0.6931^{1.1}) \approx$

$E_{1.1,0.9}(5.344) \approx 65.2$, *it means*

$$MD^{2-\zeta+\alpha_1}L = 65.2 \times 0.6931^{1.9} \times 0.1 \approx 3.24 > 1.8 = \alpha_1,$$

which contradicts with Theorem 3.

Example 2. *Supposed that* $\alpha_1 = 1.8$, $\beta_1 = \alpha_2 = \beta_2 = 0.5$. *Considering the nonlinear system as follows*

$$\begin{cases} {}^{H}D_{0^+}^{1.8,0.5;\psi}y(t) - P\,{}^{H}D_{0^+}^{0.5,0.5;\psi}y(t) = f(t,y(t)), \quad t \in (0,1], \\ D_{0^+}^{0.9;\psi}y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad I_{0^+}^{0.1;\psi}y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{cases} \tag{11}$$

where $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, $P = \begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.3 \end{bmatrix}$, $\psi(t) = e^t$ *and* $f(t,y(t)) = \begin{bmatrix} 0.1\sin(y_1(t)) \\ 0.2\cos(y_2(t)) \end{bmatrix}$, $\zeta - \nu = 1.9 - 0.75 = 1.15 > 1$.

Taking L = 0.01, $\Phi(t) = \omega(t) = \varphi(t) = 0.3$, *then H1, H2, Equation* (7), *H4, and H5 are all satisfied.*

Now, we obtain a solution

$$y(t) = \int_0^t e^s(e^t - e^s)^{0.8} E_{1.3,1.8}\left(\begin{bmatrix} 0.5 & 0.2 \\ -0.1 & 0.3 \end{bmatrix}(e^t - e^s)^{0.8}\right)\begin{bmatrix} 0.1\sin(y_1(t)) \\ 0.2\cos(y_2(t)) \end{bmatrix} ds.$$

Since $\|f(t,y(t))\| \le 0.3$, *we can transform the problem into a linear problem when considering the norm of y. One can see the norm of y in* ***Figure 1***.

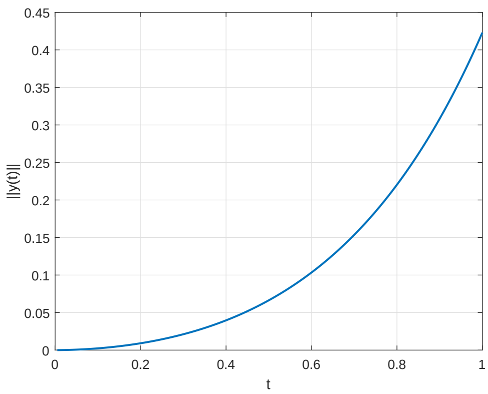

Figure 1. Norm figure of $y(t)$.

And we derive that $D = e - 1 \approx 1.7183$, $M_1 = 2.7183$, $M = E_{1.3,0.9}(0.6 \times 1.7183^{1.3}) \approx E_{1.3,0.9}(1.28) \approx 2.73$.

Supposed that $\delta = 0.15$, *we obtain* ***Table 1*** *as follows.*

Table 1. FTS results of Equation (11).

Theorem	ζ	ν	δ	T	$\|y\|_{C_{2-\zeta;\psi}}$	η	FTS
4	1.9	0.75	0.15	1	2.3859	2.39(Optimal)	Yes
5	1.9	0.75	0.15	1	2.9003	2.91	Yes
6	1.9	0.75	0.15	1	5.2059	5.21	Yes

According to Definition 7, a relative threshold η can be selected to ensure the FTS of Equation (11)*. With parameters determined as above, the selected optimal relative threshold is η = 2.39.*

Example 3. *Consider the following system:*

$$\begin{cases} {}^{H}D_{1^+}^{1.25,0.5;\psi}y_1(t) - {}^{H}D_{1^+}^{0.75,0.5;\psi}y_2(t) = 0.4, \quad 1 < t \le 2, \\ {}^{H}D_{1^+}^{1.25,0.5;\psi}y_2(t) - {}^{H}D_{1^+}^{0.75,0.5;\psi}y_1(t) = u(t) + 0.3\sin(y_2(t)), \\ D_{1^+}^{0.625;\psi}y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; I_{1^+}^{0.375;\psi}y(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{cases} \tag{12}$$

and it has

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \alpha_1 = 1.25, \quad \alpha_2 = 0.75, \quad \beta_1 = \beta_2 = 0.5, \quad \psi(t) = t^2.$$

$$f(t, y(t)) = \begin{bmatrix} 0.4 \\ 0.3\sin(y_2(t)) \end{bmatrix}.$$

We can verify easily that $\zeta - \nu = 1.625 - 0.09375 = 1.55625 > 1$, by setting $k = 0.3$, $b_1 = 0$ it has

$$\begin{aligned} \|f(t,x) - f(t,y)\| &= \begin{bmatrix} 0 \\ 0.3\,|\sin(x_2(t)) - \sin(y_2(t))| \end{bmatrix} \\ &\le 0.3 \begin{bmatrix} 0 \\ |x_2(t) - y_2(t)| \end{bmatrix}, \\ \|f(t, y(t))\| &\le 0.4 + 0.3|\sin(y_2(t))| \le b_1, \end{aligned}$$

which shows H3 and H6 stand.

The controllability Gramian matrix is

$$\begin{aligned} M &= \int_1^2 2s(4-s^2)^{0.5} E_{0.5,1.25}(P(4-s^2)^{0.5}) QQ^T E_{1/2,5/4}(P^T(4-s^2)^{0.5}) ds \\ &= \begin{bmatrix} 03247 & 0.2156 \\ 0.2156 & 0.8912 \end{bmatrix}. \end{aligned}$$

Since $\det(M) = 0.3247 \times 0.8912 - 0.2156^2 = 0.2429$*, the controllability Gramian matrix M is invertible. Therefore, the system is controllable.*

With the help of Theorem 8, for the final state $x_1 = \begin{bmatrix} 0.9 \\ 0.9 \end{bmatrix}$, take $y_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can provide the algorithm for y_n and u_n.

$$\begin{aligned} y_0(t) &= y_0, \\ y_{n+1}(t) &= \eta_1(t)c_1 + \eta_2(t)c_2 + \int_a^t \kappa(t,s)Qu_n(s) + \kappa(t,s)f(s, y_n(s))ds, \\ u_n(t) &= Q^T\kappa^T(b,t)M^{-1}\left\{x_1 - \eta_1(t)c_1 - \eta_2(t)c_2 - \int_a^b \kappa(b,s)f(s,y_n(s))ds\right\}. \end{aligned}$$

The numerical computations are performed using MATLAB. The ψ*-fractional integrals are approximated by the composite trapezoidal rule with a step size of* $h = 0.01$*. The matrix Mittag-Leffler function* $E_{\alpha,\beta}(Z)$ *is computed by truncating its defining series after* $N = 50$ *terms, which ensures convergence for the given parameters.*

After 50 iterations, the computed final state is $y(2) \approx [0.8998, 0.9001]^T$*, which is very close to the desired target* $x_1 = [0.9, 0.9]^T$*. This confirms that the terminal controllability condition is effectively achieved.*

In ***Figures 2*** *and* ***3****,* $y_p(t)$ *and* $y_q(t)$ *represent the first and second components of vector y(t), respectively. The curves labeled 1, 2, and 3 correspond to solution values at iteration steps n = 5, 25, and 50. It is evident that the solution becomes increasingly reliable and converges to the target value at t = 2 as the number of iterations increases.*

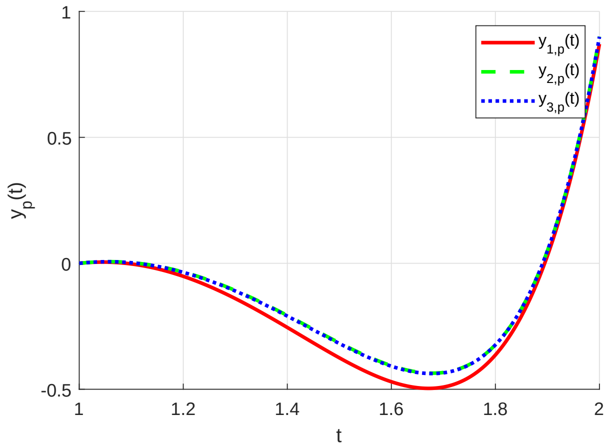

Figure 2. The image of the abscissa of $y(t)$ when n takes different values.

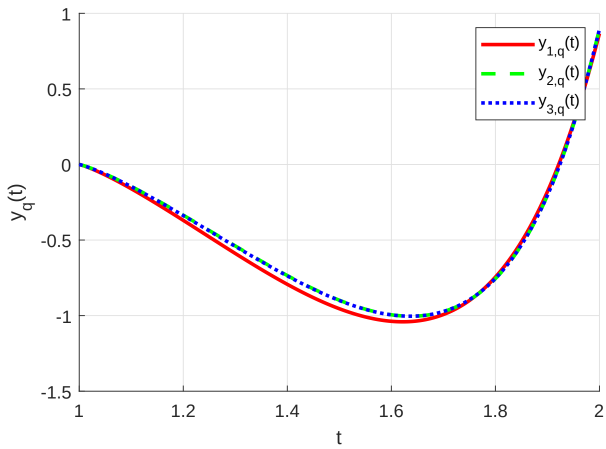

Figure 3. The image of the ordinate of $y(t)$ when n takes different values.

If we set

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad f(t,y) = \begin{bmatrix} 0 \\ 0.1\sin(y_3) \\ 0.1\cos(y_2) \end{bmatrix},$$

with the same fractional parameters $\alpha_1 = 1.25$*,* $\alpha_2 = 0.75$*,* $\beta_1 = \beta_2 = 0.5$*,* $\psi(t) = t^2$*,* $a = 1$*,* $b = 2$*, zero initial conditions,* $k = b_1 = 0.1$*, and target* $x_1 = (0.9, 0.5, 0.2)^T$*.*

The matrix P has eigenvalues 1, i, $-i$*. The controllable subspace is spanned by* $(1,0,0)^T$*, the other two modes are uncontrollable because Q only acts on the first coordinate. The Gramian is rank-1 and singular.*

Using the pseudo-inverse, the projection of target onto the controllable subspace is $(0.9,0,0)^T$*. The nonlinear terms excite the uncontrollable modes, and using the Gronwall estimate from Section 4 (Lemma 7), we bounded the uncontrollable component at* $t = 2$ *by* $\|y_{\perp}(2)\| \leq 0.1\sqrt{2}\ M_2 D^{2-\zeta}(b-a) \approx 0.0502$*. Thus, after 50 iterations,* $y_{50}(2) \approx (0.9,\ 0.0355,\ 0.0355)^T$*, with error norm* ≈ 0.4928*. This error is clearly unacceptable.*

We emphasized that this estimate explicitly uses the fractional Gronwall inequality from Section 4, thereby linking the stability analysis directly to the controllability.

7. Conclusion

This work establishes a comprehensive theoretical framework for nonlinear fractional damped systems governed by the ψ-Hilfer derivative. By developing a generalized Laplace transform combined with Mittag-Leffler function properties, it derives for the first time an explicit solution representation for a class of damped systems with fractional initial conditions. Within the weighted function space $C_{2-\zeta;\psi}(J,\mathbb{R}^n)$, we achieve a valid approach addressing the inherent singularity from fractional initializations. Furthermore, by integrating fractional Gronwall inequalities with weighted norm analysis, it constructs explicit finite-time stability criteria, providing verifiable guarantees for complex damped systems with hereditary memory effects. However, this study has several limitations. It requires more precise numerical verification. In the future, we will focus on investigating the attractivity of this system and expand it to include delayed, impulsive or stochastic systems.

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