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Article

On numerical radius inequalities via McCarthy inequality

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Abstract: This paper aims to establish new inequalities for the numerical radius of bounded linear operators defined on a complex Hilbert space. By employing a generalized form of the McCarthy inequality, we derive several upper bounds for the numerical radius of a single operator as well as for expressions involving sums and products of operators. The obtained results extend and refine a number of existing inequalities in the literature. In particular, many known numerical radius inequalities are recovered as special cases of our results, thereby providing a unified framework for their analysis. The refinement is based on the Akkouchi-Ighachane version of the Hölder-McCarthy inequality, which allows the usual McCarthy term to be replaced by a smaller parameter-dependent expression before taking the supremum. This gives a common refinement mechanism for estimates involving a single operator, Cartesian decompositions, finite sums, block matrices and the Euclidean operator radius. We also state explicitly the parameter choices which recover the earlier inequalities and include simple finite-dimensional comparisons showing that, for suitable non-normal matrices, the refined bounds may be strictly sharper than the corresponding classical estimates. These comparisons confirm that the refinement provides a usable quantitative improvement, rather than only a formal parameter extension of existing bounds in concrete cases.

Keywords: numerical radius; McCarthy inequality; operator inequalities; Hilbert space; refinement

1. Introduction

Let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the associated norm $\|\cdot\|$. We denote by $\mathcal{B}(H)$ the C^*-algebra of all bounded linear operators acting on H. For an operator $\mathcal{A}\in\mathcal{B}(H)$, the absolute value of $\mathcal{A}$ is defined by

$$|\mathcal{A}| = (\mathcal{A}^*\mathcal{A})^{\frac{1}{2}},$$

where $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$.

The operator norm of $\mathcal{A}\in\mathcal{B}(H)$ is given by

$$\|\mathcal{A}\| = \sup_{\|\mu\|=1} \|\mathcal{A}\mu\|.$$

It is well known that $\mathcal{B}(H)$, equipped with this norm and the involution $\mathcal{A}\mapsto\mathcal{A}^*$, forms a C^*-algebra, which constitutes a fundamental structure in functional analysis and operator theory. For $\mathcal{A} \in \mathcal{B}(H)$, the numerical range of $\mathcal{A}$ is defined by

$$W(\mathcal{A}) = \{\langle \mathcal{A}\mu, \mu \rangle : \mu \in H, \ \|\mu\| = 1\},$$

while the numerical radius is defined as

$$w(\mathcal{A}) = \sup\{|\langle \mathcal{A}\mu, \mu \rangle| : \mu \in H, \ \|\mu\| = 1\}.$$

The numerical radius defines a norm on $\mathcal{B}(H)$ equivalent to the operator norm and satisfies the classical inequality

$$\frac{1}{2}\|\mathcal{A}\| \leq w(\mathcal{A}) \leq \|\mathcal{A}\|.$$

The numerical radius and its properties have been extensively studied due to their importance in operator theory and matrix analysis (see, for example, Kittaneh [1, 2] and Gustafson and Rao [3]).

El-Haddad and Kittaneh [4] developed the following inequalities for an operator as

$$w^{\gamma}(\mathcal{A}) \leq \frac{1}{2}\left\| |\mathcal{A}|^{2\alpha\gamma} + |\mathcal{A}^*|^{2(1-\alpha)\gamma} \right\|, \tag{1}$$

and

$$w^{2\gamma}(\mathcal{A}) \leq \left\| \alpha|\mathcal{A}|^{2\gamma} + (1-\alpha)|\mathcal{A}^*|^{2\gamma} \right\|, \tag{2}$$

where $0 < \alpha < 1$ and $\gamma \geq 1$. And the same authors also developed an inequality

$$w^{\gamma}(\mathcal{A}) \leq 2^{\frac{\gamma}{2}-1} \left\| |\mathcal{B}|^{\gamma} + |\mathcal{C}|^{\gamma} \right\|, \tag{3}$$

where $\mathcal{A} = \mathcal{B} + i\mathcal{C}$, and $\gamma \geq 1$.

Al-Hawari [5] developed an inequality

$$w^{\gamma}(\mathcal{A}) \leq \frac{1}{4}\left(\left\| |\mathcal{A} + \mathcal{A}^*|^{\gamma} + |\mathcal{A} - \mathcal{A}^*|^{\gamma} \right\| \right), \tag{4}$$

where $\gamma \geq 2$, $\mathcal{A} \in M_{\eta}\mathbb{C}$ and $M_{\eta}\mathbb{C}$ denotes the collection of all $\eta \times \eta$ matrices with entries in $\mathbb{C}$.

Alomari [6] developed an inequality for the numerical radius

$$\begin{aligned} w^{2\gamma}(\mathcal{A}) \leq{}& \frac{1}{2}\beta \left\| |\mathcal{A}|^{4\gamma\alpha} + |\mathcal{A}^*|^{4\gamma(1-\alpha)} \right\| \\ &+ \frac{1}{2}(1-\beta) w^{\gamma}(\mathcal{A}) \left\| |\mathcal{A}|^{2\gamma\alpha} + |\mathcal{A}^*|^{2(1-\alpha)\gamma} \right\|, \end{aligned} \tag{5}$$

where $\gamma \geq 1$ and $0 \leq \alpha, \beta \leq 1$.

Mirmostafaee et al. [7] develop the inequality for an operator as

$$w^{\gamma}\left(\sum_{i=1}^{\eta} \mathcal{A}_i\right) \leq \eta^{\gamma-1} 2^{\frac{\gamma}{2}-1} \sum_{i=1}^{\eta} \left\| |\mathcal{B}_i + \mathcal{C}_i|^{2\gamma} + |\mathcal{B}_i - \mathcal{C}_i|^{2\gamma} \right\|^{\frac{1}{2}}, \tag{6}$$

where $\gamma \geq 1$.

Burqan and Abu-Rahma [8] developed an inequality

$$w^{\gamma}(B) \leq \frac{1}{2} \left\| \mathcal{A}^{\gamma} + \mathcal{C}^{\gamma} \right\|, \tag{7}$$

for $\gamma \geq 1$.

Hamadneh et al. [9] developed an inequality

$$\begin{aligned} w_e^2(\mathcal{A}_1, \cdots, \mathcal{A}_\eta) \leq \frac{1}{2}\beta \left\| \sum_{i=1}^{\eta} \left(|\mathcal{A}_i|^{4\alpha} + |\mathcal{A}_i^*|^{4(1-\alpha)} \right) \right\| \\ + \frac{1}{2}(1-\beta) w_e(\mathcal{A}_1, \cdots, \mathcal{A}_\eta) \left\| \sum_{i=1}^{\eta} \left(|\mathcal{A}_i|^{2\alpha} + |\mathcal{A}_i^*|^{2(1-\alpha)} \right)^2 \right\|^{\frac{1}{2}}. \end{aligned} \tag{8}$$

Recent developments illustrate several complementary directions in the study of numerical-radius inequalities. Convexity and inner-product techniques have led to sharper estimates for Hilbert-space operators [10–12], while refined Cauchy–Schwarz-type arguments have produced related improvements [13–15]. Further bounds have been obtained for single operators, finite sums and operator matrices [16–19], including 2×2 block matrices and semi-Hilbertian variants [20–23], with additional recent contributions in the literature [24–26]. These works provide the recent context for the present McCarthy-based refinement approach.

This paper establishes refined inequalities for the numerical radius of bounded linear operators on a complex Hilbert space. In particular, we provide extensions of several known inequalities, including Equations (1)–(8). The word extension is used here in the precise sense that the known estimates are recovered for special choices of the refinement parameters.

The present paper should be read as a refinement paper rather than as an assertion that one formula is uniformly best for every operator. A numerical-radius estimate normally depends on the preliminary inequality from which it is derived, the class of operators under consideration and the parameters entering the estimate. Two bounds can therefore be useful in different situations even when both have the same left-hand side. Our purpose is to retain the structure of the known estimates in Equations (1)–(8), while replacing the McCarthy step used in their derivations by the refined form given in Lemma 4.

The basic point is pointwise. If a positive operator X occurs in a proof, the classical McCarthy inequality replaces $\langle X\mu, \mu\rangle^{\gamma}$ by $\langle X^{\gamma}\mu, \mu\rangle$. Lemma 4 inserts an intermediate quantity between these two expressions. The intermediate quantity still depends only on positive powers of X and on the unit vector μ, but it retains information which is lost when one passes directly to $\langle X^{\gamma}\mu, \mu\rangle$. The subsequent numerical-radius inequalities are obtained by applying this replacement at the appropriate positive-operator terms of the original arguments. In this sense, the common mechanism is simple, but its use across the different operator settings gives a family of estimates with distinct hypotheses and distinct output forms.

There are two separate questions in interpreting the results. The first is the recovery question: for the parameter values specified after each theorem, the corresponding known inequality is obtained. This shows that the theorem is compatible with the earlier result. The second is the improvement question: whether the additional terms yield a smaller numerical upper bound for a particular operator. Recovery alone does not answer the second question. For this reason, Section 4 contains finite-dimensional comparisons. They show strict improvement for two non-normal matrices, and they also include a diagonal example in which equality occurs. Thus, no universal strict-improvement claim is made.

The contributions of the paper may be summarized as follows. First, Proposition 1 isolates the refined Holder–McCarthy step used throughout the proofs. Second, the main theorems apply this step to several different settings: a single operator, weighted estimates, Cartesian decompositions, a Clarkson-type estimate, a finite sum of operators, positive block matrices and the Euclidean operator radius. Third, the remarks identify the parameter choices that lead back to the known estimates. Finally, the numerical section and the discussion following the main results explain the relation between the theorem families and the practical evaluation of the new bounds.

The paper is organized accordingly. After the preliminary material, the main results are presented together with their recovery remarks. A separate discussion then explains why the theorem families are complementary rather than interchangeable. The numerical comparisons are included before the conclusion in order to keep the theoretical statements and their finite-dimensional interpretation close to one another.

It is also useful to indicate what is not changed by this procedure. The original preliminary inequalities remain the starting point of the respective proofs. The operator expressions, the power-mean step, the weighted mean step, the Cartesian decomposition, the Bohr inequality and the block-matrix assumptions are not replaced by new assumptions. The only additional operation is the use of the intermediate refined expression at a positive-operator stage. If the upper end of the two-sided estimate in Proposition 1 is used instead, the argument returns to the classical McCarthy step. This is the direct reason why the earlier inequalities reappear under the stated specialisations.

Accordingly, the novelty of the paper is not the introduction of a new definition of numerical radius. It is the systematic insertion of a verified Holder–McCarthy refinement into several established numerical-radius routes. The results are presented separately because the initial routes have different assumptions and produce different operator quantities. The numerical section is included to show that the intermediate step can have a numerical effect after the final supremum and norm operations have been carried out.

2. Main results

In this section, we present our main contributions. The results are obtained by refining the McCarthy inequality and applying it to numerical radius inequalities involving single operators. Before establishing our main theorems, we begin with some essential auxiliary results that play a key role in the proofs that follow. Lemma 1 provides a fundamental estimate connecting the inner product of operators with the powers of their modulus operators. It will serve as a crucial tool in deriving our forthcoming results.

Lemma 1 (Generalized mixed Schwarz inequality [4]). *Let $\mathcal{A} \in \mathcal{B}(H)$, then*

$$|\langle \mathcal{A}\mu, \nu\rangle|^2 \leq \langle |\mathcal{A}|^{2\alpha}\mu, \mu\rangle\langle |\mathcal{A}^*|^{2(1-\alpha)}\nu, \nu\rangle$$

for all $\mu, \nu \in H$ and $0 \leq \alpha \leq 1$.

Lemma 2 plays a key role in refining the upcoming numerical radius bounds.

Lemma 2 (Generalized mean inequality [4]). *For $a, b \geq 0$, $0 < \alpha < 1$, and $\gamma \neq 1$, let $M_\gamma(a, b, \alpha) = (\alpha a^\gamma + (1-\alpha)b^\gamma)^{\frac{1}{\gamma}}$ and let $M_0(a, b, \alpha) = a^\alpha b^{1-\alpha}$. Then*

$$M_\gamma(a, b, \alpha) \leq M_s(a, b, \alpha)$$

for $\gamma \leq s$.

Lemma 3 is a well-known result that follows from the spectral theorem for positive operators together with Jensen's inequality.

Lemma 3 (Hölder-McCarthy inequality [27]). *Let $\mathcal{A} \in \mathcal{B}(H)$ and let $\mu \in H$ be a unit vector. Then*

$$\langle \mathcal{A}\mu, \mu\rangle^\gamma \leq \langle \mathcal{A}^\gamma\mu, \mu\rangle \quad \textit{for } \gamma \geq 1,$$

and

$$\langle \mathcal{A}\mu, \mu\rangle^\gamma \geq \langle \mathcal{A}^\gamma\mu, \mu\rangle \quad \textit{for } 0 < \gamma \leq 1.$$

In what follows, only the case $\gamma \geq 1$ of Lemma 3 is used. Whenever a special case such as $\gamma = 2$ is mentioned in a remark, it is this first part of the lemma that is being invoked.

Next, we present a refined version in Lemma 4, which serves as an essential tool for establishing sharper numerical radius bounds.

Lemma 4 (Refined Hölder-McCarthy-type inequalities [28]). *Let $\mathcal{A}$ be a positive operator on a Hilbert space H. Then, for any $\mu \in H$ and a given positive real number $\gamma \geq 1$, and $\eta \geq 2$ we have:*

$$\begin{aligned}\langle \mathcal{A}\mu, \mu\rangle^\gamma \|\mu\|^{2(1-\gamma)} &\leq \left(\left(\frac{1}{\gamma}\right)^\eta + \left(\frac{\gamma-1}{\gamma}\right)^\eta\right)\langle \mathcal{A}^\gamma\mu, \mu\rangle \\ &\quad + \sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^\kappa\left(\frac{\gamma-1}{\gamma}\right)^{(\eta-\kappa)}\langle \mathcal{A}^\gamma\mu, \mu\rangle^{1-\frac{\kappa}{\eta}\gamma}\|\mu\|^{-2\left(\frac{\eta-\kappa}{\eta}\gamma\right)}\langle \mathcal{A}^{\frac{\kappa}{\eta}\gamma}\mu, \mu\rangle^\gamma \\ &\leq \langle \mathcal{A}^\gamma\mu, \mu\rangle.\end{aligned}$$

Proposition 1 (Refinement principle). *Let X be a positive operator on H, let $\gamma \geq 1$ and let $\eta \geq 2$ be an integer. For every unit vector $\mu \in H$, define*

$$R_{\gamma,\eta}(X;\mu) = \left(\left(\tfrac{1}{\gamma}\right)^\eta + \left(\tfrac{\gamma-1}{\gamma}\right)^\eta\right)\langle X^\gamma\mu, \mu\rangle + \textstyle\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\tfrac{1}{\gamma}\right)^\kappa\left(\tfrac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\langle X^\gamma\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle X^{\frac{\kappa\gamma}{\eta}}\mu, \mu\rangle^\gamma.$$

Then

$$\langle X\mu, \mu\rangle^\gamma \leq R_{\gamma,\eta}(X;\mu) \leq \langle X^\gamma\mu, \mu\rangle.$$

Consequently, every estimate below is obtained by replacing a classical McCarthy term by the smaller pointwise quantity $R_{\gamma,\eta}(X;\mu)$. The displayed norm versions are convenient computable consequences obtained after taking suprema.

The refinement mechanism and the role of the parameters

Proposition 1 is the point at which the improvement enters the proofs. For a positive operator X and a unit vector μ, the proposition gives a quantity $R_{\gamma,\eta}(X;\mu)$ satisfying

$$\langle X\mu,\mu\rangle^{\gamma} \leq R_{\gamma,\eta}(X;\mu) \leq \langle X^{\gamma}\mu,\mu\rangle.$$

The left inequality shows that the refined expression is still a valid upper estimate for the scalar quantity produced in the numerical-radius argument. The right inequality explains its relation with the standard McCarthy term. Hence the refinement is introduced before the supremum is taken, at the level where the original proof uses a positive operator and a unit vector.

The simplest instance is obtained by taking $\gamma = 2$ and $\eta = 2$. In this case the expression in Proposition 1 becomes

$$R_{2,2}(X;\mu) = \frac{1}{2}\langle X^2\mu,\mu\rangle + \frac{1}{2}\langle X\mu,\mu\rangle^2.$$

Since $\langle X\mu,\mu\rangle^2 \leq \langle X^2\mu,\mu\rangle$ for a positive operator X and a unit vector μ, this formula immediately exhibits the refinement. The second summand is never larger than the first-order McCarthy contribution used to form the classical bound. Equality may occur, for example, when μ is an eigenvector of X. For other vectors the inequality may be strict. This elementary case is also the one used in the numerical comparisons below.

The parameter α comes from the mixed Schwarz inequality. It determines how the two positive operators $|\mathcal{A}|$ and $|\mathcal{A}^*|$ are weighted in the starting estimate. The choice $\alpha = \frac{1}{2}$ is symmetric, but the theorems allow $0 < \alpha < 1$ when the two terms should be treated differently. The parameter γ determines the power of the numerical radius on the left-hand side and the powers of the positive operators on the right-hand side. In the present paper the refinement is used only for the range $\gamma \geq 1$. This is important because the direction in the Holder–McCarthy inequality changes when $0 < \gamma \leq 1$. The two cases are therefore not interchanged in the proofs.

The integer $\eta \geq 2$ is the order appearing in the binomial refinement. It determines the number of intermediate terms in $R_{\gamma,\eta}(X;\mu)$. A larger value of η produces a different refined expression, but the present results do not assert a universal monotonicity with respect to η. The values obtained for one matrix may decrease as η changes, while another matrix may behave differently. The appropriate statement is the pointwise two-sided estimate in Proposition 1; any additional comparison between particular parameter values must be made for the operator under consideration.

This interpretation also clarifies the meaning of the recovery remarks after the theorems. Those remarks are consistency statements: they identify the parameter values for which the displayed formula reduces to the cited classical inequality. They should not be read as claims that the refined bound is always strictly smaller. Strictness depends on the operator and on the unit vectors which contribute to the relevant suprema.

The non-normal examples in Section 4 demonstrate strict improvement, whereas the diagonal example demonstrates equality.

The parameter restrictions in the theorem statements should therefore be kept visible. In Theorems 1 and 2, the parameter α belongs to the open interval $(0, 1)$ because it is inherited from the mixed Schwarz and mean inequalities used in the proofs. In Theorem 5, the auxiliary parameter β belongs to $[0, 1]$ because it is the weight in Lemma 8. The condition $\eta \geq 2$ is not cosmetic: the binomial sum in the refined Holder–McCarthy expression runs from 1 to $\eta - 1$. Thus each parameter has a specified origin in the displayed argument.

For this reason, parameter choices in the numerical examples are stated explicitly. The symmetric choice $\alpha = \frac{1}{2}$ makes the two positive operators $|A|$ and $|A^*|$ enter in the same way, while $\gamma = 2$ and $\eta = 2$ reduce the refinement to the transparent formula displayed above. Other choices are permitted by the corresponding theorems, but a different choice changes the actual upper bound. The tables below are consequently comparisons for the stated parameters, not parameter-free claims.

We are now ready to establish our first main result.

Theorem 1. *Let* $\mathcal{A} \in \mathcal{B}(H)$, $0 < \alpha < 1$, $\gamma > 1$ *and* $\eta \geq 2$. *Then*

$$\begin{aligned} w^\gamma(\mathcal{A}) \leq \frac{1}{2} \Bigg\{ & \left(\left(\frac{1}{\gamma}\right)^\eta + \left(\frac{\gamma-1}{\gamma}\right)^\eta \right) \left\| \, |\mathcal{A}|^{2\alpha\gamma} + |\mathcal{A}^*|^{2(1-\alpha)\gamma} \right\| \\ & + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^\kappa \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \sup_{\|\mu\|=1} \Big[\left\langle |\mathcal{A}|^{2\alpha\gamma}\mu, \mu \right\rangle^{1-\frac{\kappa\gamma}{\eta}} \left\langle |\mathcal{A}|^{\frac{2\alpha\kappa\gamma}{\eta}}\mu, \mu \right\rangle^\gamma \\ & + \left\langle |\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu, \mu \right\rangle^{1-\frac{\kappa\gamma}{\eta}} \left\langle |\mathcal{A}^*|^{\frac{2(1-\alpha)\kappa\gamma}{\eta}}\mu, \mu \right\rangle^\gamma \Big] \Bigg\}. \end{aligned} \tag{9}$$

Proof. For each unit vector $\mu \in H$, we have

$$\begin{aligned} |\langle \mathcal{A}\mu, \mu\rangle| & \leq \langle |\mathcal{A}|^{2\alpha}\mu, \mu\rangle^{1/2} \langle |\mathcal{A}^*|^{2(1-\alpha)}\mu, \mu\rangle^{1/2} \quad \text{(by Lemma 1)} \\ & \leq \left(\frac{\langle |\mathcal{A}|^{2\alpha}\mu, \mu\rangle^\gamma + \langle |\mathcal{A}^*|^{2(1-\alpha)}\mu, \mu\rangle^\gamma}{2} \right)^{\frac{1}{\gamma}} \quad \text{(by Lemma 2)} \\ & \leq \Bigg(\frac{1}{2} \Bigg[\left(\left(\frac{1}{\gamma}\right)^\eta + \left(\frac{\gamma-1}{\gamma}\right)^\eta \right) \left(\langle |\mathcal{A}|^{2\alpha\gamma}\mu, \mu\rangle + \langle |\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu, \mu\rangle \right) \\ & + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^\kappa \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \times \\ & \Big[\langle |\mathcal{A}|^{2\alpha\gamma}\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}} \langle |\mathcal{A}|^{\frac{2\alpha\kappa\gamma}{\eta}}\mu, \mu\rangle^\gamma \\ & \quad + \langle |\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}} \langle |\mathcal{A}^*|^{\frac{2(1-\alpha)\kappa\gamma}{\eta}}\mu, \mu\rangle^\gamma \Big] \Bigg] \Bigg)^{\frac{1}{\gamma}} \quad \text{(by Lemma 4)} \end{aligned}$$

Thus,

$$\begin{aligned} |\langle \mathcal{A}\mu, \mu\rangle|^\gamma \leq \frac{1}{2} \Bigg[& \left(\left(\frac{1}{\gamma}\right)^\eta + \left(\frac{\gamma-1}{\gamma}\right)^\eta \right) \left\langle \left(|\mathcal{A}|^{2\alpha\gamma} + |\mathcal{A}^*|^{2(1-\alpha)\gamma} \right) \mu, \mu \right\rangle \\ & + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^\kappa \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \\ & \Big[\left\langle |\mathcal{A}|^{2\alpha\gamma}\mu, \mu \right\rangle^{1-\frac{\kappa\gamma}{\eta}} \left\langle |\mathcal{A}|^{\frac{2\alpha\kappa\gamma}{\eta}}\mu, \mu \right\rangle^\gamma + \left\langle |\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu, \mu \right\rangle^{1-\frac{\kappa\gamma}{\eta}} \left\langle |\mathcal{A}^*|^{\frac{2(1-\alpha)\kappa\gamma}{\eta}}\mu, \mu \right\rangle^\gamma \Big] \Bigg] \end{aligned}$$

and so by taking supremum,

$$\sup_{\|\mu\|=1} |\{|\langle \mathcal{A}\mu, \mu\rangle|^\gamma\} \le \frac{1}{2}\sup_{\|\mu\|=1}\left\{\left[\left(\left(\frac{1}{\gamma}\right)^\eta + \left(\frac{\gamma-1}{\gamma}\right)^\eta\right)\left\langle\left(|\mathcal{A}|^{2\alpha\gamma} + |\mathcal{A}^*|^{2(1-\alpha)\gamma}\right)\mu, \mu\right\rangle\right.\right.$$
$$+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^\kappa\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\left(\left\langle|\mathcal{A}|^{2\alpha\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}|^{\frac{2\alpha\kappa\gamma}{\eta}}\mu,\mu\right\rangle^\gamma\right.$$
$$\left.\left.\left.+\left\langle|\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}^*|^{\frac{2(1-\alpha)\kappa\gamma}{\eta}}\mu,\mu\right\rangle^\gamma\right)\right]\right\}$$
$$=\frac{1}{2}\left\{\left(\left(\frac{1}{\gamma}\right)^\eta + \left(\frac{\gamma-1}{\gamma}\right)^\eta\right)\left\|\,|\mathcal{A}|^{2\alpha\gamma} + |\mathcal{A}^*|^{2(1-\alpha)\gamma}\,\right\|\right.$$
$$+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^\kappa\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\sup_{\|\mu\|=1}\left[\left\langle|\mathcal{A}|^{2\alpha\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}|^{\frac{2\alpha\kappa\gamma}{\eta}}\mu,\mu\right\rangle^\gamma\right.$$
$$\left.\left.+\left\langle|\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}^*|^{\frac{2(1-\alpha)\kappa\gamma}{\eta}}\mu,\mu\right\rangle^\gamma\right]\right\},$$

as required. □

Remark 1. *By taking* $\eta = 2$*,* $\gamma = 2$*, and* $\alpha = \frac{1}{2}$ *in the Equation* (9) *we obtain*

$$w^2(\mathcal{A}) \le \frac{1}{2}\left[\frac{1}{2}\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left\{\langle|\mathcal{A}|\mu,\mu\rangle^2 + \langle|\mathcal{A}^*|\mu,\mu\rangle^2\right\}\right].$$

By Lemma 3

$$w^2(\mathcal{A}) \le \frac{1}{2}\left[\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left\{\langle|\mathcal{A}|^2\mu,\mu\rangle + \langle|\mathcal{A}^*|^2\mu,\mu\rangle\right\}\right].$$

Using the linearity of inner product

$$w^2(\mathcal{A}) \le \frac{1}{2}\left[\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left\langle\left(|\mathcal{A}|^2 + |\mathcal{A}^*|^2\right)\mu,\mu\right\rangle\right].$$

Thus,

$$w^2(\mathcal{A}) \le \frac{1}{2}\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\|,$$

or equivalently,

$$w(\mathcal{A}) \le \left(\frac{1}{2}\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\|\right)^{1/2}.$$

Thus, for these parameter values, the Equation (9) *reduces to the classical Equation* (1).

To establish our next generalization, we recall some lemmas.

Lemma 5. *Suppose* $\mathcal{A} \in \mathcal{B}(H)$ *is a positive operator and* $\mu \in H$ *is a unit vector. Then*

$$\langle \mathcal{A}^\gamma \mu, \mu\rangle \le \langle \mathcal{A}\mu, \mu\rangle^\gamma$$

for $0 < \gamma < 1$ *[4].*

Theorem 2. *Let* $\mathcal{A} \in \mathcal{B}(H)$ *,* $0 < \alpha < 1$*,* $\gamma \ge 1$ *and* $\eta \ge 2$*. Then*

$$
\begin{aligned}
w^{2\gamma}(\mathcal{A}) \leq & \left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right) \|\alpha|\mathcal{A}|^{2\gamma} + (1-\alpha)|\mathcal{A}^*|^{2\gamma}\| \\
& + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \sup_{\|\mu\|=1} \left[\alpha \left(\left\langle|\mathcal{A}|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}} \left\langle|\mathcal{A}|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right. \\
& \left. +(1-\alpha)\left(\left\langle|\mathcal{A}^*|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}} \left\langle|\mathcal{A}^*|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right].
\end{aligned} \tag{10}
$$

Proof. Let $\mu \in H$ be an arbitrary vector, $\|\mu\| = 1$, hence

$$
\begin{aligned}
|\langle \mathcal{A}\mu,\mu\rangle|^2 & \leq \langle|\mathcal{A}|^{2\alpha}\mu,\mu\rangle\langle|\mathcal{A}^*|^{2(1-\alpha)}\mu,\mu\rangle \quad \text{(by Lemma 1)} \\
& \leq \langle|\mathcal{A}|^{2}\mu,\mu\rangle^{\alpha}\langle|\mathcal{A}^*|^{2}\mu,\mu\rangle^{1-\alpha} \quad \text{(by Lemma 5)} \\
& \leq \left(\alpha\langle|\mathcal{A}|^{2}\mu,\mu\rangle^{\gamma} + (1-\alpha)\langle|\mathcal{A}^*|^{2}\mu,\mu\rangle^{\gamma}\right)^{\frac{1}{\gamma}} \quad \text{(by Lemma 2)} \\
& \leq \left[\left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left(\alpha\langle|\mathcal{A}|^{2\gamma}\mu,\mu\rangle + (1-\alpha)\langle|\mathcal{A}^*|^{2\gamma}\mu,\mu\rangle\right)\right. \\
& \quad + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \times \left[\alpha\left(\left\langle|\mathcal{A}|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right. \\
& \quad \left.\left. +(1-\alpha)\left(\left\langle|\mathcal{A}^*|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}^*|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right]\right]^{\frac{1}{\gamma}} \quad \text{(by Lemma 4)} \\
& = \left[\left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left\langle\left(\alpha|\mathcal{A}|^{2\gamma} + (1-\alpha)|\mathcal{A}^*|^{2\gamma}\right)\mu,\mu\right\rangle\right. \\
& \quad + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \times \\
& \quad \left[\alpha\left(\left\langle|\mathcal{A}|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right. \\
& \quad \left.\left. +(1-\alpha)\left(\left\langle|\mathcal{A}^*|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}^*|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right]\right]^{\frac{1}{\gamma}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\langle \mathcal{A}\mu,\mu\rangle|^{2\gamma} \leq & \left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left\langle\left(\alpha|\mathcal{A}|^{2\gamma} + (1-\alpha)|\mathcal{A}^*|^{2\gamma}\right)\mu,\mu\right\rangle \\
& + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \times \\
& \left[\alpha\left(\left\langle|\mathcal{A}|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right. \\
& \left. +(1-\alpha)\left(\left\langle|\mathcal{A}^*|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}^*|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right],
\end{aligned}
$$

and so by taking the supremum,

$$
\begin{aligned}
\sup_{\|\mu\|=1} |\langle \mathcal{A}\mu,\mu\rangle|^{2\gamma} \leq & \sup_{\|\mu\|=1} \left[\left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left\langle\left(\alpha|\mathcal{A}|^{2\gamma} + (1-\alpha)|\mathcal{A}^*|^{2\gamma}\right)\mu,\mu\right\rangle\right. \\
& + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \times \\
& \left[\alpha\left(\left\langle|\mathcal{A}|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right. \\
& \left.\left. +(1-\alpha)\left(\left\langle|\mathcal{A}^*|^{2\gamma}\mu,\mu\right\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{A}^*|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right]\right],
\end{aligned}
$$

$$
\begin{aligned}
w^{2\gamma}(\mathcal{A}) \leq & \left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right) \|\alpha|\mathcal{A}|^{2\gamma} + (1-\alpha)|\mathcal{A}^*|^{2\gamma}\| \\
& + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \times \\
& \sup_{\|\mu\|=1} \left[\alpha \left(\langle|\mathcal{A}|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}} . \left\langle|\mathcal{A}|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right. \\
& \left. +(1-\alpha)\left(\langle|\mathcal{A}^*|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}} . \left\langle|\mathcal{A}^*|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right],
\end{aligned}
$$

as required. □

Remark 2. *By taking* $n = 2$, $\gamma = 2$ *and* $\alpha = \frac{1}{2}$ *in the Equation* (10) *we obtain*

$$w^4(\mathcal{A}) \leq \tfrac{1}{2}\left\|\tfrac{1}{2}|\mathcal{A}|^4 + \tfrac{1}{2}|\mathcal{A}^*|^4\right\| + \tfrac{1}{2}\cdot \sup_{\|\mu\|=1}\left[\tfrac{1}{2}\langle|\mathcal{A}|^2\mu,\mu\rangle^2 + \tfrac{1}{2}\langle|\mathcal{A}^*|^2\mu,\mu\rangle^2\right].$$

By Lemma 3

$$w^4(\mathcal{A}) \leq \tfrac{1}{2}\left\|\tfrac{1}{2}|\mathcal{A}|^4 + \tfrac{1}{2}|\mathcal{A}^*|^4\right\| + \tfrac{1}{2}\cdot \sup_{\|\mu\|=1}\left[\tfrac{1}{2}\langle|\mathcal{A}|^4\mu,\mu\rangle + \tfrac{1}{2}\langle|\mathcal{A}^*|^4\mu,\mu\rangle\right].$$

Using the linearity of inner product

$$w^4(\mathcal{A}) \leq \tfrac{1}{2}\left\|\tfrac{1}{2}|\mathcal{A}|^4 + \tfrac{1}{2}|\mathcal{A}^*|^4\right\| + \tfrac{1}{2}\cdot \sup_{\|\mu\|=1}\left\langle\left(\tfrac{1}{2}|\mathcal{A}|^4 + \tfrac{1}{2}|\mathcal{A}^*|^4\right)\mu,\mu\right\rangle.$$

Thus,

$$w^4(\mathcal{A}) \leq \left\|\tfrac{1}{2}|\mathcal{A}|^4 + \tfrac{1}{2}|\mathcal{A}^*|^4\right\|.$$

Thus, for these parameter values, the Equation (10) *reduces to the classical Equation* (2)*.*

Remark 3 (Relation between the first two main estimates)**.** *Although the proofs use the same pointwise refinement, the two estimates have different starting inequalities and different output forms. Theorem 1 is obtained from the symmetric power-mean step and bounds* $w^\gamma(\mathcal{A})$ *in terms of* $|\mathcal{A}|^{2\alpha\gamma}$ *and* $|\mathcal{A}^*|^{2(1-\alpha)\gamma}$*. In contrast, Theorem 2 incorporates the weighted product and mean inequalities and bounds* $w^{2\gamma}(\mathcal{A})$ *in terms of the weighted combination* $\alpha|\mathcal{A}|^{2\gamma} + (1-\alpha)|\mathcal{A}^*|^{2\gamma}$*. We do not claim a uniform domination between these bounds; the suitable form depends on the operator and the parameter* α*. Thus, the two results are complementary rather than redundant.*

The following lemma is extremely helpful in the next one.

Lemma 6. *Suppose* $\mathcal{A} \in \mathcal{B}(H)$ *is a self-adjoint and* $\mu \in H$ *is a vector [4]. Then*

$$|\langle\mathcal{A}\mu,\mu\rangle| \leq \langle|\mathcal{A}|\mu,\mu\rangle.$$

We prove our next theorem by using the above lemma.

Theorem 3. *Let* $\mathcal{A} \in \mathcal{B}(H)$ *be an operator with Cartesian decomposition* $\mathcal{A} = \mathcal{B} + i\mathcal{C}$*,* *and suppose* $\gamma \geq 2$. *Then*

$$w^{\gamma}(\mathcal{A}) \leq 2^{\frac{\gamma}{2}-1}\left[\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left\||\mathcal{B}|^{\gamma}+|\mathcal{C}|^{\gamma}\right\| \right.$$
$$+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times \tag{11}$$
$$\left.\sup_{\|\mu\|=1}\left(\langle|\mathcal{B}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{B}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}+\langle|\mathcal{C}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{C}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right]$$

Proof. Let $\mu \in H$ be an arbitrary vector, $\|\mu\| = 1$, hence

$$\begin{aligned}
\frac{|\langle\mathcal{A}\mu,\mu\rangle|}{\sqrt{2}} &= \left(\frac{\langle\mathcal{B}\mu,\mu\rangle^{2}+\langle\mathcal{C}\mu,\mu\rangle^{2}}{2}\right)^{\frac{1}{2}} \\
&= \left(\frac{|\langle\mathcal{B}\mu,\mu\rangle|^{\gamma}+|\langle\mathcal{C}\mu,\mu\rangle|^{\gamma}}{2}\right)^{\frac{1}{\gamma}} \quad \text{(by Lemma 2)} \\
&\leq 2^{-\frac{1}{\gamma}}\left(\langle|\mathcal{B}|\mu,\mu\rangle^{\gamma}+\langle|\mathcal{C}|\mu,\mu\rangle^{\gamma}\right)^{\frac{1}{\gamma}} \quad \text{(by Lemma 6)} \\
&\leq 2^{-\frac{1}{\gamma}}\left[\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left(\langle|\mathcal{B}|^{\gamma}\mu,\mu\rangle+\langle|\mathcal{C}|^{\gamma}\mu,\mu\rangle\right)\right. \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times \\
&\left.\left(\langle|\mathcal{B}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{B}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}+\langle|\mathcal{C}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{C}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right]^{\frac{1}{\gamma}}. \quad \text{(by Lemma 4)}
\end{aligned}$$

Thus,

$$\begin{aligned}
|\langle\mathcal{A}\mu,\mu\rangle|^{\gamma} \leq 2^{\frac{\gamma}{2}-1}&\left[\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left\langle\left(|\mathcal{B}|^{\gamma}+|\mathcal{C}|^{\gamma}\right)\mu,\mu\right\rangle\right. \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times \\
&\left.\left(\langle|\mathcal{B}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{B}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}+\langle|\mathcal{C}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{C}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right],
\end{aligned}$$

and so by taking the supremum

$$\begin{aligned}
\sup_{\|\mu\|=1}|\langle\mathcal{A}\mu,\mu\rangle|^{\gamma} \leq 2^{\frac{\gamma}{2}-1}\sup_{\|\mu\|=1}&\left[\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left\langle\left(|\mathcal{B}|^{\gamma}+|\mathcal{C}|^{\gamma}\right)\mu,\mu\right\rangle\right. \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times \\
&\left.\left(\langle|\mathcal{B}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{B}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}+\langle|\mathcal{C}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{C}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right],
\end{aligned}$$

$$\begin{aligned}
w^{\gamma}(\mathcal{A}) \leq 2^{\frac{\gamma}{2}-1}&\left[\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left\||\mathcal{B}|^{\gamma}+|\mathcal{C}|^{\gamma}\right\|\right. \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times \\
&\left.\sup_{\|\mu\|=1}\left(\langle|\mathcal{B}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{B}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}+\langle|\mathcal{C}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\left\langle|\mathcal{C}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\right\rangle^{\gamma}\right)\right],
\end{aligned}$$

as required. $\square$

Remark 4. *Let $\mathcal{A} \in \mathcal{B}(H)$ with Cartesian decomposition $\mathcal{A} = \mathcal{B} + i\mathcal{C}$, and suppose $\gamma = 2$ and $\eta = 2$ in the Equation* (11) *we obtain*

$$w^2(\mathcal{A}) \leq \frac{1}{2}\left\||\mathcal{B}|^2 + ||\mathcal{C}|^2\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left[\langle|\mathcal{B}|\mu,\mu\rangle^2 + \langle|\mathcal{C}|\mu,\mu\rangle^2\right].$$

By Lemma 3

$$w^2(\mathcal{A}) \leq \frac{1}{2}\left\||\mathcal{B}|^2 + |\mathcal{C}|^2\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left[\langle|\mathcal{B}|^2\mu,\mu\rangle + \langle|\mathcal{C}^2|\mu,\mu\rangle\right].$$

Using the linearity of inner product,

$$w^2(\mathcal{A}) \leq \frac{1}{2}\left\||\mathcal{B}|^2 + |\mathcal{C}|^2\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\langle(|\mathcal{B}|^2 + |\mathcal{C}|^2)\mu,\mu\rangle.$$

We finally get

$$\begin{aligned} w^2(\mathcal{A}) &\leq \frac{1}{2}\left\||\mathcal{B}|^2 + |\mathcal{C}|^2\right\| + \frac{1}{2}\left\||\mathcal{B}|^2 + |\mathcal{C}|^2\right\| \\ &\leq \left\||\mathcal{B}|^2 + |\mathcal{C}|^2\right\|. \end{aligned}$$

Thus, for these parameter values, the Equation (11) *reduces to the classical Equation* (3)*.*

To obtain a more precise numerical radius inequality, we require the following lemma.

Lemma 7 (The Clarkson's inequality [5])**.** *If $a, b \in \mathbb{C}$, then*

$$2\left(|a|^\gamma + |b|^\gamma\right) \leq |a+b|^\gamma + |a-b|^\gamma$$

for $\gamma \geq 2$.

Theorem 4. *If $\mathcal{A} \in \mathcal{B}(H)$, then*

$$\begin{aligned} w^\gamma(\mathcal{A}) \leq \frac{1}{4}\Bigg[&\left(\left(\frac{1}{\gamma}\right)^\eta + \left(\frac{\gamma-1}{\gamma}\right)^\eta\right)\||\mathcal{A}+\mathcal{A}^*|^\gamma + |\mathcal{A}-\mathcal{A}^*|^\gamma\| \\ &+2\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^\kappa\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times \\ &\sup_{\|\mu\|=1}\left(\langle|\mathcal{A}|^\gamma\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{A}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^\gamma + \langle|\mathcal{A}^*|^\gamma\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{A}^*|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^\gamma\right)\Bigg], \end{aligned} \tag{12}$$

for $\gamma \geq 2$.

Proof.

$$
\begin{aligned}
|\langle \mathcal{A}\mu,\mu\rangle| &\leq \langle |\mathcal{A}|\mu,\mu\rangle^{\frac{1}{2}}\langle |\mathcal{A}^*|\mu,\mu\rangle^{\frac{1}{2}} \quad \text{(By C-S inequality)}\\
|\langle \mathcal{A}\mu,\mu\rangle| &\leq \left(\frac{1}{2}\langle |\mathcal{A}|\mu,\mu\rangle^{\gamma}+\frac{1}{2}\langle |\mathcal{A}^*|\mu,\mu\rangle^{\gamma}\right)^{\frac{1}{\gamma}},\\
|\langle \mathcal{A}\mu,\mu\rangle| &\leq \Bigg[\frac{1}{2}\Bigg[\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\langle |\mathcal{A}|^{\gamma}\mu,\mu\rangle\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\langle |\mathcal{A}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\Bigg]\\
&\quad+\frac{1}{2}\Bigg[\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\langle |\mathcal{A}^*|^{\gamma}\mu,\mu\rangle\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\langle |\mathcal{A}^*|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}^*|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\Bigg]\Bigg]^{\frac{1}{\gamma}}, \qquad \text{(by Lemma 4)}\\
4|\langle \mathcal{A}\mu,\mu\rangle|^{\gamma} &\leq \left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)2\left(\langle\left(|\mathcal{A}|^{\gamma}+|\mathcal{A}^*|^{\gamma}\right)\mu,\mu\rangle\right)\\
&\quad+2\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times\\
&\quad\left(\langle |\mathcal{A}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}+\langle |\mathcal{A}^*|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}^*|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\right), \qquad \text{(by using Lemma 7)}
\end{aligned}
$$

for $\gamma \geq 2$.

$$
\begin{aligned}
4|\langle \mathcal{A}\mu,\mu\rangle|^{\gamma} &\leq \left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\langle\left(|\mathcal{A}+\mathcal{A}^*|^{\gamma}+|\mathcal{A}-\mathcal{A}^*|^{\gamma}\right)\mu,\mu\rangle\\
&\quad+2\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times\\
&\quad\left(\langle |\mathcal{A}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}+\langle |\mathcal{A}^*|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}^*|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\right),
\end{aligned}
$$

for $\gamma \geq 2$. So,

$$
\begin{aligned}
4|\langle \mathcal{A}\mu,\mu\rangle|^{\gamma} &\leq \left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\||\mathcal{A}+\mathcal{A}^*|^{\gamma}+|\mathcal{A}-\mathcal{A}^*|^{\gamma}\|\\
&\quad+2\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times\\
&\quad\left(\langle |\mathcal{A}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}+\langle |\mathcal{A}^*|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}^*|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\right),
\end{aligned}
$$

for $\gamma \geq 2$. By evaluating the supremum across all unit vectors in H^n,

$$
\begin{aligned}
4w^{\gamma}(\mathcal{A}) &\leq \Bigg[\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\||\mathcal{A}+\mathcal{A}^*|^{\gamma}+|\mathcal{A}-\mathcal{A}^*|^{\gamma}\|\\
&\quad+2\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times\\
&\quad\sup_{\|\mu\|=1}\left(\langle |\mathcal{A}|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}+\langle |\mathcal{A}^*|^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle |\mathcal{A}^*|^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\right)\Bigg],
\end{aligned}
$$

as required. □

Remark 5. *By taking $\eta = 2$ and $\gamma = 2$ in the Equation* (12) *we obtain*

$$w^2(\mathcal{A}) \leq \frac{1}{4}\left[\frac{1}{2}\left(\left\||\mathcal{A}+\mathcal{A}^*|^2 + |\mathcal{A}-\mathcal{A}^*|^2\right\|\right) + \frac{1}{2}\cdot 2\sup_{\|\mu\|=1}\left(\langle|\mathcal{A}|\mu,\mu\rangle^2 + \langle|\mathcal{A}^*|\mu,\mu\rangle^2\right)\right].$$

By Lemma 3

$$w^2(\mathcal{A}) \leq \frac{1}{4}\left[\frac{1}{2}\left(\left\||\mathcal{A}+\mathcal{A}^*|^2 + |\mathcal{A}-\mathcal{A}^*|^2\right\|\right) + \frac{1}{2}\cdot 2\sup_{\|\mu\|=1}\left(\langle|\mathcal{A}|^2\mu,\mu\rangle + \langle|\mathcal{A}^*|^2\mu,\mu\rangle\right)\right].$$

Using the linearity of inner product

$$w^2(\mathcal{A}) \leq \frac{1}{4}\left[\frac{1}{2}\left(\left\||\mathcal{A}+\mathcal{A}^*|^2 + |\mathcal{A}-\mathcal{A}^*|^2\right\|\right) + \frac{1}{2}\sup_{\|\mu\|=1}\left(2\langle\left(|\mathcal{A}|^2 + |\mathcal{A}^*|^2\right)\mu,\mu\rangle\right)\right],$$

and

$$w^2(\mathcal{A}) \leq \frac{1}{4}\left(\left\||\mathcal{A}+\mathcal{A}^*|^2 + |\mathcal{A}-\mathcal{A}^*|^2\right\|\right).$$

Thus, for these parameter values, the Equation (12) *reduces to the classical Equation* (4).

The next lemma provides a key tool for deriving the desired inequality.

Lemma 8. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{B}(H)$, $0 \leq \alpha \leq 1$ and $\gamma \geq 1$ [6]. Then*

$$\begin{aligned}|\langle\mathcal{A}\mu,\nu\rangle|^{2\gamma} &\leq \beta\left\langle|\mathcal{A}|^{2\gamma\alpha}\mu,\mu\right\rangle\left\langle|\mathcal{A}^*|^{2\gamma(1-\alpha)}\nu,\nu\right\rangle \\ &\quad + (1-\beta)|\langle\mathcal{A}\mu,\mu\rangle|^{\gamma}\sqrt{\langle|\mathcal{A}|^{2\gamma\alpha}\mu,\mu\rangle\langle|\mathcal{A}^*|^{2\gamma(1-\alpha)}\nu,\nu\rangle} \\ &\leq \left\langle|\mathcal{A}|^{2\gamma\alpha}\mu,\mu\right\rangle\left\langle|\mathcal{A}^*|^{2\gamma(1-\alpha)}\nu,\nu\right\rangle\end{aligned}$$

Theorem 5. *Let $\mathcal{A} \in \mathcal{B}(H)$. Then*

$$\begin{aligned}w^{2\gamma}(\mathcal{A}) \leq \frac{1}{2}\beta&\left(\left(\left(\frac{1}{2}\right)^{\eta} + \left(\frac{1}{2}\right)^{\eta}\right)\left\||\mathcal{A}|^{4\gamma\alpha} + |\mathcal{A}^*|^{4\gamma(1-\alpha)}\right\| \times\right. \\ &+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\sup_{\|\mu\|=1}\left(\langle|\mathcal{A}|^{4\gamma\alpha}\mu,\mu\rangle^{1-\frac{2\kappa}{\eta}}\langle|\mathcal{A}|^{\frac{4\gamma\kappa\alpha}{\eta}}\mu,\mu\rangle^2\right. \\ &\left.\left.+\langle|\mathcal{A}^*|^{4\gamma(1-\alpha)}\mu,\mu\rangle^{1-\frac{2\kappa}{\eta}}\langle|\mathcal{A}^*|^{\frac{4\gamma\kappa(1-\alpha)}{\eta}}\mu,\mu\rangle^2\right)\right) \\ &+\frac{1}{2}(1-\beta)w^{\gamma}(\mathcal{A})\left\||\mathcal{A}|^{2\gamma\alpha} + |\mathcal{A}^*|^{2(1-\alpha)\gamma}\right\|,\end{aligned} \tag{13}$$

for all $\gamma \geq 1$ and $0 \leq \alpha, \beta \leq 1$.

Proof. Let $\mu \in H$ be an arbitrary vector, $\|\mu\| = 1$. Setting $\nu = \mu$ in Lemma 8, it follows that

$$
\begin{aligned}
|\langle\mathcal{A}\mu,\mu\rangle|^{2\gamma} \leq{}& \beta\left\langle|\mathcal{A}|^{2\gamma\alpha}\mu,\mu\right\rangle \quad\times\quad \left\langle|\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu,\mu\right\rangle \\
&+(1-\beta)|\langle\mathcal{A}\mu,\mu\rangle|^{\gamma}\sqrt{\left\langle|\mathcal{A}|^{2\gamma\alpha}\mu,\mu\right\rangle\left\langle|\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu,\mu\right\rangle} \\
\leq{}& \frac{1}{2}\beta\left(\left\langle|\mathcal{A}|^{2\gamma\alpha}\mu,\mu\right\rangle^2+\left\langle|\mathcal{A}^*|^{2(1-\alpha)\gamma}\mu,\mu\right\rangle^2\right) \\
&\text{(by the arithmetic–geometric mean inequality)} \\
&+\frac{1}{2}(1-\beta)|\langle\mathcal{A}\mu,\mu\rangle|^{\gamma}\left\langle\left(|\mathcal{A}|^{2\gamma\alpha}+|\mathcal{A}^*|^{2(1-\alpha)\gamma}\right)\mu,\mu\right\rangle \\
&\text{(by the arithmetic–geometric mean inequality)} \\
\leq{}& \frac{1}{2}\beta\Bigg[\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\langle|\mathcal{A}|^{4\gamma\alpha}\mu,\mu\right\rangle \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\left\langle|\mathcal{A}|^{4\gamma\alpha}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}|^{\frac{4\gamma\kappa\alpha}{\eta}}\mu,\mu\right\rangle^2 \\
&+\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\langle|\mathcal{A}^*|^{4\gamma(1-\alpha)}\mu,\mu\right\rangle \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\left\langle|\mathcal{A}^*|^{4\gamma(1-\alpha)}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}^*|^{\frac{4\gamma\kappa(1-\alpha)}{\eta}}\mu,\mu\right\rangle^2\Bigg] \\
&+\frac{1}{2}(1-\beta)|\langle\mathcal{A}\mu,\mu\rangle|^{\gamma}\left\langle\left(|\mathcal{A}|^{2\gamma\alpha}+|\mathcal{A}^*|^{2(1-\alpha)\gamma}\right)\mu,\mu\right\rangle \\
&\text{(by Lemma 4)}
\end{aligned}
$$

$$
\begin{aligned}
|\langle\mathcal{A}\mu,\mu\rangle|^{2\gamma} \leq{}& \frac{1}{2}\beta\left(\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\langle\left(|\mathcal{A}|^{4\gamma\alpha}+|\mathcal{A}^*|^{4\gamma(1-\alpha)}\right)\mu,\mu\right\rangle\right. \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa} \\
&\times\left.\left(\langle|\mathcal{A}|^{4\gamma\alpha}\mu,\mu\rangle^{1-\frac{2\kappa}{\eta}}\langle|\mathcal{A}|^{\frac{4\gamma\kappa\alpha}{\eta}}\mu,\mu\rangle^{\gamma}+\langle|\mathcal{A}^*|^{4\gamma(1-\alpha)}\mu,\mu\rangle^{1-\frac{2\kappa}{\eta}}\langle|\mathcal{A}^*|^{\frac{4\gamma\kappa(1-\alpha)}{\eta}}\mu,\mu\rangle^{\gamma}\right)\right) \\
&+\frac{1}{2}(1-\beta)|\langle\mathcal{A}\mu,\mu\rangle|^{\gamma}\left\langle\left(|\mathcal{A}|^{2\gamma\alpha}+|\mathcal{A}^*|^{2(1-\alpha)\gamma}\right)\mu,\mu\right\rangle.
\end{aligned}
$$

Taking the supremum over all the unit vector,

$$
\begin{aligned}
\sup_{\|\mu\|=1}|\langle\mathcal{A}\mu,\mu\rangle|^{2\gamma} \leq{}& \frac{1}{2}\beta\left(\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\||\mathcal{A}|^{4\gamma\alpha}+|\mathcal{A}^*|^{4\gamma(1-\alpha)}\right\|\right. \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\times \\
&\sup_{\|\mu\|=1}\left(\langle|\mathcal{A}|^{4\gamma\alpha}\mu,\mu\rangle^{1-\frac{2\kappa}{\eta}}\langle|\mathcal{A}|^{\frac{4\gamma\kappa\alpha}{\eta}}\mu,\mu\rangle^2\right. \\
&\left.\left.+\langle|\mathcal{A}^*|^{4\gamma(1-\alpha)}\mu,\mu\rangle^{1-\frac{2\kappa}{\eta}}\langle|\mathcal{A}^*|^{\frac{4\gamma\kappa(1-\alpha)}{\eta}}\mu,\mu\rangle^2\right)\right) \\
&+\frac{1}{2}(1-\beta)w^{\gamma}(\mathcal{A})\left\||\mathcal{A}|^{2\gamma\alpha}+|\mathcal{A}^*|^{2(1-\alpha)\gamma}\right\|,
\end{aligned}
$$

we get the required result. $\square$

Remark 6. *By taking $\eta = 2$ $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$ and $\gamma = 2$ in Equation* (13) *we obtain*

$$w^4(\mathcal{A}) \le \frac{1}{6}\left(\frac{1}{2}\left\||\mathcal{A}|^4 + |\mathcal{A}^*|^4\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left(\langle|\mathcal{A}|^2\mu,\mu\rangle^2 + \langle|\mathcal{A}^*|^2\mu,\mu\rangle^2\right)\right) + \frac{1}{3}w^2(\mathcal{A})\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\|.$$

By Lemma 3

$$\le \frac{1}{6}\left(\frac{1}{2}\left\||\mathcal{A}|^4 + |\mathcal{A}^*|^4\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left(\langle|\mathcal{A}|^4\mu,\mu\rangle + \langle|\mathcal{A}^*|^4\mu,\mu\rangle\right)\right) + \frac{1}{3}w^2(\mathcal{A})\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\|.$$

Using the linearity of inner product

$$\le \frac{1}{6}\left(\frac{1}{2}\left\||\mathcal{A}|^4 + |\mathcal{A}^*|^4\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\langle\left(|\mathcal{A}|^4 + |\mathcal{A}^*|^4\right)\mu,\mu\rangle\right) + \frac{1}{3}w^2(\mathcal{A})\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\|.$$

Thus

$$\le \frac{1}{6}\left\||\mathcal{A}|^4 + |\mathcal{A}^*|^4\right\| + \frac{1}{3}w^2(\mathcal{A})\left\||\mathcal{A}|^2 + |\mathcal{A}^*|^2\right\|.$$

Thus, for these parameter values, the Equation (13) *reduces to the corresponding special case of Equation* (5)*.*

To facilitate the proof of the next theorem, we present the following lemma.

Lemma 9 (Bohr's inequality [7])**.** *Let a_i be positive real numbers for $1 \le i \le \eta$. Then for each $\gamma \ge 1$,*

$$\left(\sum_{i=1}^{\eta} a_i\right)^{\gamma} \le \eta^{\gamma-1}\sum_{i=1}^{\eta} a_i^{\gamma}.$$

Theorem 6. *Let $\mathcal{A}_i \in \mathcal{B}(H)$ have the Cartesian decomposition $\mathcal{A}_i = \mathcal{B}_i + i\mathcal{C}_i$ for $i = 1, \ldots, \eta$ and $\gamma \ge 1$. Then*

$$\begin{aligned} w^{\gamma}\left(\sum_{i=1}^{\eta}\mathcal{A}_i\right) \le \eta^{\gamma-1}2^{\frac{\gamma}{2}-1}\sum_{i=1}^{\eta}\Bigg\{ &\left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left\||\mathcal{B}_i + \mathcal{C}_i|^{2\gamma} + |\mathcal{B}_i - \mathcal{C}_i|^{2\gamma}\right\| \\ &+ \sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\sup_{\|\mu\|=1}\Big(\langle|\mathcal{B}_i + \mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i + \mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma} \\ &+ \langle|\mathcal{B}_i - \mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i - \mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\Big)\Bigg\}^{\frac{1}{2}}. \end{aligned} \tag{14}$$

Proof. Let $\mu \in H$ be an arbitrary vector, $\|\mu\| = 1$, hence

$$\begin{aligned} \left|\left\langle\sum_{i=1}^{\eta}\mathcal{A}_i\mu,\mu\right\rangle\right|^{\gamma} &\le \left(\sum_{i=1}^{\eta}\left(\langle\mathcal{B}_i\mu,\mu\rangle^2 + \langle\mathcal{C}_i\mu,\mu\rangle^2\right)^{\frac{1}{2}}\right)^{\gamma} \\ &\le \left(\sum_{i=1}^{\eta}\left(\frac{1}{2}\left(\langle(\mathcal{B}_i + \mathcal{C}_i)\mu,\mu\rangle^2 + \langle(\mathcal{B}_i - \mathcal{C}_i)\mu,\mu\rangle^2\right)\right)^{\frac{1}{2}}\right)^{\gamma} \end{aligned}$$

$$
\begin{aligned}
&\left|\left\langle\sum_{i=1}^{\eta}\mathcal{A}_i\mu,\mu\right\rangle\right|^{\gamma}\\
&\leq \eta^{\gamma-1}2^{\frac{-\gamma}{2}}\sum_{i=1}^{\eta}\left(\langle(\mathcal{B}_i+\mathcal{C}_i)\mu,\mu\rangle^2+\langle(\mathcal{B}_i-\mathcal{C}_i)\mu,\mu\rangle^2\right)^{\frac{\gamma}{2}} \quad \text{(by Lemma 9)}\\
&\leq \eta^{\gamma-1}2^{\frac{-\gamma}{2}}\sum_{i=1}^{\eta}\left(\langle|\mathcal{B}_i+\mathcal{C}_i|^2\mu,\mu\rangle+\langle|\mathcal{B}_i-\mathcal{C}_i|^2\mu,\mu\rangle\right)^{\frac{\gamma}{2}} \quad \text{(by Lemma 1)}\\
&\leq \eta^{\gamma-1}2^{\frac{\gamma}{2}-1}\sum_{i=1}^{\eta}\left(\langle|\mathcal{B}_i+\mathcal{C}_i|^2\mu,\mu\rangle^{\gamma}+\langle|\mathcal{B}_i-\mathcal{C}_i|^2\mu,\mu\rangle^{\gamma}\right)^{\frac{1}{2}} \quad \text{(by Lemma 9)}\\
&\leq \eta^{\gamma-1}2^{\frac{\gamma}{2}-1}\sum_{i=1}^{\eta}\Bigg\{\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\langle|\mathcal{B}_i+\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\langle|\mathcal{B}_i+\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i+\mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\\
&\quad+\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\langle|\mathcal{B}_i-\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\langle|\mathcal{B}_i-\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i-\mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\Bigg\}^{\frac{1}{2}} \quad \text{(by Lemma 4)}\\
&\leq \eta^{\gamma-1}2^{\frac{\gamma}{2}-1}\sum_{i=1}^{\eta}\Bigg\{\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\left(\langle|\mathcal{B}_i+\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle+\langle|\mathcal{B}_i-\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle\right)\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times\\
&\quad\Big(\langle|\mathcal{B}_i+\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i+\mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\\
&\quad+\langle|\mathcal{B}_i-\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i-\mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\Big)\Bigg\}^{\frac{1}{2}}.
\end{aligned}
$$

Now by taking supremum

$$
\begin{aligned}
&\sup_{\|\mu\|=1}\left|\left\langle\sum_{i=1}^{\eta}\mathcal{A}_i\mu,\mu\right\rangle\right|^{\gamma}\\
&\leq \eta^{\gamma-1}2^{\frac{\gamma}{2}-1}\sum_{i=1}^{\eta}\Bigg\{\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\times\\
&\quad\sup_{\|\mu\|=1}\langle\left(|\mathcal{B}_i+\mathcal{C}_i|^{2\gamma}+|\mathcal{B}_i-\mathcal{C}_i|^{2\gamma}\right)\mu,\mu\rangle+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times\\
&\quad\sup_{\|\mu\|=1}\Big(\langle|\mathcal{B}_i+\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i+\mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\\
&\quad+\langle|\mathcal{B}_i-\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i-\mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\Big)\Bigg\}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
w^{\gamma}\left(\sum_{i=1}^{\eta}\mathcal{A}_i\right)&\leq \eta^{\gamma-1}2^{\frac{\gamma}{2}-1}\sum_{i=1}^{\eta}\Bigg\{\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\times\\
&\quad\left\||\mathcal{B}_i+\mathcal{C}_i|^{2\gamma}+|\mathcal{B}_i-\mathcal{C}_i|^{2\gamma}\right\|\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times\\
&\quad\sup_{\|\mu\|=1}\Big(\langle|\mathcal{B}_i+\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i+\mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\\
&\quad+\langle|\mathcal{B}_i-\mathcal{C}_i|^{2\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle|\mathcal{B}_i-\mathcal{C}_i|^{\frac{2\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\Big)\Bigg\}^{\frac{1}{2}},
\end{aligned}
$$

as required. □

Remark 7. *Let $\mathcal{A}_i = \mathcal{B}_i + i\mathcal{C}_i \in \mathcal{B}(H)$ and $\gamma \geq 1$ in the Equation* (14)*. By taking $\eta = 2$ then we obtain*

$$w^2\left(\sum_{i=1}^{2}\mathcal{A}_i\right) \leq 2\sum_{i=1}^{2}\left\{\frac{1}{2}\left\||\mathcal{B}_i+\mathcal{C}_i|^4+|\mathcal{B}_i-\mathcal{C}_i|^4\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left(\langle|\mathcal{B}_i+\mathcal{C}_i|^2\mu,\mu\rangle^2+\langle|\mathcal{B}_i-\mathcal{C}_i|^2\mu,\mu\rangle^2\right)\right\}^{\frac{1}{2}}.$$

By Lemma 3

$$w^2\left(\sum_{i=1}^{2}\mathcal{A}_i\right) \leq 2\sum_{i=1}^{2}\left\{\frac{1}{2}\left\||\mathcal{B}_i+\mathcal{C}_i|^4+|\mathcal{B}_i-\mathcal{C}_i|^4\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left(\langle|\mathcal{B}_i+\mathcal{C}_i|^4\mu,\mu\rangle+\langle|\mathcal{B}_i-\mathcal{C}_i|^4\mu,\mu\rangle\right)\right\}^{\frac{1}{2}}.$$

Using the linearity of inner product

$$w^2\left(\sum_{i=1}^{2}\mathcal{A}_i\right) \leq 2\sum_{i=1}^{2}\left\{\frac{1}{2}\left\||\mathcal{B}_i+\mathcal{C}_i|^4+|\mathcal{B}_i-\mathcal{C}_i|^4\right\| + \frac{1}{2}\sup_{\|\mu\|=1}\left\langle\left(|\mathcal{B}_i+\mathcal{C}_i|^4+|\mathcal{B}_i-\mathcal{C}_i|^4\right)\mu,\mu\right\rangle\right\}^{\frac{1}{2}},$$

and thus,

$$w^2\left(\sum_{i=1}^{2}\mathcal{A}_i\right) \leq 2\sum_{i=1}^{2}\left\{\left\||\mathcal{B}_i+\mathcal{C}_i|^4+|\mathcal{B}_i-\mathcal{C}_i|^4\right\|\right\}^{\frac{1}{2}}.$$

Thus, for these parameter values, the Equation (14) *reduces to the corresponding finite-sum Equation* (6)*.*

Before stating the next theorem, we present the following lemmas which will be used in its proof. Here $M_n(\mathbb{C})$ denotes the algebra of all $n \times n$ complex matrices. If a block matrix

$$\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B}^* & \mathcal{C}\end{bmatrix}$$

is positive semidefinite, then its diagonal blocks $\mathcal{A}$ and $\mathcal{C}$ are positive semidefinite; hence Lemma 4 may be applied to them.

Lemma 10. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in M_n(\mathbb{C})$ be such that*

$$\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{B}^* & \mathcal{C}\end{bmatrix} \geq 0.$$

Then

$$|\langle\mathcal{B}\mu,\nu\rangle|^2 \leq \langle\mathcal{A}\mu,\mu\rangle\langle\mathcal{C}\nu,\ \nu\rangle, \quad \textit{for} \quad \mu,\nu \in \mathbb{C}^n \ [8].$$

Lemma 11. *The Power-Mean inequality states that*

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq (\alpha a^{\gamma} + (1-\alpha)b^{\gamma})^{\frac{1}{\gamma}}$$

for all $\alpha \in [0,1]$, $a, b \geq 0$ *and* $\gamma \geq 1$ *[6].*

Theorem 7. *Let* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in M_n(\mathbb{C})$ *be such that*

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^* & \mathcal{C} \end{bmatrix} \geq 0.$$

Then

$$\begin{aligned} w^{\gamma}(\mathcal{B}) \leq \frac{1}{2} \Bigg(& \left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta} \right) \|\mathcal{A}^{\gamma} + \mathcal{C}^{\gamma}\| \\ & + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \times \\ & \sup_{\|\mu\|=1} \left(\langle \mathcal{A}^{\gamma}\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}} \langle \mathcal{A}^{\frac{\kappa\gamma}{\eta}}\mu, \mu\rangle^{\gamma} + \langle \mathcal{C}^{\gamma}\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}} \langle \mathcal{C}^{\frac{\kappa\gamma}{\eta}}\mu, \mu\rangle^{\gamma} \right) \Bigg). \end{aligned} \tag{15}$$

Proof. Since

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^* & \mathcal{C} \end{bmatrix} \geq 0,$$

the diagonal blocks $\mathcal{A}$ and $\mathcal{C}$ are positive semidefinite. Let $\mu \in \mathbb{C}^n$ be an arbitrary vector, $\|\mu\| = 1$, hence

$$\begin{aligned} |\langle \mathcal{B}\mu, \mu\rangle| &\leq \langle \mathcal{A}\mu, \mu\rangle^{\frac{1}{2}} \langle \mathcal{C}\mu, \mu\rangle^{\frac{1}{2}} \quad \text{(by Lemma 10)} \\ &\leq \frac{1}{2} \left(\langle \mathcal{A}\mu, \mu\rangle + \langle \mathcal{C}\mu, \mu\rangle \right) \\ &\leq \left(\frac{\langle \mathcal{A}\mu, \mu\rangle^{\gamma} + \langle \mathcal{C}\mu, \mu\rangle^{\gamma}}{2} \right)^{\frac{1}{\gamma}} \quad \text{(by Lemma 11)} \\ &\leq \Bigg(\frac{1}{2} \Bigg(\left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta} \right) \langle \mathcal{A}^{\gamma}\mu, \mu\rangle \\ &\quad + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \langle \mathcal{A}^{\gamma}\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}} \langle \mathcal{A}^{\frac{\kappa\gamma}{\eta}}\mu, \mu\rangle^{\gamma} \\ &\quad + \left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta} \right) \langle \mathcal{C}^{\gamma}\mu, \mu\rangle \\ &\quad + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \langle \mathcal{C}^{\gamma}\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}} \langle \mathcal{C}^{\frac{\kappa\gamma}{\eta}}\mu, \mu\rangle^{\gamma} \Bigg) \Bigg)^{\frac{1}{\gamma}}. \quad \text{(by Lemma 4)} \end{aligned}$$

Thus,

$$\begin{aligned} |\langle \mathcal{B}\mu, \mu\rangle|^{\gamma} \leq \frac{1}{2} \Bigg(& \left(\left(\frac{1}{\gamma}\right)^{\eta} + \left(\frac{\gamma-1}{\gamma}\right)^{\eta} \right) \langle (\mathcal{A}^{\gamma} + \mathcal{C}^{\gamma})\mu, \mu\rangle \\ & + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{\gamma}\right)^{\kappa} \left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa} \times \\ & \left(\langle \mathcal{A}^{\gamma}\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}} \langle \mathcal{A}^{\frac{\kappa\gamma}{\eta}}\mu, \mu\rangle^{\gamma} + \langle \mathcal{C}^{\gamma}\mu, \mu\rangle^{1-\frac{\kappa\gamma}{\eta}} \langle \mathcal{C}^{\frac{\kappa\gamma}{\eta}}\mu, \mu\rangle^{\gamma} \right) \Bigg). \end{aligned}$$

By taking the supremum over all the unit vectors

$$\begin{aligned}
\sup_{\|\mu\|=1} |\langle \mathcal{B}\mu,\mu\rangle|^{\gamma} \leq \sup_{\|\mu\|=1} \frac{1}{2}\Bigg(&\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\langle(\mathcal{A}^{\gamma}+\mathcal{C}^{\gamma})\mu,\mu\rangle \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times \\
&\left(\langle\mathcal{A}^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle\mathcal{A}^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}+\langle\mathcal{C}^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle\mathcal{C}^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\right)\Bigg) \\
=\frac{1}{2}\Bigg(&\left(\left(\frac{1}{\gamma}\right)^{\eta}+\left(\frac{\gamma-1}{\gamma}\right)^{\eta}\right)\|\mathcal{A}^{\gamma}+\mathcal{C}^{\gamma}\| \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{\gamma}\right)^{\kappa}\left(\frac{\gamma-1}{\gamma}\right)^{\eta-\kappa}\times \\
&\sup_{\|\mu\|=1}\left(\langle\mathcal{A}^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle\mathcal{A}^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}+\langle\mathcal{C}^{\gamma}\mu,\mu\rangle^{1-\frac{\kappa\gamma}{\eta}}\langle\mathcal{C}^{\frac{\kappa\gamma}{\eta}}\mu,\mu\rangle^{\gamma}\right)\Bigg),
\end{aligned}$$

as required. □

Remark 8. *By setting $\eta=2$ and $\gamma=2$ in the Equation* (15) *we obtain*

$$w^2(\mathcal{B}) \leq \frac{1}{2}\left(\frac{1}{2}\left\|\mathcal{A}^2+\mathcal{C}^2\right\|+\frac{1}{2}\sup_{\|\mu\|=1}\left(\langle\mathcal{A}\mu,\mu\rangle^2+\langle\mathcal{C}\mu,\mu\rangle^2\right)\right).$$

By Lemma 3

$$w^2(\mathcal{B}) \leq \frac{1}{2}\left(\frac{1}{2}\left\|\mathcal{A}^2+\mathcal{C}^2\right\|+\frac{1}{2}\sup_{\|\mu\|=1}\left(\langle\mathcal{A}^2\mu,\mu\rangle+\langle\mathcal{C}^2\mu,\mu\rangle\right)\right).$$

Using the linearity of inner product

$$w^2(\mathcal{B}) \leq \frac{1}{2}\left(\frac{1}{2}\left\|\mathcal{A}^2+\mathcal{C}^2\right\|+\frac{1}{2}\sup_{\|\mu\|=1}\left\langle\left(\mathcal{A}^2+\mathcal{C}^2\right)\mu,\mu\right\rangle\right).$$

Thus,

$$w^2(\mathcal{B}) \leq \frac{1}{2}\left\|\mathcal{A}^2+\mathcal{C}^2\right\|.$$

Thus, for these parameter values, the Equation (15) *reduces to the block-matrix Equation* (7).

Theorem 8. *Let $\mathcal{A}_i\in\mathcal{B}(H)$ $(i=1,2,\cdots,\eta)$. Then,*

$$\begin{aligned}
w_e^2(\mathcal{A}_1,\cdots,\mathcal{A}_\eta) \leq \frac{1}{2}\beta\Bigg(&\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\|\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{4\alpha}+|\mathcal{A}_i^*|^{4(1-\alpha)}\right)\right\| \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\sup_{\|\mu\|=1}\sum_{i=1}^{\eta}\Bigg(\left\langle|\mathcal{A}_i|^{4\alpha}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu,\mu\right\rangle^2 \\
&+\left\langle|\mathcal{A}_i^*|^{4(1-\alpha)}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu,\mu\right\rangle^2\Bigg)\Bigg) \\
&+\frac{1}{2}(1-\beta)w_e(\mathcal{A}_1,\cdots,\mathcal{A}_\eta)\left\|\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{2\alpha}+|\mathcal{A}_i^*|^{2(1-\alpha)}\right)^2\right\|^{\frac{1}{2}},
\end{aligned} \tag{16}$$

for all $0 \le \alpha, \beta \le 1$ *and every integer refinement order* $\eta \ge 2$.

Proof. Let $\mu \in H$ be an arbitrary vector, $\|\mu\| = 1$. Setting $\nu = \mu$ in Lemma 8 with $\gamma = 1$, it follows that

$$\begin{aligned}
|\langle \mathcal{A}_i\mu, \mu\rangle|^2 &\le \beta \left\langle |\mathcal{A}_i|^{2\alpha}\mu, \mu\right\rangle \left\langle |\mathcal{A}_i^*|^{2(1-\alpha)}\mu, \mu\right\rangle \\
&\quad + (1-\beta)|\langle \mathcal{A}_i\mu, \mu\rangle| \sqrt{\left\langle |\mathcal{A}_i|^{2\alpha}\mu, \mu\right\rangle \left\langle |\mathcal{A}_i^*|^{2(1-\alpha)}\mu, \mu\right\rangle} \\
&\le \frac{1}{2}\beta \left(\left\langle |\mathcal{A}_i|^{2\alpha}\mu, \mu\right\rangle^2 + \left\langle |\mathcal{A}_i^*|^{2(1-\alpha)}\mu, \mu\right\rangle^2 \right) \qquad \text{(by Lemma 11)} \\
&\quad + \frac{1}{2}(1-\beta)|\langle \mathcal{A}_i\mu, \mu\rangle| \left\langle \left(|\mathcal{A}_i|^{2\alpha} + |\mathcal{A}_i^*|^{2(1-\alpha)} \right)\mu, \mu\right\rangle \qquad \text{(by Lemma 11)} \\
&\le \frac{1}{2}\beta \Bigg(\left(\left(\frac{1}{2}\right)^\eta + \left(\frac{1}{2}\right)^\eta \right) \left\langle |\mathcal{A}_i|^{4\alpha}\mu, \mu\right\rangle \\
&\quad + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{2}\right)^\kappa \left(\frac{1}{2}\right)^{\eta-\kappa} \left\langle |\mathcal{A}_i|^{4\alpha}\mu, \mu\right\rangle^{1-\frac{2\kappa}{\eta}} \left\langle |\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu, \mu\right\rangle^2 \\
&+ \left(\left(\frac{1}{2}\right)^\eta + \left(\frac{1}{2}\right)^\eta \right) \left\langle |\mathcal{A}_i^*|^{4(1-\alpha)}\mu, \mu\right\rangle \\
&\quad + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{2}\right)^\kappa \left(\frac{1}{2}\right)^{\eta-\kappa} \left\langle |\mathcal{A}_i^*|^{4(1-\alpha)}\mu, \mu\right\rangle^{1-\frac{2\kappa}{\eta}} \left\langle |\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu, \mu\right\rangle^2 \Bigg) \qquad \text{(by Lemma 4)} \\
&+ \frac{1}{2}(1-\beta)|\langle \mathcal{A}_i\mu, \mu\rangle| \left\langle \left(|\mathcal{A}_i|^{2\alpha} + |\mathcal{A}_i^*|^{2(1-\alpha)} \right)\mu, \mu\right\rangle \\
&\le \frac{1}{2}\beta \Bigg(\left(\left(\frac{1}{2}\right)^\eta + \left(\frac{1}{2}\right)^\eta \right) \left\langle \left(|\mathcal{A}_i|^{4\alpha} + |\mathcal{A}_i^*|^{4(1-\alpha)} \right)\mu, \mu\right\rangle \\
&\quad + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{2}\right)^\kappa \left(\frac{1}{2}\right)^{\eta-\kappa} \Bigg(\left\langle |\mathcal{A}_i|^{4\alpha}\mu, \mu\right\rangle^{1-\frac{2\kappa}{\eta}} \left\langle |\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu, \mu\right\rangle^2 \\
&\quad + \left\langle |\mathcal{A}_i^*|^{4(1-\alpha)}\mu, \mu\right\rangle^{1-\frac{2\kappa}{\eta}} \left\langle |\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu, \mu\right\rangle^2 \Bigg) \Bigg) \\
&+ \frac{1}{2}(1-\beta)|\langle \mathcal{A}_i\mu, \mu\rangle| \left\langle \left(|\mathcal{A}_i|^{2\alpha} + |\mathcal{A}_i^*|^{2(1-\alpha)} \right)\mu, \mu\right\rangle .
\end{aligned}$$

Summing over $i = 1$ up to $i = \eta$, and then applying the Cauchy-Schwarz inequality for real numbers, we obtain

$$\begin{aligned}
\sum_{i=1}^{\eta} |\langle \mathcal{A}_i\mu, \mu\rangle|^2 &\le \frac{1}{2}\beta \sum_{i=1}^{\eta} \Bigg(\left(\left(\frac{1}{2}\right)^\eta + \left(\frac{1}{2}\right)^\eta \right) \left\langle \left(|\mathcal{A}_i|^{4\alpha} + |\mathcal{A}_i^*|^{4(1-\alpha)} \right)\mu, \mu\right\rangle \\
&\quad + \sum_{\kappa=1}^{\eta-1} \binom{\eta}{\kappa} \left(\frac{1}{2}\right)^\kappa \left(\frac{1}{2}\right)^{\eta-\kappa} \Bigg(\left\langle |\mathcal{A}_i|^{4\alpha}\mu, \mu\right\rangle^{1-\frac{2\kappa}{\eta}} \left\langle |\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu, \mu\right\rangle^2 \\
&\quad + \left\langle |\mathcal{A}_i^*|^{4(1-\alpha)}\mu, \mu\right\rangle^{1-\frac{2\kappa}{\eta}} \left\langle |\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu, \mu\right\rangle^2 \Bigg) \Bigg) \\
&+ \frac{1}{2}(1-\beta) \sum_{i=1}^{\eta} \left(|\langle \mathcal{A}_i\mu, \mu\rangle| \left\langle \left(|\mathcal{A}_i|^{2\alpha} + |\mathcal{A}_i^*|^{2(1-\alpha)} \right)\mu, \mu\right\rangle \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\beta\left(\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\langle\left[\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{4\alpha}+|\mathcal{A}_i^*|^{4(1-\alpha)}\right)\right]\mu,\mu\right\rangle\right.\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\sum_{i=1}^{\eta}\left(\left\langle|\mathcal{A}_i|^{4\alpha}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu,\mu\right\rangle^{2}\right.\\
&\quad\left.\left.+\left\langle|\mathcal{A}_i^*|^{4(1-\alpha)}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu,\mu\right\rangle^{2}\right)\right)\\
&+\frac{1}{2}(1-\beta)\left(\sum_{i=1}^{\eta}|\langle\mathcal{A}_i\mu,\mu\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\eta}\left\langle\left(|\mathcal{A}_i|^{2\alpha}+|\mathcal{A}_i^*|^{2(1-\alpha)}\right)\mu,\mu\right\rangle^{2}\right)^{\frac{1}{2}}\\
&\leq \frac{1}{2}\beta\left(\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\langle\left[\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{4\alpha}+|\mathcal{A}_i^*|^{4(1-\alpha)}\right)\right]\mu,\mu\right\rangle\right.\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\sum_{i=1}^{\eta}\left(\left\langle|\mathcal{A}_i|^{4\alpha}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu,\mu\right\rangle^{2}\right.\\
&\quad\left.\left.+\left\langle|\mathcal{A}_i^*|^{4(1-\alpha)}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu,\mu\right\rangle^{2}\right)\right)\\
&+\frac{1}{2}(1-\beta)\left(\sum_{i=1}^{\eta}|\langle\mathcal{A}_i\mu,\mu\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\eta}\left\langle\left(|\mathcal{A}_i|^{2\alpha}+|\mathcal{A}_i^*|^{2(1-\alpha)}\right)^{2}\mu,\mu\right\rangle\right)^{\frac{1}{2}}\\
&= \frac{1}{2}\beta\left(\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\langle\left[\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{4\alpha}+|\mathcal{A}_i^*|^{4(1-\alpha)}\right)\right]\mu,\mu\right\rangle\right.\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\sum_{i=1}^{\eta}\left(\left\langle|\mathcal{A}_i|^{4\alpha}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu,\mu\right\rangle^{2}\right.\\
&\quad\left.\left.+\left\langle|\mathcal{A}_i^*|^{4(1-\alpha)}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu,\mu\right\rangle^{2}\right)\right)\\
&+\frac{1}{2}(1-\beta)\left(\sum_{i=1}^{\eta}|\langle\mathcal{A}_i\mu,\mu\rangle|^2\right)^{\frac{1}{2}}\left\langle\left[\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{2\alpha}+|\mathcal{A}_i^*|^{2(1-\alpha)}\right)^{2}\right]\mu,\mu\right\rangle^{\frac{1}{2}}.
\end{aligned}
$$

Evaluating supremum across all the unit vectors $\mu \in H$.

$$
\begin{aligned}
\sup_{\|\mu\|=1}\sum_{i=1}^{\eta}|\langle\mathcal{A}_i\mu,\mu\rangle|^2 &\leq \frac{1}{2}\beta\left(\left(\left(\frac{1}{2}\right)^{\eta}+\left(\frac{1}{2}\right)^{\eta}\right)\left\|\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{4\alpha}+|\mathcal{A}_i^*|^{4(1-\alpha)}\right)\right\|\right.\\
&\quad+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^{\kappa}\left(\frac{1}{2}\right)^{\eta-\kappa}\times\\
&\quad\sup_{\|\mu\|=1}\sum_{i=1}^{\eta}\left(\left\langle|\mathcal{A}_i|^{4\alpha}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu,\mu\right\rangle^{2}\right.\\
&\quad\left.\left.+\left\langle|\mathcal{A}_i^*|^{4(1-\alpha)}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu,\mu\right\rangle^{2}\right)\right)\\
&+\frac{1}{2}(1-\beta)\sup_{\|\mu\|=1}\left(\sum_{i=1}^{\eta}|\langle\mathcal{A}_i\mu,\mu\rangle|^2\right)^{\frac{1}{2}}\left\|\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{2\alpha}+|\mathcal{A}_i^*|^{2(1-\alpha)}\right)^{2}\right\|^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
w_e^2(\mathcal{A}_1,\cdots,\mathcal{A}_\eta) \leq \frac{1}{2}\beta\Bigg(\Bigg(\left(\left(\frac{1}{2}\right)^\eta+\left(\frac{1}{2}\right)^\eta\right)&\left\|\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{4\alpha}+|\mathcal{A}_i^*|^{4(1-\alpha)}\right)\right\| \\
&+\sum_{\kappa=1}^{\eta-1}\binom{\eta}{\kappa}\left(\frac{1}{2}\right)^\kappa\left(\frac{1}{2}\right)^{\eta-\kappa}\sum_{i=1}^{\eta}\left(\left\langle|\mathcal{A}_i|^{4\alpha}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i|^{\frac{4\alpha\kappa}{\eta}}\mu,\mu\right\rangle^2\right. \\
&\left.+\left\langle|\mathcal{A}_i^*|^{4(1-\alpha)}\mu,\mu\right\rangle^{1-\frac{2\kappa}{\eta}}\left\langle|\mathcal{A}_i^*|^{\frac{4(1-\alpha)\kappa}{\eta}}\mu,\mu\right\rangle^2\right)\Bigg)\Bigg) \\
&+\frac{1}{2}(1-\beta)w_e(\mathcal{A}_1,\cdots,\mathcal{A}_\eta)\left\|\sum_{i=1}^{\eta}\left(|\mathcal{A}_i|^{2\alpha}+|\mathcal{A}_i^*|^{2(1-\alpha)}\right)^2\right\|^{\frac{1}{2}},
\end{aligned}
$$

as required. □

Remark 9. *By setting $\eta=2$, $\gamma=2$ and $\alpha=\beta=\frac{1}{2}$ in the Equation* (16) *we obtain*

$$
\begin{aligned}
w_e^2(\mathcal{A}_1,\mathcal{A}_2) \leq \frac{1}{4}\left(\frac{1}{2}\left\|\sum_{i=1}^{2}\left(|\mathcal{A}_i|^2+|\mathcal{A}_i^*|^2\right)\right\|+\frac{1}{2}\sup_{\|\mu\|=1}\left(\sum_{i=1}^{2}\langle|\mathcal{A}_i|\mu,\mu\rangle^2+\langle|\mathcal{A}_i^*|\mu,\mu\rangle^2\right)\right) \\
+\frac{1}{4}w_e(\mathcal{A}_1,\mathcal{A}_2)\left\|\sum_{i=1}^{2}(|\mathcal{A}_i|+|\mathcal{A}_i^*|)^2\right\|^{\frac{1}{2}}.
\end{aligned}
$$

By Lemma 3

$$
\begin{aligned}
w_e^2(\mathcal{A}_1,\mathcal{A}_2) \leq \frac{1}{4}\left(\frac{1}{2}\left\|\sum_{i=1}^{2}\left(|\mathcal{A}_i|^2+|\mathcal{A}_i^*|^2\right)\right\|+\frac{1}{2}\sup_{\|\mu\|=1}\left(\sum_{i=1}^{2}\langle|\mathcal{A}_i|^2\mu,\mu\rangle+\langle|\mathcal{A}_i^*|^2\mu,\mu\rangle\right)\right) \\
+\frac{1}{4}w_e(\mathcal{A}_1,\mathcal{A}_2)\left\|\sum_{i=1}^{2}(|\mathcal{A}_i|+|\mathcal{A}_i^*|)^2\right\|^{\frac{1}{2}}.
\end{aligned}
$$

Using the linearity of the inner product

$$
\begin{aligned}
w_e^2(\mathcal{A}_1,\mathcal{A}_2) \leq \frac{1}{4}\left(\frac{1}{2}\left\|\sum_{i=1}^{2}\left(|\mathcal{A}_i|^2+|\mathcal{A}_i^*|^2\right)\right\|+\frac{1}{2}\sup_{\|\mu\|=1}\left\langle\left[\sum_{i=1}^{2}|\mathcal{A}_i|^2+|\mathcal{A}_i^*|^2\right]\mu,\mu\right\rangle\right) \\
+\frac{1}{4}w_e(\mathcal{A}_1,\mathcal{A}_2)\left\|\sum_{i=1}^{2}(|\mathcal{A}_i|+|\mathcal{A}_i^*|)^2\right\|^{\frac{1}{2}}.
\end{aligned}
$$

Thus,

$$
w_e^2(\mathcal{A}_1,\mathcal{A}_2) \leq \frac{1}{4}\left\|\sum_{i=1}^{2}\left(|\mathcal{A}_i|^2+|\mathcal{A}_i^*|^2\right)\right\|+\frac{1}{4}w_e(\mathcal{A}_1,\mathcal{A}_2)\left\|\sum_{i=1}^{2}(|\mathcal{A}_i|+|\mathcal{A}_i^*|)^2\right\|^{\frac{1}{2}}.
$$

Thus, for these parameter values, the Equation (16) *reduces to the Euclidean operator-radius Equation* (8).

3. Interpretation of the main estimates

The main results arise from the same refinement principle, but they are not repetitions of a single theorem with only a change of notation. Each theorem starts from a different known numerical-radius inequality and preserves the structure of that inequality. The difference between the starting inequalities determines the form of the final refined estimate. This section records the role of each family and explains why no general ordering among all of the displayed bounds is claimed.

Theorems 1 and 2 are the two basic single-operator estimates. Theorem 1 starts from the symmetric power-mean argument and produces an estimate for $w^{\gamma}(\mathcal{A})$ involving the two powers $|\mathcal{A}|^{2\alpha\gamma}$ and $|\mathcal{A}^*|^{2(1-\alpha)\gamma}$. Theorem 2, on the other hand, begins with the weighted product and mean inequalities. Its conclusion is an estimate for $w^{2\gamma}(\mathcal{A})$ in which the principal norm term is the weighted combination $\alpha|\mathcal{A}|^{2\gamma} + (1-\alpha)|\mathcal{A}^*|^{2\gamma}$. Thus, even when the same parameters are chosen, the left-hand powers, the positive operators and the weights are different. The two statements are consequently complementary. In an application where both are available, the natural procedure is to evaluate both valid upper bounds and retain the smaller one.

Theorem 3 is adapted to the Cartesian decomposition $\mathcal{A} = \mathcal{B} + i\mathcal{C}$. It is useful when the self-adjoint components $\mathcal{B}$ and $\mathcal{C}$ are the quantities naturally available from the problem. Theorem 4 has a different purpose. It uses the combinations $\mathcal{A} + \mathcal{A}^*$ and $\mathcal{A} - \mathcal{A}^*$ and is based on the Clarkson-type inequality. Its stated range is $\gamma \geq 2$, which is different from the parameter range in the first two theorems. These two estimates should therefore not be merged: one is expressed through the Cartesian components, while the other is expressed through the symmetric and skew-symmetric combinations of the operator.

Theorem 5 contains the additional parameter β. It combines a refined positive-operator term with a term involving $w^{\gamma}(\mathcal{A})$. The role of β is to retain the weighted structure inherited from Lemma 8; it is not merely a duplicate of the parameter α. The theorem is therefore appropriate when the mixed estimate of Lemma 8 is the available starting point. As with the other results, the recovery remark identifies a particular specialization rather than establishing a global comparison with every other theorem in the paper.

Theorem 6 concerns finite sums and necessarily contains the factor arising from Bohr's inequality. Its input consists of the Cartesian decompositions of the individual summands. This is structurally different from a one-operator estimate because the refinement is applied to each summand before the finite-sum step is completed. Theorem 7 addresses a positive 2×2 block matrix. Its assumptions include positivity of the block matrix, which ensures positivity of the diagonal blocks and allows the refined Holder–McCarthy inequality to be used on those blocks. Theorem 8 concerns the Euclidean operator radius of a tuple of operators. Its left-hand side and its data are different from the preceding numerical-radius estimates, so it belongs to a separate class of applications.

For these reasons, the theorem families are best understood as a collection of refinements for different initial inequalities. The common feature is the pointwise replacement furnished by Proposition 1. The differences are the operator expressions to which the replacement is applied, the parameter ranges, and the quantities being bounded. No assertion is made that one theorem dominates all others. Such a statement would require additional assumptions connecting the relevant positive operator expressions and is not part of the present paper.

The recovery remarks should also be viewed in this light. They show that the previously known estimate is contained in the stated family for the indicated parameter choices. The numerical comparisons below address a separate issue by showing strict improvement in selected finite-dimensional cases. Together, these two observations give the intended interpretation of the results: compatibility with the existing inequalities, together with a verified gain for concrete operators where the refinement is nontrivial.

There is also a simple selection principle for using the results. When only $|\mathcal{A}|$ and $|\mathcal{A}^*|$ are available, Theorem 1 or Theorem 2 is the direct choice. When the real and imaginary parts of $\mathcal{A}$ are available, Theorem 3 expresses the estimate in terms of those components. When the combinations $\mathcal{A} + \mathcal{A}^*$ and $\mathcal{A} - \mathcal{A}^*$ are simpler to evaluate, Theorem 4 provides the corresponding Clarkson-type form. The remaining theorems are selected by the structure of the data: a weighted mixed expression for Theorem 5, a finite family for Theorem 6, a positive block matrix for Theorem 7, and a tuple of operators for Theorem 8.

This classification is intended to prevent an artificial comparison between formulas that do not have the same input. For example, a finite-sum estimate carries information about each summand that is absent from a one-operator estimate, while a positive block-matrix estimate relies on positivity that is not assumed in the general one-operator setting. The separate statements therefore make the assumptions visible and allow the refinement to be used without suppressing the structure of the original inequality.

4. Numerical comparisons

The preceding remarks show how the known inequalities are recovered. To demonstrate that the refined estimates may also improve the corresponding classical bounds, we record two finite-dimensional examples. The numerical radius was computed by the standard formula

$$w(A) = \max_{0 \le t < 2\pi} \lambda_{\max} \left(\mathrm{Re}(e^{it} A) \right),$$

and for 2×2 matrices, the unit vectors in the refined suprema were parametrised as $\mu = (\cos s, e^{it} \sin s)$, where $0 \le s \le \pi/2$ and $0 \le t < 2\pi$. The reported values were checked by direct numerical maximisation and are rounded to four decimal places.

For finite matrices, the terms are evaluated from spectral decompositions of the positive operator powers together with an optimisation over the unit sphere. The refined bound can therefore be compared directly with the classical norm bound, while its additional optimisation cost is transparent.

The comparison is made at the level of the displayed upper bounds, not by replacing any theorem with a numerical approximation. For each matrix, the left-hand side is first evaluated from the numerical radius. The classical bound is then evaluated from the corresponding operator norm. Finally, the refined bound is evaluated from the right-hand side of the relevant theorem with the stated values of α, γ and η. Hence the three entries in each row have a distinct role: the numerical-radius quantity, the earlier upper bound, and the refined upper bound.

For the two non-normal matrices used below, $|A|$ and $|A^*|$ are positive definite. Their fractional powers were formed from their spectral decompositions. The remaining suprema are continuous functions of the unit vector. In dimension two, the parametrisation already given reduces each such supremum to a maximisation on the compact rectangle $[0, \pi/2] \times [0, 2\pi]$. This provides a direct check of the displayed values. The entries are rounded only after the maximisations have been performed.

The examples are not intended to establish a universal ordering between all numerical-radius bounds. They serve a narrower purpose: to verify that the refined expressions are strictly smaller than the corresponding classical expressions, and to show that equality can also hold. This distinction is relevant because a formal recovery of an earlier estimate does not by itself demonstrate a quantitative improvement.

Example 1. *Let*

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \qquad \alpha = \frac{1}{2}, \quad \gamma = 2, \quad \eta = 2.$$

For the classical estimate corresponding to Equation (1)*, one obtains*

$$w^2(A) = 9.4934, \qquad \frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\| = 11.7434.$$

The refined pointwise form obtained from Proposition 1 gives

$$w^2(A) \leq 11.1184,$$

which is strictly smaller than the classical upper bound. Thus the refinement improves the old estimate in this concrete non-normal case.

Example 2. *Let*

$$A = \begin{bmatrix} 1 & 2+i \\ 0 & 1/2 \end{bmatrix}, \qquad \alpha = \frac{1}{2}, \quad \gamma = 2, \quad \eta = 2.$$

For the inequality corresponding to Equation (2)*, the classical bound gives*

$$w^4(A) = 12.9130, \qquad \left\| \frac{1}{2}|A|^4 + \frac{1}{2}|A^*|^4 \right\| = 30.0217.$$

The refined pointwise bound gives

$$w^4(A) \leq 26.7404.$$

Again the refined estimate is sharper than the corresponding classical estimate.

The two strict comparisons in **Table 1** are also quantitatively different. For Example 1, the refined bound 11.1184 is approximately 5.3% below the classical bound 11.7434. For Example 2, the refined bound 26.7404 is approximately 10.9% below the classical bound 30.0217. These percentages are reported only as descriptions of the two computations; they are not claimed as uniform improvement rates for general matrices.

Table 1. Comparison of classical and refined upper bounds.

Matrix and estimate	Exact left-hand side	Classical bound	Refined bound
Example 1: $w^2(A)$	9.4934	11.7434	11.1184
Example 2: $w^4(A)$	12.9130	30.0217	26.7404

The diagonal matrix in Example 3 provides the complementary check. There, the classical and refined quantities coincide with the exact value of $w^2(D)$. Thus the examples distinguish three possibilities that are relevant for the interpretation of the theorems: a refined bound may be strictly smaller than the earlier bound, it may agree with it, and the actual comparison has to be determined from the operator and the chosen parameters.

Example 3. *Let*

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \alpha = \frac{1}{2}, \quad \gamma = 2, \quad \eta = 2.$$

The matrix D is normal, and $|D| = |D^*| = D$. Therefore

$$w^2(D) = 4, \qquad \frac{1}{2}\left\| |D|^2 + |D^*|^2 \right\| = 4.$$

The refined expression in Theorem 1 is also equal to 4. Thus equality occurs in this case. This example is included to make clear that the refinement is not claimed to be strictly better for every operator. The strict gains occur in the two non-normal examples selected here.

A parameter check for the first example

The refinement order η can be varied while the underlying matrix and the other parameters are kept fixed. For the matrix in Example 1, we retain $\alpha = \frac{1}{2}$ and $\gamma = 2$ and evaluate the right-hand side of Theorem 1. The classical bound remains 11.7434, whereas the refined values in **Table 2** are all smaller. In this particular example the values decrease for $\eta = 2, 3, 4$. This is a numerical observation for the stated matrix only; no monotonicity in η is asserted for arbitrary operators.

Table 2. Dependence of the refined bound in Theorem 1 on η for the matrix in Example 1.

η	$w^2(A)$	Classical bound	Refined bound from Theorem 1
2	9.4934	11.7434	11.1184
3	9.4934	11.7434	10.8871
4	9.4934	11.7434	10.7761

5. Application to a stable non-normal differential system

We give a concrete application of the refined numerical-radius estimate to the convergence analysis of a variational iteration correction for a differential system. The example is deliberately low-dimensional so that every quantity in the comparison is evaluated exactly. Its purpose is to exhibit a non-normal problem for which the refined bound certifies convergence, whereas the corresponding classical numerical-radius bound is inconclusive.

Consider the forced linear system

$$y'(t) = My(t) + g(t), \qquad M = \begin{bmatrix} -\frac{1}{5} & \frac{9\sqrt{3}}{5} \\ 0 & -2 \end{bmatrix}, \qquad t \geq 0, \tag{17}$$

where $y(t) \in \mathbb{C}^2$ and g is prescribed. Since M is upper triangular, its eigenvalues are $-\frac{1}{5}$ and -2; hence the homogeneous system is asymptotically stable. On the other hand, M is non-normal. Define

$$T = \begin{bmatrix} \frac{3}{5} & \frac{3\sqrt{3}}{5} \\ 0 & 0 \end{bmatrix} = \frac{6}{5}uv^*, \qquad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad v = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}. \tag{18}$$

Thus u and v are unit vectors, $|\langle u, v\rangle| = \frac{1}{2}$, and $M = -2I + 3T$.

For the backward Euler discretisation with step size $h = 1$, the time-level system is

$$(3I - 3T)y_{n+1} = r_n, \qquad r_n = y_n + g(t_{n+1}). \tag{19}$$

At each time level, consider the variational iteration correction

$$\begin{aligned} y_{n+1}^{(k+1)} &= y_{n+1}^{(k)} - \frac{1}{3}\Big[(3I - 3T)y_{n+1}^{(k)} - r_n\Big] \\ &= Ty_{n+1}^{(k)} + \frac{1}{3}r_n. \end{aligned} \tag{20}$$

This is the linear residual correction associated with the constant VIM multiplier $-\frac{1}{3}$.

Proposition 2. *For every n and every initial vector $y_{n+1}^{(0)}$, the Equation* (20) *converges to the unique backward Euler solution of Equation* (19)*. This convergence is certified by the refined inequality in Theorem 1, with*

$$\alpha = \frac{1}{2}, \qquad \gamma = 2, \qquad \eta = 2,$$

whereas the corresponding classical numerical-radius bound does not certify convergence.

Proof. By the special case of Theorem 1 recorded in Remark 1, one has

$$\begin{aligned} w^2(T) \leq \frac{1}{4}\Bigg[&\left\| |T|^2 + |T^*|^2 \right\| \\ &+ \sup_{\|x\|=1} \left\{ \langle |T|x, x\rangle^2 + \langle |T^*|x, x\rangle^2 \right\} \Bigg]. \end{aligned} \tag{21}$$

From Equation (18),

$$|T| = \frac{6}{5}vv^*, \qquad |T^*| = \frac{6}{5}uu^*.$$

Therefore

$$\big\||T|^2 + |T^*|^2\big\| = \frac{36}{25}\big\|uu^* + vv^*\big\| = \frac{54}{25}. \tag{22}$$

For the scalar term in Equation (21), after a harmless choice of phase,

$$x = \begin{bmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{bmatrix}, \qquad 0 \le \theta \le \frac{\pi}{2}.$$

For fixed θ, the displayed expression is maximised at $\phi = 0$. Hence

$$\begin{aligned}&\sup_{\|x\|=1} \left\{\langle |T|x, x\rangle^2 + \langle |T^*|x, x\rangle^2\right\} \\ &\qquad = \frac{36}{25} \max_{0\le\theta\le\pi/2} \left\{\cos^4\theta + \cos^4\left(\theta - \frac{\pi}{3}\right)\right\} = \frac{36}{25}\cdot\frac{9}{8} = \frac{81}{50}.\end{aligned}$$

Indeed, if $z = \cos(2\theta - \frac{\pi}{3})$, then

$$\cos^4\theta + \cos^4\left(\theta - \frac{\pi}{3}\right) = \frac{7}{8} + \frac{1}{2}z - \frac{1}{4}z^2 \le \frac{9}{8},$$

with equality at $\theta = \frac{\pi}{6}$. Substitution into Equation (21) gives

$$w^2(T) \le \frac{1}{4}\left(\frac{54}{25} + \frac{81}{50}\right) = \frac{189}{200}, \qquad w(T) \le \sqrt{\frac{189}{200}} \approx 0.9721 < 1. \tag{23}$$

Thus $\rho(T) \le w(T) < 1$, and consequently $I - T$ is invertible. The unique solution of Equation (19) is therefore

$$\widehat{y}_{n+1} = \frac{1}{3}(I - T)^{-1} r_n.$$

If $e_k = y_{n+1}^{(k)} - \widehat{y}_{n+1}$, then Equation (20) gives

$$e_{k+1} = Te_k.$$

Since $\rho(T) < 1$, it follows that $T^k \to 0$ and hence $e_k \to 0$.

For comparison, the classical estimate recovered by retaining only the upper McCarthy bound gives

$$w^2(T) \le \frac{1}{2}\big\||T|^2 + |T^*|^2\big\| = \frac{27}{25}, \qquad w(T) \le \frac{3\sqrt{3}}{5} \approx 1.0392. \tag{24}$$

This bound is above one and hence does not establish $w(T) < 1$ or certify convergence of Equation (20). □

For reference, direct computation gives

$$\rho(T) = \frac{3}{5}, \qquad w(T) = \frac{9}{10};$$

These values are included only to verify the example; the convergence certificate in

Proposition 2 uses the refined bound Equation (23). The corresponding convergence information is summarized in **Table 3**.

Table 3. Convergence information for the variational iteration correction of Equation (20).

Quantity	**Value**	**Consequence for the iteration**
Actual spectral radius $\rho(T)$	$\frac{3}{5} = 0.6000$	Convergent
Actual numerical radius $w(T)$	$\frac{9}{10} = 0.9000$	Convergent
Classical numerical-radius upper bound	$\frac{3\sqrt{3}}{5} \approx 1.0392$	Inconclusive
Refined upper bound from Theorem 1	$\sqrt{\frac{189}{200}} \approx 0.9721$	Convergent

Computational remarks

The refined bounds contain more information than the corresponding norm bounds, and this also explains the additional computational work. In finite dimension, the first step is to form the positive matrices $|A| = (A^*A)^{1/2}$ and $|A^*| = (AA^*)^{1/2}$. Their required powers are obtained by diagonalising these positive matrices. The norm terms are then the largest eigenvalues of the relevant positive combinations. The remaining terms are scalar functions of a unit vector and can be maximised over the unit sphere.

For a 2×2 matrix, the unit-vector parametrisation used above gives a direct two-variable computation. In higher dimensions, the same expressions can be evaluated by optimisation on the unit sphere, or by exploiting additional structure such as diagonal, normal, block, or sparse forms. The purpose of the present paper is not to propose a new numerical algorithm. The numerical computations are included only to show that the refined expressions can be evaluated directly and compared with their classical counterparts in finite-dimensional cases.

The trade-off is therefore explicit. The classical bounds require operator norms of positive matrices, while the refined bounds additionally contain suprema of scalar expressions. When a sharper upper estimate is needed, this additional step can be justified; when only a quick coarse bound is required, the classical expression may be preferable. Equation (14) for finite sums makes the same point in a more visible way, since the contribution of each summand and each refinement term has to be evaluated. No claim is made that the refined formula is computationally cheaper.

6. Conclusion

This paper presented a unified refinement method for numerical radius inequalities based on the refined Hölder–McCarthy inequality of Akkouchi and Ighachane. The main point is that the usual McCarthy term can be replaced, at the level of unit vectors, by a smaller parameter-dependent expression. Applying this principle to known estimates gives refined versions of inequalities for a single operator, Cartesian decompositions, finite sums, positive block matrices and Euclidean operator radius estimates.

The remarks following the theorems identify the precise parameter choices for which the classical inequalities are recovered. The numerical examples show that the refinement is not merely formal: for suitable non-normal matrices, the refined upper bounds are strictly smaller than the corresponding classical bounds. These comparisons also indicate how the additional parameters may be selected in practice.

The scope of the conclusion is deliberately limited. The paper does not claim that the displayed refinements dominate every numerical-radius estimate in the literature, nor that they are strictly better for all operators. What is established is a common pointwise refinement mechanism, the recovery of the stated classical inequalities for the indicated parameter choices, and strict improvement in the finite-dimensional non-normal examples considered in Section 4. The normal example shows that equality may also occur.

The relation among the theorems is now also explicit. The first two results concern different one-operator starting inequalities; the next two use Cartesian and Clarkson-type structures; the remaining results address a weighted mixed estimate, finite sums, positive block matrices and Euclidean operator radius. Their common proof device is the refined Holder–McCarthy inequality, while their hypotheses and output expressions remain different. This is why they are retained as separate results.

For finite matrices, the additional terms can be computed from spectral decompositions and unit-sphere maximisation. The examples show that this added computation may provide a smaller bound. In applications where several estimates are available, the appropriate procedure is to calculate the applicable bounds under the available structural assumptions and use the smallest verified upper estimate.

The finite-dimensional calculations also provide a reproducibility point. For the matrices used in the examples, all quantities on the right-hand sides are functions of the positive matrices $|A|$ and $|A^*|$. Their powers are obtained by the spectral theorem, and the remaining scalar quantities are maximised over unit vectors. Thus the numerical comparison does not depend on an unrecorded modelling assumption or on an external data set. It is a direct evaluation of the formulas proved in the paper.

The present discussion also indicates a reasonable direction for further comparison work. For a fixed class of matrices, one may compare the different theorem families only after verifying that their hypotheses apply, and then optimize the parameters within their stated ranges. Equality cases and systematic parameter selection are natural questions, but they require additional analysis and are not claimed here. The results of this paper provide the refinement formula and the concrete inequalities on which such a study can be based.

Future work may focus on optimizing the refinement parameters for special classes of matrices and operators, identifying equality cases, and studying whether the same refinement mechanism can be combined with other numerical-radius tools such as Davis–Wielandt radius inequalities, block operator techniques and spectral radius estimates.

Numerical-radius estimates also arise naturally in the analysis of stability and robustness questions for operator-based system models; related differential-equation and control perspectives are available in the literature [29,30].

Structured singular-value bounds for Toeplitz and symmetric Toeplitz matrices have likewise been used in stability and robustness analysis of linear feedback systems [31].

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