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Article

Multiple positive solutions to a nonlocal problem with critical and supercritical nonlinear terms

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Abstract: Nonlocal equations have emerged as a prominent research frontier in the field of nonlinear partial differential equations, while simultaneously posing pervasive challenges in theoretical modeling across disciplines including elastic vibrations and geometric analysis. To date, the academic community has developed a robust and comprehensive theoretical framework for such equations, with existing results encompassing both critical and supercritical nonlinearities. Nevertheless, when employing variational methods to investigate multiple positive solutions of nonlocal equations defined on the four-dimensional ball, the loss of embedding compactness induced by critical terms remains a core bottleneck hindering further progress. This paper explores multiple positive solutions of nonlocal equations with both critical and supercritical nonlinear terms on the four-dimensional spherical domain. To rigorously establish the existence of multiple positive solutions, we integrate the Nehari manifold framework with advanced variational techniques. We harness the Brézis–Lieb lemma to circumvent the compactness deficiency arising from critical nonlinearities, and draw upon potential function analysis to compensate for the failure of compactness conditions caused by supercritical terms, thereby rigorously proving the existence of k distinct positive solutions for the equation. This result not only generalizes some existing conclusions in the literature but also offers new insights for further research on high-dimensional nonlocal problems.

Keywords: nonlocal problem; critical exponent; supercritical growth; Nehari method

1. Introduction and main results

We are concerned with the nonlocal problem stated as follows

$$\begin{cases} -\Big(a-b\displaystyle\int_{B_r}|\nabla u|^2dx\Big)\Delta u = V(x)|u|^2u+\mu\dfrac{|x|^\gamma}{1+|x|^\gamma}|u|^{q-2}u, & \text{in } B_r\subset\mathbb{R}^4,\\ u=0, & \text{on } \partial B_r,\end{cases} \tag{1}$$

where a, b, r, γ and q are constants, μ is a parameter, $B_r=\{x\in\mathbb{R}^4: |x|<r\}$, $V(x)$ denotes a positive continuous function on $\overline{B}_r$ meeting the condition below:

(V_1) Let $x_1, x_2, ..., x_k$ be interior points of B_r and $V_M > 0$ be given number. Assume that

$$V(x_i)=V_M=\max_{x\in\overline{B}_r}V(x),\ \forall\, 1\le i\le k,$$

and

$$V(x)=V(x_i)+o(|x-x_i|^{\gamma_0})\ as\ x\to x_i,$$

where $2\le\gamma_0<4$.

In 2017, Wang, Suo and Lei [1] studied the nonlocal problem stated below:

$$-\Big(a-b\int_{\mathbb{R}^4}|\nabla u|^2dx\Big)\Delta u=|u|^2u+\mu f(x)$$

with $\mu>0$ enough small and $0<f(x)\in L^{4/3}(\mathbb{R}^4)$, they obtained two solutions via the variational methods. Moreover, for every $\mu>0$, Wang gave the third solution [2]. Thereafter, a great many scholars have carried out investigations on the following problem:

$$\begin{cases} -\Big(a-b\int_{\Omega}|\nabla u|^2dx\Big)\Delta u=f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Indeed, it corresponds to the steady-state counterpart of the dynamic equation [3]

$$\rho h\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=f(x,u)$$

with negative modulus, the necessary explanation can be found in reference [1]. When the modulus is positive, there are many papers study the results of the existence and properties for the solutions. For instance, Naimen [4] studied

$$\begin{cases} -\Big(a+b\int_{\Omega}|\nabla u|^2dx\Big)\Delta u=\mu u^3+\lambda u^q, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases}$$

some existence results were obtained via variational arguments together with the concentration compactness framework, under the assumptions $\lambda,\mu>0$ and $1\le q<3$. Wang and Wei [5] considered optimal control of this kind of nonlocal problems, the feasible and tangent directions were established using cone theory, and subsequently derived the necessary optimality conditions by incorporating the descent direction of the performance indicators. Furthermore, they introduced the strong maximum principle and weak maximum principle, and provided information on cost free bangbang control. In 2022, under some certain assumptions with $0<\beta<1$, $6<p<6+2\beta$ and $M(\cdot)$, Lei and Liao [6] have studied the following equation:

$$\begin{cases} M\Big(\int_{B_r}|\nabla u|^2dx\Big)\Delta u+u^5+\dfrac{|x|^\beta}{1+|x|^\beta}u^{p-1}=0, & \text{in } B_r\subset\mathbb{R}^3,\\ u=0, & \text{on } \partial B_r. \end{cases}$$

Obviously, this problem has the critical and supercritical nonlinear terms. With an energy functional decomposition scheme combined with refined analysis, the authors determined a threshold value, which further enables the verification of positive solution existence. Lei and Suo [7] investigated a Schrödinger–Poisson system in three-dimensional space, where the nonlinearity consists of a critical growth term and a supercritical perturbation term. By employing variational methods together with the Nehari method, they prove the existence of k positive solutions, provided that the coefficient $Q(x)$ is positive and continuous.

Over recent years, substantial progress has been achieved in the study of Kirchhoff-type equations and related nonlocal problems from multiple angles.

Abolhassanifar et al. [8] deal with a p-Laplacian Kirchhoff problem involving critical growth and degenerate coefficients. Through variational arguments and a refined mountain pass scheme, they demonstrate the existence of ground state solutions and thoroughly characterize the limiting profiles of solutions as the parameter λ approaches a critical value, with special attention paid to the role of the degenerate coefficient $K(x)$. On the other hand, Cunha, Faraci and Silva [9] study a critical nonlocal problem with strong singularity in high-dimensional settings. Nonsmooth analysis is applied to the energy functional, and by exploring the topological structure of the sublevel sets associated with its regular part, the authors establish the existence of precisely three weak solutions, two of which are local minima and the third one corresponds to a critical point of mountain-pass type. Within the context of fractional Kirchhoff equations, Yu et al. [10] investigate the high perturbation scenario with critical nonlinearities. Applying the concentration compactness theory, the mountain pass theorem, together with truncation approaches, they establish the existence of solutions. Moreover, using the symmetric version of the mountain pass theorem, they derive multiplicity, and also discuss the absence of positive solutions as well as their asymptotic behavior. Next, Guefaifia et al. [11] address a nonlocal singular coupled system driven by the N-Laplacian, where the source term includes several exponential growth contributions controlled by the Trudinger-Moser inequality together with a singular term at the origin. With the aid of variational methods and the Trudinger-Moser inequality, the authors derive existence results concerning positive solutions, analyze the parametric dependence, and show that no solution exists when the spatial dimension equals two. Moreover, Guefaifia and Boulaaras [12] extend the investigation to anisotropic Finsler-Laplacian operators, considering a generalized Kirchhoff-type elliptic system that contains both singular terms and nonlinearities of subcritical to critical exponential growth. Owing to the anisotropy and exponential nature, classical Sobolev embeddings are no longer valid. Using variational tools, Galerkin approximation schemes and the sharp anisotropic Trudinger-Moser inequality, the authors derive at least one positive weak solution subject to reasonable conditions, and these conclusions are generalized within the complete anisotropic Finsler-Laplacian framework. Besides, they carry out finite element numerical simulations to verify the theoretical forecasts, which confirm solution existence, positivity and stability. Meanwhile, the numerical outcomes explicitly demonstrate the regularizing role played by the nonlocal Kirchhoff term as well as the impact stemming from singular parameters.

Turning to the result by Jiao and Pei [13], they are concerned with fractional p-Kirchhoff equations involving critical Sobolev exponents. By adopting the concentration compactness argument together with the symmetric mountain pass theorem, the authors establish the existence as well as multiplicity of solutions. The study by Gao et al. [14] deals with a nonlinear coupled Kirchhoff system with a purely Sobolev critical exponent. By introducing an appropriate transformation, the original system is recast as an equivalent coupling of a critical Schrödinger system and an algebraic system. In dimensions three and four, the authors obtain existence and classification results for positive ground states; for the degenerate case, a complete classification is provided in arbitrary dimensions. This represents the first systematic classification of ground states for Kirchhoff systems, and it partially extends and complements the earlier findings of Lü and Peng on subcritical coupled Kirchhoff systems and those of Chen and Zou on phase separation for critical Schrödinger equations. As for the study by Jiang [15], the work studies a p-Kirchhoff equation with critical exponent under suitable assumptions on the coefficient $M(t)$ and the nonlinearity $f(x,t)$. Making use of the dual fountain theorem together with the concentration compactness argument, they circumvent the loss of compactness originating from critical growth and obtain infinitely many solutions. Cheng and Bai [16] consider a parameter-dependent Kirchhoff double-phase Dirichlet problem involving Hardy-Sobolev exponents. Based on variational approaches and critical point arguments, they establish the existence of solutions, depending on different growth conditions on the nonlinearity. Notably, they extend the Sobolev-Hardy inequality from the classical space $W_0^{1,p}$ to $W_0^{1,\mathcal{H}}$, thereby bridging the singular term with the norm of the Musielak-Orlicz-Sobolev space. Tordecilla [17] treats a singular nonlocal elliptic equation of N-Kirchhoff type. Its nonlinearity consists of an exponential growth term (subcritical, critical or supercritical) compatible with the Trudinger-Moser inequality, as well as a singular term defined at the origin. By combining variational arguments with that inequality, they prove the existence of positive solutions, establish nonexistence in two dimensions, and analyze the asymptotic dependence on the parameter. Finally, Liu et al. [18] construct a new nonlocal Q-Laplacian problem over the Heisenberg group, with a nonlinear term of exponential growth. By virtue of the mountain pass theorem together with Ekeland's variational principle, the authors obtain existence results, and these findings are novel even in the Euclidean case.

This naturally leads to the consideration of the following problem:

$$-\Big(a-b\int_{\Omega}|\nabla u|^2dx\Big)\Delta u = f(x,u), \tag{3}$$

where Ω denotes either a smooth bounded domain in $\mathbb{R}^N$ or $\Omega=\mathbb{R}^N$, and $a,b>0$. The existence as well as multiplicity of solutions for problem (3) have been widely studied in recent years. The study by Wang et al. [19] addresses a class of weighted nonlocal transmission equations and investigates the multiplicity of their solutions. By combining variational methods with algebraic analysis and combinatorial approximation techniques, the authors establish the existence of one positive solution. Furthermore, through a further combination of algebraic arguments with known results, they show that infinitely many solutions exist. Later, Lapa [20] is devoted to studying the existence and asymptotic characteristics of solutions for nonlinear Kirchhoff-type equations. Under a suitably small initial energy condition, the Tartar method is employed together with refined a priori estimates involving $|\Delta u(t)|$, leading to a global existence result and an explicit exponential decay estimate. In Qian's study [21], the focus shifts to a nonlocal problem with critical exponent. Using variational tools and delicate energy estimates, the authors demonstrate that provided that λ is sufficiently small, there exist no fewer than k positive solutions to the problem. Meanwhile, Qian and Shi [22] investigate a nonlocal problem equipped with rapidly increasing weights and critical nonlinear exponents. With the aid of Ekeland's variational principle and the concentration compactness argument, the authors prove the existence of two positive solutions, and moreover point out that at least one solution is a positive ground state solution. The work by Chu et al. [23] concerns a new type of nonlocal problem with critical growth, where infinitely many sign-changing solutions are obtained via the method of invariant sets for descending flows, combined with perturbation techniques and suitable estimates.

In a related direction, Chu and Liu [24] study a new $p(x)$-Kirchhoff equation and establish existence as well as multiplicity of solutions via perturbation techniques, variational methods, and the invariant set approach for descending flows. The study by Zhang et al. [25] is concerned with the multiplicity as well as asymptotic properties of solutions to a degenerate Kirchhoff-type problem. By employing genus theory, the symmetric mountain pass lemma together with truncation techniques, the authors obtain an infinite sequence of solutions provided that the nonlinearity g is odd. Moreover, according to whether g behaves superlinearly or sublinearly near the origin, they demonstrate that the associated solution sequences either possess energy concentrated at the critical level $b^2/(4c)$ or converge to zero in the H_0^1 norm. Yaremenko [26] works within the variable exponent framework to establish a logarithmic Sobolev inequality and then applies it to a hyperbolic differential equation with a logarithmic source term. By deriving a generalized parametric logarithmic Sobolev inequality of Gagliardo-Nirenberg type, they establish weak solution existence. Finally, He and You [27] are devoted to exploring existence and multiplicity properties of positive solutions for Kirchhoff-type equations under critical frequency conditions. By means of variational approaches, the authors confirm that the problem has at least two positive solutions, and their approach offers the advantage of being applicable to a broader range of values for both the fractional parameter s and the spatial dimension N, in contrast to most existing results in the literature.

As for the study by Rădulescu and Vetro [28], it deals with an anisotropic Navier–Kirchhoff problem involving convection and Laplacian dependence. An appropriate Nemitsky map is constructed to handle the nonlinearity, and then Galerkin approximations together with Brouwer's fixed point theorem are employed to prove existence in the weak sense. In the nondegenerate case, the theory of pseudomonotone operators is invoked, accompanied by a corresponding asymptotic analysis. Wu et al. [29] are concerned with multiple positive solutions for a Kirchhoff-type problem equipped with critical and supercritical nonlinearities. Employing the Nehari approach and Ljusternik-Schnirelmann category theory, they point out an inherent relation: the number of positive solutions is closely associated with how many local maxima the coefficient function from the critical term possesses, which yields corresponding multiplicity outcomes.

As an extension, Chu and Xiao [30] considered the problem (3) with variable exponents and governed by the $p(x)$-Laplacian operator. With the aid of the Ekeland variational principle and mountain pass argument, they prove the existence of two nontrivial weak solutions. In particular, Wu et al. [31] explore the existence together with multiple positive solutions to a singular nonlocal problem that contains critical exponents, which reads:

$$\begin{cases} -\Big(a-b\displaystyle\int_{\mathbb{R}^4}|\nabla u|^2dx\Big)\Delta u=\lambda f(x)|u|^{-\gamma}+Q(x)|u|^2u, & \text{in } \mathbb{R}^4,\\ u>0, & \text{on } \mathbb{R}^4,\end{cases}$$

in which $a, b > 0$, $0 < \gamma < 1$, and $\lambda > 0$ is a parameter, while $f(x)$ and $Q(x)$ denote nonnegative continuous functions. Imposing appropriate hypotheses upon $Q(x)$ and $f(x)$, the authors prove the existence of no fewer than $k+1$ positive solutions via the Nehari method together with the variational method.

Based on our previously mentioned work, we consider multiple positive solutions for the problem (1) with critical and supercritical. Since the critical growth of problem (1) leads to the lack of compactness for the embedding $H^1_{0,rad}(B_r) \hookrightarrow L^4(B_r)$, we overcome this difficulty with the aid of Brézis–Lieb's Lemma. In addition, we point out that the compactness of the supercritical term is obtained by using the potential function.

The main conclusion of this work is stated below:

Theorem 1. *Let $a, b > 0$, $0 < \gamma < 1$ and $4 < q < 4 + 2\gamma$. If condition (V_1) is satisfied, then there is $\mu_0 > 0$ for which problem* (1) *has at least k positive solutions for all $\mu \in (0, \mu_0)$.*

Remark 1. *Compared with Lei and Liao [6], we consider the new nonlocal problem in $\mathbb{R}^4$. By virtue of the Nehari method combined with the variational method, we obtain that there exist k positive solutions to this problem. Compared with Wu et al. [31], the present work deals with a Kirchhoff-type equation incorporating critical and supercritical nonlinear terms posed on a ball.*

2. Preliminaries

Throughout the present work, we employ the notations given as follows:

$H = H^1_{0,rad}(B_r)$ denotes the first-order Sobolev space consisting of radial functions, which is furnished with an inner product and an associated norm:

$$\langle u, v\rangle = \int_{B_r}\nabla u\nabla v dx,\ \ \|u\| = \Big(\int_{B_r}|\nabla u|^2dx\Big)^{\frac{1}{2}}.$$

$|\cdot|_p$ as the usual norm of space $L^p(B_r)$. The symbols $C, C_1, C_2, \ldots$ represent diverse positive constants whose values can differ in different lines. We use S to denote the best Sobolev constant, i.e.,

$$S = \inf_{u\in H\setminus\{0\}} \frac{\int_{B_r}|\nabla u|^2dx}{\big(\int_{B_r}|u|^4dx\big)^{\frac{1}{2}}}. \tag{4}$$

The energy functional associated with problem (1) is given by

$$I_\mu(u) = \tfrac{a}{2}\|u\|^2 - \tfrac{b}{4}\|u\|^4 - \tfrac{1}{4}\textstyle\int_{B_r}V(x)|u|^4dx - \tfrac{\mu}{q}\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^qdx,$$

clearly the functional $I \in C^1(H, \mathbb{R})$.

We call that $u \in H$ is a weak solution to the problem (1) if and only if

$$\left(a-b\int_{B_r}|\nabla u|^2dx\right)\int_{B_r}\nabla u\nabla\varphi dx-\int_{B_r}V(x)|u|^3\varphi dx-\mu\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^{q-1}\varphi dx=0$$

for any $\varphi\in H$.

To verify our main conclusions, we first investigate the functional restricted to the Nehari manifold:

$$\mathcal{N}_\mu=\left\{u\in H\backslash\{0\}:a\|u\|^2-b\|u\|^4-\int_{B_r}V(x)|u|^4dx-\mu\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^qdx=0\right\}.$$

Lemma 1. *Let $a,b,\mu>0$, $0<\gamma<1$ and $4<q<4+2\gamma$. If $\{u_n\}$ is bounded and $u_n\rightharpoonup u_*$ in H, then*

$$\lim_{n\to\infty}\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^qdx=\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_*|^qdx.$$

Proof. By virtue of Lemma 2.2 presented in the reference by Do Ó et al. [32], we obtain

$$|u_n(x)|\le C\|u_n\|\frac{1}{|x|}\ \ a.e.\ x\in B_r.$$

According to the boundedness of $\{u_n\}$ and $4<q<4+2\gamma$, we have

$$\begin{aligned}\int_{|x|\le1}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^qdx&\le C\int_{|x|\le1}\frac{|x|^\gamma}{(1+|x|^\gamma)|x|^q}dx\\&\le C\int_{|x|\le1}\frac{1}{|x|^{q-\gamma}}dx\\&=C\int_0^1\frac{1}{t^{q-\gamma-3}}dt\\&<\infty\ \ (\text{as } q<4+2\gamma),\end{aligned}$$

and

$$\begin{aligned}\int_{|x|\ge1}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^qdx&\le C\int_{|x|\ge1}\frac{|x|^\gamma}{(1+|x|^\gamma)|x|^q}dx\\&\le C\int_{|x|\ge1}\frac{1}{|x|^q}dx\\&=C\int_1^{+\infty}\frac{1}{t^{q-3}}dt\\&<+\infty\ \ (\text{as } q>4).\end{aligned}$$

Hence, from the Lebesgue dominated convergence theorem, one has

$$\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^qdx\to\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_*|^qdx,$$

as $n\to\infty$. The proof of Lemma 1 is hereby concluded. □

Lemma 2. *Suppose that $a,b,\mu>0$, $0<\gamma<1$ and $4<q<4+2\gamma$, then $\mathcal{N}_\mu\neq\varnothing$.*

Proof. If $u\in H\backslash\{0\}$, we define the fiber map J_u: $t\to I_\mu(tu)$ for $t>0$, one has

$$J_u(t)=\frac{a}{2}t^2\|u\|^2-\frac{b}{4}t^4\|u\|^4-\frac{t^4}{4}\int_{B_r}V(x)|u|^4dx-\frac{\mu}{q}t^q\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^qdx.$$

Then $J_u(0) = 0$ and $J_u(t) \to -\infty$ as $t \to +\infty$. Furthermore,

$$J_u'(t) = t\left[a\|u\|^2 - bt^2\|u\|^4 - t^2\int_{B_r} V(x)|u|^4dx - \mu t^{q-2}\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx\right].$$

For every $u \in H\backslash\{0\}$, there exists exactly one $t_u > 0$ satisfying

$$J_u'(t_u) = t_u a\|u\|^2 - t_u^3 b\|u\|^4 - t_u^3\int_{B_r} V(x)|u|^4dx - \mu t_u^{q-1}\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx = 0.$$

Thus $t_u u \in \mathcal{N}_\mu$, that is $\mathcal{N}_\mu \neq \varnothing$. The proof is thus finished. □

Lemma 3. *Assume that* $0 < \gamma < 1$ *and* $4 < q < 4+2\gamma$*, then the functional* I_μ *satisfies the* $(PS)_c$ *condition with*

$$c < \frac{a^2S^2}{4(bS^2 + V_M)}.$$

Proof. Suppose that $\{u_n\} \subset H$ forms a $(PS)_c$ sequence associated with the functional I_μ, namely,

$$I_\mu(u_n) \to c, \quad I_\mu'(u_n) \to 0, \quad as\ n \to \infty. \tag{5}$$

Our first assertion is that $\{u_n\}$ is bounded in H. In fact, from Equation (5) we have

$$\begin{aligned} c + 1 + o(\|u_n\|) &\geq I_\mu(u_n) - \frac{1}{4}\langle I_\mu'(u_n), u_n\rangle \\ &= a\left(\frac{1}{2} - \frac{1}{4}\right)\|u_n\|^2 + \mu\left(\frac{1}{4} - \frac{1}{q}\right)\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^q dx \\ &\geq \frac{a}{4}\|u_n\|^2, \end{aligned}$$

which yields the boundedness of $\{u_n\}$ in H. Passing to a subsequence, still labeled as $\{u_n\}$, there exists $u_* \in H$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } H, \\ u_n \to u_*, & \text{strongly in } L^p(B_r)\ (1 \leq p < 4), \\ u_n(x) \to u_*(x), & \text{a.e. in } B_r. \end{cases}$$

Let $w_n = u_n - u_*$, then $\|w_n\| \to 0$. Otherwise, passing to a subsequence (again denoted by $\{w_n\}$) we may assume $\lim_{n\to\infty}\|w_n\| = l > 0$. Combining Brézis–Lieb's Lemma and Lemma 1, one has

$$\begin{aligned} &a\|w_n\|^2 + a\|u_*\|^2 - b\|w_n\|^4 - b\|u_*\|^4 - 2b\|u_*\|^2\|w_n\|^2 - \int_{B_r} V(x)|w_n|^4dx \\ &- \int_{B_r} V(x)|u_*|^4dx - \mu\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_*|^q dx = o(1). \end{aligned} \tag{6}$$

Since $\lim_{n\to\infty}\langle I_\mu'(u_n), u_*\rangle = 0$, we have

$$a\|u_*\|^2 - b\|u_*\|^4 - b\|u_*\|^2\|w_n\|^2 - \int_{B_r} V(x)|u_*|^4dx - \mu\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_*|^q dx = 0. \tag{7}$$

It follows from Equations (6) and (7) that

$$a\|w_n\|^2 - b\|w_n\|^4 - b\|u_*\|^2\|w_n\|^2 - \int_{B_r} V(x)|w_n|^4 dx = o(1). \tag{8}$$

Then set $n \to \infty$ in Equation (8) and by Equation (4), one has

$$al^2 - bl^4 - bl^2\|u_*\|^2 \leq V_M S^{-2} l^4,$$

which implies that

$$l^2 \geq \frac{S^2(a - b\|u_*\|^2)}{bS^2 + V_M} > 0. \tag{9}$$

On one hand, using Equation (7), we obtain

$$\begin{aligned}
I_\mu(u_*) =& \frac{a}{2}\|u_*\|^2 - \frac{b}{4}\|u_*\|^4 - \frac{1}{4}\int_{B_r} V(x)|u_*|^4 dx - \frac{\mu}{q}\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u_*|^q dx \\
& - \frac{a}{4}\|u_*\|^2 + \frac{b}{4}\|u_*\|^4 + \frac{b}{4}\|u_*\|^2\|w_n\|^2 + \frac{1}{4}\int_{B_r} V(x)|u_*|^4 dx \\
& + \frac{\mu}{4}\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u_*|^q dx \\
=& \frac{a}{4}\|u_*\|^2 + \frac{b}{4} l^2 \|u_*\|^2 + \mu\left(\frac{1}{4} - \frac{1}{q}\right)\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u_*|^q dx \\
\geq& \frac{a}{4}\|u_*\|^2 + \frac{b}{4} l^2 \|u_*\|^2 \\
\geq& \frac{aV_M}{4(bS^2+V_M)}\|u_*\|^2 + \frac{abS^2}{2(bS^2+V_M)}\|u_*\|^2 - \frac{b^2S^2}{4(bS^2+V_M)}\|u_*\|^4 \\
\geq& \frac{abS^2}{2(bS^2+V_M)}\|u_*\|^2 - \frac{b^2S^2}{4(bS^2+V_M)}\|u_*\|^4.
\end{aligned}$$

On the other hand, combining Equations (8) and (9), one has

$$\begin{aligned}
c + o(1) =& I_\mu(u_n) \\
=& \frac{a}{2}\|u_n\|^2 - \frac{b}{4}\|u_n\|^4 - \frac{1}{4}\int_{B_r} V(x)|u_n|^4 dx - \frac{\mu}{q}\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^q dx \\
=& \frac{a}{2}\|u_*\|^2 - \frac{b}{4}\|u_*\|^4 - \frac{1}{4}\int_{B_r} V(x)|u_*|^4 dx - \frac{\mu}{q}\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u_*|^q dx \\
& + \frac{a}{2}\|w_n\|^2 - \frac{b}{4}\|w_n\|^4 - \frac{b}{2}\|u_*\|^2\|w_n\|^2 - \frac{1}{4}\int_{B_r} V(x)|w_n|^4 dx + o(1) \\
=& I_\mu(u_*) + \frac{a - b\|u_*\|^2}{4} l^2 + o(1) \\
\geq& I_\mu(u_*) + \frac{a^2S^2}{4(bS^2+V_M)} - \frac{abS^2}{2(bS^2+V_M)}\|u_*\|^2 + \frac{b^2S^2}{4(bS^2+V_M)}\|u_*\|^4 + o(1) \\
\geq& \frac{a^2S^2}{4(bS^2+V_M)} + o(1).
\end{aligned}$$

This contradicts the hypothesis $c < \frac{a^2S^2}{4(bS^2+V_M)}$, so we deduce $l = 0$, which yields $u_n \to u_*$ in H as $n \to \infty$. This concludes the proof of Lemma 3. □

Lemma 4. *Let $0 < \gamma < 1$ and $4 < q < 4 + 2\gamma$. Then the functional $I_\mu(u)$ is coercive and bounded below on $\mathcal{N}_\mu$.*

Proof. For $u \in \mathcal{N}_\mu$, we obtain

$$\begin{aligned} I_\mu(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{4}\int_{B_r} V(x)|u|^4 dx - \frac{\mu}{q}\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx \\ &= a\left(\frac{1}{2} - \frac{1}{4}\right)\|u\|^2 + \mu\left(\frac{1}{4} - \frac{1}{q}\right)\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx \\ &\geq \frac{a}{4}\|u\|^2. \end{aligned}$$

Hence, the functional $I_\mu(u)$ turns out to be coercive and bounded from below on $\mathcal{N}_\mu$. This finishes the proof of Lemma 4. □

Lemma 5. *Assume u_0 to be a local minimizer associated with the functional I_μ restricted to $\mathcal{N}_\mu$. Then $I'_\mu(u_0) = 0$ in H^{-1}.*

Proof. Let u_0 be a local minimizer of I_μ on $\mathcal{N}_\mu$. Then there exists a neighborhood N of u_0 in H, such that

$$I_\mu(u_0) = \min_{u \in \mathcal{N}_\mu} I_\mu(u) = \min_{u \in N\backslash\{0\}, J'_u(1)=0} I_\mu(u),$$

and

$$J'_u(1) = a\|u\|^2 - b\|u\|^4 - \int_{B_r} V(x)|u|^4 dx - \mu\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx.$$

Therefore, according to Lemma 2 and $J'_u(1) = 0$, one has

$$\begin{aligned} J''_u(1) &= a\|u\|^2 - 3b\|u\|^4 - 3\int_{B_r} V(x)|u|^4 dx - (q-1)\mu\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx \\ &= (1-3)a\|u\|^2 - (q-1-3)\mu\int_{B_r} \frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx \\ &< 0. \end{aligned}$$

In view of the Lagrange multiplier rule, one can find a real number τ satisfying $I'_\mu(u_0) = \tau J''_{u_0}$. Since $u_0 \in \mathcal{N}_\mu$, we have

$$0 = \langle I'_\mu(u_0), u_0\rangle = \tau\langle J''_{u_0}(1), u_0\rangle,$$

which show that $\tau = 0$. Thus we get $I'_\mu(u_0) = 0$ in H^{-1}. The proof is complete. □

Select $\zeta > 0$ small enough such that $0 \notin B_\zeta(x_i)$, $\cup_{i=1}^k \overline{B_\zeta(x_i)} \subset B_r$ and $\overline{B_\zeta(x_i)} \cap \overline{B_\zeta(x_j)} = \varnothing$ whenever $i \neq j$ with $i, j = 1, 2, \cdots, k$, where $\overline{B_\zeta(x_i)} = \left\{x \in \mathbb{R}^4 | \ |x - x_i| \leq \zeta\right\}$. To minimize the functional I_μ over some submanifolds of $\mathcal{N}_\mu$, we introduce a barycenter map $g : H\backslash\{0\} \to \mathbb{R}^4$ as

$$g(u) = \frac{\int_{B_r} x|u|^4 dx}{\int_{B_r} |u|^4 dx}.$$

Set

$$\begin{aligned} \mathcal{N}_\mu^i &= \{u \in \mathcal{N}_\mu : |g(u) - x_i| < \zeta\}, \quad 1 \leq i \leq k, \\ \Theta_\mu^i &= \{u \in \mathcal{N}_\mu : |g(u) - x_i| = \zeta\}, \quad 1 \leq i \leq k. \end{aligned}$$

In accordance with Lemma 2, we know that $\mathcal{N}_\mu \neq \varnothing$. Then from Lemma 4, we can define

$$c_\mu^i = \inf_{u\in\mathcal{N}_\mu^i} I_\mu(u), \quad \tilde{c}_\mu^i = \inf_{u\in\Theta_\mu^i} I_\mu(u), \quad 1\leq i\leq k.$$

As is common knowledge, the function

$$U(x) = \frac{8^{\frac{1}{2}}}{1+|x|^2}, \;\; x\in\mathbb{R}^4,$$

satisfies

$$-\Delta U = U^3 \quad \text{in } \mathbb{R}^4.$$

Pick δ sufficiently small such that $0\notin B_\delta(x_i)\subset B_r$ for every $i=1,2,\cdots,k$. Let $\varphi_i(x)\in C_0^\infty$ denote a cut-off function satisfying $0\leq\varphi_i(x)\leq 1$ for all $x\in\mathbb{R}^4$ and

$$\varphi_i(x) = \begin{cases} 1, & |x-x_i|\leq\delta/2, \\ 0, & |x-x_i|\geq\delta, \end{cases}$$

We define

$$u_\varepsilon^i(x) = \varepsilon^{-1}\varphi_i(x)U(\frac{x-x_i}{\varepsilon}) = \frac{\varphi_i(x)8^{\frac{1}{2}}\varepsilon}{\varepsilon^2+|x-x_i|^2}.$$

It is known by Brezis and Nirenberg [33] that

$$\|u_\varepsilon^i\|^2 = S^2+O(\varepsilon^2), \qquad |u_\varepsilon^i|_4^4 = S^2+O(\varepsilon^4). \tag{10}$$

Lemma 6. *Assume that $a,b,\mu>0$, $0<\gamma<1$ and $4<q<4+2\gamma$. If (V_1) holds then*

$$c_\mu^i < \frac{a^2S^2}{4(bS^2+V_M)},$$

for $1\leq i\leq k$.

Proof. From (V_1) for all $\rho>0$, there exists $\eta>0$ such that

$$|V(x)-V(x_i)| < \rho|x-x_i|^{\gamma_0}, \quad 0<|x-x_i|<\eta<\delta/2. \tag{11}$$

According to Lemma 2, there exists $t_\varepsilon^i>0$ such that $t_\varepsilon^i u_\varepsilon^i\in\mathcal{N}_\mu$ and $I_\mu(t_\varepsilon^i u_\varepsilon^i) = \sup_{t\geq 0} I_\mu(tu_\varepsilon^i)$. By the symmetry of u_ε^i about x_i, we further obtain $g(t_\varepsilon^i u_\varepsilon^i)\in B_{r_0}(x_i)$. Therefore, in order to complete the proof for this lemma, it is sufficient to check that

$$\sup_{t\geq 0} I_\mu(tu_\varepsilon^i) < \frac{a^2S^2}{4(bS^2+V_M)}.$$

On one hand, assume there exists $t_1>0$, satisfying $t_\varepsilon^i>t_1>0$ for all sufficiently small $\varepsilon>0$. If this fails, we can take a sequence $\varepsilon_n\to 0^+$ such that $t_{\varepsilon_n}^i\to 0$. In view of the continuity of I_μ and the boundedness of $\{u_{\varepsilon_n}^i\}$, we have

$$\sup_{t\geq 0} I_\mu(tu_{\varepsilon_n}^i) = I_\mu(t_{\varepsilon_n}^i u_{\varepsilon_n}^i) \to 0 < \tfrac{a^2S^2}{4(bS^2+V_M)},$$

this shows that the proof is finished. On the other hand, $I_\mu(tu_\varepsilon^i) \to -\infty$ as $t \to +\infty$ guarantees the existence of some $t_2 > 0$ satisfying $t_\varepsilon^i < t_2$. Consequently, $0 < t_1 < t_\varepsilon^i < t_2$ holds for all sufficiently small $\varepsilon > 0$. For further analysis, define

$$h(t) = \tfrac{at^2}{2}\|u_\varepsilon^i\|^2 - \tfrac{bt^4}{4}\|u_\varepsilon^i\|^4 - \tfrac{t^4}{4}\int_{B_r} V_M|u_\varepsilon^i|^4dx.$$

Direct computation shows that $h(t)$ reaches its maximum at

$$t_{max} = \left(\frac{a\|u_\varepsilon^i\|^2}{b\|u_\varepsilon^i\|^4 + \int_{B_r} V_M|u_\varepsilon^i|^4dx}\right)^{\frac{1}{2}} = \left(\frac{aS^2 + O(\varepsilon^2)}{bS^4 + V_M S^2 + O(\varepsilon^2)}\right)^{\frac{1}{2}} = \left(\frac{aS^2}{bS^4 + V_M S^2}\right)^{\frac{1}{2}} + O(\varepsilon^2),$$

and

$$h(t_{max}) = \tfrac{a^2S^2}{4(bS^2+V_M)} + O(\varepsilon^2).$$

When $\varepsilon > 0$ small enough, by Equation (11) one has

$$\begin{aligned}
\frac{(t_\varepsilon^i)^4}{4}\int_{B_r}|V(x)-V(x_i)||u_\varepsilon^i|^4dx =& \frac{(t_\varepsilon^i)^4}{4}\int_{B_{\delta/2}(x_i)}|V(x)-V(x_i)||u_\varepsilon^i|^4dx\\
<& \frac{(t_\varepsilon^i)^4}{4}\int_{B_\eta(x_i)}\rho|x-x_i|^{\gamma_0}\frac{(8^{\frac{1}{2}}\varepsilon)^4}{(\varepsilon^2+|x-x_i|^2)^4}dx\\
&+ 2V(x_i)\frac{(t_\varepsilon^i)^4}{4}\int_{B_{\delta/2}(x_i)/B_\eta(x_i)}\frac{(8^{\frac{1}{2}}\varepsilon)^4}{(\varepsilon^2+|x-x_i|^2)^4}dx\\
\le& C\rho\varepsilon^4\int_0^\eta \frac{r^{3+\gamma_0}}{(\varepsilon^2+r^2)^4}dr + C\varepsilon^4\int_\eta^{\delta/2}\frac{r^3}{(\varepsilon^2+r^2)^4}dr\\
\le& C\rho\varepsilon^{\gamma_0}\int_0^{\frac{\eta}{\varepsilon}}\frac{r^{3+\gamma_0}}{(1+r^2)^4}dr + C\varepsilon^4\int_\eta^{\delta/2} r^{-5}dr\\
\le& C_1\rho\varepsilon^{\gamma_0} + C_2\varepsilon^4.
\end{aligned}$$

We give the following inequality

$$|a+b|^m \le 2^m(|a|^m + |b|^m), \tag{12}$$

for all $a, b \in \mathbb{R}$, $m > 0$. From the definition of u_ε^i and Equation (12), we infer that

$$\begin{aligned}
\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_\varepsilon^i|^q dx \ge& C\int_{|x-x_i|\le\frac{\delta}{2}}\frac{|x|^\gamma}{1+|x|^\gamma}\frac{\varepsilon^q}{(\varepsilon^2+|x-x_i|^2)^q}dx\\
\ge& C\varepsilon^q\int_{|x-x_i|\le\frac{\delta}{2}}\frac{|x|^\gamma}{1+(|x_i|+\frac{\delta}{2})^\gamma}\frac{1}{(\varepsilon^2+|x-x_i|^2)^q}dx\\
\ge& C\varepsilon^q\int_{|x-x_i|\le\frac{\delta}{2}}\frac{|x|^\gamma+|x-x_i|^\gamma-|x-x_i|^\gamma}{(\varepsilon^2+|x-x_i|^2)^q}dx\\
\ge& C\varepsilon^q\int_{|x-x_i|\le\frac{\delta}{2}}\frac{2^{-\gamma}|x_i|^\gamma}{(\varepsilon^2+|x-x_i|^2)^q}dx - C\varepsilon^q\int_{|x-x_i|\le\frac{\delta}{2}}\frac{|x-x_i|^\gamma}{(\varepsilon^2+|x-x_i|^2)^q}dx\\
\ge& C\varepsilon^q\int_0^{\frac{\delta}{2}}\frac{r^3}{(\varepsilon^2+r^2)^q}dr - C\varepsilon^q\int_0^{\frac{\delta}{2}}\frac{r^{3+\gamma}}{(\varepsilon^2+r^2)^q}dr\\
=& C\varepsilon^{4-q}\int_0^{\frac{\delta}{2\varepsilon}}\frac{t^3}{(1+t^2)^q}dt - C\varepsilon^{4+\gamma-q}\int_0^{\frac{\delta}{2\varepsilon}}\frac{t^{3+\gamma}}{(1+t^2)^q}dt\\
\ge& C\varepsilon^{4-q}\int_0^1 t^3dt - C\varepsilon^{4+\gamma-q}\int_0^1 t^{3+\gamma}dt - C\varepsilon^{4+\gamma-q}\int_1^{+\infty}t^{3+\gamma-2q}dt\\
\ge& C\varepsilon^{4-q}\int_0^1 t^3dt - C\varepsilon^{4+\gamma-q}\left(\frac{1}{4+\gamma} + \frac{1}{4+\gamma-2q}t^{4+\gamma-2q}\Big|_1^{+\infty}\right)\\
\ge& C_3\varepsilon^{4-q}.
\end{aligned}$$

Through direct calculation, we obtain

$$\begin{aligned}
\sup_{t\geq 0} I_\mu(tu_\varepsilon^i) &= I_\mu(t_\varepsilon^i u_\varepsilon^i)\\
&= h(t_\varepsilon^i) + \frac{(t_\varepsilon^i)^4}{4}\int_{B_r}|V(x)-V(x_i)||u_\varepsilon^i|^4dx - \frac{(t_\varepsilon^i)^q}{q}\mu\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_\varepsilon^i|^q dx\\
&\leq h(t_{max}) + \frac{(t_\varepsilon^i)^4}{4}\int_{B_r}|V(x)-V(x_i)||u_\varepsilon^i|^4dx - \frac{(t_\varepsilon^i)^q}{q}\mu\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_\varepsilon^i|^q dx\\
&\leq \frac{a^2S^2}{4(bS^2+V_M)} + O(\varepsilon^2) + C_1\rho\varepsilon^{\gamma_0} + C_2\varepsilon^4 - C_3\mu\varepsilon^{4-q}\\
&< \frac{a^2S^2}{4(bS^2+V_M)},
\end{aligned}$$

which completes the proof. □

Lemma 7. *Suppose that* (V_1) *holds, there exists* $\mu_0 > 0$ *for all* $0 < \mu < \mu_0$*, then*

$$\tilde{c}_\mu^i > \tfrac{a^2S^2}{4(bS^2+V_M)},$$

for $1 \leq i \leq k$.

Proof. We proceed by contradiction. Let $\{\mu_n\}$ be a positive sequence tending to zero as $n \to \infty$, for which $\{u_n\} \in \Theta_\mu^i$ and

$$I_{\mu_n}(u_n) \to c \leq \tfrac{a^2S^2}{4(bS^2+V_M)},$$

and

$$a\|u_n\|^2 - b\|u_n\|^4 - \int_{B_r}V(x)|u_n|^4dx - \mu_n\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^q dx = 0. \tag{13}$$

By Lemma 4 we know that $\{u_n\}$ is bounded in H. Moreover, combining Equation (4), Equation (13) and Lemma 1, one has

$$\begin{aligned}
a\|u_n\|^2 &= b\|u_n\|^4 + \int_{B_r}V(x)|u_n|^4dx + \mu_n\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^q dx\\
&\leq b\|u_n\|^4 + V_MS^{-2}\|u_n\|^4 + \mu_nC_4\|u_n\|^q,
\end{aligned}$$

it follow that $\|u_n\| \geq C_5 > 0$ for all $\mu_n \in (0,\mu_0)$. As $\mu_n \to 0$, we conclude from Equation (13) that there exists some positive constant C_6 satisfying

$$0 < C_6 < a\|u_n\|^2 = b\|u_n\|^4 + \int_{B_r}V(x)|u_n|^4dx,$$

for all $u_n \in \mathcal{N}_\mu$. Therefore, one may select $t_n > 0$ so that $w_n = t_nu_n$ fulfills

$$a\|w_n\|^2 = b\|w_n\|^4 + \int_{B_r}V_M|w_n|^4dx. \tag{14}$$

Then by the Sobolev inequality, we get $\|w_n\|^2 \geq \frac{aS^2}{bS^2+V_M}$. Moreover,

$$t_n = \left(\tfrac{b\|u_n\|^4+\int_{B_r}V(x)|u_n|^4dx+\mu_n\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^qdx}{b\|u_n\|^4+\int_{B_r}V_M|u_n|^4dx}\right)^{\frac{1}{2}},$$

we deduce that t_n is uniformly bounded. Suppose $\lim\limits_{n\to\infty} t_n = t_0$, since $\{u_n\}$ is bounded in H, $V(x) \le V_M$ and $\mu_n \to 0$, we have $t_0 \le 1$. In fact $t_0 = 1$. This follows easily from

$$\begin{aligned}\frac{a^2S^2}{4(bS^2+V_M)} &\le \lim_{n\to\infty}\frac{a}{4}\|w_n\|^2 = \lim_{n\to\infty}\frac{a}{4}t_n^2\|u_n\|^2\\ &= \lim_{n\to\infty} t_n^2\Big[(\frac{1}{2}-\frac{1}{4})\Big(a\|u_n\|^2 - b\|u_n\|^4 - \mu_n\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^q dx\Big)\\ &\quad +(\frac{1}{2}-\frac{1}{4})b\|u_n\|^4 + (\frac{1}{2}-\frac{1}{q})\mu_n\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^q dx\Big]\\ &= \lim_{n\to\infty} t_n^2 I_{\mu_n}(u_n)\\ &= t_0^2 c \le \frac{a^2S^2}{4(bS^2+V_M)}.\end{aligned}$$

The inequalities above also show that

$$c = \tfrac{a^2S^2}{4(bS^2+V_M)}, \quad \lim_{n\to\infty}\|w_n\|^2 = \lim_{n\to\infty}\|u_n\|^2 = \tfrac{aS^2}{bS^2+V_M}. \tag{15}$$

Set $v_n = \frac{w_n}{|w_n|_4}$, then $|v_n|_4 = 1$. In addition, combining with Equations (14) and (15), we have

$$\lim_{n\to\infty}\|v_n\|^2 = \lim_{n\to\infty}\tfrac{\|w_n\|^2}{|w_n|_4^2} = \lim_{n\to\infty}\frac{\|w_n\|^2}{V_M^{-\frac{1}{2}}(a\|w_n\|^2 - b\|w_n\|^4)^{\frac{1}{2}}} = S.$$

Accordingly, $\{v_n\}$ is a sequence that minimizes S. We find a point $x_0 \in \overline{B}_r$ using the result from Lions [34], such that

$$|\nabla v_n|^2 \rightharpoonup d\tilde{\mu} = S\delta_{x_0}, \ \ |v_n|^4 \rightharpoonup d\tilde{\nu} = \delta_{x_0}. \tag{16}$$

This convergence is weak in the sense of measures, with $\tilde{\mu}, \tilde{\nu}$ standing for finite measures, and δ_{x_0} representing the Dirac mass supported at x_0. Then we get

$$g(u_n) = \frac{\int_{B_r} x|u_n|^4 dx}{\int_{B_r}|u_n|^4 dx} = \frac{\int_{B_r} x|v_n|^4 dx}{\int_{B_r}|v_n|^4 dx} \to x_0,$$

as $n \to \infty$. This yields $x_0 \notin \{x_i | i = 1, 2, \cdots, k\}$. Using Equations (14) and (16), we derive that

$$\begin{aligned}\lim_{n\to\infty} I_{\mu_n}(u_n) &= \lim_{n\to\infty} t_n^2\Big[(\frac{1}{2}-\frac{1}{4})\Big(a\|u_n\|^2 - b\|u_n\|^4 - \mu_n\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^q dx\Big)\\ &\quad +(\frac{1}{2}-\frac{1}{4})b\|u_n\|^4 + (\frac{1}{2}-\frac{1}{q})\mu_n\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u_n|^q dx\Big]\\ &= \lim_{n\to\infty}\frac{1}{4}\int_{B_r} V(x)|u_n|^4 dx + \lim_{n\to\infty}\frac{1}{4}b\|u_n\|^4\\ &= \lim_{n\to\infty}\frac{1}{4}\int_{B_r} V(x)|w_n|^4 dx + \lim_{n\to\infty}\frac{1}{4}b\|w_n\|^4\\ &= \frac{V(x_0)}{4V_M}\lim_{n\to\infty}\int_{B_r} V_M|w_n|^4 dx + \lim_{n\to\infty}\frac{1}{4}b\|w_n\|^4\\ &= \frac{V(x_0)}{4V_M}\lim_{n\to\infty}(a\|w_n\|^2 - b\|w_n\|^4) + \lim_{n\to\infty}\frac{1}{4}b\|w_n\|^4\\ &= \frac{V(x_0)}{4V_M}\frac{a^2S^2}{bS^2+V_M} + \lim_{n\to\infty}\frac{b}{4}(1-\frac{V(x_0)}{V_M})\|w_n\|^4\\ &< \frac{a^2S^2}{4(bS^2+V_M)}.\end{aligned}$$

This contradicts relation (15), which completes the proof. □

Lemma 8. *Given arbitrary $u \in \mathcal{N}_\mu^i$ with $1 \leq i \leq k$, there exists $\eta > 0$ together with a differentiable functional $f : B_\eta(0) \subset H \to \mathbb{R}$ satisfying $f(0) = 1$, $f(v)(u - v) \in \mathcal{N}_\mu^i$ for all $v \in B_\eta(0)$ and*

$$\langle f'(0), \varphi\rangle = \frac{(2a-4b\|u\|^2)\int_{B_r}\nabla u\nabla\varphi dx - 4\int_{B_r}V(x)|u|^3\varphi dx - \mu q\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^{q-1}\varphi dx}{-2a\|u\|^2-(q-4)\mu\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx}.$$

Proof. For any $u \in \mathcal{N}_\mu^i (1 \leq i \leq k)$, we introduce the mapping $F : \mathbb{R}^+ \times H \to \mathbb{R}$ via

$$F(t, v) = at\|u - v\|^2 - bt^3\|u - v\|^4 - t^3\int_{B_r}V(x)|u - v|^4 dx - \mu t^{q-1}\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u - v|^q dx.$$

It is easily obtained that $F(1, 0) = \langle I'_\mu(u), u\rangle = 0$ and

$$\begin{aligned}F_t(1, 0) &= F_t(1, 0) - 3\langle I'_\mu(u), u\rangle \\ &= (1 - 3)a\|u\|^2 - (q - 1 - 3)\mu\int_{B_r}\frac{|x|^\gamma}{1 + |x|^\gamma}|u|^q dx \\ &< 0.\end{aligned}$$

Therefore, the implicit function theorem applies, which guarantees the existence of a sufficiently small $\eta > 0$ together with a differentiable functional $f : B_\eta(0) \subset H \to \mathbb{R}$ satisfying $f(0) = 1$ and $F(f(v), v) = 0$ for all $v \in B_\eta(0)$. Then we conclude that

$$\begin{aligned}0 =& f(v)F(f(v), v) \\ =& af^2(v)\|u - v\|^2 - bf^4(v)\|u - v\|^4 \\ &- f^4(v)\int_{B_r}V(x)|u - v|^4 dx - \mu f^q(v)\int_{B_r}\frac{|x|^\gamma}{1 + |x|^\gamma}|u - v|^q dx,\end{aligned}$$

which implies that $f(v)(u - v) \in \mathcal{N}_\mu^i$. Furthermore, one has

$$\begin{aligned}\langle F_v(1, 0), \varphi\rangle &= \lim_{s\to 0}\frac{F(1, 0 + s\varphi) - F(1, 0)}{s} \\ &= -(2a - 4b\|u\|^2)\int_{B_r}\nabla u\nabla\varphi dx + 4\int_{B_r}V(x)|u|^3\varphi dx + \mu q\int_{B_r}\frac{|x|^\gamma}{1 + |x|^\gamma}|u|^{q-1}\varphi dx,\end{aligned}$$

therefore

$$\begin{aligned}\langle f'(0), \varphi\rangle &= -\frac{\langle F_v(1, 0), \varphi\rangle}{F_t(1, 0)} \\ &= \frac{(2a - 4b\|u\|^2)\int_{B_r}\nabla u\nabla\varphi dx - 4\int_{B_r}V(x)|u|^3\varphi dx - \mu q\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^{q-1}\varphi dx}{-2a\|u\|^2 - (q - 4)\mu\int_{B_r}\frac{|x|^\gamma}{1+|x|^\gamma}|u|^q dx}.\end{aligned}$$

This finishes the proof of Lemma 8. □

3. Proof of Theorem 1

Proof of Theorem 1. By Lemmas 6 and 7, we can conclude that

$$c_\mu^i < \frac{a^2S^2}{4(bS^2 + V_M)} < \tilde{c}_\mu^i,$$

for all $\mu \in (0, \mu_0)$, $i = 1, 2, \cdots, k$. From this we deduce that

$$c_\mu^i = \inf\left\{I_\mu(u) | u \in \mathcal{N}_\mu^i \cup \Theta_\mu^i\right\}.$$

By applying Ekeland's variational principle [35], we construct a minimizing sequence $\{u_n^i\} \subset \mathcal{N}_\mu^i \cup \Theta_\mu^i$ which satisfies the properties given in the following

$$I_\mu(u_n^i) \leq c_\mu^i + \frac{1}{n}, \quad I_\mu(u_n^i) \leq I_\mu(w) + \frac{1}{n}\|w - u_n^i\|, \text{ for all } w \in \mathcal{N}_\mu^i \cup \Theta_\mu^i. \quad (17)$$

We now aim to verify $I'_\mu(u_n^i) \to 0$ as $n \to \infty$. Owing to the equality $I_\mu(|u_n^i|) = I_\mu(u_n^i)$, we deduce $u_n^i \geq 0$. Applying Lemma 8 for $u = u_n^i$, there exists $\eta_n^i > 0$ together with a differentiable functional $f_n^i : B_{\eta_n^i}(0) \subset H \to \mathbb{R}$ satisfying $f_n^i(0) = 1$ and $f_n^i(v)(u_n^i - v) \in \mathcal{N}_\mu^i$ for all $v \in B_{\eta_n^i}(0)$. Take $0 < \eta < \eta_n^i$ and set $v_\eta = \eta\varphi$ with $\|\varphi\| = 1$. Fix n and let $w_\eta = f_n^i(v_\eta)(u_n^i - v_\eta)$, we get $v_\eta \in B_{\eta_n^i}(0)$ and $w_\eta \in \mathcal{N}_\mu^i$. By Equation (17) one has

$$I_\mu(w_\eta) - I_\mu(u_n^i) \geq -\frac{1}{n}\|w_\eta - u_n^i\|,$$

it follows from the Taylor expansion that

$$\langle I'_\mu(u_n^i), w_\eta - u_n^i\rangle + o(\|w_\eta - u_n^i\|) \geq -\frac{1}{n}\|w_\eta - u_n^i\|.$$

Therefore

$$\langle I'_\mu(u_n^i), (u_n^i - v_\eta) + (f_n^i(v_\eta) - 1)(u_n^i - v_\eta) - u_n^i\rangle \geq -\frac{1}{n}\|w_\eta - u_n^i\| + o(\|w_\eta - u_n^i\|),$$

that is

$$-\eta\langle I'_\mu(u_n^i), \varphi\rangle + (f_n^i(v_\eta) - 1)\langle I'_\mu(u_n^i), u_n^i - v_\eta\rangle \geq -\frac{1}{n}\|w_\eta - u_n^i\| + o(\|w_\eta - u_n^i\|).$$

By $f_n^i(v_\eta)(u_n^i - v_\eta) \in \mathcal{N}_\mu^i$, then $\langle I'_\mu(w_\eta), f_n^i(v_\eta)(u_n^i - v_\eta)\rangle = 0$, we obtain

$$\begin{aligned}
-\eta\langle I'_\mu(u_n^i), \varphi\rangle &+ \frac{(f_n^i(v_\eta) - 1)}{f_n^i(v_\eta)}\langle I'_\mu(w_\eta), f_n^i(v_\eta)(u_n^i - v_\eta)\rangle \\
&+ (f_n^i(v_\eta) - 1)\langle I'_\mu(u_n^i) - I'_\mu(w_\eta), u_n^i - v_\eta\rangle \\
&\geq -\frac{1}{n}\|w_\eta - u_n^i\| + o(\|w_\eta - u_n^i\|).
\end{aligned}$$

Consequently

$$\langle I'_\mu(u_n^i), \varphi\rangle \leq \tfrac{\|w_\eta - u_n^i\|}{n\eta} + \tfrac{o(\|w_\eta - u_n^i\|)}{\eta} + \tfrac{f_n^i(v_\eta) - 1}{\eta}\langle I'_\mu(u_n^i) - I'_\mu(w_\eta), u_n^i - v_\eta\rangle. \quad (18)$$

From Lemma 8 and the boundedness of $\{u_n^i\}$, we have $\|w_\eta - u_n^i\| \leq C_7|f_n^i(v_\eta) - 1| + \eta$ and

$$\lim_{\eta \to 0} \frac{|f_n^i(v_\eta) - 1|}{\eta} \leq \|(f_n^i)'(0)\| \leq C_8.$$

For fixed n and let $\eta \to 0$ in Equation (18), one has

$$\langle I'_\mu(u_n^i), \varphi \rangle \leq \frac{C_9}{n}.$$

From this we deduce $I'_\mu(u_n^i) \to 0$ as $n \to \infty$, which means $\{u_n^i\}$ is a Palas–Smale sequence for I_μ at energy level c_μ^i. In view of Lemma 3 and $c_\mu^i < \frac{a^2 S^2}{4(bS^2+V_M)}$, one can extract a subsequence of $\{u_n^i\}$ (still labeled as $\{u_n^i\}$) and a function $u^i \in H$ satisfying

$$u_n^i \to u^i \quad \text{strongly in } H,$$

consequently $u^i \geq 0$ with $1 \leq i \leq k$. Lemma 7 yields $\|u_n^i\| \geq C > 0$, so $\lim\limits_{n\to\infty} I_\mu(u_n^i) = I_\mu(u^i) = c_\mu^i > 0$, which ensures $u^i \not\equiv 0$. By the strong maximum principle, we conclude that $u^i > 0 (1 \leq i \leq k)$. Moreover, the solutions are mutually distinct, since the $B_\varsigma(x_i)$ are pairwise different and $g(u^i) \in B_\varsigma(x_i)$ for every i. Accordingly, problem (1) possesses at least k positive solutions for all $\mu \in (0, \mu_0)$. This finishes the proof. □

4. Discussion and conclusion

The proof procedure for the Kirchhoff equations with critical and supercritical nonlinear terms on the four-dimensional ball involves notable complexity. Critical terms disrupt the compactness of Sobolev embeddings, rendering classical variational methods inapplicable directly; thus, we must combine the Nehari manifold, Brézis–Lieb lemma, and stratified potential function analysis to restore compactness conditions, with multiple condition verifications significantly increasing the difficulty of derivation. Distinct from the findings by Lei and Liao [6], this paper investigates a new class of nonlocal equations with four dimensions. Specifically, we conduct theoretical analysis by combining variational methods with the Nehari manifold approach, proving the existence of k positive solutions for the equation. In addition, compared to the Kirchhoff model with only critical nonlinear terms by Wu et al. [31], this paper constructs an equation with both critical and supercritical nonlinearities, while effectively addressing the loss of compactness caused by critical nonlinearities using the Nehari manifold method. Notably, the compactness analysis framework established in this paper possesses generality: it can be extended to smooth bounded domains of arbitrary dimensions and adapted to analogous elliptic equations with various combinations of critical, subcritical, and supercritical nonlinearities, as the entire argumentation framework can be reused merely by adjusting the potential function. However, this study has certain limitations: all theoretical conclusions only apply to regular spherical domains, and irregular boundary cases remain unanalyzed. Research on multiple positive solutions under complex boundaries can serve as a direction for future expansion.

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