Advances in Differential Equations and Control Processes 2026, 33(3), 4549.
doi: https://doi.org/10.59400/adecp4549

Article

On a class of bivalued hybrid cyclic and non-cyclic contractive self-mappings on unions of possibly disjoint subsets in metric spaces

Manuel De la Sen[1,*], **Asier Ibeas**[2], **Hasanen A. Hammad**[3,4]

[1] Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country (EHU), 48940 Leioa, Spain

[2] Department of Telecommunications and Systems Engineering, Universitat Autònoma de Barcelona (UAB), 08193 Barcelona, Spain

[3] Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

[4] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

*** Corresponding authors:** Manuel De la Sen, manuel.delasen@ehu.eus

CITATION

De la Sen M, Ibeas A, Hammad HA. On a class of bivalued hybrid cyclic and non-cyclic contractive self-mappings on unions of possibly disjoint subsets in metric spaces. Advances in Differential Equations and Control Processes. 2026; 33(3): 4549. https://doi.org/10.59400/adecp4549

ARTICLE INFO

Received: 22 June 2026
Revised: 21 July 2026
Accepted: 27 July 2026
Available online: 30 July 2026

Abstract: This paper presents and analyzes a class of self-mappings in metric spaces which are endowed of mixed characteristics between that of contractive cyclic self-mappings on union of subsets and that of contractive mappings within the individual subsets. Stability analysis of dynamic systems can be focused on by formalizing the relationship between cyclic contractive mappings over non-necessarily intersecting subsets and the stability criteria for switched systems with cyclic dynamics. The case of disjoint set structures necessitates the application of best proximity theory. Therefore, this manuscript introduces the mentioned class of hybrid cyclic/non-cyclic bivalued contractive self-mappings which can operate iteration-by-iteration in either a cyclic operation mode (the selected image lies in the next adjacent subset) or in an intra mode operation (the selected image lies in the current subset). Unlike cyclic operators, these proposed self-mappings allow dynamic switches to either the current subset or the next adjacent one in a cyclic disposal, governed by an iteration-dependent image selection sequence. This "modus-operandi" is possible since one of the images of each point at each iteration is activated for the cyclic operation mode while the other one is activated for an intra mode mode within some subset. Under the key assumptions that the subsets are closed, at least one best proximity set is a singleton, and the subset itself is boundedly compact, new boundedness and convergence theorems are established. It is emphasized how cyclic contractive properties are useful in the context of asymptotic stability of hybrid systems, validating the theoretical framework through illustrative examples.

Keywords: cyclic/non-cyclic hybrid bivalued contractive mappings; cyclic mappings; best proximity points

1. Introduction

It is well-known that the stability analysis of complex dynamic systems constitutes one of the fundamental pillars of modern control engineering and certain areas of applied mathematics. At the core of this field, lies the search for conditions that guarantee that the trajectories of a system converge toward an equilibrium state or that they remain confined within acceptable bounds. Historically, stability properties have been focused on in the frameworks of two deeply interconnected mathematical perspectives, namely, stability in the sense of the qualitative theory of dynamic systems, and stability from the viewpoint of contraction theory. The former approach relies on the existence of energy or Lyapunov functions or bounded trajectories while contraction theory provides a constructive and quantitative approach in the sense that, if an operator governing the system evolution reduces the distance between states at each iteration, then convergence toward a unique solution is intrinsically guaranteed. The behavior of the system under study might be rigorously focused on within a customary metric space (X, d). When the space X contains a set of subsets that do not necessarily intersect, the system dynamics requires sometimes the definition of cyclic mappings that act alternately on these subsets, specifically when the dynamics for parameterizations associated with the various subsets behaves in a cyclical manner. Since the absence of mutual intersection of the mentioned subsets precludes the existence of a fixed point, contraction theory evolves into the best proximity point theory. The pioneering work of Eldred and Veeramani [1] established the foundations for identifying the existence and convergence of these points, minimizing the geometric distance between the involved sets. Following this milestone, the literature expanded toward the study of p-cyclic contractions [2], with $p \geq 2$ in general, and generalizations of Meir-Keeler conditions applied to cyclic frameworks [3]. Subsequently, even broader contractive conditions were explored by incorporating cyclic multivalued operators [4] and specific existence theorems under generalized contractions.

The recent development of this theory has followed several directions. Two main ones consist of refining iterative schemes operated in the classical metric space (X, d) and of abstracting these principles into more generalized metric spaces. Within the standard space framework, studies have been focused on for a class of LBS-type mappings in extended variants of the metrics [5]. In Ali and Ahmad [6], a best proximity point generalization of fixed point is given which is useful for the case when the contraction map is not a self-map. In particular, new types of proximal contraction for cyclic mapping were proposed in strong b-metric spaces. In Wu and Weng [7], and based on extended b-metric spaces, a new structure of cyclic mapping, a ϕ_δ-type cyclic mapping was proposed. The fixed-point theorem for it was established and proved. In Gabeleh et al. [8], a Takahashi-type convex structure was considered on (probabilistic) Menger PM-spaces which was used to characterize the existence of best proximity points for weak cyclic Kannan contractions. It was also proved that every compact and convex pair in metrically convex Menger PM-spaces is endowed with the probabilistic proximal quasi-normal structure. In Hafshejani [9], the notion of strongly cyclic Sehgal-type φ-contraction of type one was focused on as a generalization of those of cyclic φ-contraction maps in the sense of Păcurar-Rus and cyclic contraction maps in the Suzuki-Kikawa-Vetro sense.

In parallel, a highly active line of research has shifted the behavior of these mappings towards more abstract topological structures to relax the triangular inequality. In this context, the study of cyclic contractions and proximity points has been consolidated in G-metric spaces [10,11], and also in Meir–Keeler cyclic maps [11–13]. On the other hand, in Pasupathi et al. [14], certain generalizations of the Banach contraction principle for cyclic contractions were considered. For the application to the fractal field, iterated function systems (IFS) were proposed which consist of cyclic contractions and special cyclic contraction. Furthermore, a discussion was developed on some special properties of the Hutchinson operator associated with the cyclic (c)-comparison IFS. The existence and uniqueness of the optimal proximity point were focused on in Ishtiaq and Batool [15] for several classes of generalized cyclic enriched contractions and convergence results were given. Conditions have also been given such that iterative method can provide the optimal proximity point. Additionally, best proximity results have been obtained within the framework of S-metric spaces, that is involving distances on three points, for cyclic mappings [16, 17]. Those approaches demonstrate that cyclic contractive properties directly lead to asymptotic convergence and confinement of system orbits. In Chaipornjareansri [18], the concept of MT-function was proposed to establish a best proximity point results for a certain class of proximal cyclic contractive mappings in S-metric spaces. In Rada [19], a fixed point result for cyclic contractive mappings in complete b-G-metric spaces was proved by means of accelerated lamda-iteration scheme. Working in a convex b-G-metric setting endowed with a metric-affine convex structure, a typical iterative process was studied with the formal determination of the existence and uniqueness of fixed points.

This solid mathematical foundation establishes a direct bridge to the stability of switched systems. A switched system, by its very nature, evolves by alternating between multiple modes or subsystems according to a switching law. When these modes operate under state constraints represented by disjoint subsets in (X, d) and the dynamics alternates cyclically in-between the various associated parameterizations, the switched system can be formally modeled as a cyclic mapping [20]. Thus, the stability of the switched system becomes linked to the existence of best proximity points or fixed points of the global cyclic contraction. Crucially, this analytical equivalence allows solving complex control challenges. Among them, the classic analysis of exponential asymptotic stability in linear switched systems presenting constant discrete time delays stands out. See De La Sen and Ibeas [20] and some of the references therein. Similarly, it allows addressing scenarios where global stability is guaranteed through simultaneous triangularization and the explicit calculation of a common Lyapunov function. See, for instance, Ibeas and De La Sen [21]. In particular, this theoretical connection sheds light on the long-term dynamic behavior of complex non-linear models, enabling the precise determination of basins of attraction in higher-order difference equations governing biological dynamics, as for instance, in a type of generalized Beverton-Holt equation [22].

Recently, some results have been obtained on interpolative contractions which merge classical fixed point theory with interpolation theory. In that context, the contractive rule can use distinct exponent coefficients as powers for the checked pairs of points of the previous values in the iteration chain weighted also by the contractive constant. Known interpolative rules of these types are Kannan-type interpolative contractions and Hardy-Rogers-type interpolative contractions. For instance, in Kessy and Wangwe [23], a generalization was given of interpolative results related to Kannan-type contractions. See also, Abbas et al. [24], Devi and Debnath [25], Nazam et al. [26], and Tomar [27] for further related results.

The main purpose of this article is to formalize the relationship between contraction theory for cyclic mappings, which can also operate in a non-cyclic manner, over non-necessarily intersecting subsets in a customary metric space (X, d). A second purpose is to apply the results to examples in the context of stability of switched systems whose dynamics evolve cyclically in-between a set of $p(\geq\ 2)$ subsets of the metric space. Throughout the manuscript, new boundedness and convergence theorems are introduced, based on some previous results from the literature. These results are applied "ad hoc" to the proposed, and so-called bivalued hybrid cyclic/noncyclic self-mappings. The main reason is that they can operate in both such modes by selecting one of the two images at each iteration. In particular, the proposed self-mappings are not purely cyclic since they allow switches of the relevant self-mapping which defines the dynamics to both each current subset of the metric space and the next adjacent one in the cyclic disposal. Therefore, the self-mappings under study are referred to as cyclic/ non-cyclic hybrid bivalued contractive self-mappings since there are two potential images of each element in the iteration, namely, one in each current subset and another one in the next adjacent one. A defined, so-called, image selection sequence, which is iteration-dependent and which monitors the cyclic/noncyclic hybrid mapping at hand, selects if the next element in the iteration lies in the current subset or in the next adjacent one. It is also seen how cyclic contractive properties translate operationally into analytical tools to evaluate and guarantee the asymptotic stability of the class of studied hybrid cyclic/non-cyclic self-mappings while some illustrative examples are also discussed.

Unlike conventional multivalued cyclic contractions, the proposed bivalued mapping incorporates a selection signal that determines which of the two admissible images is activated at each iteration. This additional degree of freedom provides greater flexibility than a purely cyclic framework. In addition, it allows the iteration to switch between the two sets according to the prescribed signal rather than following a fixed cyclic pattern. It can be pointed out that, in the intra mode case, the convergence rate is linear since the proposed mapping satisfies a Banach contraction condition with contraction constant less than unity. In the cyclic mode, the subsequences of the main sequence that run within each of the subsets of the cyclic disposal have also a linear convergence rate.

The paper is organized as follows. Section 2 describes the notation and nomenclature issues to be then used. Section 3 gives the central definition of the hybrid cyclic/non-cyclic self-mapping to be then formally discussed together with the set of assumptions to be then invoked in the results. A central hypothesis is that the subsets of the metric spaces are closed and that one of the best proximity sets of at least one subset of the cyclic disposal to its adjacent one is a singleton. Other hypotheses are that such a subset is boundedly compact and that both images of best proximity points are best proximity points, one located in each current subset of the cyclic disposal and the other one located in the next adjacent subset. It might also be assumed that, if the subsets of the cyclic disposal do not intersect, then the fixed point of the self-mapping restricted to each subset, which is necessarily a strict contraction by nature, is one of the best proximity points to its next adjacent subset. The above assumption is not needed for derivation of the main mathematical proofs. However, it is an eventual suitable option in some practical applications in the sense that a limit final point in trapped operation intra mode can be also one of the points of limit cycles in cyclic operation mode. On the other hand, it is not assumed a priori that the subsets are bounded. This section also establishes and proves some preliminary results previous to the main results, to be given and proved in the next section, as well as some clarifying remarks on the preliminary results and the assumptions. The main results are stated and proved in Section 4. In particular, it is proved that the image selection sequence cannot alternate infinitely many times between images in the current subset or switches to the next adjacent one when building any sequence. In other words, after a finite number of iterations, the image selection sequence remains constant implying either: a) a selection of the current subset as final host where the sequence is being contained in subsequent iterations; or b) a selection of continuing taking values ahead in a purely cyclic mode by alternating the successive sequence values between the various subsets of the cyclic disposal. If the distance between the elements of a generated sequence is smaller than the distance between the adjacent subsets then no more switches are allowed for successive elements of the sequence to continue running in cyclic mode. Such a constraint is a direct result from the contractive characteristic of the hybrid cyclic/non-cyclic self-mapping. The section also proves the main results on convergence to fixed points in one of the subsets or to limit cycles defining by best proximity points, one per subset of the cyclic disposal. Three illustrative examples are discussed in Section 5. The examples visualize, in particular, the duality of the iteration-by-iteration choice between the intra mode operation and the cyclic operation mode in the evolution of the sequences of distances. One of the examples is devoted to describe a dynamic system with four disjoint subsets for eventual state confinement in trapped intra mode. The whole solution trajectory can work in both cyclic and non-cyclic modes with mutual switches. Furthermore, it is shown that global stability is preserved. Finally, some conclusions and a description of potential future work end the paper.

2. Notation and nomenclature

The following notation is used in the following:

- $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\mathbf{C}$ are, respectively, the sets of integer, real and complex numbers.
- $\boldsymbol{Z}_+$ and $\mathbf{R}_+$ are, respectively, the sets of positive integer and real numbers.
- $\boldsymbol{Z}_-$ and $\mathbf{R}_-$ are, respectively, the sets of negative integer and real numbers.
- $\boldsymbol{Z}_{0+} = \boldsymbol{Z}_+ \cup \{0\}$; $\boldsymbol{Z}_{0-} = \boldsymbol{Z}_- \cup \{0\}$.
- $\boldsymbol{R}_{0+} = \boldsymbol{R}_+ \cup \{0\}$; $\boldsymbol{R}_{0-} = \boldsymbol{R}_- \cup \{0\}$; $\bar{n} = \{1, 2, \cdots, n\}$.
- I_n is the identity matrix of order n.
- $p(\geq 2) \in \boldsymbol{Z}_+$ is the number of nonempty subsets A_i for $i \in \bar{p}$ in the cyclic disposal with $A_{np+i} = A_i$; $\forall i \in \bar{p}$, $\forall n \in \boldsymbol{Z}_{0+}$. The subsets A_i and A_{i+1} are said to be adjacent and A_{i+1} is the next adjacent subset to A_i; $\forall i \in \bar{p}$ and A_{i0} is the best proximity set in A_i to A_{i+1}, that is, $A_{i0} = \{a \in A_i : d(A_i, A_{i+1}) = d(A_{i0}, A_{i+1}) = d(a,\ A_{i+1})\}$; $\forall i \in \bar{p}$.
- The same above criterion is used to denote cyclic repetitions for the contractive constants K_i and the distances D_i between adjacent subsets A_i and A_{i+1}, that is, $K_{np+i} = K_i$ and $D_{np+i} = D_i$; $\forall i \in \bar{p}$, $\forall n \in \boldsymbol{Z}_{0+}$.

- If $T: \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ is a $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive self-mapping then the sets $A_1, A_{2,}, \dots, A_p$ of X in the above order, or more precisely, in any order $A_i, A_{i+1}, \dots A_{i+p-1} = A_{i-1}$ for any arbitrary $i \in \bar{p}$ are referred to as a cyclic disposal.
- The image selection sequence $\{\sigma_n\}_{n=1}^{\infty} \in \{0,\ 1\}^{\boldsymbol{Z}_+}$ of elements $\sigma_n = \sigma_n\left(T^{n-1}x,\ T^{n-1}y\right) \in \{0,\ 1\}$; $\forall x, y \in A_i \cup A_{i+1}$, $\forall i \in \bar{p}$, $\forall n \in \boldsymbol{Z}_+$ selects if the chosen images in the iteration process $T^n x$ and $T^n y$ at the n-th step are both located within the same subset of the cyclic disposal $(\sigma_n = 0)$ or in distinct adjacent subsets $(\sigma_n = 1)$.
- χ_0 is the infinite cardinal of a denumerable set with infinite elements.
- Note that the sequence $\{\sigma_n\}_{n=1}^{\infty} \in \{0,\ 1\}^{\boldsymbol{Z}_+}$ is always the countable union of disjoint binary sets of elements 0 and 1 as follows:

  a) If $\{\sigma_n\}_{n=1}^{\infty}$ has at least two distinct elements then

  $$\{0,\ 1\}^{\boldsymbol{Z}_+} \ni \{\sigma_n\}_{n=1}^{\infty} = \bigcup\nolimits_{i \in \bar{k}} \left[\{\sigma_i\}^{k_{i+1}-k_i},\ \{\sigma_{i+1}\}^{k_{i+2}-k_{i+1}}\right),$$

  where $\{k_i\}_{i=1}^{k}$ is strictly increasing with $k_1 = 1$ and k is either finite, then $card\left(\bar{k}\right) < +\infty$ and $k_{k+1} = +\infty$, or $\bar{k} = \boldsymbol{Z}_+$ so that $card\left(\bar{k}\right) = \chi_0$.This means that, if k is finite, then both elements of the distance of the pair of sequences through the iteration process remain in the same adjacent subsets after k iterations and $\sigma_n = 0$ for $n(\geq k) \in \boldsymbol{Z}_+$;

  b) If $\{\sigma_n\}_{n=1}^{\infty}$ is constant then $\{\sigma_n\}_{n=1}^{\infty} = \{0\}^{\boldsymbol{Z}_+} \vee \{1\}^{\boldsymbol{Z}_+}$. If $\{\sigma_n\}_{n=1}^{\infty} = \{0\}^{\boldsymbol{Z}_+}$ then for the given $x, y \in A_i \cup A_{i+1}$, $\{T^n x\}_{n=1}^{\infty}$ and $\{T^n y\}_{n=1}^{\infty}$ are both within either the subset A_i or within its adjacent subset A_{i+1}. If $\{\sigma_n\}_{n=1}^{\infty} = \{1\}^{\boldsymbol{Z}_+}$ then $T^n x$ and $T^n y$ are both in distinct adjacent subsets A_i and A_{i+1}; $i \in \bar{p}$ of the cyclic disposal for any $n \in \boldsymbol{Z}_+$. A consequent restriction is that, if x and y are in adjacent subsets A_i and A_{i+1} for any $i \in \bar{p}$ and $\{\sigma_n\}_{n=1}^{\infty} = \{0\}^{\boldsymbol{Z}_+}$, then $\{T^n x\}_{n=1}^{\infty} \subset A_{i+1}$ and $\{T^n y\}_{n=1}^{\infty} \subset A_{i+1}$.

- If $x \in A_i$ then $Tx \in \boldsymbol{T}x = \left\{(Tx)^0,\ (Tx)^1\right\}$, where $(Tx)^0 \in A_i$, $(Tx)^1 \in A_{i+1}$; $\forall i \in \bar{p}$, and $\boldsymbol{T}x$ is the binary set to which Tx belongs to.
- The concatenated strip of the history of selection of images in each generated sequence by the self-mapping T is: $\widehat{\sigma}_n = \widehat{\sigma}_{1n} = \left(\widehat{\sigma}_{n-1}\right)\sigma_n = \sigma_1.\sigma_2.\dots.\sigma_n \in \{0,\ 1\}^n$; $\forall n \in \boldsymbol{Z}_+$ which results in:

  $$\widehat{\sigma}_n = \widehat{\sigma}_{n-1}.\ \theta \in \widehat{\boldsymbol{\sigma}}_{\boldsymbol{n}} = \{\widehat{\sigma}_{n-1}.0,\ \widehat{\sigma}_{n-1}.1\};$$

  $$\forall n \in \boldsymbol{Z}_+\ with\ \theta \in \{0,\ 1\}, \widehat{\sigma}_1 = \widehat{\sigma}_{11} = \sigma_1$$

  such that

  $$\widehat{\sigma}_{(m+1)n} = \sigma_{m+1}.\sigma_{m+2}.\dots.\sigma_n$$

  so that

  $$\widehat{\sigma}_n = \left(\widehat{\sigma}_m\right)\left(\widehat{\sigma}_{(m+1)n}\right); \forall m, n(> m) \in \boldsymbol{Z}_{0+}.$$

  The number of values "1" in the strip $\widehat{\sigma}_n$ denotes the number of times that the image of T has been selected in the next adjacent subset, up-till the n-th iteration, to build the sequence $\{T^n x\}_{n=1}^{\infty}$ for a given $x \in \bigcup_{i\in\bar{p}} A_i$. The number of values "0" denotes the number of times that the image of T has been selected in the same subset up-till the n-th iteration. The order of the number of times for the image to stay in the current subset or to switch to the next adjacent one is also reflected in the ordering of the elements in the strip $\widehat{\sigma}_n$. Each element $T^n x$ of the sequence $\{T^n x\}_{n=1}^{\infty} = \left\{(T^n x)^{\widehat{\sigma}_n}\right\}_{n=1}^{\infty}$ is in the binary set $\boldsymbol{T}\left(T^{n-1}x\right)^{\widehat{\sigma}_{n-1}} = \left\{(T^n x)^{\widehat{\sigma}_{n-1}.0},\ (T^n x)^{\widehat{\sigma}_{n-1}.1}\right\}$ so that one has the following recursion associated to any element of the sequence $\{T^n x\}_{n=1}^{\infty}$:

$$T^n x = (T^n x)^{\widehat{\sigma}_n} = \left(T\left(T^{n-1}x\right)^{\widehat{\sigma}_{n-1}}\right)^{\sigma_n} =$$

$$\left(T\left(T\left(T^{n-2}x\right)^{\widehat{\sigma}_{n-2}}\right)^{\sigma_{n-1}}\right)^{\sigma_n} = (T^n x)^{\widehat{\sigma}_n} = \left(T^{n-m}\left(T^m x\right)^{\widehat{\sigma}_m}\right)^{\widehat{\sigma}_{(m+1)n}}$$

$$\in \boldsymbol{T}\left(T^{n-1}x\right)^{\widehat{\sigma}_{n-1}} = \left\{(T^n x)^{\widehat{\sigma}_{n-1}.0},\ (T^n x)^{\widehat{\sigma}_{n-1}.1}\right\};\forall n \in \boldsymbol{Z}_+$$

- The generated sequence $\{T^n x\}_{n=1}^{\infty}$ from an initial $x \in A_i$ is built iteration-to-iteration from the multisequence $\{\boldsymbol{T}^n x\}_{n=1}^{\infty}$ which is a set of cardinal 2^n for each $n \in \boldsymbol{Z}_+$.
- $f_n = \sum_{i=1}^{n} \sigma_n (\leq n) \in \boldsymbol{Z}_{0+}$ is the number of iterations of the sequence $\{T^n x\}_{n=1}^{\infty}$ with images from a subset to the next adjacent subset, implying unity values of the corresponding $\sigma_{(.)}$ such that if $x \in A_i$ for $i \in \bar{p}$ then $T^n x \in A_{i+j}$ where $j = j(n) = f_n - p\,[f_n/p]$ and $[f_n/p]$ is the integer part of f_n/p
- $Fix(T)$ is the set of fixed points of the self-mapping T.

Remark 1. *In order to avoid a cumbersome notation, the notation* $T^n x = (T^n x)^{\widehat{\sigma}_n} \in \boldsymbol{T}\left(T^{n-1}x\right)^{\widehat{\sigma}_{n-1}} = \left\{(T^n x)^{\widehat{\sigma}_{n-1}.0},\ (T^n x)^{\widehat{\sigma}_{n-1}.1}\right\}$*;* $\forall n \in \boldsymbol{Z}_+$*, will be simplified by denoting explicitly just the value* σ_n *of the last iteration in the historic concatenation* $\widehat{\sigma}_n$*, when no confusion is expected, as* $T^n x \in \boldsymbol{T}\left(T^{n-1}x\right) = \left\{(T^n x)^0,\ (T^n x)^1\right\}$.

3. Basic definition and assumptions with some preliminary results and illustrative remarks

Let (X, d) be a metric space with a set of subsets $A_i (\neq \varnothing) \subset X$ with $A_{np+j+1} = A_{j+1}$; $\forall j \in \overline{p-1} \cup \{0\}$, such that $D = D_i = d\left(A_i, A_{i+1}\right) > 0$; $\forall i \in \bar{p}$ is the distance between any two adjacent subsets A_i and A_{i+1}. It can be pointed out that a common distance D between adjacent subsets for cyclic mappings on the union of non-disjoint sets has been addressed in Lemma 3.3 of Karpagam and Agrawal [2] based on a need derived from the fact that such mappings are necessarily nonexpansive. Consider also a so-called hybrid cyclic/non-cyclic self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ which allows to fix the image of any point either in its same subset A_i for any given $i \in \bar{p}$ or in its next adjacent one A_{i+1} due to its bivalued nature of the self-mapping, in the sense that each point in A_i for all $i \in \bar{p}$ has two images, one in each of the above subsets.

More formally, we state the subsequent definition:

Definition 1. *Let (X, d) be a metric space and $A_i \subset X$ for $i \in \bar{p}$ are nonempty subsets of X. Then, a self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is said to be $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive if it satisfies Conditions C.1–C.4 below:*

C.1: *$T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is such that $T(A_i) \subset A_i \cup A_{i+1}$ and $D = D_i = d(A_i, A_{i+1})$; $\forall i \in \bar{p}$, and if $x \in A_i$ then $Tx \in \boldsymbol{T}x = \left\{(Tx)^0,\ (Tx)^1\right\}$, where $(Tx)^0 \in A_i$, $(Tx)^1 \in A_{i+1}$; $\forall i \in \bar{p}$.*

As above, we will denote in the sequel in bold font the set of two images of x and in regular font the particular element in the generated sequence.

C.2: *If $x \in A_{i0}$ then $(Tx)^0 \in A_{i0}$ and $(Tx)^1 \in A_{i+1,0}$; $\forall i \in \bar{p}$.*

Note that C.1 relies on the fact that there are two images for each point, one in the same subset as that of the pre-image and another one in the next adjacent subset. This is the main foundation of referring the self-mapping as a bivaluated hybrid mapping contrarily to the usual contractive single-valued cyclic contractive self-mappings in the literature. The hybrid cyclic/non-cyclic nature of the map relies on the locations of such images [1–3]. On the other hand, C.2 states that the two images referred to in C.1 for best proximity points are also best proximity points one in the same subsets of the pre-image and the other in the adjacent subset.

C.3: *$T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ satisfies the contractive condition:*

$$d\left(T^n x,\ T^n y\right) \leq Kd\left(T^{n-1}x,\ T^{n-1}y\right) + (1-K)\sigma_n D \tag{1}$$

The self-mapping T is monitored by a iteration image selection binary sequence $\{\sigma_n\}_{n=0}^{\infty}$ of elements $\sigma_n = \sigma_n\left(T^{n-1}x,\ T^{n-1}y\right) \in \{0,\ 1\}$; $\forall x, y \in A_i \cup A_{i+1}$, $\forall i \in \bar{p}$, $\forall n \in \boldsymbol{Z}_+$ for some $K \in [0,\ 1)$. Such a selection sequence chooses the images of the iterations in the built sequences by fixing each image point either in the same subset as the original set or in its adjacent subset (see C.1–C.2).

C.4: *If $\sigma_n = 0$ for any $n \in \boldsymbol{Z}_+$ then $T^n x,\ T^n y \in A_i$ if $T^{n-1}x,\ T^{n-1}y \in A_i$ for any $i \in \bar{p}$, and, if $T^{n-1}x \in A_i, T^{n-1}y \in A_{i+1}$ for any $i \in \bar{p}$, then $T^n x \in A_i, T^n y \in A_{i+1}$.*

if $\sigma_n = 1$ then if $T^{n-1}x \in A_i$, $T^{n-1}y \in A_i$ for any $i \in \bar{p}$ then $T^n x \in A_{i+1}$, $T^n y \in A_{i+1}$, and if $T^{n-1}x \in A_i$, $T^{n-1}y \in A_{i+1}$ for any $i \in \bar{p}$ then $T^n x \in A_{i+1}$, $T^n x \in A_{i+2}$.

Note that C.4 specifies how the contractive condition C.3 is implemented by the choice of the image selection binary sequence in order to fulfil C.2 when building each concrete sequence depending on the locations of the points whose distances are evaluated in the contractive condition. Note also that, if the distance takes a value of D, then either $\sigma_n = 1$ or, if $\sigma_n = 0$, then $T^n x,\ T^n y \in A_i$ if $T^{n-1}x,\ T^{n-1}y \in A_i$ since the alternative option for $\sigma_n = 0$ is not feasible since at the next iteration the distance lies below D for points in adjacent subsets whose mutual distance is D. Since the mapping is also non expansive a further activation of the cyclic mode is not possible. More additional conditions, C.5, C.5′and C.6 which are not required by Definition 1 are given below. They will be invoked later on for $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive self-mappings in the derivation of some of the results concerned with the convergence properties to fixed points and best proximity points:

C.5: *The nonempty subsets $A_j \subset X$; $j \in \bar{p}$ are all closed and at least one subset A_i is boundedly compact for some $i \in \bar{p}$ and its best proximity set is a singleton, that is, $A_{i0} = \{z_i\}$.*

C.5 is used for inducing the convergence to best proximity points of the sequences of points from previously proving the convergence of distances to the distance between adjacent subsets. The boundedly compactness of one of the subsets is a usual requirement [1], when cyclic mapping operate on just two subsets ($p = 2$). Here, it is invoked on one of the subsets for any $p \geq 2$ which, in addition, it is required to have a single best proximity point. This will lead to prove convergence of sequences in cyclic mode to such a point on such a subset (from the previous proof of convergence of sequences. As a result, all the successive images in adjacent subsets would converge to one of the best proximity points of each of the remaining subsets.

***C.5′** (extended C.5 if C.2 is omitted)**:** The nonempty subsets $A_j \subset X$; $j \in \bar{p}$ are all closed and at least one A_i is boundedly compact for some $i \in \bar{p}$, and its best proximity set is a singleton, that is, $A_{i0} = \{z_i\}$ and, furthermore, $(Tz_i)^0 = z_i$ and $(Tz_i)^1 \in A_{i+1,0}$.*

C.5′ is required, if C.2 is not invoked for all subsets, as an ad-hoc extension of C.5 (via C.2) for the single best proximity point required for one of the subsets. In this way, once the sequence reaches the cyclic mode, it stays in cyclic mode with each successive images located in the next adjacent subset.

C.6: *If (X, d) is complete and $D > 0$, for each $i \in \bar{p}$, there is a $z_i \in A_{i0}$ such that $z_i = T_{ri} z_i$ where $T_{ri} : A_i \to A_i$ is the restriction of $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ in A_i, that is, the unique fixed point of $T \left(\bigcup_{i \in \bar{p}} A_i \right) | A_i \to \left(\bigcup_{i \in \bar{p}} A_i \right) | A_i$ is one of the best proximity points of A_i to A_{i+1}.*

C.6 is not necessary for the main results from a mathematical point of view. In fact, the fixed points of the restricted mappings to which sequences converge in trapped intra mode in each subset could be potentially different from best proximity points. However, this condition can be useful in some applications where cyclic mode converges to a sequence of best proximity points while if the sequence is trapped in some of the subsets then it will converge (under this assumption) to some point of the above mentioned sequence which is precisely the fixed point of the restricted mapping.

Assertion 1. *If $D = 0$ and $C.5$ holds for some $i \in \bar{p}$ then $\{z_i\} = A_{i0} \cap A_{i+1} = A_i \cap A_{i+1}$.*

Proof. Since $D = 0$ and all subsets A_j for $j \in \bar{p}$ are nonempty and closed, and since $A_{i0} = \{z_i\}$, then $\{z_i\} \cap A_{i+1} = \{z_i\} \neq \varnothing$, since otherwise, $[\{z_i\} \cap A_{i+1} = A_{i0} \cap A_{i+1} = \varnothing] \Rightarrow [D > 0]$, a contradiction. It has been proved that $\{z_i\} = A_{i0} \cap A_{i+1}$. Now assume that there is a proper set inclusion (i.e., without identity of sets in the set inclusion) in $A_i \cap A_{i+1} \supset A_{i0} \cap A_{i+1} = \{z_i\}$. Since the above set inclusion is non-proper then $A_i \cap A_{i+1} = \{z_i, \ \widehat{z}_i\}$ with $\widehat{z}_i \neq z_i$. Since $D = 0$, $\{z_i, \ \widehat{z}_i\}$ are both best proximity points from A_i to A_{i+1} so that A_{i0} is not a singleton. Hence a contradiction follows. As a result, $\{z_i\} = A_{i0} \cap A_{i+1} = A_i \cap A_{i+1}$ and the assertion is proved. $\square$

Assertion 2. *If $\bigcap_{i\in\bar{p}} A_i \neq \varnothing$, the subsets $A_i \subset X$; $i \in \bar{p}$ are all nonempty and closed, (X, d) is complete and the self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is subject to C.3 then $Fix(T) = \{z\} \in \bigcap_{i\in\bar{p}} A_i$.*

If, in addition to C.3, C.5 holds for some $i \in \bar{p}$, then $z = z_i = \{z_i\} \in A_{j0}$; $\forall j \in \bar{p}$ even if (X, d) is not complete.

Proof. If $\bigcap_{i\in\bar{p}} A_i \neq \varnothing$ so that $D = 0$ then all the pairs of adjacent subsets intersect, that is, $A_i \cap A_{i+1} = A_{i0} \cap A_{i+1,0} \neq \varnothing$. Furthermore, if C.3 holds, from Equation (1), T is a strict contraction and the Banach principle applies so that there is a unique fixed point z of T in $\bigcap_{j\in\bar{p}} A_j = \bigcap_{j\in\bar{p}} A_{j0}$ if (X, d) is complete.

If, in addition to C.3, $C.5$ also holds then $z_i = \{z_i\} = z = \{z\} = Fix(T) = A_{i0} \cap A_{i+1,0}$, for some $i \in \bar{p}$ since $D = 0$ and it has to be a best proximity point of all the subsets $A_i \subset X$; $\forall i \in \bar{p}$ so that it is coincident with the unique best proximity point by virtue of (1) since then $\bigcap_{j\in\bar{p}} A_j = \bigcap_{j\in\bar{p}} A_{j0} = \{z_i\} = Fix(T)$ since, otherwise, a) either $\bigcap_{j\in\bar{p}} A_j$ is empty and then (X, d),which is complete, would not have a unique fixed point what is impossible since T is contractive; or b) $\bigcap_{j\in\bar{p}} A_j$ would consist of more than one point z_i which is impossible since $A_i \cap A_{i+1} = \{z_i\}$. Therefore, it would not exist some $x \in A_j$ for some $j(\neq i) \in \bar{p}$ and a convergent sequence in A_j to some $z_j\ (\neq z_i = z) \in A_{j0}$. Therefore, $z_i(= z) \in \bigcap_{j\in\bar{p}} A_{j0} = \bigcap_{j\in\bar{p}} A_{j0}$. In this case, it is not needed the completeness of (X, d) since the metric subspace (A_i, d) is complete (since A_i is boundedly compact) and $\{z_i\} = Fix(T) \subset \bigcap_{j\in\bar{p}} A_i = \bigcap_{j\bar{\in}p} A_{j0} = \{z_i\}$. □

It is of interest to give a view of the relations between the joint conditions C.3 and C.5′ and C.3 and C.5 in the case when all the subsets of the cyclic disposal intersect.

Assertion 3. *Let (X, d) be a metric space, $A_j \subset X$ are nonempty and closed; $\forall j \in \bar{p}$. Assume that $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is subject to Conditions $C.1$ and $C.3$ such that A_i is boundedly compact for some $i \in \bar{p}$ then $A_{j0} \neq \varnothing$; $\forall j \in \bar{p}$.*

Proof. It is a consequence of the fact that any two consecutive values of any sequence $\{T^n x\}_{n=0}^{\infty}$ in cyclic mode converge to $D = d\,(A_i, A_{i+1})$; $i \in \bar{p}$; $\forall x \in \bigcup_{i\in\bar{p}} A_i$ (since T is contractive) and the fact that any subsequence $\left\{T^{n_k p+j} x\right\}_{k=0}^{\infty} \subset \{T^n x\}_{n=0}^{\infty}$ for some $j = j(i) \in \overline{p-1} \cup \{0\}$ converges to a best proximity point $p_i \in A_{i0}$ (then $A_{i0} \neq \varnothing$) in the subset which is boundedly compact together with $\left\{d\left(T^{n_k p+\ell} x,\ T^{n_k p+\ell+1} x\right)\right\}_{k=0}^{\infty} \to D$; $\forall \ell \in \bar{p}$. Since C.3 implies also that T is nonexpansive, thus Lipschitz-continuous and all the subsets are closed, then $\left\{T^{n_k p+j+\ell} x\right\}_{k=0}^{\infty} \to p_{i+\ell}\,(\in A_{i+\ell,0})$; $\forall \ell \in \bar{p}$ and then $A_{j0} \neq \varnothing$; $\forall j \in \bar{p}$ as a result. □

Remark 2. *Assertion 3 concludes that a bivariate contractive hybrid cyclic/non-cyclic self-mapping which satisfies the first part of Assumption C.5 (that is, one of the subsets of X is boundedly compact) and all the subsets of the cyclic disposal are nonempty and closed guarantees that all the best proximity sets of all the subsets are non-empty. The result also holds for a standard single-valued cyclic contraction.*

Remark 3. *Provided that all the subsets of X the cyclical disposal are nonempty and closed and if $C.3$ holds, C.5 is weaker than C.6 while C.5′ implies C.5. Note that in fact C.5′ requires for all proximity points to be singletons in pairs of each subset and its next adjacent one rather than for uniqueness of one of them as C.5 asks for. In particular, if $D = 0$ then the all those proximity sets between each subset and its next adjacent one are singletons and if C.3 and C.5′ hold together then the unique fixed point is coincident with all the best proximity points which are the same singleton. If C.3 and C.5 hold together then the unique fixed point is coincident with the singleton best proximity point between A_i and A_{i+1}. Such a point is also a best proximity point between each A_j and A_{j+1} but, for $j \neq i$, the best proximity sets are not necessarily singletons. On the other hand, since all generated subsequences are convergent under C.3 with, Equation (1), then bounded, $D = 0$ then all the subsets of the cyclical disposal are boundedly compact so that this part of C.5 holds directly in a strongest way, namely, for all the subsets and without the need for imposing it.*

Proposition 1. *Let (X, d) be a metric space and $A_i \subset X$ for $i \in \bar{p}$ are nonempty subsets of X and assume that the self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive and that $\sigma_n = 1$ for some $n \in \boldsymbol{Z}_+$ then $d\left(T^n x,\ T^n y\right) \leq d\left(T^{n-1}x,\ T^{n-1}y\right)$; $\forall x, y \in A_i$ and $\forall (x, y) \in (A_i \times A_{i+1}) \cup (A_{i+1} \times A_i)$; $\forall i \in \bar{p}$.*

Proof. Case a: Assume that $\left(T^{n-1}x, T^{n-1}y\right) \in (A_i \times A_{i+1}) \cup (A_{i+1} \times A_i)$ and $(T^n x, T^n y) \in (A_{i+1} \times A_{i+2}) \cup (A_{i+2} \times A_{i+1})$ for any given $i \in \bar{p}$.

Then, min $\left(d\left(T^{n-1}x, T^{n-1}y\right),\ d\left(T^{n-1}x, T^{n-1}y\right)\right) \geq D$. Now, assume that $d\left(T^n x,\ T^n y\right) > d\left(T^{n-1}x,\ T^{n-1}y\right)$ so that

$$d\left(T^{n-1}x,\ T^{n-1}y\right) < d\left(T^n x,\ T^n y\right) \leq Kd\left(T^{n-1}x,\ T^{n-1}y\right) + (1-K)D.$$

This implies that $(1-K)d\left(T^{n-1}x,\ T^{n-1}y\right) < (1-K)D$, equivalently, $d\left(T^{n-1}x,\ T^{n-1}y\right) < D$, a contradiction.

Case b: Assume that $d\left(T^{n-1}x, T^{n-1}y\right) \geq D$, $d\left(T^n x, T^n y\right) > d\left(T^{n-1}x, T^{n-1}y\right)$ and either $T^{n-1}x, T^{n-1}y, T^n x \in A_i$ and $T^n y \in A_{i+1}$, or $T^{n-1}x, T^{n-1}y, T^n x \in A_{i+1}$ and $T^n y \in A_i$, or $T^{n-1}x, T^{n-1}y, T^n y \in A_i$ and $T^n x \in A_{i+1}$, or $T^{n-1}x, T^{n-1}y, T^n y \in A_{i+1}$ and $T^n x \in A_i$.

$$D \leq d\left(T^{n-1}x,\ T^{n-1}y\right) < d\left(T^n x,\ T^n y\right) \leq Kd\left(T^{n-1}x,\ T^{n-1}y\right) + (1-K)D$$

which implies that $(1-K)d\left(T^{n-1}x,\ T^{n-1}y\right) < (1-K)D$, equivalently, $d\left(T^{n-1}x,\ T^{n-1}y\right) < D$, a contradiction. □

The next result relies on the fact that the cyclic/non-cyclic hybrid bivalued contractive mapping is nonexpansive irrespective of the sequences of values that define the image selection binary sequence.

Proposition 2. *Let (X, d) be a metric space and $A_i \subset X$ for $i \in \bar{p}$ are nonempty subsets of X and assume that the self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is $p(\geq 2)$*

cyclic/non-cyclic hybrid bivalued contractive and that $\sigma_n = 0$ *for some* $n \in \boldsymbol{Z}_+$. *Then,* $d(T^n x,\ T^n y) \leq d\left(T^{n-1}x,\ T^{n-1}y\right)$*;* $\forall x, y \in A_i$*,* $\forall i \in \bar{p}$ *and also for* $(x, y) \in (A_i \times A_{i+1}) \cup (A_{i+1} \times A_i)$*;* $\forall i \in \bar{p}$ *with* $d(x, y) \geq D$*.*

Proof. Two cases can arise, namely,

Case a: $x, y, Tx, Ty \in A_i$ for some $i \in \bar{p}$, then $d(T^n x,\ T^n y) \leq Kd\left(T^{n-1}x,\ T^{n-1}y\right) < d\left(T^{n-1}x,\ T^{n-1}y\right)$ if $d\left(T^{n-1}x,\ T^{n-1}y\right) > 0$ and $d(T^n x,\ T^n y) = d\left(T^{n-1}x,\ T^{n-1}y\right) = 0$, and the result is proved.

Case b: $d(x, y) \geq D$ and $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ for some $i \in \bar{p}$. Then, $d(T^n x,\ T^n y) - D \leq K\left(d\left(T^{n-1}x,\ T^{n-1}y\right) - D\right) < d\left(T^{n-1}x,\ T^{n-1}y\right) - D$ if $d\left(T^{n-1}x,\ T^{n-1}y\right) > D$ so that $d(T^n x,\ T^n y) < d\left(T^{n-1}x,\ T^{n-1}y\right)$, and $d(T^n x,\ T^n y) = d\left(T^{n-1}x,\ T^{n-1}y\right) = D$ if $d\left(T^{n-1}x,\ T^{n-1}y\right) = D$, and the result is proved.

□

Proposition 3. *Let* (X, d) *be a metric space and* $A_i \subset X$ *for* $i \in \bar{p}$ *are nonempty subsets of* X *and assume that the self-mapping* $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ *is* $p(\geq 2)$ *cyclic/non-cyclic hybrid bivalued contractive. For any iteration image selection binary sequence* $\{\sigma_n\}_{n=0}^{\infty} \subset \{0,\ 1\}$*, it follows that* $d(T^n x,\ T^n y) \leq d(x, y)$*;* $\forall x, y \in A_i$*,* $\forall i \in \bar{p}$*;* $\forall n \in \boldsymbol{Z}_+$ *and also for* $(x, y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$*;* $\forall i \in \bar{p}$ *with* $d(x, y) \geq D$*.*

Proof. It follows directly by complete induction from $j = 1$ to $j = n$ for any $n \in \boldsymbol{Z}_+$ from Proposition 1 and Proposition 2. □

The subsequent result establishes that the sequences generated by a cyclic/non-cyclic hybrid bivalued contractive which satisfies Condition C.5 are bounded.

Proposition 4. *Let* (X, d) *be a metric space and* $A_i \subset X$ *for* $i \in \bar{p}$ *are nonempty subsets of* X *and assume that the self-mapping* $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ *is* $p(\geq 2)$ *cyclic/non-cyclic hybrid bivalued contractive and that it satisfies Condition C.5. Then,* $\{T^n x\}_{n=0}^{\infty}$ *is bounded for any* $x \in \bigcup_{i\in\bar{p}} A_i$*.*

Proof. Take any $x \in A_i$ for any arbitrarily chosen $i \in \bar{p}$ with $Fix\,(T_{ri}) = z_i$. The following cases can arise: Case 1: $\sigma_n = 1$ for $n\,(\geq \vartheta_0) \in \boldsymbol{Z}_+$ and some finite $\vartheta_0 \in \boldsymbol{Z}_+$. Then, there exists some finite $\vartheta\,(\geq \vartheta_0) \in \boldsymbol{Z}_+$ such that $\{T^{np}x\}_{n=\vartheta}^{\infty} \subset A_{i0}\,(\subset A_i)$ and $\left\{T^{np}T^k x\right\}_{n=\vartheta}^{\infty} \subset A_{i+k,0}\,(\subset A_{i+k})$ for any arbitrary $k \in \overline{p-1} \cup \{0\}$. If $k = j - i$ if $i < j \leq p$ or $k = p + j - i$, otherwise, then $\{T^{np}\bar{x}\}_{n=\vartheta}^{\infty} \subset A_{j0}\,(\subset A_j)$ where $\bar{x} = T^k x$ since all the best proximity sets are nonempty from C.5 (Assertion 3). For any given $k \in \overline{p-1} \cup \{0\}$ (note that $\bar{x} = x$ if $k = 0$ and $j = i$), it always exists $j = j(k, i) \in \bar{p}$ such that the above relation holds for $x \in A_i$ and $\bar{x} = T^k x \in A_{i+k}$.

Case 2: $\sigma_n = 0$ for $n\,(\geq \vartheta_0) \in \boldsymbol{Z}_+$ and some finite $\vartheta_0 \in \boldsymbol{Z}_+$.Then, there exists some finite $\vartheta\,(\geq \vartheta_0) \in \boldsymbol{Z}_+$ such that $j = j(k, i) \in \bar{p}$ for some $j = j(k, i) \in \bar{p}$.

Case 3: $\{\sigma_n\}_{n=0}^{\infty} \in \{0,\ 1\}^{\boldsymbol{Z}_+}$ is not identically equal to unity or zero after a finite number of iterations, that is, there are infinitely many values of zero and unity distributed alternately in connected intervals of finite measures. It turns out that this case is impossible from Propositions 1–3 since the self-mapping T is nonexpansive and locally strictly contractive under null values of the sequence $\{\sigma_n\}_{n=0}^{\infty}$. Then once distances are less than D with $\sigma_n = 0$ such sequence cannot switch again to $\sigma_n = 1$ which requires for the sequence of distances to be non smaller than D. Then, Case 3 is not feasible.

As a result, from Cases 1–2, $x \in A_i$ for any $x \in A_i$, any $i \in \bar{p}$ and any given $j \in \overline{p-1} \cup \{0\}$ and also from Condition C.5 and Proposition 3, one has that $d\left(T^{np+\ell}\bar{x}, \quad z_j\right) \leq d\left(T^{np+\ell}\bar{x}, \quad z_j\right) = d\left(T^{np+\ell}\bar{x}, \quad T^{np}z_j\right)$

$$\leq K^{\ell} d\left(T^{np}\bar{x}, \quad T^{np}z_j\right) + \left(1 - K^{\ell}\right) D \leq K^{\ell}\left(d\left(\bar{x}, \ z_j\right) - D\right) + D$$

$$\begin{cases} < d\left(\bar{x}, z_j\right) & if\ d\left(\bar{x}, z_j\right) > D \\ = D = d\left(\bar{x}, z_j\right) & otherwise \end{cases} ; \forall j \in \bar{p}$$

such that $\left\{T^{np+\ell}\bar{x}\right\}_{n=\vartheta}^{\infty} \subset A_{j+\ell}$; $\forall \ell \in \overline{p-1} \cup \{0\}$ irrespective of the sequence $\{\sigma_n\}_{n=0}^{\infty}$ for any $x \in A_i$ and any $i \in \bar{p}$. This implies that $\left\{T^{np+\ell}x\right\}_{n=\vartheta}^{\infty}$ is also bounded. Assume that this is not the case. Then $+\infty = \limsup_{n \to \infty} d\left(T^{np+\ell}x, T^{np+\ell}\bar{x}\right) \leq d(x, \bar{x}) < +\infty$ from nonexpansivity would lead to a contradiction. Thus, $\left\{T^{np+\ell}\bar{x}\right\}_{n=\vartheta}^{\infty}$ is bounded for any given $x \in A_i$ for any given arbitrary $i \in \bar{p}$ and any $\ell \in \overline{p-1} \cup \{0\}$ so that $\left\{T^{np+j}x\right\}_{n=0}^{\infty}$ is also bounded for any given $x \in \bigcup_{i \in \bar{p}} A_i$, since a subsequence which differs only in a finite number of terms is bounded. As result, $\{T^n x\}_{n=0}^{\infty}$ is bounded as well for any given $x \in \bigcup_{i \in \bar{p}} A_i$. □

Remark 4. *The following key features follow from Conditions C.1–C.5´:*

4.1: *Condition C.1 states that $T : \bigcup_{i \in \bar{p}} A_i \to \bigcup_{i \in \bar{p}} A_i$ is bivalued such that each point in $\bigcup_{i \in \bar{p}} A_i$ which is in some subset A_i of X has one image in A_i and another one in its adjacent set A_{i+1}. One of both images is selected to build the next point of the iteration sequence.*

4.2: *Condition C.2 states that the two point images of best proximity points are also best proximity points in a pair of adjacent subsets.*

4.3: *Conditions C.3 and C.4 establish that the distance between any two points is nonexpansive y at a particular iteration either a standard contraction if $\sigma_n = 0$ or a cyclic one if $\sigma_n = 1$ subject to the following constraints:*

a) *If, for any $n \in \boldsymbol{Z}_+$ and any $i \in \bar{p}$, $\sigma_n = 1$ then*

a1) *if $\left(T^{n-1}x, T^{n-1}y\right) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$ then $(T^n x, T^n y) \in A_{i+1} \times A_{i+2} \cup A_{i+2} \times A_{i+1}$;*

a2) *if $d\left(T^{n-1}x, T^{n-1}y\right) \geq D$ and either $T^{n-1}x, T^{n-1}y, T^n x \in A_i$ and $T^n y \in A_{i+1}$, or $T^{n-1}x, T^{n-1}y, T^n x \in A_{i+1}$ and $T^n y \in A_i$, or $T^{n-1}x, T^{n-1}y, T^n y \in A_i$ and $T^n x \in A_{i+1}$, or $T^{n-1}x$, $T^{n-1}y, T^n y \in A_{i+1}$ and $T^n x \in A_i$,* *and in both cases:*

$$d\left(T^{n}x,\ T^{n}y\right) \leq Kd\left(T^{n-1}x,\ T^{n-1}y\right) + (1-K)D$$

b) *If, for any $n \in \boldsymbol{Z}_{+}$ and any $i \in \bar{p}$, $\sigma_n = 0$ and $x, y \in A_i$ then $Tx, Ty \in A_i$, and*

$$d\left(T^{n}x,\ T^{n}y\right) \leq Kd\left(T^{n-1}x,\ T^{n-1}y\right).$$

Note that the image of any point through T is selected to build the iteration sequence depending of the current value of the iteration image selection binary sequence $\{\sigma_n\}_{n=0}^{\infty}$. Concretely, if $\sigma_n = 0$ in (1), then $T^n x = \left(T^n x\right)^0 \in \boldsymbol{T^n}x$ is in the same set as $T^{n-1}x$, otherwise, if $\sigma_n = 1$ in (1), then $T^n x = \left(T^n x\right)^1 \in \boldsymbol{T^n}x$ is in the next adjacent set in the cyclic disposal to the one which contains $T^{n-1}x$.

To fix the above main scheme´s subjacent ideas in few words, note that:

a) *If (x, y) are in the same subset A_i of the cyclic disposal for any $i \in \bar{p}$ and $\sigma_1 = 0$ then (Tx, Ty) can be either in A_i or in A_{i+1}, but, in this second case, it is needed that $d(x, y) \geq D$ (Propositions 1–2). That is, is, in the first case, $Tx = (Tx)^0 \vee (Tx)^1 \in \boldsymbol{T}x$ and $Tx = (Tx)^1 \in \boldsymbol{T}x$ and $Ty = (Ty)^1 \in \boldsymbol{T}y$; and in the second case, $Tx = (Tx)^1 \in \boldsymbol{T}x$ and $Ty = (Ty)^1 \in \boldsymbol{T}x$.*

b) *If $(Tx, \rightleftarrows Ty) \in (A_i \times A_{i+1}) \cup (A_{i+1} \times A_i)$ for any $i \in \bar{p}$ then $\sigma_1 = 1$ irrespective of if x, y are both either in A_i or in A_{i+1} or if $(x, y) \in (A_i \times A_{i+1}) \cup (A_{i+1} \times A_i)$.*

4.4: *Condition C.5 will be used later on for the proof of cauchyness and convergence of some relevant sequences. By now, note that the unique best proximity point z_i in one of the sets A_i to A_{i+1} for some $i \in \bar{p}$ is also the unique fixed point of the hybrid self-mapping T restricted to A_i and that $\boldsymbol{T z_i} = \left\{(Tz_i)^0,\ (Tz_i)^1\right\} = \left\{z_i, (Tz_i)^1\right\} \in A_{i0} \cup A_{i+1,0}$ from C.2 since $A_{i0} = \{z_i\}$, and from C.2, $A_{i0} = \{z_i\} = Fix\left(T_{ri}\right)$ for such an $i \in \bar{p}$ and note also that all the best proximity sets are nonempty if the mapping is contractive (Assertion 3). It can be pointed out that if all the subsets A_i intersect for all $i \in \bar{p}$ then $z_i = z; \forall i \in \bar{p}$ is the unique fixed point of the $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive self-mapping $T : \bigcup_{i\in\bar{p}} A_i \rightarrow \bigcup_{i\in\bar{p}} A_i$, $D = D_i = 0$ and $A_{i0} = \{z\} = Fix\left(T_{ri}\right)$; $\forall i \in \bar{p}$.*
In this case, if all the subsets intersect in a singleton intersection set $\{z\}$ then $z \in Fix(T)$ and $\boldsymbol{T}z = Tz = \{z\} = z$ so that the set $\boldsymbol{T}z$ is single valued.

4.5: *Condition C.6 implies that if $T : \bigcup_{i\in\bar{p}} A_i \rightarrow \bigcup_{i\in\bar{p}} A_i$ restricted to any subset A_i of X, that is T_{ri}, is necessarily a strict contraction; $\forall i \in \bar{p}$, namely, $K \in [0,\ 1)$ and $\sigma_n = 0$; $\forall n \in \boldsymbol{Z}_{0+}$ so that it has a unique fixed point z_i in A_i then such a fixed point is assumed to be one of the best proximity points to its adjacent subset A_{i+1}. In other words, each subset has a unique fixed point since the self-mapping is contractive (since the image selection sequence is identically null) and restricted to such a subset and such a fixed point is also in the set of best proximity sets to* *its adjacent subset A_{i+1}.*

It has to be pointed out that this condition is not relevant for the theoretical results of boundedness and convergence of sequences since the fact that restricted mappings within each nonempty and closed subset are strict contractions implies the existence of a fixed point located each subset to which Cauchy sequences converge. However, in practical situations, it can be convenient that such a fixed point be one of the best proximity points so that the sequences can track points forming the limit cycles reached in the cyclic mode of operation which can be coincident with fixed points in subsets if the sequences remain in such subsets after a certain number of iterations in the non-hybrid mode.

***4.6**: The image selection sequence does not play any role and it can be removed from the formal framework if all the subsets intersect since then $D = 0$ so that such a binary sequence is not relevant for the contractive condition (1).*

Remark 5. *Note that if $\sigma_n = 1$ then, for some $i \in \bar{p}$, $(T^n x,\ T^n y) \in (A_i \times A_{i+1}) \cup (A_{i+1} \times A_i)$, so that $d\left(T^{n-1}x, T^{n-1}y\right) \geq D$, and $\left(T^{n-1}x, T^{n-1}y\right) \in (A_{i-1} \times A_i) \cup (A_i \times A_{i-1}) \cup (A_{i-1} \times A_{i-1}) \cup (A_i \times A_i)$.*

In the case when $\left(T^{n-1}x, T^{n-1}y\right) \notin (A_{i-1} \times A_i) \cup (A_i \times A_{i-1})$, that is, if both elements of the above pair are in the same subset, either A_{i-1} or A_i, it is required that $d\left(T^{n-1}x, T^{n-1}y\right) \geq D$, since otherwise, the contradiction $D < D$ would follow under the joint switching of the two elements of the pair in one of the subsets to its adjacent subset (since $\sigma_n = 1$). Also, if $d\left(T^n x, T^n y\right) = D$ all subsequent switches to adjacent subsets (i.e., $\sigma_k = 1$ with $k\left(\in [n,\ n_1]\right); n, n_1 \in \boldsymbol{Z}_+$) will keep the above identity, that is, $d\left(T^{n+k}x, T^n y\right) = D$; $\forall k \in [n, n_1] \cap \boldsymbol{Z}_+$. In other words, the iteration image selection sequence $\{\sigma_n\}_{n=0}^{\infty}$ can assign to a pair in the same subset images in adjacent subsets only if the first pair has a distance being non less than the one between adjacent subsets. In the case when $\left(T^{n-1}x, T^{n-1}y\right) \in (A_{i-1} \times A_i) \cup (A_i \times A_{i-1})$ and $(T^n x,\ T^n y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$, we have the usual cyclic contraction for this local iteration. The value $\sigma_n = 1$ is not allowed in the upper-bound $\left(K d\left(T^{n-1}x,\ T^{n-1}y\right) + (1-K)\sigma_n D\right)$ of the distance $d\left(T^n x, T^n y\right)$ if $d\left(T^{n-1}x, T^{n-1}y\right) < D$, which necessary implies that both $T^{n-1}x, T^{n-1}y$ are in the same subset A_i for some $i \in \bar{p}$ of the cyclic disposal. In this case, the images through T of the next iterate have to be selected in A_i.

In the above case, the $n{-}th$ value of the iteration image selection sequence σ_n has to be set to zero so that there is no switch of the involved points in the same subset of the cyclic disposal to its adjacent one, namely, $T^n x = (T^n x)^1 = T\left(T^{n-1}x\right)$, $T^n y = (T^n y)^1 = T\left(T^{n-1}y\right)$ and $T^{n-1}x, T^{n-1}y$ are located in the same subset A_i for some $i \in \bar{p}$.

4. Main results

The following two results are relevant concerning the convergence of the sequences generated by the cyclic-non-cyclic contractions:

Theorem 1. *Let (X,d) be a metric space and $A_i \subset X$ for $i \in \bar{p}$ are nonempty subsets of X and assume that the self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive. Then, the following two properties hold if $(x,y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$; $\forall i \in \bar{p}$:*

(i)

$$d\left(T^n x,\ T^n y\right) \leq K^n d(x,\ y) + (1-K)D\left(\sum\nolimits_{k=1}^{n} \sigma_k K^{n-k}\right) \tag{2}$$

and

$$\lim_{n\to\infty} sup d\left(T^n x, T^n y\right) \leq D \tag{3}$$

irrespective of the sequence $\{\sigma_n\}_{n=0}^{\infty}$.

(ii) *There exist both the limit $\lim\limits_{n\to\infty} d\left(T^n x, T^n y\right) = D$ if $\sigma_n = 1$ for all $n > k$ and some finite k, and the limit $\lim\limits_{n\to\infty} d\left(T^n x, T^n y\right) = 0$ if $\sigma_n = 0$ for all $n > k$ and some finite k.*

Proof. Proof of Property (i). For any $(x,y) \in A_i \times A_{i+1} \cup A_{i+1} \times A_i$; $\forall i \in \bar{p}$, one has irrespective of the sequence $\{\sigma_n\}_{n=0}^{\infty}$:

$$\begin{aligned}
d\left(T^n x,\ T^n y\right) &\leq K d\left(T^{n-1} x,\ T^{n-1} y\right) + (1-K)\sigma_n D \\
&\leq K\left(K d\left(T^{n-2} x,\ T^{n-2} y\right) + (1-K)\sigma_{n-1} D\right) + (1-K)\sigma_n D \\
&= K^2 d\left(T^{n-2} x,\ T^{n-2} y\right) + \left(\left(\sigma_{n-1} - \sigma_n\right) K + \left(\sigma_n - K^2 \sigma_{n-1}\right)\right) D \\
&= K^2 d\left(T^{n-2} x,\ T^{n-2} y\right) + \left(\sigma_n + \sigma_{n-1} K\right)(1-K) D \\
&\leq K^3 d\left(T^{n-2} x,\ T^{n-2} y\right) + \left(\sigma_n + \sigma_{n-1} K + \sigma_{n-2} K^2\right)(1-K) D
\end{aligned} \tag{4}$$

which proves (2). Now, define

$$\begin{gathered}
LS\sigma(0,1) = \lim_{n\to\infty} \sup\ d\left(T^n x, T^n y\right)\Big]_{\{\sigma_n\}_{n=1}^{\infty} \in \{0,1\}^{\boldsymbol{Z}_+}} = \\
(1-K)D\left(\sum\nolimits_{k=1}^{\infty} \sigma_k K^{n-k}\right)
\end{gathered} \tag{5}$$

It is now proved that $\lim\limits_{n\to\infty} \sup\ d\left(T^n x, T^n y\right) \leq D$. Assume that this property is not true. Then, from (2), the following contradiction is got:

$$\begin{aligned}
D = (1-K)D/(1-K) &= (1-K)D\left(\sum\nolimits_{k=1}^{\infty} K^{n-k}\right) \\
&\geq (1-K)D\left(\sum\nolimits_{k=1}^{\infty} \sigma_k K^{n-k}\right) > D
\end{aligned}$$

so that $\lim\limits_{n\to\infty} sup\ d\left(T^n x, T^n y\right) \leq D$. Property (i) has been proved. □

Proof. Proof of Property (ii). Now, define the following amounts depending on the values of the zero or unity values of $\{\sigma_n\}_{n=0}^{\infty}$ in finite connected intervals:

$$\begin{gathered}
LS\sigma(0) = \lim_{n\to\infty} d\left(T^n x, T^n y\right)\Big]_{\{\sigma_n\}_{n=1}^{\infty} \in \{0\} Z_+} = (1-K)D\left(\sum\nolimits_{k=1}^{\infty} \sigma_k K^{n-k}\right) = \\
\left(\lim_{n\to\infty} K^n\right) d\left(T^{k-1} x, T^{k-1} y\right) = 0;
\end{gathered}$$

$$LS\sigma(1) = \lim_{n\to\infty} d\left(T^n x,\ T^n y\right)\Big]_{\{\sigma_n\}_{n=1}^{\infty} \in \{1\}^{\boldsymbol{Z}_+}} \leq (d(x,y) - D)\left(\lim_{n\to\infty} K^n\right) + D = D;$$

$$LS\sigma\left(1[1,k],0[k+1,\infty]\right)) = \lim_{n\to\infty} d\left(T^n x, T^n y\right)\,]_{\{\sigma_n\}_{n=1}^{k}\in\{1\}^k,\{\sigma_n\}_{n=k+1}^{\infty}\in\{0\}^{\boldsymbol{Z}_+}}$$
$$= \left(\lim_{n\to\infty} K^{n-k}\right) d\left(T^{k-1}x, T^{k-1}y\right) = 0;\ \forall k\,(\in \boldsymbol{Z}_+) \le \bar{k}, \forall k\,(\in \boldsymbol{Z}_+) < \infty;$$

$$LS\sigma\left(0[1,k],1[k+1,\infty]\right) = \lim_{n\to\infty} d\left(T^n x, T^n y\right)\Big]_{\{\sigma_n\}_{n=1}^{k}\in\{0\}^k,\{\sigma_n\}_{n=k+1}^{\infty}\in\{1\}^{Z_+}} =$$
$$(d(x,y)-D)\left(\lim_{n\to\infty} K^{n-k}\right) + D = D$$

so that for any finite $k, k_1 \in \boldsymbol{Z}_+$:

$$\begin{aligned} 0 = LS\sigma\left(1[1,k],0[k+1,\infty]\right) = LS\sigma(0) \le LS\sigma(0,1) \le LS\sigma(1) = \\ LS\sigma\left(0\left[1,k_1\right],1\left[k_1+1,\infty\right]\right) = D \end{aligned} \tag{6}$$

Property (ii) follows from (6). $\square$

The above result leads directly to the subsequent one which concludes that $\lim_{n\to\infty} d\left(T^n x, T^n y\right)$ always exist and it is either zero or D depending on the values of the iteration image selection binary sequence.

Theorem 2. *Let (X,d) be a metric space and $A_i \subset X$ for $i \in \bar{p}$ are nonempty subsets of X and assume that the self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is $p\ (\ge 2)$ cyclic/non-cyclic hybrid bivalued contractive. Then, the following properties hold:*

(i) *The image selection sequence has a finite number of connected components $\left[\{\sigma_i\}^{k_{i+1}-k_i},\ \{\sigma_{i+1}\}^{k_{i+2}-k_{i+1}}\right)$ where $\{k_i\}_{i=1}^{k}$ is strictly increasing with $k_1 = 1$; $k \in \boldsymbol{Z}_+$ is finite and $k_{k+1} = +\infty$ so that $\{0,\ 1\}^{\boldsymbol{Z}_+} \ni \{\sigma_n\}_{n=1}^{\infty} = \bigcup_{i\in\bar{k}}\left[\{\sigma_i\}^{k_{i+1}-k_i},\ \{\sigma_{i+1}\}^{k_{i+2}-k_{i+1}}\right)$.*

(ii) *The limit $\lim_{n\to\infty} d\left(T^n x, T^n y\right)$ always exists and either $\lim_{n\to\infty} d\left(T^n x, T^n y\right) = D$ or $\lim_{n\to\infty} d\left(T^n x, T^n y\right) = 0$; $\forall x, y \in A_i \cup A_{i+1}$ for any given arbitrary $i \in \bar{p}$ depending on the values of the image selection sequence.*

(iii) *If $\bigcap_{i\in\bar{p}} A_i \ne \varnothing$ then the self-mapping T is asymptotically regular.*

Proof. Proof of Properties [(i)–(ii)]. We first prove that the number k of switches between 0 and 1 cannot be infinity in the image selection sequence. From Theorem 1 (i), Equation (3), for any given real constant $\varepsilon > 0$, there exists some finite $k = k(\varepsilon,\ i) \in \boldsymbol{Z}_+$ such that $d\left(T^{k-1}x,\ T^{k-1}y\right) \le D + \varepsilon$ for any given $x, y \in A_i \cup A_{i+1}$ for any arbitrary $i \in \bar{p}$.

Case 1: Assume that $\sigma_k = 0$ then $d\left(T^k x,\ T^k y\right) \le Kd\left(T^{k-1}x,\ T^{k-1}y\right) \le K(D+\varepsilon) < D$

by fixing $\varepsilon < \left(K^{-1}-1\right)D$. Since $d\left(T^k x,\ T^k y\right) < D$ then $\{T^m x\}_{m=k}^{\infty} \subset A_j$; $\{T^m y\}_{m=k}^{\infty} \subset A_j$ for some $j = j(i) \in \bar{p}$ and then $d\left(T^n x,\ T^n y\right) \le K^{n-k} d\left(T^k x,\ T^k y\right)$ so that $\sigma_n = 0$ for any $n(\ge k) \in \boldsymbol{Z}_+ d\left(T^n x,\ T^n y\right) \to 0$ as $n \to \infty$.

Case 2: Assume that $\sigma_j = 1$ for $j \in [k,\ k+\ell]$ for some arbitrary real constant $\varepsilon > 0$ finite $k = k(\varepsilon, i) \in \boldsymbol{Z}_+$ and some finite $\ell = l(k,\varepsilon,i) \in \boldsymbol{Z}_+$ and that $\sigma_{k+\ell+1} = 0$, then $d\left(T^{k+\ell}x,\ T^{k+\ell}y\right) \le K^{\ell} d\left(T^k x,\ T^k y\right) + \left(1-K^{\ell}\right)D \le K^{\ell}(D +$

$\varepsilon) + \left(1 - K^{\ell}\right) D = K^{\ell}\varepsilon + D$

$$d\left(T^{k+\ell+1}x,\ T^{k+\ell+1}y\right) \leq K^{\ell+1}\varepsilon + KD < D$$

by fixing $\varepsilon < \left(K^{-1} - 1\right) D$. Then, one concludes that there exists a finite integer $k+\ell$ such that $\sigma_n = 0$ for any $n\,(> k+\ell) \in \boldsymbol{Z}_+$ and we are in Case 1 by replacing $k \to k+\ell+1$, and again $d\left(T^n x,\ T^n y\right) \to 0$ as $n \to \infty$.

Since $d\left(T^k x,\ T^k y\right) < D$ then $\{T^m x\}_{m=k}^{\infty} \subset A_j$; $\{T^m y\}_{m=k}^{\infty} \subset A_j$ for some $j = j(i) \in \bar{p}$ and then $d\left(T^n x,\ T^n y\right) \leq K^{n-k} d\left(T^k x,\ T^k y\right)$ so that $\sigma_n = 0$ for any $n(\geq k) \in \boldsymbol{Z}_+ d\left(T^n x,\ T^n y\right) \to 0$ as $n \to \infty$.

Case 3: $\sigma_n = 1$; $\forall n(\geq k) \in \boldsymbol{Z}_+$ and some finite $k \in \boldsymbol{Z}_+$. Then,

$d\left(T^n x, T^n y\right) \leq K^{n-k} d\left(T^k x, T^k y\right) + \left(1 - K^{n-k}\right) D$; $\forall n(\geq k) \in \boldsymbol{Z}_+$ and $d\left(T^n x, T^n y\right) \to D$ as $n \to \infty$, $d\left(T^{np+j}x, T^{np+j}y\right) \to D$ as $n \to \infty$, $\left(\left\{T^{pn+j}x\right\}_{n=0}^{\infty}, \left\{T^{pn+j}y\right\}_{n=0}^{\infty}\right) \subset \left(A_{\ell+j} \times A_{\ell+j+1}\right) \cup \left(A_{\ell+j+1} \times A_{\ell+j}\right)$; for some $\ell \in \bar{p}$. $\forall j \in \overline{p-1} \cup \{0\}$.

From the above cases, it is concluded that $\{\sigma_n\}_{n=0}^{\infty}$ remains constant after a finite number of switches between 0 and 1 or vice-versa. Properties [(i)–(ii)] have been proved.

Proof of Property (iii). Property (iii) is a direct consequence of Property (ii) for the case when $\bigcap_{i\in\bar{p}} A_i \neq \varnothing$, that implies that the distance between each pair of adjacent subsets is $D = 0$ so that $\lim\limits_{n\to\infty} d\left(T^n x, T^n y\right) = 0$; $\forall x, y \in A_i \cup A_{i+1}$ irrespective of $i \in \bar{p}$ and then the mapping is asymptotically regular. $\square$

Remark 6. *Note that Theorem 2 states that the $p(\geq 2)$ cyclic/ non-cyclic hybrid bivalued contractive self-mappings are either asymptotically regular or their sequences of distances converge to the distance between each pair of adjacent subsets D depending on the iteration image selection binary sequence. Both alternatives depend in turn on the values of the image selection sequence. In other words, the pairs in the iterated sequences either remain in the same subset of the cyclic disposal after a finite number of iterations and the self-mapping is asymptotically regular or it has infinitely many switches between adjacent subsets. Note also that Theorem 2 specifies Theorem 1 to the existence of either the limit $\lim\limits_{n\to\infty} d\left(T^n x,\ T^n y\right) = D$ or that of the limit $\lim\limits_{n\to\infty} d\left(T^n x,\ T^n y\right) = 0$ depending on the iteration image selection binary sequence. In particular:*

a) $\lim\limits_{n\to\infty} d\left(T^n x,\ T^n y\right) = D$ if $\{\sigma_n\}_{n=1}^{\infty}$ has infinitely many unity values. This implies also that $\{\sigma_n\}_{n=1}^{\infty}$ has only a finite number of zero values and that σ_n is unity after a finite number of iterations.

b) $\lim\limits_{n\to\infty} d\left(T^n x,\ T^n y\right) = 0$ if $\{\sigma_n\}_{n=1}^{\infty}$ has infinitely many zero values. This implies also that $\{\sigma_n\}_{n=1}^{\infty}$ has only a finite number of unity values and that σ_n is zero after a finite number of iterations.

This happens since infinitely many alternations cannot happen because of the nonexpansivity of the cyclic/non-cyclic hybrid bivalued contractive mapping. Such a property avoids new switching actions to the corresponding adjacent subset of any element of the pair of distances once the distances between elements of such pairs *located in the same subsets are less than D. In other words, $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive self-maps, the pairs of sequences in the sequences of distances either remain in the same subset A_i for some $i \in \bar{p}$ after a finite number of iterations, i.e., $\sigma_n = 0$; $\forall n\,(\in \boldsymbol{Z}_+) \geq k$, for some finite $k \in \boldsymbol{Z}_+$ or they behave cyclically on the whole disposal of subsets A_i; $\forall i \in \bar{p}$ after a finite number of iterations, i.e., $\sigma_n = 1$; $\forall n\,(\in \boldsymbol{Z}_+) \geq k$ for some finite $k \in \boldsymbol{Z}_+$. As a result, the iteration image selection binary sequence cannot contain infinitely many switches between zero and unity.*

Remark 7. *The idea that the number of switches of the image selection sequence is finite, as proved in Theorem 2, may be figured out and summarized in few steps as follows:*

7.1. *The contractive mapping is always non-expansive. In the same set, the result is obvious, while in adjacent subsets, one has: $0 \leq d(Tx, Ty) - D \leq K\,(d(x, y) - D) < d(x, y)$ if $d(x, y) \geq D$ and x, y are in adjacent subsets.*

This implies that $D \leq d(Tx, Ty) \leq d(x, y)$ if $d(x, y) \geq D$ in cyclic mode and $d(Tx, Ty) < d(x, y)$ in trapped intra mode if $d(x, y) > 0$. But note that, in both modes, $d(Tx, Ty) \leq d(x, y)$ because the self-mapping T is nonexpansive by nature.

7.2. *If $d(Tx, Ty) < d(x, y)$, then the sequence of distances is necessarily operating in trapped intra mode since the cyclic mode requires that $d(x, y) \geq D$. But, if it is in trapped mode at a current iteration with $d(x, y) < D$, then the distance continuous to decrease since $d(Tx, Ty) \leq Kd(x, y)$ because both points are in the same subset with null $\sigma_{(.)}$ as a result and it cannot switch again to cyclic mode with unity value of $\sigma_{(.)}$. So, the image selection sequence would not switch anymore to a cyclic mode operation.*

7.3. *The self-mapping is nonexpansive so that after a finite number of switches the distance is not larger than $(D + \varepsilon)$ for any finite positive real constant ε. At this iteration step, the number of switches of the binary image selection has been finite, although dependent on ε, since the number of iterations has been finite. Since ε is arbitrary, it can be chosen small enough such that if a new switch happens to trapped intra mode, the distance is below ε for the next (still finite) iteration and then still under a finite number of previous switches between trapped intra mode and cyclic mode. In fact, it suffices for that to be verified to choose $\varepsilon < (1-K)D$. If no new switch happens to trapped intra mode from the current cyclic mode then the image selection sequence continues without adding new switches to its current finite value to be accounted for.*

7.4. *If all the pairs of adjacent subsets intersect then $D = 0$ then the best proximity points are those of the intersection set. If, in addition, all the subsets intersect and the metric space is complete then the unique fixed point of the mapping lies in that intersection. Then, any sequence of distances will enter the intersection set after a finite number of iterations.*

The following result is direct concerning the best proximity points of a $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ under the additional conditions C.5–C.6:

Lemma 1. *Let (X,d) be a metric space and $A_i \subset X$ for $i \in \bar{p}$ are nonempty subsets of X and assume that the self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive which satisfies in addition, Condition C.5 for some $i \in \bar{p}$. Then, the following properties hold:*

(i) *There is at least a best proximity point in A_i to A_{i+1}, $z_i \in A_{i0}\,(=\{z_i\})$ such that $\boldsymbol{T}z_i = \left\{z_i \left(= \ (Tz_i)^0\right),\ (Tz_i)^1\right\} \in A_{i0} \times A_{i+1,0}$ for some $i \in \bar{p}$.*

(ii) *There are at most p best proximity points in A_i to A_{i+1}, $z_i \in A_{i0}$ such that $\boldsymbol{T}z_i = \left\{z_i \left(= \ (Tz_i)^0\right),\ z_{i+1}\left(= (Tz_i)^1\right)\right\} \in A_{i0}\times A_{i+1,0}$; $\forall i \in \bar{p}$ with $A_{i0} = \{z_i\}$; $\forall i \in \bar{p}$.*

(iii) *If (X,d) is complete and $\bigcap_{i\in\bar{p}} A_i \neq \varnothing$ then there is a unique fixed point $z \in \bigcap_{i\in\bar{p}} A_i$ such that $\boldsymbol{T}z = (Tz)^0 = (Tz)^1 = z = z_i = \{z_i\} = A_{i0}$.*

Proof. Proof of Property (i). Since $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is a mapping according to Definition 1 that satisfies, furthermore, the C.5, it jointly satisfies Conditions C.1–C.5.From C.5 all the subsets A_i are closed and there exists $z_i \in A_{i0}\,(=\{z_i\})$ for some $i \in \bar{p}$ which is a best proximity point in A_i to A_{i+1} which is a singleton. From C.2 the two images of T of any best proximity point z_j in A_j to A_{j+1} satisfy $\boldsymbol{T}z_j = \left\{(Tz_j)^0,\ (Tz_j)^1\right\} \in A_{j0} \times A_{j+1,0}$; $\forall j \in \bar{p}$. $z_i \in A_{i0}\,(=\{z_i\})$ is unique then $\boldsymbol{T}z_i = \left\{z_i \left(= \ (Tz_i)^0\right),\ (Tz_i)^1\right\} \in A_{i0} \times A_{i+1,0}$ and Property (i) has been proved. □

Proof. Proof of Property (ii). Property (ii) is a direct extension of Property (i) since there are at most p best proximity points which are singletons one per subset of the cyclic disposal. □

Proof. Proof of Property (iii). If $\bigcap_{i\in\bar{p}} A_i \neq \varnothing$ then $D = 0$. From Condition C.3, Equation (1) with $D = 0$ implies that the self-mapping T is a strict contraction which has a fixed point which is unique and which is also the confluence of all the best proximity points from C.5 but since A_{i0} is the singleton $\{z_i\}$ then $\bigcap_{j\in\bar{p}} A_{j0} = A_{i0} = \{z_i\}$. This leads directly to Property (iii). □

The subsequent result relies on convergence and cauchyness of sequences generated by a cyclic/non-cyclic hybrid bivalued contractive which satisfies, in addition, Condition C.5.

Theorem 3. *Let (X,d) be a metric space and $A_i \subset X$ for $i \in \bar{p}$ are nonempty closed subsets of X and assume that the self-mapping $T : \bigcup_{i\in\bar{p}} A_i \to \bigcup_{i\in\bar{p}} A_i$ is $p(\geq 2)$ cyclic/non-cyclic hybrid bivalued contractive which satisfies, in addition, Condition C.5. Then, the following properties hold:*

(i) *If $\sigma_n = 1$ for $n(\geq \vartheta) \in \boldsymbol{Z}_{0+}$ and some finite $\vartheta \in \boldsymbol{Z}_{0+}$ then $\left\{\left(T^{np+\ell+s}x\right)^1\right\}_{n=0}^{\infty} \to (T^s z_i)^1\,(\in A_{i+s,0})$; $\forall s \in \bar{p}$ for any given $x \in A_j$ for any arbitrarily given $j \in \bar{p}$, and some $\ell = l(i,j) \in \bar{p}$, where A_i for $i \in \bar{p}$ satisfying C.5 such that $A_{i0} = \{z_i\}$. In addition, $\left\{\left(T^{np+\ell+s}x\right)^1\right\}_{n=0}^{\infty}$ is a Cauchy sequence; $\forall s \in \bar{p}$ for any given $x \in A_j$ for any arbitrarily given $j \in \bar{p}$, and some $\ell = \ell(i,j) \in \bar{p}$.*

Furthermore, $\left\{(T^{n+s}x)^1\right\}_{n=0}^{\infty}$ *converges to a unique point which belongs to a unique* p *-tuple in the best proximity set* $A_{10} \times ... \times A_i\,(=\{z_i\}) \times ... \times A_{p0}$ *for some* $s \in \overline{p-1} \bigcup \{0\}$ *which defines that the above set product is ran in the given order. For null* s *the set product involves the same subsets but not necessarily in the natural ordering.*

(ii) *If* $\sigma_n = 0$ *for* $n(\geq \vartheta) \in \boldsymbol{Z}_{0+}$ *and some finite* $\vartheta \in \boldsymbol{Z}_{0+}$ *then* $\left\{\left(T^{np+\ell}x\right)^0\right\}_{n=0}^{\infty} \to \left(T^{\ell}z_i\right)^0 (\in A_{i+\ell,0})$ *for some* $\ell = l(i,j) \in \bar{p}$*, any given* $x \in A_j$ *and any arbitrarily given* $j \in \bar{p}$*. Furthermore,* $\left\{\left(T^{n+k}x\right)^0\right\}_{n=0}^{\infty}$ *converges to a unique best proximity point* $z_j \in A_{j0}$ *from some subset* A_j *of* X *to its next adjacent subset* A_{j+1} *of* X *for some* $k = k(j,i) \in \bar{p}$*. Such a point is also a fixed point to the restricted mapping* T *to* A_j*. If* $j = i$ *then* $z_j = \{z_j\} = A_{j0}$*. Furthermore,* $\left\{\left(T^{np+\ell}x\right)^0\right\}_{n=0}^{\infty}$ *and* $\left\{\left(T^{n+k}x\right)^0\right\}_{n=0}^{\infty}$ *are Cauchy sequences for any given* $x \in A_j$ *for any arbitrarily given* $j \in \bar{p}$*, and some* $\ell = l(i,j) \in \bar{p}$ *and* $k = k(j,i) \in \bar{p}$*.*

(iii) *If* $\bigcap_{i\in\bar{p}} A_i \neq \varnothing$ *then* $\sigma_n = 0$ *for any* $n \in \boldsymbol{Z}_{0+}$ *and there is a unique* $z \in \bigcap_{i\in\bar{p}} A_i$ *such that* $\boldsymbol{T}z = (Tz)^0 = z$*, that is,* z *is the unique fixed point of* T *and* $\{T^n x\}_{n=0}^{\infty}$ *is a Cauchy sequence.*

Proof. Proof of Property (i). Note that it has been proved (Theorem 2 (i)) that the image selection sequence has a finite number of connected components what means that after a finite number of switchings between its two possible values, the switching process ends. Note also that, from Theorem 2 (ii), $\lim_{n\to\infty} d\left(T^n x, T^n y\right) = D$ or $\lim_{n\to\infty} d\left(T^n x, T^n y\right) = 0$; $\forall x, y \in A_i \cup A_{i+1}$ for any given arbitrary $i \in \bar{p}$ depending on the values of the image selection sequence. According to Remark 6, and supposed that $D > 0$, one has that $\lim_{n\to\infty} d\left(T^n x, T^n y\right) = D$ if $\sigma_n = 1$ for any iteration after a certain a finite number of iterations and $\lim_{n\to\infty} d\left(T^n x, T^n y\right) = 0$ if $\sigma_n = 0$ for all iterations exceeding a certain finite threshold of iterations. If $D = 0$ then, from Theorem 2 (iii), $\lim_{n\to\infty} d\left(T^n x, T^n y\right) = 0$. From Theorem 2(ii), infinitely many switchings between the values "0" and "1" cannot occur in the image selection sequence. The above results also imply directly that either $\lim_{n\to\infty} d\left(T^{np+j}x, T^{np+j}y\right) = D$ or $\lim_{n\to\infty} d\left(T^{np+j}x, T^{np+j}y\right) = 0$; $\forall x, y \in A_i \cup A_{i+1}$ for any given arbitrary $i \in \bar{p}$; $\forall j \in \overline{p-1} \cup \{0\}$ if either σ_n is unity or zero after a finite number of iterations if $D > 0$ and that $\lim_{n\to\infty} d\left(T^{np+j}x, T^{np+j}y\right) = 0$; $\forall x, y \in A_i \cup A_{i+1}$ for any given arbitrary $i \in \bar{p}$; $\forall j \in \overline{p-1} \cup \{0\}$ if $D = 0$. Note also from the contractive condition (1) that the self-mapping T is Lipschitz-continuous with unity Lipschitz constant since it is nonexpansive (Proposition 2).

Assume that $\sigma_n = 1$ for $n(\geq \vartheta) \in \boldsymbol{Z}_{0+}$ for some finite $\vartheta \in \boldsymbol{Z}_{0+}$. Then, $\lim_{n\to\infty} d\left(T^{np+l}x, T^{np+l}y\right) = D$; $\forall x, y \in A_j \cup A_{j+1}$ for any given arbitrary $j \in \bar{p}$; $\forall \ell \in \overline{p-1} \cup \{0\}$. Thus, there is some $\ell = l(j,i) \in \overline{p-1} \cup \{0\}$ for each $j \in \bar{p}$ with $x, Tx \in A_j \cup A_{j+1}$ such that $\lim_{n\to\infty} d\left(T^{np+l}x, T^{np+l}y\right) = D$, with $D > 0$, with $\left\{\left(T^{np+\ell}x\right)^0\right\}_{n=0}^{\infty} \subset A_i$, $\left\{\left(T^{np+\ell}x\right)^1\right\}_{n=0}^{\infty} \subset A_{i+1}$ such that A_i is boundedly compact with $A_{i0} = \{z_i\}$ from Condition C.5, there is a subsequence $\left\{\left(T^{n_k p+\ell}x\right)^0\right\}_{k=0}^{\infty} \subset \left\{\left(T^{np+\ell}x\right)^0\right\}_{n=0}^{\infty} \subset A_i$ such that $\left\{\left(T^{n_k p+\ell}x\right)^0\right\}_{k=0}^{\infty} \to z_i\,(=\{z_i\} = A_{i0})$ since A_i is boundedly compact with a singleton best proximity set

$\{z_i\}$ to A_{i+1} and $\left\{\left(T^{np+\ell}x\right)^0\right\}_{n=0}^{\infty}$ is bounded from Proposition 4. Otherwise, if the above subsequence converges to another point z_{i1} (it has to from the bounded compactness property), then such a limit point is not in A_{i0} since $\{z_i\} = A_{i0}$ and then $d\left(z_{i1},\ (Tz_{i1})^1\right) > D$, hence a contradiction. Also, since T is nonexpansive then it is Lipschitz-continuous with unity Lipschitz constant and $\left\{\left(T^{n_k p+\ell}x\right)^1\right\}_{k=0}^{\infty} \to (Tz_i)^1\ (\in A_{i+1,0})$. Proceeding in the same way, one concludes that $(T^s z_i)^1 = z_{i+s}\ (\in A_{i+j,0})$; $\forall s \in \overline{p}$ and then

$$\left\{\left(T^{n_k p+\ell+s}x\right)^1\right\}_{k=0}^{\infty} \to z_{i+s}\left(= (T^s z_i)^1\ (\in A_{i+s,0})\right); \forall s \in \overline{p}.$$

It is now proved that this convergence property of the subsequence extends to the whole sequence. Proceed by contradiction by taking a claimed to exist strictly increasing subsequence $\{m_k\}_{k=0}^{\infty} \subset \boldsymbol{Z}_+$ with $m_k \in (n_k, n_{k+1}] \cap \boldsymbol{Z}_{0+}$ such that, for some $j \in \bar{p}$, $\left\{\left(T^{n_k p+\ell+s}x\right)^1\right\}_{k=0}^{\infty}$ does not converge to $(T^s z_i)^1$; $\forall s \in \bar{p}$, that is, one has from $C.3$, that

$$D+\varepsilon \le d\left(\left(T^{m_k p+\ell+s+1}x\right)^1, \left(T^{m_k p+\ell+s}x\right)^1\right) \le$$
$$K^{(m_k-n_k)p}\left(d\left(T^{n_k p+\ell+s+1}x\right)^1,\ \left(T^{n_k p+\ell+s}x\right)^1 - D\right) + D$$

for some real $\varepsilon > 0$ and then, since $\left\{d\left(T^{n_k p+\ell+s+1}x\right)^1,\ \left(T^{n_k p+\ell+s}x\right)^1\right\}_{k=0}^{\infty} \to D$, one gets the following contradiction:

$$D+\varepsilon \le \underset{k\to\infty}{lim\ inf} d\left(\left(T^{m_k p+\ell+s+1}x\right)^1, \left(T^{m_k p+\ell+s}x\right)^1\right)$$
$$\le \underset{k\to\infty}{lim\ sup}\left[K^{(m_k-n_k)p}\left(d\left(T^{n_k p+\ell+s+1}x\right)^1, \left(T^{n_k p+\ell+s}x\right)^1 - D\right) + D\right]$$
$$= \lim_{k\to\infty}\left(K^{(m_k-n_k)}p\right)\lim_{k\to\infty}\left(d\left(T^{n_k p+\ell+s+1}x\right)^1, \left(T^{n_k p+\ell+s}x\right)^1 - D\right)$$
$$+D < \lim_{k\to\infty}\left(d\left(T^{n_k p+\ell+s+1}x\right)\right)^1, \left(T^{n_k p+\ell+s}x\right)^1 - D) + D = D$$

Thus, one has that

$$\left\{\left(T^{np+\ell+s}x\right)^1\right\}_{n=0}^{\infty} \to (T^s z_i)^1\ (\in A_{i+s,0})\,; \forall s \in \overline{p}$$

for the whole sequence that for some $\ell = \ell(i,j) \in \overline{p-1} \cup \{0\}$, where A_i is boundedly compact and it has a singleton best proximity set to its next adjacent subset for some $i \in \bar{p}$ and A_j for some $j \in \bar{p}$ is the arbitrary subset of X where the sequence is initialized. Also, any $\left\{\left(T^{np+\ell+s}x\right)^1\right\}_{n=0}^{\infty}$ is a Cauchy sequence since it is convergent. Furthermore, it holds trivially that $\left\{\left(T^{n+s}x\right)^1\right\}_{n=0}^{\infty}$ converges to a unique point which belongs to a unique p-tuple in the set $A_{10} \times \ldots \times A_i\ (= \{z_i\}) \times \ldots \times A_{p0}$ for each $s \in \overline{p-1} \bigcup \{0\}$ since the two images of each best proximity point, one of them in the current subset and another one in the next adjacent subset are unique and one of the best proximity sets $A_{i0} \subset A_i$ is a singleton $\{z_i\}$ to which all sequences converge within the corresponding subset A_i of X. Property (i) has been proved irrespective of the initial point $x \in A_j$ for any arbitrary $j \in \bar{p}$. $\square$

Proof. Proof of Property (ii). The proof of Property (ii) is similar to that of Property (i) by taking into account that if σ_n is identically zero after a finite number ϑ of iterations then the hybrid cyclic/noncyclic contraction becomes after such a finite number of iterations a strict contraction of the mapping T being restricted to one of the subsets of the cyclic disposal to itself. From C.5, all restriction of T to itself within any of the subsets A_j of X for $j \in \bar{p}$ of the cyclic disposal has a unique fixed point which is one of the best proximity points to its next adjacent subset. $\square$

Proof. Proof of Property (iii). Property (iii) follows directly since if all the subsets of the cyclic disposal intersect then $\{\sigma_n\}_{n=0}^{\infty}$ is a null sequence and then the self-mapping T on $\bigcup_{i\in\bar{p}} A_i$ is a strict contraction with a unique fixed point in $\bigcap_{i\in\bar{p}} A_i$. Such a fixed point is trivially also a best proximity point in each A_j to A_{j+1}; $\forall j \in \bar{p}$. $\square$

Remark 8. *Since Theorem 2 establishes that the image selection sequence is fixed either to zero or to unity in a finite number of iterations, it turns out that Condition C.5 has to hold in Theorem 3 to cover any of the two situations.*

Remark 9. *Note that Theorem 3 does not require the completeness of the metric space (X, d) since the invoked argument for the convergence and Cauchyness of sequences is that the subsets of the cyclic disposal are closed and one of them is boundedly compact. Therefore, each bounded sequence (that is, all those which are generated by the mapping irrespective of the image selection sequence from Proposition 4) has a convergent subsequence. Such a subsequence converges to a limit which is the singleton best proximity set to its next adjacent subset. This occurs since, otherwise, the distances of this sequence and that in the adjacent subset would exceed at the limit the distances between sets lying in a contradiction. As a further result, it is proved that the whole sequences, which are already bounded, are also convergent so that they are also Cauchy sequences.*

5. Examples

Some illustrative examples are now discussed:

5.1. Example 1

Consider the metric space $(\boldsymbol{R}, d)$, where the distance d is the Euclidean metric, that is, $d(x, y) = |x - y|$. Consider the two nonempty disjoint closed subsets $A = \{0\} \cup [4, 5\ ]$, which is not perfect since it has an adherent isolated point $x = 0$ which is not an accumulation point, and $B = [1, 2]$. The Euclidean distance between them is $D = d(A,\ B) = 1$. The best proximity sets are $A_{10} = \{0\}$ and $B_{10} = \{1\}$. The self-mapping T on $A \cup B$ is firstly considered single-valued and defined by:

a) $T0 = 1, T1 = 0$.
b) First tracking zone from A to B: $Tx = 1.8$ if $x \in [4, 4.5)$.
c) Absorption zone in B from a first switch from A to B: $Tx = 1.2$ if $x \in [4.5, 5]$.

d) First return zone from B to A: $Ty = 4.5$ if $y \in [1.5,\ 2]$.

e) Trapped zone in B from B: $Ty = 1.2$ if $y \in (1, 1.5)$.

There are three cyclic iterations to absorption according to the following orbit tracking pattern:

1) Iteration 1 from A to B: $x_0 = 4.2 \in A$, $x_1 = T4.2 = 1.8 \in B$.
2) Iteration 2 from B to A: $x_2 = T1.8 = 4.5 \in A$.
3) Iteration 3 from A to B: $x_3 = T4.5 = 1.2 \in B$
4) Iteration 4 and beyond which implies a permanent absorption in B : $x_{n+1} = Tx_n = 1.2 \in B$ for $n \geq 3$, with $x_3 = 1.2$.

It turns out that a contractive constant $K = 0.8$ is valid for the definition of this contraction T. **Figure 1** illustrates the sequence of iterates for different initial conditions close to 4.2. From the definition of the operator T, the sequence becomes trapped at the value 1.2 from the third iteration onward, as shown in **Figure 1**.

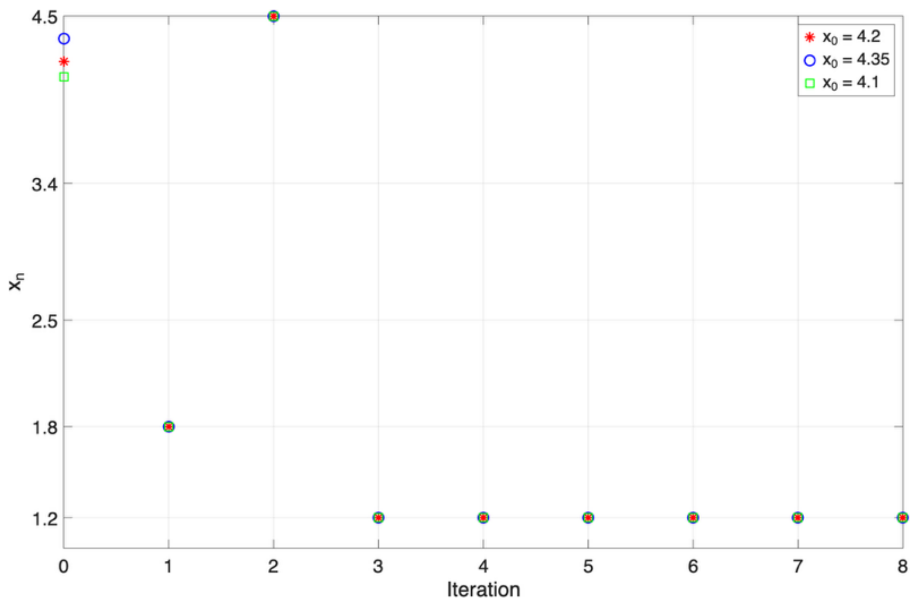


Figure 1. Evolution of the sequence x_n according to the single-valued operator T from Example 1. The sequence converges to 1.2 for any initial condition close to $x_0 = 4.2$.

Now T is cyclic/non-cyclic and bivariate with the image set $\boldsymbol{T}x = \{(Tx)^0, (Tx)^1\}$ and the built sequences satisfy $Tx \in \boldsymbol{T}x$. Now, we have:

For points in A:

a) Isolated point: $(T0)^0 = 0 \in A$; $(T0)^1 = 1 \in B$.

b) First tracking zone for $x \in [4,\ 4.5]$: $(Tx)^0 = 4.2 \in A$; $(Tx)^1 = 1.8 \in B$.

c) Absorption zone for $x \in (4.5, 5]$: $(Tx)^0 = 4.6 \in A$; $(Tx)^1 = 1.2 \in B$.

- For points in B:

d) Best proximity point: $(T1)^0 = 1.2 \in B$; $(T1)^1 = 0 \in A$.

e) First return zone for $y \in [1.5,\ 2]$: $(Ty)^0 = 1.2 \in B$; $(Ty)^1 = 4.5 \in A$.

f) Trapped zone for $y \in (1, 1.5]$: $(Ty)^0 = 1.2 \in B$; $(Ty)^1 = 4.2 \in A$.

There are three cyclic iterations to absorption of the sequence $x_{n+1} =$

$(Tx_n)^{\sigma_n}$ with $\sigma_n \in \{0,\ 1\}$ for $n \in \boldsymbol{Z}_{0+}$ according to the following orbit tracking pattern:

1) Iteration 1 from A to B (cyclic): $n = 0$, $\sigma_0 = 1$, $x_0 = 4.2 \in [4,\ 4.5] \subset A$, $x_1 = (T4.2)^1 = 1.8 \in B$.
2) Iteration 2 from B to A (cyclic): $n = 1$, $\sigma_1 = 1$, $x_1 = 1.8 \in [1.5,\ 2] \subset B$, $x_2 = (T1.8)^1 = 4.5 \in A$.
3) Iteration 3 from A to B (cyclic): $n = 2$; $\sigma_2 = 1$, $x_2 = 4.5 \in [4, 4.5] \subset A$. $x_3 = (T4.5)^1 = 1.8 \in B$
4) Iteration 4 and beyond which implies a permanent absorption (non-cyclic) in B: $n \geq 3$, $\sigma_n = 0$, $x_3 = 1.8 \in [1.5,\ 2] \subseteq B$, $x_{n+1} = (Tx_n)^0 = 1.2 \in B$ for $n \geq 3$.

It turns out that a contractive constant $K = 0.8$ is valid for the definition of this 2-hybrid cyclic/non-cyclic contraction T since

$$d\left((Tx)^1, (Ty)^1\right) \leq Kd(x,y) + (1-K)D \text{ for } x \in A, y \in B$$

$$d\left((Tx)^0, (Ty)^0\right) \leq Kd(x,y) \text{ for } x, y \in B \text{ and } \text{ for } x, y \in A$$

Figure 2 illustrates the evolution of the sequence of iterates x_n according to the pattern specified by the selected image selection sequence $\{\sigma_n\}_{n=1}^{\infty}$, for different initial conditions close to 4.2. The sequences converge to 1.2, as previously discussed, where they remain from iteration 4 onward.

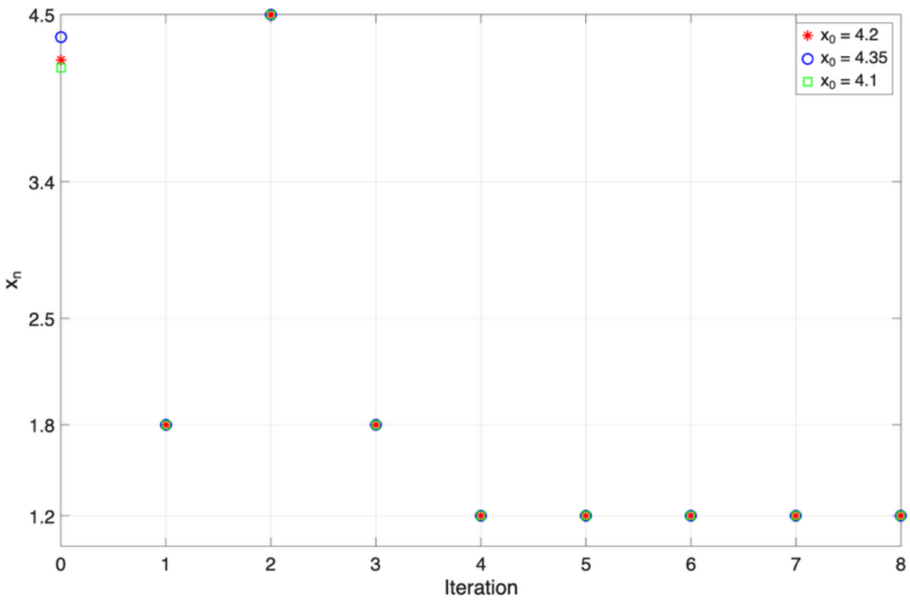
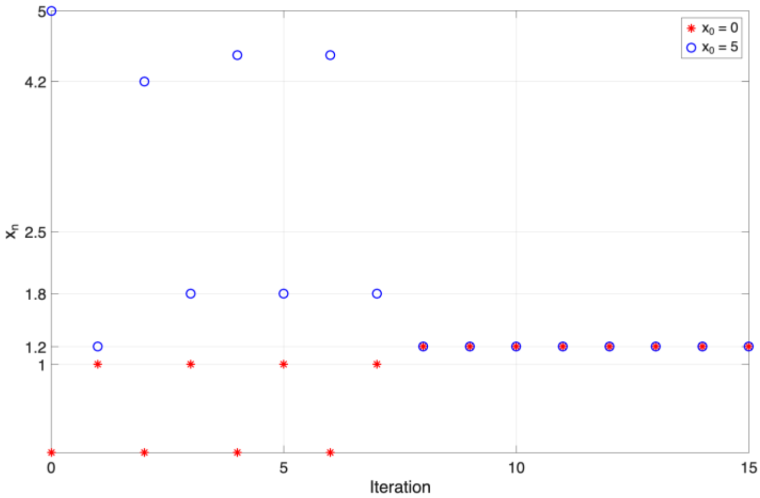
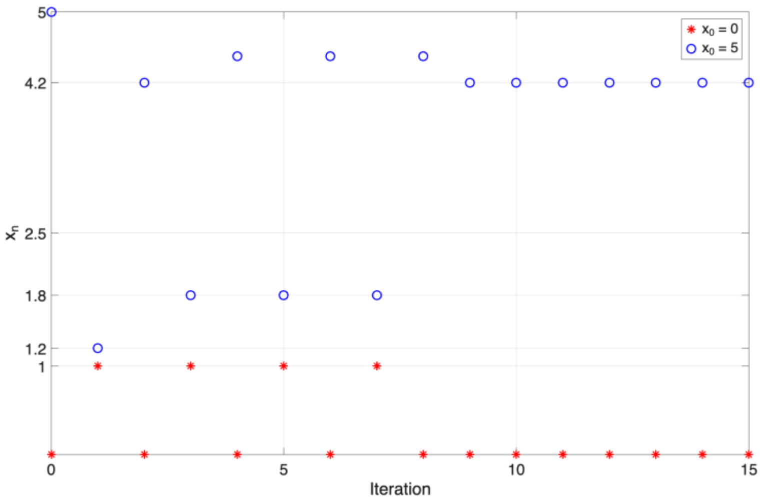


Figure 2. Evolution of the sequence x_n according to the bi-valued operator T from Example 1. The sequence converges to 1.2.

On the other hand, it turns out that if x, y belong to A then maximum number of iterations for any trapped zone in A for the best case of allowed entering conditions in A is $n_A = 7$ iterations since for the maximum Euclidean distance in A assumed initially is $|x-y| = 5$ for $K = 0.8$ then $\left|(T^n x)^0 - (T^n y)^0\right| \leq 0.8^n |x-y| < D = 1$ avoids a jump to cyclic mode. So, the maximum number of iterations staying in A is at most seven iterations which happens if the permanence period starts from the larger possible distance for initial conditions x, y, that is, $d(x,y) = |x-y| = \max_{a,b\in A} |a-b| = diam(A) = 5$. If the iteration fulfils $n \geq 8 > n_A = 7$ then no jump is allowed to the cyclic mode an the sequence will converge to $z = 0$ which is the fixed point of the self-mapping restricted to A. **Figure 3** illustrates the evolution of the iterates when the initial conditions $x_0 = 0$ and $x_0 = 5$ are considered, corresponding to the maximal separation between the elements of the set A. On the one hand, **Figure 3a** displays the iterates when the image selection sequence is one up to iteration 7 and zero onward. In this case, the sequence of iterates converges to the fixed point 1.2 in B. On the other hand, **Figure 3b** displays the iterates when the image selection sequence is one up to iteration 8 and zero onward. The sequence with initial condition at $x_0 = 0$ converges to the fixed point x = 0 while the one with initial condition at $x_0 = 5$ converges to 4.2, which are the fixed points of T in A. Thus, the image selection sequence plays a crucial role in the convergence properties.



(a)



(b)

Figure 3. Evolution of the sequence x_n according to the bi-valued operator T from Example 1 when the initial points are $x_0 = 0$ and $x_0 = 5$. **(a)** The image selection sequence is one up to iteration 7 and zero onward. Both sequences converge to 1.2 from iteration 8 onward. This is the fixed point of T in B; **(b)** The image selection sequence is one up to iteration 8 and zero onward. The sequence starting at $x_0 = 0$ converges to 0 and the one starting at $x_0 = 5$ converges to 4.2. The values of 0 and 4.2 are fixed points for T in A.

If the trapped (or staying) mode is claimed to be scheduled in B, and since $diam(B) = 1$, then no iteration is allowed within the subset B allowing further jumps later on to the cyclic mode for any initial conditions of entering in B.

5.2. Example 2

Consider an aerial drone that periodically inspects two disjoint regions, denoted by A and B, which may correspond, for example, to different industrial facilities or geographically separated natural areas. The objective is to repeatedly alternate between both regions so as to detect potential hazardous situations as early as possible. This naturally leads to the problem of designing cyclic trajectories between two sets in the plane. The control architecture of the drone is assumed to follow the standard hierarchical structure consisting of two time scales. The inner-loop controller operates at a high frequency and is responsible for stabilizing the vehicle attitude and position while accurately tracking the reference trajectory and rejecting external disturbances. The outer-loop controller operates at a slower time scale and generates the reference trajectories to be followed by the inner loop. Assuming that the inner-loop controller is sufficiently fast and robust, the tracking error can be neglected, so that the actual position of the vehicle is approximated by the reference position, i.e., $(x, y) \approx (x_{ref},\ y_{ref})$. Under this assumption, the outer-loop guidance law is modeled as $(\dot{x},\ \dot{y}) = v(x, y)$, for a velocity function v. Applying a forward Euler discretization with sampling period $h > 0$ yields the discrete-time dynamics $(x_{k+1}, y_{k+1}) = (x_k, y_k) + h\, v(x_k, y_k) = T(x_k, y_k)$, where $T(x, y)$ denotes the discrete-time map induced by the vector field v. This Example studies different discrete-time maps and the corresponding cyclic trajectories generated by the associated discrete-time dynamics for the drone.

Now, consider $B = \{(x, y) \in \boldsymbol{R}^2 :\ -1 \le x \le 0; |y| \le x + 1\}$; $A = A_1 \cup A_2$, $A_1 = [2,\ 3] \times [1, 1.8]$, $A_2 = [2,\ 3] \times [-1.8, -1]$ which are subsets of the metric space $(\boldsymbol{R}^2,\ d)$ where the distance $d : \boldsymbol{R}^2 \to \boldsymbol{R}_{0+}$ is the Euclidean metric. In this case, $D = \sqrt{(2-0)^2 + (1-0)^2} = \sqrt{5} \cong 2.236$. Consider the single-valued cyclic contractive self-mapping $T : A \cup B \to A \cup B$ of contraction constant $K = 0.8$:

$$T(x, y) = \begin{cases} [(-0.8x + 2,\quad 0.8y + sgn{*}y)] \quad if \quad (x, y) \in B \backslash \{0, 0\}\,;\ T(0, 0) = (2, 1) \\ [(-0.8x + 1.6,\quad 0.2y - 0.2sgn{*}y)] \quad if \quad (x, y) \in A \end{cases}$$

where $sgn^*(y) = 1$ if $y \ge 0$ and $sgn^*(y) = -1$ if $y < 0$. This situation corresponds to the case in which the drone position alternates cyclically between the two regions. Note that T is not continuous but it is upper-semicontinuous at $(0, 0)$. The best proximity sets are the singleton $B_0 = \{(0, 0)\}$ and $A_0 = \{(2, 1),\ (2, -1)\}$ with $\{(2, 1)\} \in A_1 \subset A$ and $\{(2, -1)\} \in A_2 \subset A$.

Note that the best proximity points apply into best proximity points since $T\,(B_0) = T\,(\{0, 0\}) = \{2, 1\} \in A_0 \subset A_1 \subset A$, $T\,(\{2, 1\}) = B_0 = \{0, 0\}$ and $T\,(\{2, -1\}) = B_0 = \{0, 0\}$. This fact is illustrated in **Figure 4**, which demonstrates how the image of a best proximity point remains a best proximity point.

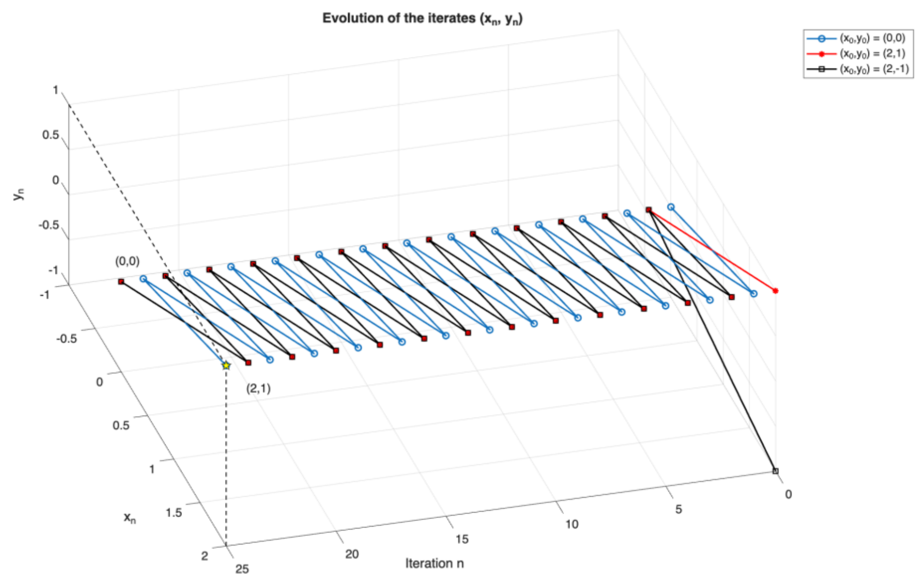
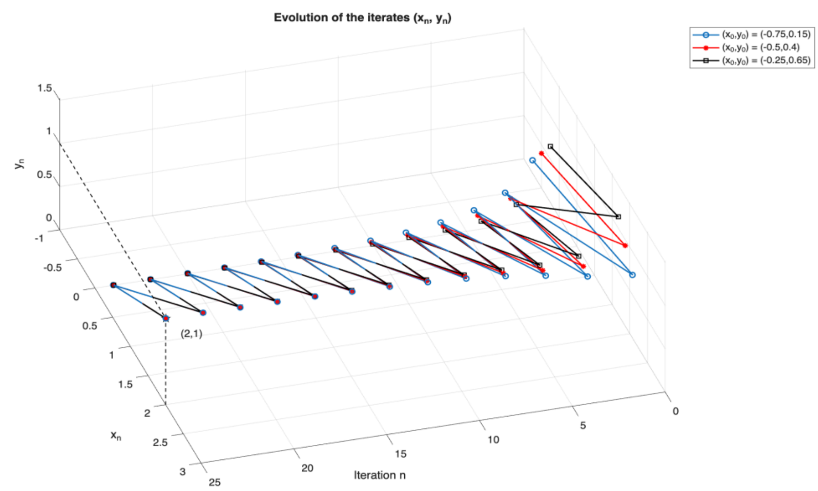


Figure 4. Images of best proximity points under the single-valued operator *T*.

The contraction condition is

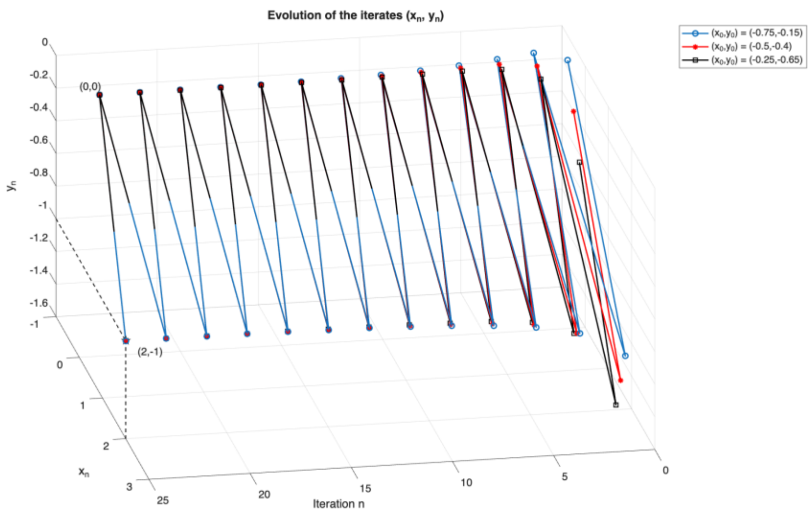
$$\|Tx - Ty\| \leq K\,\|x - y\| + (1-K)D = 0.8\,\|x - y\| + (1-0.8)\sqrt{5} = 0.8\,\|x - y\| + 0.2\sqrt{5}$$

In the cyclic mode, the distance $\|T^n x - T^n y\| \to \sqrt{5}$ as $n \to \infty$ and the sequence to a limit cycle of two best proximity points $\{(0,0) \ \times (2,1)\} \cup \{(2,1) \ \times (0,0)\}$ if $(x,y) \in (A_1 \times B) \cup (B \times A_1)$ and to the limit cycle of best proximity points $\{(0,0) \ \times (2,-1)\} \cup \{(2,-1) \ \times (0,0)\}$ if $(x,y) \in (A_2 \times B) \cup (B \times A_2)$, one being in both cases the singleton B_0 as expected. **Figure 5** illustrates the evolution of the iterates for various initial conditions. When the initial conditions are in set B with positive vertical-axis values, the cyclic behavior of the operator causes the iterates to oscillate between the best proximity points (0, 0) and (2,1) (**Figure 5a**). If the initial conditions lie in B but the y_0 component is negative, the operator causes the iterates to oscillate between (0,0) and (2, −1) (**Figure 5b**). Finally, if the initial conditions are in A, the limit cycle occurs between the best proximity points (0, 0) and (2, 1) (**Figure 5c**).

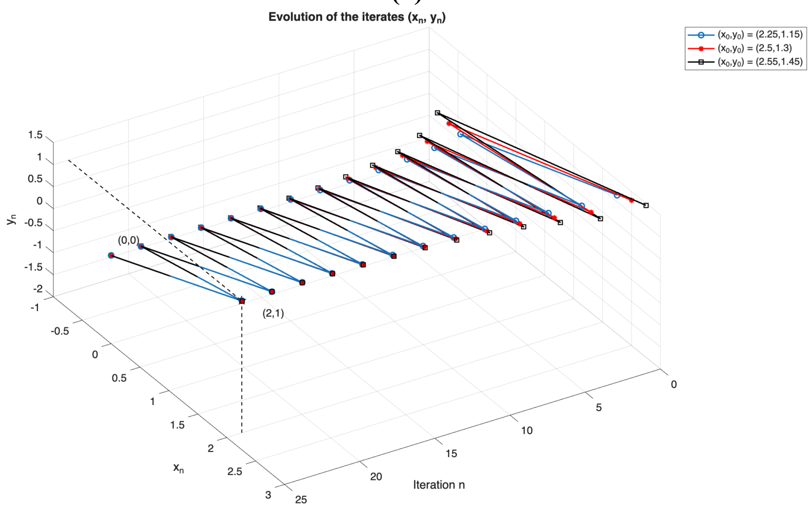


(a)

Figure 5. *Cont.*



(b)



(c)

Figure 5. Iterates of the single-valued operator T for various initial conditions. In all cases, the operator approaches a limit cycle, alternating between best proximity points. **(a)** Initial conditions in the set B with a positive value y_0; **(b)** Initial conditions in the set B with a negative value y_0; **(c)** Initial conditions in the set A.

Several modifications of the self-mapping to introduce the dual cyclic/non-cyclic operation modes are discussed. These cases correspond to scenarios in which, instead of alternating cyclically between the two regions at every iteration, the drone remains within one region for a certain number of iterations before moving to the other. This behavior may arise, for example, when the drone detects a potentially hazardous situation and must gather additional information before resuming its cyclic inspection of the two regions. The decision to either remain in the current region or transition to the other one is determined by the switching signal σ. In all cases, the theory developed in this paper makes it possible to characterize the properties of the generated reference trajectories. The cases to be now describes explicitly are:

Case 1: The mapping is modified as follows in order to define a bivariate one which is an hybrid bivariate cyclic/non-cyclic hybrid contractive self-mapping, with one of the images of each point in the same subset and which is cyclic/non-cyclic hybrid contractive:

$$T(x,y)=\begin{cases} [(-0.8x+2,\ 0.8y+sgn*y)]\vee[(0.8x,\ 0)] \quad if \quad (x,y)\in B\backslash\{0,0\}\,;T(0,0)=(2,1) \\ [(-0.8x+1.6,\ 0.2y-0.2sgn*y)]\vee[T_a(x,y)] \quad if \quad (x,y)\in A \end{cases}$$

where the above expressions after the logic disjunction apply when the built sequences stay in the same subset for the next iteration (non-cyclic mode) while the expression prior to the logic disjunction apply for the cyclic mode, such that

$$T_a(x,y)=\begin{cases}(0.8x+0.5,\ 0.8y+0.25)\ if \ \ (x,y)\in A_1 \\ (0.8x+0.5,\ 0.8y-0.25)\ if \ \ (x,y)\in A_2\end{cases}$$

The best proximity points for the cyclic mode are $B_0=\{(0,0)\}$ and $A_0=\{(2,1),\ (2,-1)\}$. The fixed points are $Fix\,(T|B)=B_0=\{0,0\}$ in B, $Fix\,(T|A_1)=Fix\,(T_a|A_1)=\{2.5,1.25\}$ in A_1 and $Fix\,(T|A_2)=Fix\,(T_a|A_2)=\{2.5,-1.25\}$ in A_2. The existence of two fixed points in A, one in each connected component A_1 and A_2, is coherent with the fact that there are no jumps for any sequence from A_1 to A_2 or vice-versa in the trapped intra mode (or non-cyclic mode) mode in A.

The best proximity points apply into best proximity points since $(T\,(B_0))^2=(T\,(\{0,0\}))^2=\{2,1\}\in A_0$, $(T\,(\{2,1\}))^2=B_0=\{0,0\}$ and $(T\,(\{2,-1\}))^2=B_0=\{0,0\}$.

Furthermore, $\boldsymbol{T}\,(B_0)=\left\{(T\,(B_0))^0\,,\ (T\,(B_0))^1\right\}=\{(0,0),\ \{2,1\}\}$ (that is, one of the images of the singleton best proximity set is itself), and

$$\boldsymbol{T}\,(\{2,1\})=\left\{(T\,(\{2,1\}))^0\,,\ (T\,(\{2,1\}))^1\right\}=\{(2.1,1.05),\ \{0,0\}\}\in A_1\cup B_0;$$

$$\boldsymbol{T}\,(\{2,-1\})=\left\{(T\,(\{2,-1\}))^0\,,\ (T\,(\{2,-1\}))^1\right\}=\{(2.1,-1.05),\ \{0,0\}\}\in A_2\cup B_0.$$

Note that Condition $C.2$ is fulfilled only in part since one of the images of the best proximity points $\{2,1\}\in A_1$ and $\{2,-1\}\in A_2$ are in the same respective subsets but they are not best proximity points in A to B, that is, $(T\,(\{2,\pm 1\}))^0\notin A_0$ while $(T\,(\{2,\pm 1\}))^1\in B_0$. The fact that A has two fixed points, one in each connected component A_1 and A_2 and that two of the images of the best proximity points in A to B are not best proximity points affects to the scheme´s cyclic mode operation in the sense that the limit cycle which is the asymptotic trajectory of any built sequence in cyclic mode operates only in the best proximity sets B_0 and A_{10}. **Figure 6** illustrates the evolution of the iterates for the set-valued operator T considering initial conditions in B with a positive y_0 value as an example. **Figure 6a** displays the case where σ equals zero for the first ten iterations and then takes the value one permanently. **Figure 6b** shows the opposite scenario, in which σ takes the value one for the first ten iterations and then becomes zero. **Figure 6c** presents the case where σ switches continuously, taking a random value of either zero or one at each iteration. When σ takes the permanent value of one, the system converges to the limit cycle described in Example 2, which oscillates between the best proximity points (0,0) and (2,1), as shown in **Figure 6a**. When σ stabilizes at zero, the system remains in the same set (intra-mode) and converges to its corresponding best proximity point, which is $(0,0)$, as shown in **Figure 6b**. This behaviour confirms the Theorem 3 result from Section 4. Finally, if σ never ceases to switch, no pattern emerges in the sequence of iterates, as shown in **Figure 6c**. In this case, the sequence of iterates does not satisfy the contractive condition (1) and, therefore, it does not converge to any of the best proximity points. As a consequence, the infinite switching case should be avoided in order to guarantee the convergence of the iterates, as commented in Remark 6 after Theorem 2.

Case 2: $T(x,y)$ is kept with the same above structure by replacing $T(0,0) = (2,1)$ with $(T(0,0))^1 = (2,1)$ and $T_a(x,y)$ being modified as follows:

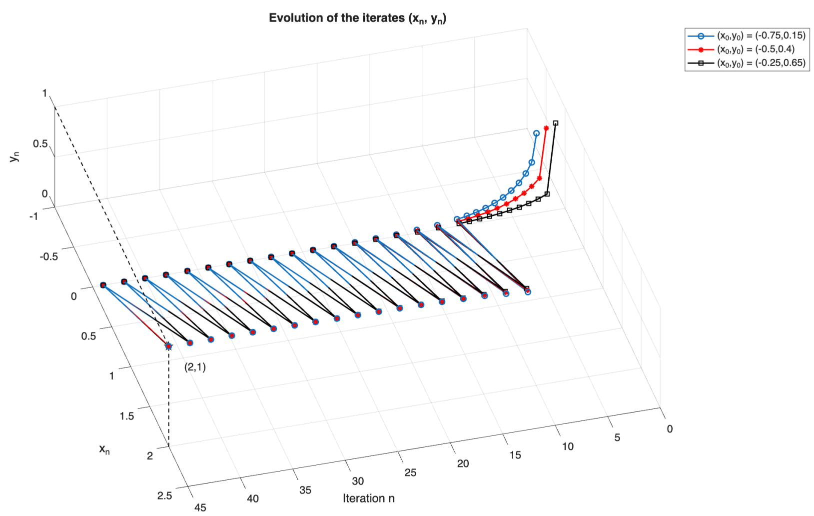
$$T_a(x,y) = \begin{cases} (0.8x+0.4,\ -0.8y-0.2)\ if\ \ (x,y)\in A_1 \\ (0.8x+0.4,\ -1.25y-0.25)\ if\ \ (x,y)\in A_2 \end{cases}$$

Now, the images of the best proximity points of A become:

$$\boldsymbol{T}\left(\{2,1\}\right) = \left\{\left(T(2,1)\right)^0,\ \left(T(2,1)\right)^1\right\} = \{(2,-1),\ \{0,0\}\} \in A_{20}\cup B_0 \subset A_0\cup B_0 \text{ for } \{2,1\}\in A_{10};$$

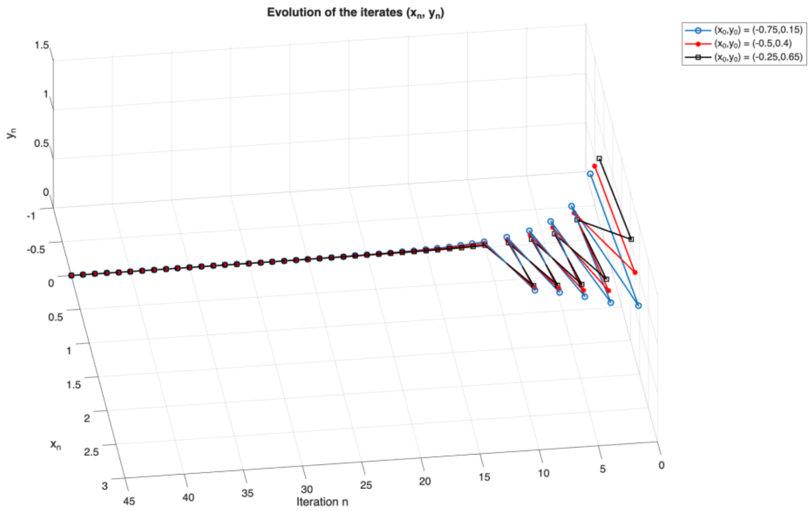
$$\boldsymbol{T}\left(\{2,-1\}\right) = \left\{\left(T(2,-1)\right)^0, \left(T(2,-1)\right)^1\right\} = \{(2,1),\{0,0\}) \in A_{10}\cup B_0 \in A_0\cup B_0\ \{2,-1\}\in A_{20}.$$

Then, the images in A of the best proximity points of A change from A_{10} to A_{20} and vice-versa. The fixed point is unique $Fix(T) = Fix\left(T|B\right) = B_0 = \{0,0\}$ in B since checking for fixed points in A_1 and A_2, it is found that they lie outside those sets so that they do not exist. This means that the intra mode for trapping sequences in each of the sets A and B in the non-cyclic operation mode is only feasible in B. This case is illustrated in **Figure 7**, which depicts the evolution of the iterates when σ takes the value zero (intra mode), corresponding to initial conditions in both sets *A* and *B*. **Figure 7a** shows the scenario when the system becomes trapped in *B*. The only possible fixed point is the value (0,0) and all trajectories with initial conditions in *B* converge to it. For the best proximity points (2, 1) and (2, −1) in A, the trajectories oscillate from one to the other, as depicted in **Figure 7b**.

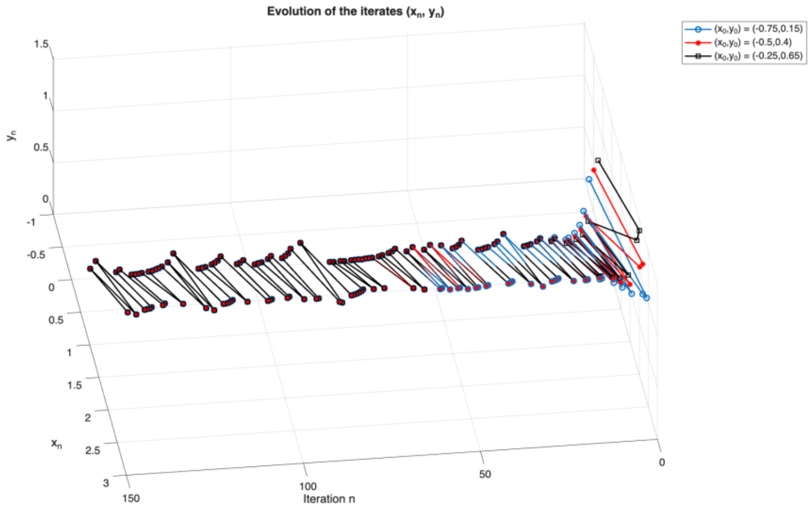


(a)

Figure 6. *Cont.*

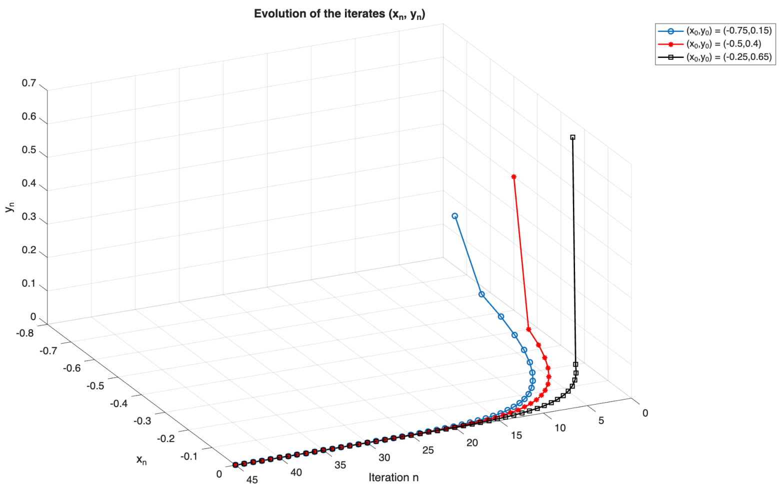


(b)



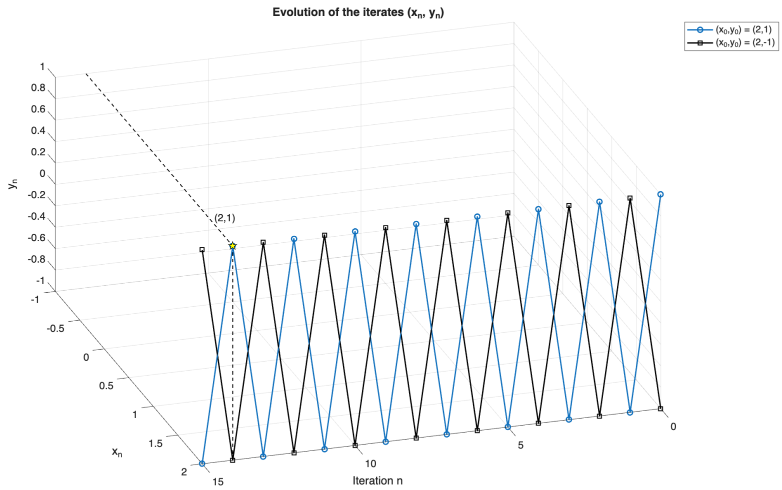
(c)

Figure 6. Evolution of the sequence of iterates for the set-valued operator T from Example 2-Case 1. **(a)** Scenario where σ takes the value zero for the first ten iterations and subsequently remains at one; **(b)** Scenario where σ takes the value one for the first ten iterations and subsequently remains at zero; **(c)** Scenario where σ takes random values between zero and one at each iteration.



(a)

Figure 7. *Cont.*



(b)

Figure 7. Evolution of the iterates when σ takes the value zero, corresponding to the intra mode, for the operator defined in Example 2-Case 2. **(a)** The initial conditions are in the set B and the only possible fixed point is (0, 0); **(b)** The initial conditions are (2, 1) and (2, 1) while the iterates oscillate between them.

As in Case 1, and since $\boldsymbol{T}(\{0,0\}) = \left\{(T(0,0))^0,\ (T(0,0))^1\right\} = \{(0,0),(2,1)\} \in B_0 \cup A_{10}$ the asymptotic cyclic mode can only involve the points $\{0,0\}$ and $\{2,1\}$, namely, all sequences in cyclic mode converge to limit cycles defined by the sequence of points $\{0,0\},\{2,1\},\{0,0\},\{2,1\}\ldots$ This situation is represented in **Figure 8**. In this case, σ takes the value zero, corresponding to the intra mode, up to iteration 10, and then takes the value one permanently, corresponding to the cyclic mode. **Figure 8** shows the iterates corresponding to three different initial conditions in sets A and B. In this scenario, all the iterates oscillate exclusively between the best proximity points (0,0) and (2,1) in the asymptotic cyclic mode, as predicted theoretically.

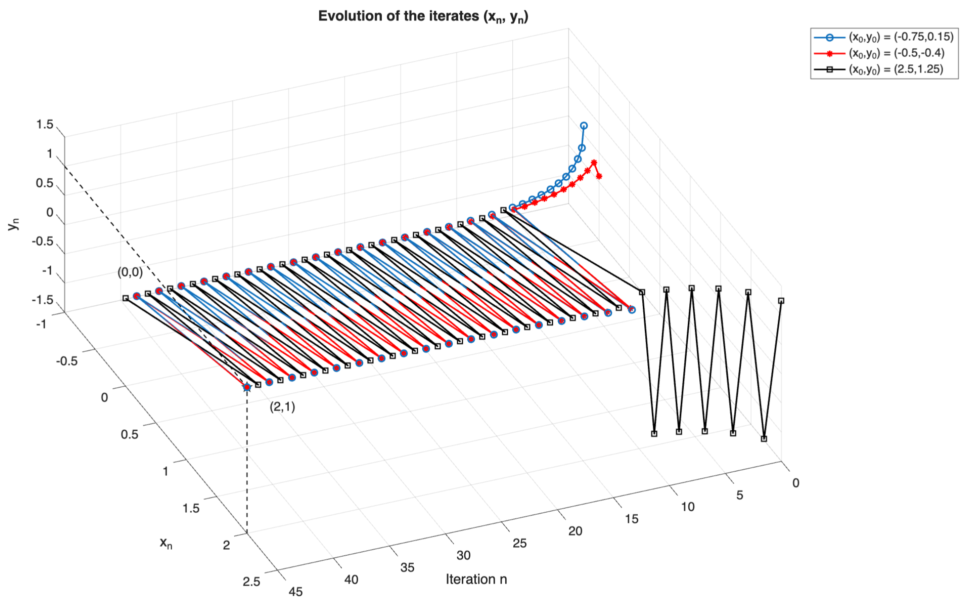


Figure 8. Evolution of the iterates for different initial conditions in the sets A and B, corresponding to the asymptotic cyclic behavior. It is observed that the oscillation involves only the best proximity points (0,0) and (2,1).

Case 3: $T(x, y)$ is kept with the same above structure with a modified $T_a(x, y)$ as follows:

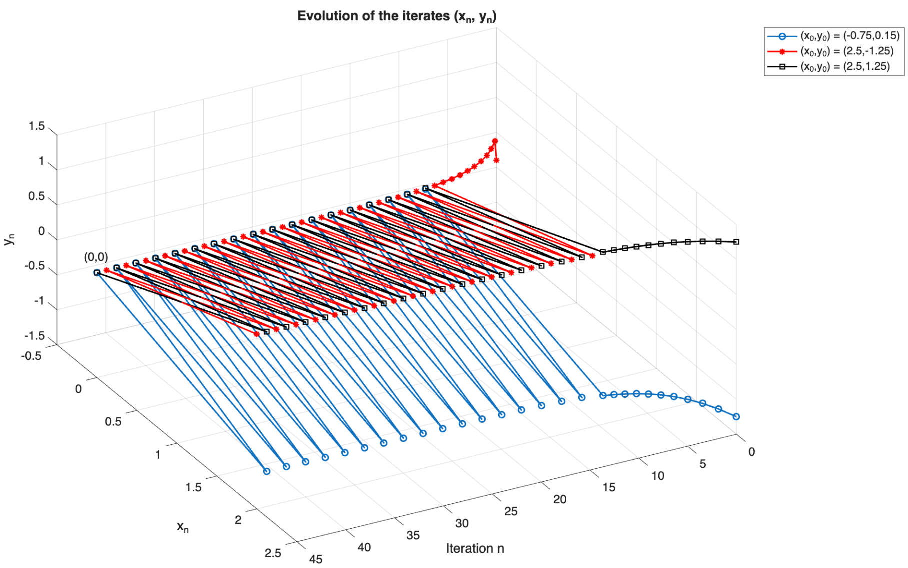
$$T_a(x,y) = \begin{cases} (0.8x + 0.4,\ 0.8y + 0.2)\ if\ \ (x,y) \in A_1 \\ (0.8x + 0.4,\ 0.8y - 0.2)\ if\ \ (x,y) \in A_2 \end{cases}$$

Now, the images of the best proximity points of A become:

$$\boldsymbol{T}\left(\{2,1\}\right) = \left\{(T(2,1))^0,\ (T(2,1))^1\right\} = \{(2,1),\ \{0,0\}\} \in A_{10} \cup B_0 \subset A_0 \cup B_0 \text{ for } \{2,1\} \in A_{10};$$

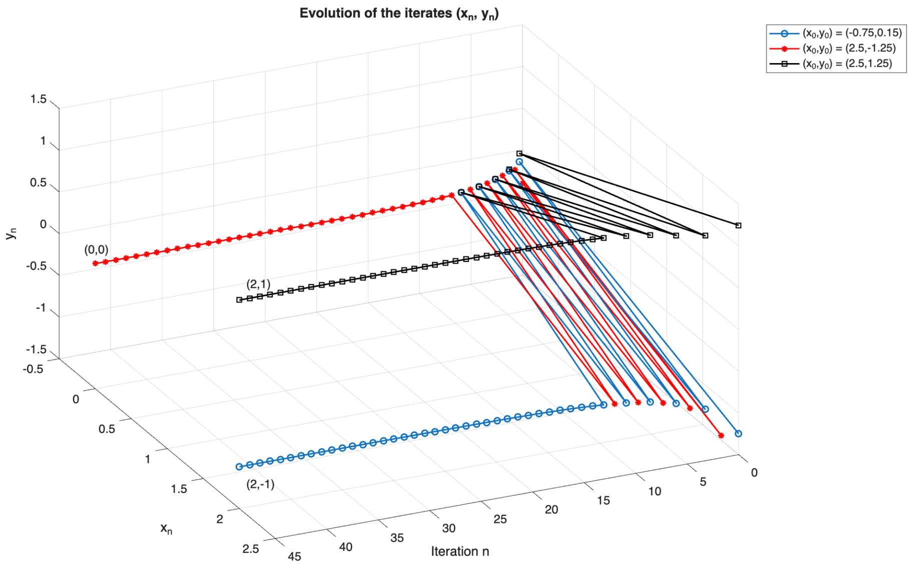
$$\boldsymbol{T}\left(\{2,-1\}\right) = \left\{(T(2,-1))^0, (T(2,-1))\right\} = \{(2,-1),\{0,0\}\} \in A_{20} \cup B_0 \in A_0 \cup B_0\ \{2,-1\} \in A_{20}.$$

Thus, the images in A of the best proximity points of A stay respectively in the same best proximity subsets of A_0, that is, A_{10} and A_{20}, as those where their pre-images are located. The fixed points are coincident with the best proximity points, that is, $Fix\,(T|B) = B_0 = \{0,0\}$ in B, $Fix\,(T|A_1) = Fix\,(T_a|A_1) = \{2,1\} = A_{10} \subset A_1$ and $Fix\,(T|A_2) = Fix\,(T_a|A_2) = \{2,-1\} = A_{20} \subset A_2$. The intra mode in this case is feasible in both subsets B and A and, in particular, in any of the connected components A_1 and A_2 of A. Since we have again $\boldsymbol{T}\left(\{0,0\}\right) = \left\{(T(0,0))^0,\ (T(0,0))^1\right\} = \{(0,0),(2,1)\} \in B_0 \cup A_{10}$ the cyclic mode operates asymptotically as in Case 2 and in Case 1. This situation is illustrated in **Figure 9**. Thus, **Figure 9a** displays the case when σ takes the value zero for the first ten iterations and subsequently remains at one permanently, corresponding to the asymptotic cyclic mode. The sequences of iterates oscillate between the best proximity points (0, 0) and (2, 1) or (0, 0) and (2, −1), as shown in **Figure 9a**. On the other hand, **Figure 9b** depicts the case when σ takes the value one for the first ten iterations and subsequently remains at zero permanently, corresponding to the asymptotic intra mode. The sequence of iterates converges in this case to the fixed point of each set, A or B, as observed in **Figure 9b**.



(a)

Figure 9. *Cont.*



(b)

Figure 9. Evolution of the iterates for the operator T from Example 2-Case 3 under different initial conditions: **(a)** σ takes the value zero for the first ten iterations and subsequently remains at one permanently. The sequences of iterates oscillate between the best proximity points of the sets A and B; **(b)** σ takes the value one for the first ten iterations and subsequently remains at zero permanently. The sequences of iterates converge to the fixed point of each set A and B.

5.3. Example 3

It can be pointed out that switched systems are receiving relevant attention in the background literature in multiple contexts like, for instance, in classical continuous-time systems, in discrete-time systems, in Markovian or in stochastic systems, etc. [20, 21, 28–31]. Their importance comes basically from the fact that many dynamic systems might have different operation points along their full evolution through time with potential switches to their respective domains of influence. Sometimes the parameterizations are distinct for the various modes and usually the state levels skip at the moment that switching is activated. This can cause instability behavior even if the various modes are individually stable.

This example is devoted to the application of the results established in Sections 3 and 4 to the stability analysis of a switched discrete-time dynamical system. To this end, consider the metric space (R^2, d) where d denotes the Euclidean distance in the plane. In this space, let us consider the four pairwise disjoint regions A_1, A_2, A_3 and A_4 shown in **Figure 10**. The region A_1 is defined as the circular sector with vertex at (a, a), where $a = \sqrt{2}$, radius equal to 10 units, and central angle ranging from 20 to 70 degrees. The regions A_2, A_3 and A_4 are obtained by applying three successive counterclockwise rotations of 90 degrees to A_1.

The best proximity points are located at the vertices of these regions and have coordinates (a, a), (−a, a), (−a, −a) and (a, −a) for A_1, A_2, A_3 and A_4, respectively. The distance between consecutive sets is $D = 2\sqrt{2} \approx 2.83$, as indicated in **Figure 10**. We define the discrete-time dynamical system by

$$X_k^T = [x_k \, , y_k] \, ; X_{k+1} = \widetilde{X}_k + \alpha \left(V\left(\widetilde{X}_k\right) - \widetilde{X}_k \right)$$

$$\widetilde{X}_k = R_{\sigma_k} X_k; R_{\sigma_k} = \begin{bmatrix} cos\left(\sigma_k \frac{\pi}{2}\right) & -sin\left(\sigma_k \frac{\pi}{2}\right) \\ sin\left(\sigma_k \frac{\pi}{2}\right) & cos\left(\sigma_k \frac{\pi}{2}\right) \end{bmatrix}$$

$$V\left(\widetilde{X}_k\right) = \begin{cases} [a,a]^T if\ \widetilde{x}_k > 0,\ \widetilde{y}_k > 0 \\ [-a,a]^T if\ \widetilde{x}_k < 0,\ \widetilde{y}_k > 0 \\ [-a,-a]^T if\ \widetilde{x}_k < 0,\ \widetilde{y}_k < 0 \\ [a,-a]^T if\ \widetilde{x}_k > 0,\ \widetilde{y}_k < 0 \end{cases}$$

where $\alpha \in (0,1)$ and $X_0 \in A_i$ for some i = 1, 2, 3, 4. From an intuitive point of view, this dynamical system has two operating modes. When it operates cyclically (σ equals one), the system rotates a point from one set to the next and moves it closer to the vertex of the new set. When it operates in the intra-set mode (that is, σ equals zero), the operator moves the point toward the vertex of the set to which it belongs without rotating it to the next set (that is, the sequence point remains within the same set). In both cases, the motion toward the vertex takes place along the ray joining the point to the corresponding vertex. This discrete dynamical system maps best proximity points to best proximity points. If $X_k = P_i$ and σ is unity then $\widetilde{X}_k = R_{\sigma_k} X_k = P_{i+1}$ and $V(P_{i+1}) - P_{i+1} = P_{i+1} - P_{i+1} = 0$ leading to $X_{k+1} = \widetilde{X}_k = P_{i+1}$. In addition, if σ is zero then $X_{k+1} = \widetilde{X}_k = X_k = P_i$. That is, the image of a best proximity point is a best proximity point.

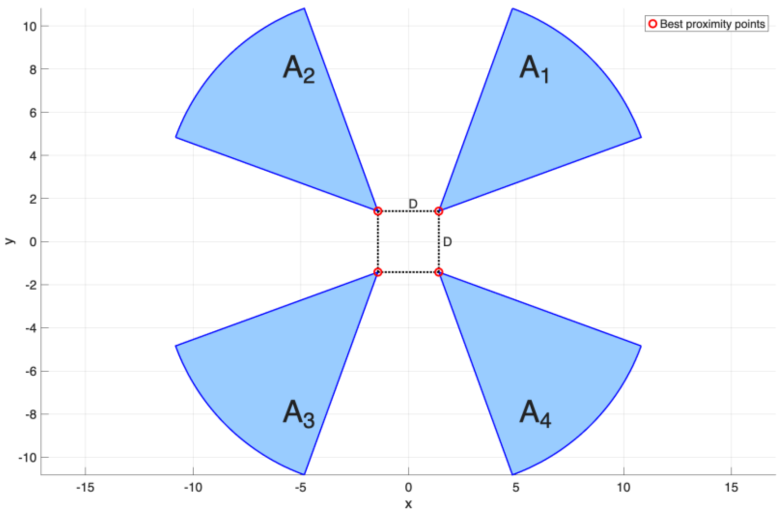


Figure 10. Representation of the pairwise disjoint sets A_1, A_2, A_3 and A_4 in *(R^2, d)*.

It can be verified that the operator defining the discrete-time dynamical system satisfies the contractive condition (1). To this end, consider two sequences $X_k = (x_{1k}, y_{1k})$ and $Y_k = (x_{2k}, y_{2k})$ at least one of them being different from a best proximity point evolving according to the discrete-time system under the same switching law, that is,

$$X_{k+1} = (1-\alpha)\widetilde{X}_k + \alpha V\left(\widetilde{X}_k\right)$$

$$Y_{k+1} = (1-\alpha)\widetilde{Y}_k + \alpha V\left(\widetilde{Y}_k\right)$$

By subtracting both expressions, we obtain:

$$X_{k+1} - Y_{k+1} = (1-\alpha)\left(\widetilde{X}_k - \widetilde{Y}_k\right) + \alpha\left(V\left(\widetilde{X}_k\right) - V\left(\widetilde{Y}_k\right)\right)$$

We take the Euclidean norms and apply the triangle inequality:

$$\|X_{k+1}-Y_{k+1}\|_2 \leq (1-\alpha)\left\|\widetilde{X}_k-\widetilde{Y}_k\right\|_2+\alpha\left\|V\left(\widetilde{X}_k\right)-V\left(\widetilde{Y}_k\right)\right\|_2$$

Since the matrices R_{σ_k} are orthogonal, they preserve the Euclidean norm, and thus, $\|X_k-Y_k\|_2=\left\|\widetilde{X}_k-\widetilde{Y}_k\right\|_2$ so that:

$$\|X_{k+1}-Y_{k+1}\|_2 \leq (1-\alpha)\|X_k-Y_k\|_2+\alpha\left\|V\left(\widetilde{X}_k\right)-V\left(\widetilde{Y}_k\right)\right\|_2$$

If X_k and Y_k belong to the same sector and one of them is distinct of a best proximity point then $V\left(\widetilde{X}_k\right)=V\left(\widetilde{Y}_k\right)$, and

$$\|X_{k+1}-Y_{k+1}\|_2 \leq (1-\alpha)\|X_k-Y_k\|_2 .$$

Notice that this equation holds because at least one of the points is not a best proximity point. Otherwise, there is no contraction as best proximity points are mapped into another ones. On the other hand, if X_k and Y_k belong to adjacent sets, and at least one of them is different from one of the vertices, we have

$$D=\left\|V\left(\widetilde{X}_k\right)-V\left(\widetilde{Y}_k\right)\right\|_2=\|V(X_k)-V(Y_k)\|_2 \leq \delta\|X_k-Y_k\|_2<\|X_k-Y_k\|_2$$

with $0<\delta=\delta(X_k,Y_k)\leq 1$ (it is unity only if both points are best proximity points) since the vertices are the best proximity points between adjacent sets and, therefore, the distance between any pair of points belonging to adjacent sets with one of them being different from a best proximity point is larger than the distance between sets. The value of δ depends on the specific location of points X_k and Y_k in the sets.

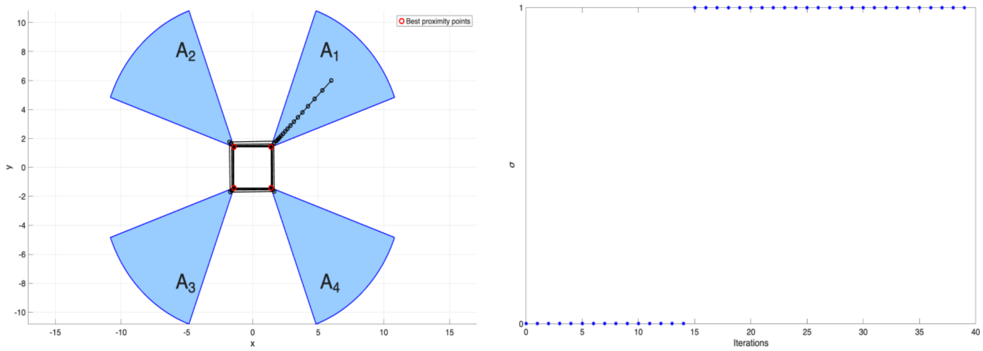
Therefore, when at least one of the points X_k or Y_k is different from a best proximity point:

$$\|X_{k+1}-Y_{k+1}\|_2 \leq (1-\alpha)\|X_k-Y_k\|_2+\alpha\delta\|X_k-Y_k\|_2=(1-\alpha+\alpha\delta)\|X_k-Y_k\|_2$$

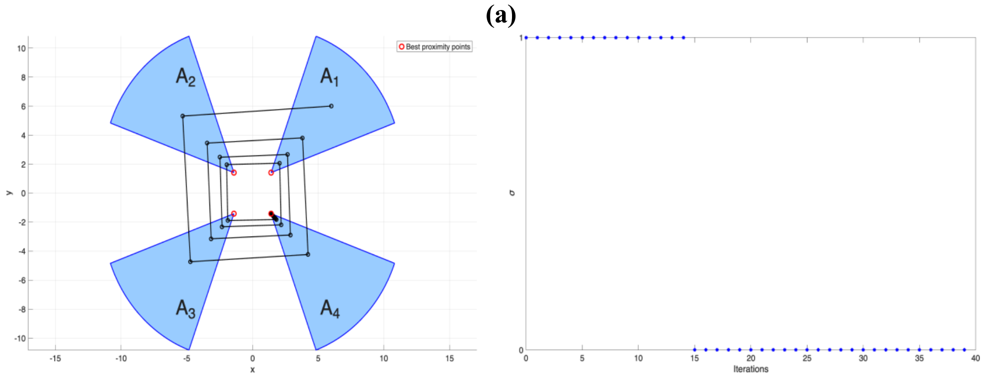
which verifies the condition (1) since

$$\|X_{k+1}-Y_{k+1}\|_2 \leq (1-\alpha+\alpha\delta)\|X_k-Y_k\|_2+(\alpha-\alpha\delta)\sigma_k D$$

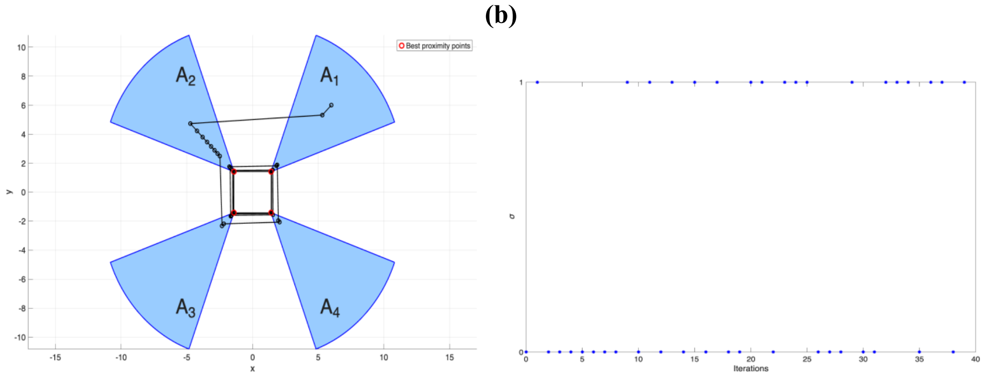
for $K=K(X_k,Y_k)=1-\alpha+\alpha\delta(X_k,Y_k)$. Therefore, Assertion 3 guarantees the existence of best proximity points in each set and Theorem 3 ensures that the sequences of points generated by the dynamical system converge to the best proximity points. In this way, the norm is monotonously decreasing until both sequences reach the best proximity points, moment at which the dynamic system makes the iterates either to oscillate from one proximity point to another one or to remain in it. Consequently, boundedness of the dynamical system and convergence of the trajectories to the best proximity points can be established via this result, highlighting the usefulness of fixed point tools in the stability analysis of dynamical systems. **Figure 11** illustrates the behavior of the dynamical system when $\alpha=0.15$ for different image selection sequences and initial condition (6, 6). In all cases shown in **Figure 11**, the iterates are observed to converge to the best proximity points independently of the value of σ, in accordance with Theorem 3. Moreover, Theorem 3 ensures that the iterates within each set form a Cauchy sequence whenever the image selection sequence remains constant from some finite index onward. Likewise, **Figure 12** shows the evolution of the dynamical system when σ permanently takes the value one, for two different initial conditions: in one case, both initial conditions belong to the same quadrant, whereas in the other case they belong to different quadrants. Alongside these trajectories, the evolution of $\|X_k - Y_k\|_2$ is also displayed. When both initial conditions lie in the same sector, the norm converges to zero; when they lie in adjacent sectors, the norm value converges to the separation distance between the sets.



(a)

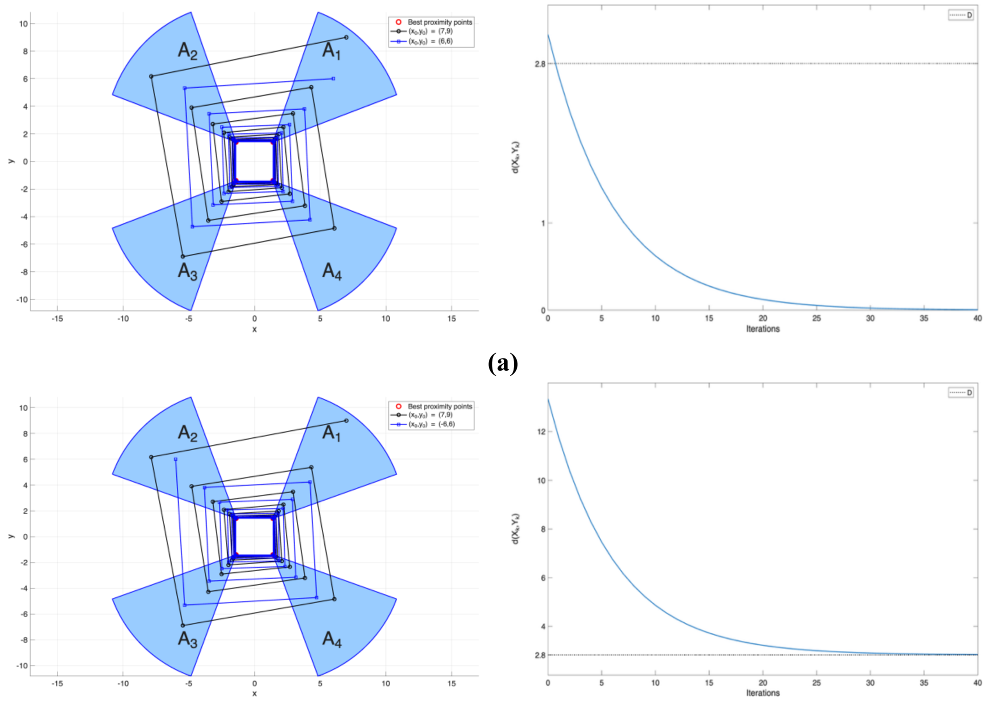


(b)



(c)

Figure 11. Evolution of the iterates for different values of σ: **(a)** the iteration converges to the cycle formed by the best proximity points after an initial intra mode evolution; **(b)** the iteration converges to the best proximity point of A_4 after an initial cyclic evolution; **(c)** the iterates evolve in either the intra mode or the cyclic mode depending on the random value taken by the signal σ.



(a)

(b)

Figure 12. Evolution of the trajectories of two distinct points X_k and Y_k when $\sigma = 1$: **(a)** both initial points lie in the same sector; **(b)** the initial points lie in adjacent sectors.

It can be pointed out that this problem exhibits global stability (that is, boundendness of all solution sequences) in both cyclic mode, intra mode and mixed operation between intra mode and cyclic mode. In some cases, global asymptotic stability is also achieved (that is, boundedness with additional convergence to the equilibrium point), as for instance, in the case of intersecting subsets and also in that of non-intersecting ones with final trapped intra mode operation.

It can be also pointed out that one could potentially explore AI-enhanced mathematical derivation and multi-modal neural symbolic reasoning algorithms to establish automatically derived convergence theorems and optimal approximate convergence theorems when applying the above formal results to dynamic systems. These strategies have been successfully tried on in the solutions of certain non-trivial differential systems. In that context, it can be pointed out that some AI techniques were used in Zhao and Zhang [32] to derive a solution of Benjamin-Bona-Mahony equation avoiding numerical approaches to find an approximate solution. In Wang et al. [33], a disposal of fractional sub-equation neural networks were first proposed to construct exact solutions of space-time fractional partial differential equations while a neuro-symbolic intelligent algorithm was used in Zhang et al. [34] for solving the Boiti-Leon-Manna-Pempinelli equation about the motion of waves in fluids.

6. Conclusions and potential future related work

This work has introduced and formalized a new class of so-called hybrid cyclic/non-cyclic bivalued contractive self-mappings that accommodate dynamic switches within defined subsets of metric spaces. The contractive mappings are defined on the union on a set of subsets of a metric space which are assumed disjoint in the most general case. Such mappings may operate either in the standard cyclic mode in the sense that the next iteration in a sequence of the bivariate mapping is selected in the adjacent subset, or in the non-cyclic one, such that the next iteration is selected within the current subset. An image selection sequence parameterizes the contractive condition iteration by iteration which makes possible to select one of the two above mentioned operation modes as the image of each current point for the next iteration. Key assumptions invoked in the performed study are that the subsets of the metric spaces which define the cyclic disposal are non-empty and closed, non-necessarily bounded or intersecting, that at least one best proximity set in one of the subsets is a singleton, and that such a subset itself is boundedly compact.

The contractive condition, which includes the mentioned image selection sequence, operates (as a result of its own definition) in such a way that it is not possible to switch from the trapped intra mode mode to the cyclic mode once the distance between points along the iteration lies below the distance in-between adjacent subsets provided that the compared points are already both in the same subset. In this way, if distances between compared points in the sequence of distances exceed the threshold defined by the distance between adjacent subsets then the whole sequence can operate in both cyclic and non-cyclic modes while if the distance between points lie below the threshold defined by the distance between adjacent subsets, the iteration remains trapped for future iterations in the current subset of the cyclic disposal. A consequence of that property is that the sequence of distances can only switch from the non-cyclic mode to the cyclic one only after a finite number of switches when it is staying in the non-cyclic mode. In the case that it remains operating in the non-cyclic mode the sequences converge to a fixed point while if they stay in cyclic mode, the whole trajectory solution converges to a limit cyclic formed by best proximity points, one per subset of the cyclic disposal.

Three worked examples have been also discussed. One of them is concerned with an application to the stability and convergence of the trajectory solution of a dynamic discrete system to a limit cycle in the case when the subsets of the cyclic disposal do not intersect. This leads to a scenario where traditional fixed points do not exist due to the non-intersecting sets which define constraints on the state trajectory solutions. Also, the trajectory solution is allowed to operate in intra mode in some of the subsets of the state confinement along certain iterations.

On the other hand, and based on the role of the image selection sequence, it is foreseen a broader modelling applicability of potential usefulness in applied contexts. In this way, the proposed methodology offers a foundation which is potentially extendable in the future to evaluate a long-term dynamic behavior in typical switched systems, for instance, in those with time delays appearing in nature as well as in complex higher-order non-linear biological models or in industrial processes with several sequentially alternated operation modes.

In the theoretical context, it is also foreseen to try to build a theoretical theorem which, combined with neural network numerical verification mixed analysis framework, could be used for formal derivations of proofs in complex high-dimensional partitioned mappings defined in metric spaces. Also, one can draw on AI-enhanced mathematical derivation and multi-modal neural symbolic reasoning algorithm, so as to achieve an automatic derivation of convergence theorems and optimal nearest point conditions while potentially reducing the workload of manual proof and of eventual use to verify the manual derivations. Furthermore, and combined with the neural network symbolic calculation method, that strategy might automatically solve the explicit optimal nearest point and limit cycle analytical expressions of complex mappings, especially, in the formal contexts of differential equations and dynamic systems.

Author contributions: Conceptualization, MDlS and AI; methodology, MDlS, AI and HAH; software, AI; validation, AI and HAH; formal analysis, MDlS; investigation, MDlS, AI and HAH; resources, MDlS; data curation, AI; writing—original draft preparation, MDlS and AI; writing—review and editing, MS, AI and HAH; visualization, AI; supervision, AI and HAH; project administration, MDlS; funding acquisition, MDlS. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no funding.

Institutional review board statement: Not applicable.

Informed consent statement: Not applicable.

Data availability statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflict of interest: The authors declare no conflicts of interest.

AI use statement: The authors declare that no artificial intelligence (AI) tools were used in the preparation of this manuscript.

References

1. Eldred AA, Veeramani P. Existence and convergence of best proximity points. Journal of Mathematical Analysis and Applications. 2006; 323(2): 1001–1006. doi: 10.1016/j.jmaa.2005.10.081
2. Karpagam S, Agrawal S. Best Proximity Point Theorems for p-Cyclic Meir-Keeler Contractions. Fixed Point Theory and Applications. 2009; 2009(1): 197308. doi: 10.1155/2009/197308
3. Karpagam S, Agrawal S. Existence of best proximity points of p-cyclic contractions. Fixed Point Theory. 2012; 13(1): 99–105. Available online: https://www.math.ubbcluj.ro/~nodeacj/download.php?f=121Karpagam-final.pdf
4. De La Sen M, Singh SL, Gordji ME, et al. Best proximity and fixed point results for cyclic multivalued mappings under a generalized contractive condition. Fixed Point Theory and Applications. 2013; 2013(1): 324. doi: 10.1186/1687-1812-2013-324
5. Weng S, Xiao H. Related Fixed Point Results for LB_3S_2-Type Cyclic Mapping in Extended b-Metric Spaces With an Application. Journal of Mathematics. 2025; 2025(1): 8867702. doi: 10.1155/jom/8867702
6. Ali WF, Ahmad BAA. Best Proximity Point for Some Type of Cyclic Mappingin Strong b̲-Metric Space. Iraqi Journal of Science. 2025; 1996–2002. doi: 10.24996/ijs.2025.66.5.18
7. Wu D, Weng S. Results for ϕ_δ-type cyclic mapping on extended b-metric space. Journal of Physics: Conference Series. 2025; 2964(1): 012082. doi: 10.1088/1742-6596/2964/1/012082
8. Gabeleh M, Uyanık Ekici E, Aphane M. Best Proximity Theory in Metrically Convex Menger PM-Spaces via Cyclic Kannan Maps. Symmetry. 2025; 17(9): 1549. doi: 10.3390/sym17091549
9. Hafshejani AS. On the error estimates for the sequence of successive approximations for cyclic φ–contractions in

metric spaces. Carpathian Journal of Mathematics. 2024; 41(1): 193–204. doi: 10.37193/CJM.2025.01.13
10. Ghosh S, Bhardwaj R, Narayan S, et al. A result concerning best proximity point from the perspective of G-metric spaces using control functions with an application. Advances in Mathematical Sciences and Application. 2025; 34(1): 33–46. Available online: https://www.researchgate.net/publication/391346551_A_Result_Concerning_Best_Proximity_Point_from_the_perspective_of_G-Metric_Spaces_using_Control_Functions_with_an_Application_Adv_Math_Sci_Appl34133-46ISSN_1343-4373
11. Unni AS, Pragadeeswarar V. Proximal iterated function systems using cyclic Meir-Keeler contractions and an application to fractal theory. Fixed Point Theory and Algorithms for Sciences and Engineering. 2025; 2025(1): 3. doi: 10.1186/s13663-025-00784-7
12. Di Bari C, Suzuki T, Vetro C. Best proximity points for cyclic Meir–Keeler contractions. Nonlinear Analysis: Theory, Methods & Applications. 2008; 69(11): 3790–3794. doi: 10.1016/j.na.2007.10.014
13. Pasupathi R, Chand AKB, Navascués MA. Cyclic Meir-Keeler Contraction and Its Fractals. Numerical Functional Analysis and Optimization. 2021; 42(9): 1053–1072. doi: 10.1080/01630563.2021.1937215
14. Pasupathi R, Chand AKB, Navascués MA. Cyclic iterated function systems. Journal of Fixed Point Theory and Applications. 2020; 22(3): 58. doi: 10.1007/s11784-020-00790-9
15. Ishtiaq T, Batool A. Approximating Fixed Points of Generalized Cyclic Enriched Contraction Mapping Using Ishi Iteration Scheme with Application. Global Integrated Mathematics. 2026; 2(1): 28–40. doi: 10.64229/e7pt2x51
16. Saluja GS. Some fixed point theorems via cyclic contractive conditions in s-metric spaces. Facta Universitatis, Series: Mathematics and Informatics. 2021; 377. doi: 10.22190/FUMI200811028S
17. Gupta A. Cyclic contraction on an S-metric space. International Journal of Analysis and Applications. 2013; 3(2): 119–130. Available online: https://www.etamaths.com/index.php/ijaa/article/view/84
18. Chaipornjareansri S. Some best proximity point results for MT-rational cyclic contractions in S- metric space. Communications in Mathematics and Applications. 2020; 11(4): 587–600. Available online: https://rgnpublications.com/journals/index.php/cma/article/download/1413/1123
19. Rada E. Novel Fixed Point Theorems and Stability Analysis for Cyclic Contractions in b-G-metric Spaces via λ-Iteration. European Journal of Pure and Applied Mathematics. 2026; 19(1): 6907. doi: 10.29020/nybg.ejpam.v19i1.6907
20. De La Sen M, Ibeas A. Stability Results for Switched Linear Systems with Constant Discrete Delays. Mathematical Problems in Engineering. 2008; 2008(1): 543145. doi: 10.1155/2008/543145
21. Ibeas A, De La Sen M. Exponential stability of simultaneously triangularizable switched systems with explicit calculation of a common Lyapunov function. Applied Mathematics Letters. 2009; 22(10): 1549–1555. doi: 10.1016/j.aml.2009.03.023
22. Kulenović MRS, Sullivan R. Basins of Attraction for Third-Order Sigmoid Beverton Holt Difference Equation. Mathematics. 2025; 13(18): 2990. doi: 10.3390/math13182990
23. Kessy JA, Wangwe L. On a fixed point theorem of Karapinar for interpolative contractions. Research in Mathematics. 2025; 12(1): 2526218. doi: 10.1080/27684830.2025.2526218
24. Abbas A, Ali A, Al Sulami H, et al. Recent Advancements in KRH-Interpolative-Type Contractions. Symmetry. 2023; 15(8): 1515. doi: 10.3390/sym15081515
25. Devi D, Debnath P. Fixed points of two interpolative cyclic contractions in b-metric spaces. Heliyon. 2025; 11(1): e41667. doi: 10.1016/j.heliyon.2025.e41667
26. Nazam M, Aydi H, Hussain A. Existence theorems for (Ψ,Φ)-orthogonal interpolative contractions and an application to fractional differential equations. Optimization. 2023; 72(7): 1899–1929. doi: 10.1080/02331934.2022.2043858
27. Tomar A, Rana US, Kumar V. Fixed point, its geometry, and application via ω-interpolative contraction of Suzuki type mapping. Mathematical Methods in the Applied Sciences. 2024; 47(5): 3507–3528. doi: 10.1002/mma.8871
28. Zeng HB, Chen YJ, He Y, et al. A delay-derivative-dependent switched system model method for stability analysis of linear systems with time-varying delay. Automatica. 2025; 175: 112183. doi: 10.1016/j.automatica.2025.112183
29. Xu L, Bao B, Hu H. Stability of impulsive delayed switched systems with conformable fractional-order derivatives. International Journal of Systems Science. 2025; 56(6): 1271–1288. doi: 10.1080/00207721.2024.2421454
30. Liu S, Zhao N, Zhang L, et al. Adaptive neural hierarchical sliding mode control for uncertain switched underactuated nonlinear systems against unmodeled dynamics and input delay. Asian Journal of Control. 2025; 27(3): 1552–1569. doi: 10.1002/asjc.3528
31. Chen H, Zong G, Shen M, et al. Finite-Time Resilient Control of Networked Markov Switched Nonlinear Systems: A Relaxed Design. IEEE Transactions on Systems, Man, and Cybernetics: Systems. 2025; 55(4): 2569–2579. doi:

10.1109/TSMC.2025.3526321

32. Zhao ZL, Zhang RF. Artificial Intelligence-Enhanced Mathematical Derivation method: Exact solutions of the Benjamin–Bona–Mahony equation. Engineering Applications of Artificial Intelligence. 2026; 173: 114464. doi: 10.1016/j.engappai.2026.114464
33. Wang J, Liu Y, Yan L, et al. Fractional sub-equation neural networks (fSENNs) method for exact solutions of space–time fractional partial differential equations. Chaos: An Interdisciplinary Journal of Nonlinear Science. 2025; 35(4): 043110. doi: 10.1063/5.0259937
34. Zhang H, Zhang R, Liu Q. A Novel Multi-Modal Neurosymbolic Reasoning Intelligent Algorithm for BLMP Equation. Chinese Physics Letters. 2025; 42(10): 100002. doi: 10.1088/0256-307X/42/10/100002