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Article

Mathematical justification of stabilized 2D piezoelectric plates with electromagnetic feedback

Munyabugingo Valens[1,2], Banzi Wellars[1], Abdou Sène[3,*]

[1] Department of Mathematics, School of Science, College of Science and Technology, University of Rwanda, Kigali P.O. Box 3900, Rwanda
[2] Department of Mathematics and Computer Science Education, School of Mathematics and Science Education, College of Education, University of Rwanda, Rwamagana P.O. Box 55, Rwanda
[3] Pôle d'Innovation et d'Expertise pour le Développement (PIED), Université Numérique Cheikh Hamidou Kane (UNCHK), Fann, Dakar BP 15126, Sénégal
*** Corresponding author:** Abdou Sène, abdou.sene@unchk.edu.sn

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Abstract: In this paper, we provide a rigorous mathematical justification for a simplified model of a piezoelectric plate stabilized around a steady state. Asymptotic analysis of a 3D piezoelectric materials model with linear feedback control laws is performed as the thickness h of the plate tends to zero. We derive a 2D piezoelectric plate model which is consistent and stable for an approximation of the 3D model, ensuring its validity for the design of thin electromechanical devices. The energy decay for the 2D and 3D systems is established. Such a dimension reduction is very important because, when the plate thickness is very small, it simplifies numerical calculations and, above all, avoids the numerical calculation problems caused by distortion between the plate dimensions. Furthermore, the study reveals that when the thickness of a piezoelectric plate is very small, we no longer need the restrictions to only eleven stabilizable types of piezoelectric materials. The core contribution of this work is the direct integration of this fully coupled dimensional reduction with control theory.

Keywords: piezoelectricity; asymptotic analysis; scaling; feedback control

1. Introduction

Despite being predicted by Coulomb and discovered by Becquerel in 1819, the piezoelectric effect was not properly explained experimentally until 1880 by the brothers Jacques and Pierre Curie. An overview of the history of piezoelectricity can be read in the study by Bao and Zhang [1]. Since then, various rigorous mathematical models have been developed to describe piezoelectric behavior more accurately [2–5]. Piezoelectricity is an electromechanical interaction. Piezoelectric materials are dielectrics that deform under the effect of an electric field and generate polarization under the effect of deformations. The latter phenomenon is called the "direct effect" for purely historical reasons, given its reversible and symmetrical nature [6]. This behavior is expressed by laws connecting the mechanical stress tensor and the electric field vector on the one hand, and the strain tensor and the polarization vector or the electrical displacement vector on the other.

In recent years, the engineering application of smart materials has undergone significant diversification, heavily driven by breakthroughs in piezoelectric energy harvesting systems, self-powered sensors, and active bio-implantable components. To maximize power conversion efficiency, modern configurations frequently exploit complex structural mechanisms, broadband frequencies, and intentionally designed nonlinear vibrations. The precise analysis of these systems demands deep insights into structural optimization, nonlinear plate dynamics, and chaotic responses under large external excitations [7, 8]. However, a critical gap persists between applied control engineering and foundational mathematical physics. The literature extensively covers the optimization of harvesting circuits or numerical modeling of chaotic attractors [9], these models routinely rely on the quasi-static approximation, completely neglecting the transient dynamics of the underlying electromagnetic fields. Prior to addressing full nonlinear or chaotic regimes, a mathematically airtight baseline must be established. This study bridges this gap by executing a rigorous 3D to 2D asymptotic reduction that retains full electromagnetic field coupling under an isotropic baseline, directly synthesizing linear feedback control laws that prove exactly how the coupled mechanical and electromagnetic energy dissipate over time. This provides the exact structural and control foundation required before secure scaling into complex nonlinear operations can take place.

The application of asymptotic analysis of piezoelectric materials models is particularly vital when transitioning from complex 3D structures to simplified 2D models for thin plates or films. Modern research has utilized these techniques to explore analysis in piezoelectric semiconductors, specifically focusing on ensuring that crack-face boundary conditions remain energetically consistent [10]. Such consistency is paramount when modeling the structural integrity of smart materials under high stress. This mathematical foundation supports the development of next-generation devices, such as high-effectiveness energy harvesting gyroscopes, which rely on precise comparative modeling to maximize voltage output from ambient vibrations [11]. As the field moves toward the integration of ultrathin 2D materials, international research initiatives continue to push the boundaries of how these materials are modeled and applied in industrial contexts [12].

In the industrial sector, these characteristics are employed to regulate specific elastic structures actively. In this regard, by using a dispersed network of sensors and actuators embedded in or on the material, structural vibrations can be reduced or even stopped. In addition, they are employed in the control of the shapes of telescope mirrors, airplane wings, propellers, and artificial organs in biomechanics, among many other applications [2, 13, 14].

A coupled system of partial differential equations (PDEs) that characterizes the relationship between electromagnetic and mechanical displacement fields mathematically models piezoelectric systems. The mathematical and numerical modeling of these systems becomes much more difficult when they are embedded in thin geometries like plates, beams, or shells. The governing equations become stiff due to the small thickness, necessitating the use of asymptotic techniques to produce accurate and computationally efficient reduced models [4, 15, 16].

Typically, these models include elliptic and hyperbolic equations that are analyzed in the Sobolev and Lebesgue spaces, especially in Hilbert space settings such as $H^1(\Omega)$, $H_0^1(\Omega)$, and $L^2(\Omega)$. The displacement field and the magnetic potential are sought in $(H^1(\Omega))^3$, while the electric potential belongs in $H^1(\Omega)$. The application of variational techniques, energy estimates, and asymptotic analysis produces models that are accurate and computationally efficient. The use of such spaces guarantees the mathematical well-posedness of weak formulations [17, 18]. Piezoelectric systems must be controlled and stabilized to guarantee dependability and performance, especially in precision and vibration suppression applications [19, 20]. Due to their robustness under real-world constraints, simplicity, and ease of implementation, linear feedback controls continue to be attractive. However, there is still a lack of a thorough understanding of how these linear controls behave under asymptotic limits, especially as the domain's thickness tends to zero. This paper addresses this gap by performing a detailed asymptotic analysis of a piezoelectric system equipped with linear feedback controls as the thickness parameter $h \to 0$. Starting from the full three-dimensional model, we derive a reduced-order model that captures the essential electromechanical coupling in the thin-structure limit. We establish convergence results using compactness arguments and prove uniform energy decay under the proposed control strategy. Importantly, we show that the linear feedback control laws remain effective at the limit, providing a theoretical foundation for their use in thin piezoelectric structures. Moreover, electromagnetic control of thin piezoelectric structures is made possible by the involvement of the magnetic field in feedback control laws. The mathematical and control frameworks establish a direct practical relevance for the design, simulation, and operational reliability of piezoelectric Micro-Electromechanical Systems (MEMS). Modern thin-film piezoelectric materials applications such as film bulk acoustic resonators (FBARs), piezoelectric micro-machined ultrasonic transducers (PMUTs), radio-frequency (RF) filters, and micro-scale energy harvesters operate under increasingly high frequencies and rapid transient excitations [8]. At these micro-scales, traditional modeling and simulation workflows almost universally invoke the quasi-static approximation, treating the electric field as a stationary potential gradient ($\mathbf{E} = -\nabla\phi$) and entirely neglecting the magnetic field. However, as MEMS devices scale down and operational frequencies rise into the megahertz and gigahertz, these neglected electromagnetic dynamics introduce unmodeled energy losses, phase shifts, and parasitic electromagnetic couplings. These unmodeled phenomena present a severe challenge for structural control. Although the current controlled piezoelectric beam model with dynamic electromagnetic effects is developed and a rigorous framework for stabilization analysis is provided by Özer and Morris [21], the magnetic field is mainly treated as a passive component of the coupled system. The explicit use of magnetic field dynamics for control design, exact controllability, and robust stabilization remains largely unexplored, motivating further investigation into magnetic field based control strategies for piezoelectric structures. Instead, the control community routinely designs feedback laws based on electrostatic effects, making them blind to dynamic magnetic induction and transient electrical fields, which can trigger localized destabilization and spillover in high-frequency MEMS applications. This study directly bridges these engineering and mathematics aspects by performing a rigorous 3D to 2D asymptotic reduction that preserves the complete, coupled Maxwell piezoelectric equations for an isotropic material. We provide MEMS engineers with a computationally efficient analytical model that circumvents exhaustive 3D multiscale finite element simulations.

The main contribution of this article is a rigorous justification of a stabilized 2D piezoelectric plate model via the electromagnetic field. Furthermore, the asymptotic analysis of the piezoelectric materials model carried out in the present work removes the restriction imposed by Bidouan et al. [22] to eleven types of materials, and reveals stabilization or strong stabilization of its perturbations when the thickness h of a piezoelectric plate is very small, and this holds true for any piezoelectric material.

The rest of the document is organized as follows. The control framework and the governing equations are presented in Section 2. The preliminary results of stabilization and the existence of a unique solution are tackled in Section 3. The convergence results, the asymptotic limit process, the limit controlled system, and its energy decay characteristics are covered in Section 4. Finally, concluding remarks are made in Section 5.

2. Notations and problem statement

Let $\Omega = \omega \times]-1, 1[$ be a reference cylindrical domain of $\mathbb{R}^3$, where the middle surface ω, is a domain of $\mathbb{R}^2$. Let the boundary of Ω be denoted by $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma_\ell$, where $\Gamma^+ = \omega \times \{+1\}$ is the upper surface, and $\Gamma^- = \omega \times \{-1\}$ is the lower surface, and $\Gamma_\ell = \partial\omega \times]-1, 1[$ is the lateral surface. For $T > 0$ fixed, let $Q = [0, T] \times \Omega$. The Neumann boundary conditions are imposed on $\Gamma_N = \Gamma^+ \cup \Gamma^-$. The Dirichlet boundary conditions are imposed on Γ_ℓ. The domain and its boundary components stated above are clarified in **Figure 1**.

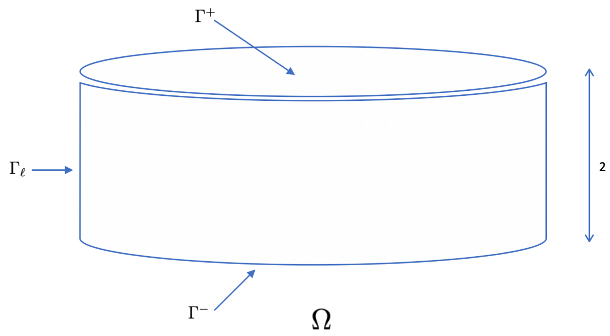

Figure 1. Cylindrical domain.

Note that for the vector field $\mathbf{v} = (v_1, v_2, v_3)$, the operators curl and div are defined as follows: $\operatorname{curl} \mathbf{v} := \nabla \times \mathbf{v} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix} = \nabla \wedge \mathbf{v}$, $\operatorname{div}(\mathbf{v}) := \nabla \cdot \mathbf{v} = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3$, $\partial_i v_j = \dfrac{\partial v_j}{\partial x_i}$, $1 \leq i, j \leq 3$.

The electromagnetic fields are given in terms of the scalar and vector potentials by:

$$
\begin{cases}
\mathbf{E} = -\nabla \phi_p - \dfrac{\partial \mathbf{A}_p}{\partial t}, & (1) \\
\mathbf{B} = \nabla \times \mathbf{A}_p, & (2) \\
\mathbf{H} = \dfrac{1}{\mu_1} \mathbf{B}, & (3)
\end{cases}
$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic flux density, $\mathbf{H}$ is the magnetic field intensity, $\mathbf{A}_p$ is the magnetic vector potential, ϕ_p is the electric potential, and μ_1 is the magnetic permeability. For a linear piezoelectric medium, the constitutive equations are

$$\begin{cases} \sigma = Ce(\mathbf{v}_p) - P\mathbf{E}, & (4) \\ \mathbf{D} = Pe(\mathbf{v}_p) + \epsilon\mathbf{E}, & (5) \end{cases}$$

where σ denotes the stress tensor representing internal forces within the material, $\mathbf{v}_p$ is the displacement vector of the material, $\mathbf{D}$ is the electric displacement vector accounting for both free and bound charges, $C = (C_{ijkl})$ is the fourth order elasticity tensor, $P = (P_{ijk})$ is the third order piezoelectric tensor, $\epsilon = (\epsilon_{ij})$ is the second order dielectric tensor, and $e(\mathbf{v}_p) = (e_{ij}(\mathbf{v}_p)) = \frac{1}{2}\left(\frac{\partial v_{pj}}{\partial x_i} + \frac{\partial v_{pi}}{\partial x_j}\right), 1 \leq i, j \leq 3$ is the material strain tensor. Expressions $P\mathbf{E}$ and Pe for a given vector $\mathbf{E}$ or a given second order tensor e are defined by:

$$(P\mathbf{E})_{ij} = P_{kij}\mathbf{E}_k, \quad (Pe)_i = P_{ikl}e_{kl}.$$

The material is supposed to be homogeneous. All coefficients are constant through the material. The tensors C and ϵ are elliptic in the sense that

$$\exists\, c > 0, \forall e \in \mathbb{R}^9, C_{ijkl}e_{kl}e_{ij} > c\sum_{i,j=1}^{3}(e_{ij})^2, \forall\, \mathbf{E} \in \mathbb{R}^3, \epsilon_{ij}\mathbf{E}_j\mathbf{E}_i > c\sum_{i=1}^{3}(\mathbf{E}_i)^2. \tag{6}$$

The balance linear momentum reads as follows:

$$\rho\frac{\partial^2 \mathbf{v}_p}{\partial t^2} - \nabla \cdot \sigma = \mathbf{f} \quad \text{in} \quad Q, \tag{7}$$

where ρ is the density of the piezoelectric material occupying the domain, and $\mathbf{f}$ is the density of the volumetric force.

The Gauss law is given by:

$$\nabla \cdot \mathbf{D} = q_p \quad \text{in} \quad Q, \tag{8}$$

where q_p is the density of the volumetric electric charge.

The Maxwell-Ampère and Maxwell-Faraday laws are given respectively by:

$$\nabla \times \left(\frac{1}{\mu_1}\nabla \times \mathbf{A}_p\right) = \mathbf{J}_p + \frac{\partial \mathbf{D}}{\partial t} \quad \text{in} \quad Q,$$

and

$$\nabla \times \mathbf{E} = -\tfrac{\partial \mathbf{B}}{\partial t} \quad \text{in} \quad Q,$$

where $\mathbf{J}_p$ is the density of the electric current.

Remark 1. *Note that if one chooses to work with the magnetic vector potential* $\mathbf{A}$ *and the electric potential* ϕ*, as is the case in this article, one is faced with the non-uniqueness of the potentials. Indeed, if one expresses the electric field and the magnetic field in terms of the potentials*

$$\mathbf{B} = \nabla \times \mathbf{A}_p, \qquad \mathbf{E} = -\nabla\phi_p - \tfrac{\partial \mathbf{A}_p}{\partial t},$$

then the gauge transformation

$$\widetilde{\mathbf{A}} = \mathbf{A}_p + \nabla\chi, \qquad \widetilde{\phi} = \phi_p - \tfrac{\partial\chi}{\partial t}$$

leaves **E** *and* **B** *unchanged. Therefore, the potentials are not unique: an infinite number of pairs* $(\phi_p, \mathbf{A}_p)$ *describe the same physical fields. Choosing a gauge (Coulomb, Lorenz, etc.) serves to fix this freedom in order to obtain a unique representation.*

However, in what follows, in the residual piezoelectric system to be controlled, it can be noted that the problem of non-uniqueness of the potential pair does not arise due to the choice of feedback control which allows to establish an existence and uniqueness theorem in Section 3.2.

Then, the piezoelectric materials model with initial and boundary conditions is given by:

$$\begin{cases} \text{(a)} \quad \rho\mathbf{v}_p'' - \operatorname{div}[Ce(\mathbf{v}_p) - P(-\nabla\phi_p(\mathbf{v}_p) - \mathbf{A}_p')] = \mathbf{f} & \text{in} \quad Q, \\ \text{(b)} \quad \operatorname{div}(Pe(\mathbf{v}_p) + \epsilon(-\nabla\phi_p(\mathbf{v}_p) - \mathbf{A}_p')) = q_p & \text{in} \quad Q, \\ \text{(c)} \quad \dfrac{1}{\mu_1}\operatorname{curl}\operatorname{curl}\mathbf{A}_p = \mathbf{J}_p + \dfrac{\partial}{\partial t}\Big[Pe + \epsilon(-\nabla\phi_p(\mathbf{v}_p) - \mathbf{A}_p')\Big] & \text{in} \quad Q, \\ \text{(d)} \quad [Ce(\mathbf{v}_p) - P(-\nabla\phi_p(\mathbf{v}_p) - \mathbf{A}_p')]\mathbf{n} = 0 & \text{on} \quad \Gamma_N, \\ \text{(e)} \quad \operatorname{curl}\mathbf{A}_p \wedge \mathbf{n}/\mu_1 = -\mathbf{j}_p & \text{on} \quad \Gamma_N, \\ \text{(f)} \quad \mathbf{v}_p = 0 & \text{on} \quad \Gamma_\ell, \\ \text{(g)} \quad \phi_p = \phi_{p0} & \text{on} \quad \Gamma, \\ \text{(h)} \quad \mathbf{A}_p = 0 & \text{on} \quad \Gamma_\ell, \end{cases} \tag{9}$$

where q_p is the density of the electric charge and $\mathbf{j}_p$ is the density of the surface current. Equation (9d) enforces a homogeneous Neumann condition on Γ_N, signifying that the plate faces are traction-free, where the combined mechanical and electromechanical stress vanishes. On the same surface, Equation (9e) prescribes the tangential magnetic field via surface current density $\mathbf{j}_p$, representing external electromagnetic excitation. In contrast, the lateral boundary Γ_ℓ is subject to the Dirichlet condition in Equation (9f,h), physically representing a clamped edge where mechanical displacement and magnetic vector potential are zero. Finally, the electrical state is regulated on the boundary Γ through Equation (9g), where the Dirichlet condition $\phi_p = \phi_{p0}$ defines the interface for the prescribed voltage. Considering a reference steady state, we have:

$$\begin{cases} \text{(a)} \quad -\operatorname{div}(Ce(\mathbf{v}_r) + P\nabla\phi_r(\mathbf{v}_r)) = \mathbf{f} & \text{in} \quad Q, \\ \text{(b)} \quad \operatorname{div}(Pe(\mathbf{v}_r) - \epsilon\nabla\phi_r(\mathbf{v}_r)) = q_r & \text{in} \quad Q, \\ \text{(c)} \quad \dfrac{1}{\mu_1}\operatorname{curl}\operatorname{curl}\mathbf{A}_r = \mathbf{J}_r & \text{in} \quad Q, \\ \text{(d)} \quad (Ce(\mathbf{v}_r) + P\nabla\phi_r(\mathbf{v}_r))\mathbf{n} = 0 & \text{on} \quad \Gamma_N, \\ \text{(e)} \quad \operatorname{curl}\mathbf{A}_r \wedge \mathbf{n}/\mu_1 = -\mathbf{j}_r & \text{on} \quad \Gamma_N, \\ \text{(f)} \quad \mathbf{v}_r = 0 & \text{on} \quad \Gamma_\ell, \\ \text{(g)} \quad \phi_r = \phi_{r0} & \text{on} \quad \Gamma, \\ \text{(h)} \quad \mathbf{A}_r = 0 & \text{on} \quad \Gamma_\ell, \end{cases} \tag{10}$$

where the variables $\mathbf{A}_r, \phi_r, \mathbf{v}_r$ for reference steady state and parameter $\mathbf{n}$ are defined as follows:

- $\mathbf{A}_r$ is the magnetic vector potential within the material,
- ϕ_r is the electric potential,

- $\mathbf{v}_r$ is the displacement vector of the material,
- $\mathbf{n}$ is the unit outer normal vector to Γ.

Remark 2. *It should be noted that, for the reasons outlined in Remark 1, a gauge could be added to Equation (10), but this has no bearing on the design of the feedback laws for the residual system to be controlled.*

Tensors C, P and ϵ satisfy the following properties using the Einstein summation convention.

1. Given its symmetric nature, the stiffness tensor C is defined as

$$C = \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2222} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{pmatrix},$$

and satisfies the coercivity condition: $\exists\, \mathscr{X} > 0$ such that, for any real and symmetric matrix M of order 3, we have

$$C_{ijkl} M_{kl} M_{ij} \geq \mathscr{X} \sum_{i,j=1}^{3} (M_{ij})^2. \tag{11}$$

Additionally, C complies with the following symmetry properties:

$$C_{ijkl} = C_{klij} = C_{jikl}. \tag{12}$$

2. The dielectric constant tensor (also called dielectric tensor or permittivity tensor) ϵ defined as

$$\epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix},$$

satisfies the condition: $\exists\, \mathscr{N} > 0$ such that, for any vector $\theta \in \mathbb{R}^3$, we have the coercivity condition:

$$\epsilon_{kl} \theta_k \theta_l \geq \mathscr{N} \sum_{j=1}^{3} \theta_j^2, \tag{13}$$

and the symmetry property:

$$\epsilon_{kl} = \epsilon_{lk}. \tag{14}$$

3. The piezoelectric tensor P is defined as

$$P = \begin{pmatrix} P_{111} & P_{122} & P_{133} & P_{123} & P_{113} & P_{112} \\ P_{211} & P_{222} & P_{233} & P_{223} & P_{213} & P_{212} \\ P_{311} & P_{322} & P_{333} & P_{323} & P_{313} & P_{312} \end{pmatrix},$$

and satisfies the following symmetry property:

$$P_{mkl} = P_{mlk}. \tag{15}$$

Let $(\mathbf{u}, \overline{\mathbf{A}}, \Psi)$ be the trajectory resulting from the perturbation $(\mathbf{v}^0, \mathbf{v}^1, \mathbf{A}^0, \mathbf{A}^1)$ of the reference state $(\mathbf{v}_r, \mathbf{A}_r, \phi_r)$ with initial conditions

$$(\mathbf{u}^0, \mathbf{u}^1) = (\mathbf{v}_r^0 + \mathbf{v}^0, \mathbf{v}_r^1 + \mathbf{v}^1), \quad (\overline{\mathbf{A}}^0, \overline{\mathbf{A}}^1) = (\mathbf{A}_r^0 + \mathbf{A}^0, \mathbf{A}_r^1 + \mathbf{A}^1).$$

The new perturbed state is then a solution of the piezoelectric system below:

$$\begin{cases} \text{(a)} \quad \rho \mathbf{u}'' - \operatorname{div}\left(Ce(\mathbf{u}) + P\nabla\Psi(\mathbf{u}) + P\overline{\mathbf{A}}'\right) = \mathbf{f} & \text{in} \quad Q, \\ \text{(b)} \quad \operatorname{div}\left(Pe(\mathbf{u}) - \epsilon\nabla\Psi(\mathbf{u}) - \epsilon\overline{\mathbf{A}}'\right) = q_r + q & \text{in} \quad Q, \\ \text{(c)} \quad (-Pe(\mathbf{u}) + \epsilon\nabla\Psi(\mathbf{u}))' + \epsilon\overline{\mathbf{A}}'' + \dfrac{1}{\mu_1}\operatorname{curl}\operatorname{curl}\overline{\mathbf{A}} = \mathbf{J}_r + \mathbf{J} & \text{in} \quad Q, \end{cases}$$

with boundary conditions

$$\begin{cases} \text{(d)} \quad (Ce(\mathbf{u}) + P\nabla\Psi(\mathbf{u}) + P\overline{\mathbf{A}}')\mathbf{n} = \mathbf{s} & \text{on} \quad \Gamma_N, \\ \text{(e)} \quad \operatorname{curl}\overline{\mathbf{A}} \wedge \mathbf{n}/\mu_1 = -\mathbf{j}_r - \mathbf{j} & \text{on} \quad \Gamma_N, \\ \text{(f)} \quad \mathbf{u} = 0 & \text{on} \quad \Gamma_\ell, \\ \text{(g)} \quad \Psi = \phi_{r0} & \text{on} \quad \Gamma, \\ \text{(h)} \quad \overline{\mathbf{A}} = 0 & \text{on} \quad \Gamma_\ell, \end{cases} \tag{16}$$

and initial conditions

$$\begin{cases} \text{(i)} \quad \mathbf{u}(0) = \mathbf{u}^0(x), \quad \mathbf{u}'(0) = \mathbf{u}^1(x) & \text{on} \quad \Omega, \\ \text{(j)} \quad \overline{\mathbf{A}}(0) = \overline{\mathbf{A}}^0(x), \quad \overline{\mathbf{A}}'(0) = \overline{\mathbf{A}}^1(x) & \text{on} \quad \Omega, \end{cases}$$

here, components (a)–(j) are collectively referred to as a single system denoted by Equation (16), where:

- q stands for the internal charge density,
- **j** stands for the surface current density vector,
- **J** stands for the internal current density vector,
- **s** stands for the surface force on Γ_N.

Let $(\mathbf{u}, \overline{\mathbf{A}}, \Psi) = (\mathbf{v}_r + \mathbf{v}, \mathbf{A}_r + \mathbf{A}, \phi_r + \phi)$ in Equation (16). Then, the residual state $(\mathbf{v}, \mathbf{A}, \phi)$ satisfies the system:

$$\begin{cases} \text{(a)} \quad \rho \mathbf{v}'' - \operatorname{div}(Ce(\mathbf{v}) + P\nabla\phi(\mathbf{v}) + P\mathbf{A}') = 0 & \text{in} \quad Q, \\ \text{(b)} \quad \operatorname{div}(Pe(\mathbf{v}) - \epsilon\nabla\phi(\mathbf{v}) - \epsilon\mathbf{A}') = q & \text{in} \quad Q, \\ \text{(c)} \quad (-Pe(\mathbf{v}) + \epsilon\nabla\phi(\mathbf{v}))' + \epsilon\mathbf{A}'' + \dfrac{1}{\mu_1}\operatorname{curl}\operatorname{curl}\mathbf{A} = \mathbf{J} & \text{in} \quad Q, \end{cases}$$

with boundary conditions

$$\begin{cases} \text{(d)} \quad (Ce(\mathbf{v}) + P\nabla\phi(\mathbf{v}) + P\mathbf{A}')\mathbf{n} = \mathbf{s} & \text{on} \quad \Gamma^+, \\ \text{(e)} \quad (Ce(\mathbf{v}) + P\nabla\phi(\mathbf{v}) + P\mathbf{A}')\mathbf{n} = 0 & \text{on} \quad \Gamma^-, \\ \text{(f)} \quad \operatorname{curl}\mathbf{A} \wedge \mathbf{n}/\mu_1 = -\mathbf{j} & \text{on} \quad \Gamma^-, \\ \text{(g)} \quad \operatorname{curl}\mathbf{A} \wedge \mathbf{n}/\mu_1 = 0 & \text{on} \quad \Gamma^+, \\ \text{(h)} \quad \mathbf{v} = 0 & \text{on} \quad \Gamma_\ell, \\ \text{(i)} \quad \phi = 0 & \text{on} \quad \Gamma, \\ \text{(j)} \quad \mathbf{A} = 0 & \text{on} \quad \Gamma_\ell, \end{cases} \tag{17}$$

and initial conditions

$$\begin{cases} \text{(k)} \quad \mathbf{v}(0) = \mathbf{v}^0(x), \quad \mathbf{v}'(0) = \mathbf{v}^1(x) & \text{on} \quad \Omega, \\ \text{(l)} \quad \mathbf{A}(0) = \mathbf{A}^0(x), \quad \mathbf{A}'(0) = \mathbf{A}^1(x) & \text{on} \quad \Omega, \end{cases}$$

here, components (a)–(l) are collectively referred to as a single system denoted by

Equation (17).

We assume that the perturbed system retains the same boundary conditions for the potential. In other words, we have chosen not to control the potential over the boundary. The problem is to define scaled linear feedback control laws $q, \mathbf{J}, \mathbf{j}$ and $\mathbf{s}$ such that the energy of the scaled residual system decreases with respect to time, to study the limit behavior of solutions if the thickness h of the plate tends to zero, and to derive the effective limit model governing the limit solutions. Let us define the functional spaces where a strong solution can be found for the Equation (17).

2.1. Functional spaces

Consider the following spaces:

$$\begin{aligned} &\mathbb{V}(\Omega) = \left\{ \mathbf{v} \in (H^1(\Omega))^3 : \mathbf{v}|_{\Gamma_\ell} = 0 \right\}, \\ &\Psi(\Omega) = H_0^1(\Omega). \end{aligned} \tag{18}$$

The spaces above, each endowed with the norms derived from $H^1(\Omega)$ respectively, are all Hilbert spaces. To learn more about those spaces, see the relevant literature [23].

2.2. Classical solution

Definition 1. *Let $T > 0$ be an arbitrary real number, $(\mathbf{v}^0, \mathbf{A}^0) \in (\mathbb{V}(\Omega))^2$ and $(\mathbf{v}^1, \mathbf{A}^1) \in (L^2(\Omega))^3 \times (L^2(\Omega))^3$, the triplet $(\mathbf{v}, \mathbf{A}, \boldsymbol{\phi})$ is a classical solution of the Equation (17) on $[0, T)$ if:*

1. $\mathbf{v} \in L^2(0, T; \mathbb{V}(\Omega)), \mathbf{v}' \in L^2(0, T; (L^2(\Omega))^3)$,
2. $\mathbf{A} \in L^2(0, T; \mathbb{V}(\Omega)), \mathbf{A}' \in L^2(0, T; (L^2(\Omega))^3)$,
3. $\boldsymbol{\phi} \in L^2(0, T; \Psi(\Omega))$,
4. *and the Equation (17) holds in the sense of distributions.*

3. Stabilization results

3.1. Preliminary results

In order to obtain the decrease of the perturbation energy, we consider the following linear feedback control laws:

$$\begin{aligned} &q = -\operatorname{div}\left(\varepsilon \mathbf{A}'\right), \quad \mathbf{s} = -\left(\xi \mathbf{v}'' + \beta \mathbf{v}'\right), \\ &\mathbf{j} = -\left(\vartheta \mathbf{A}'' + \gamma \mathbf{A}' + \varkappa \nabla \mathbf{A} \cdot \mathbf{n}\right), \quad \mathbf{J} = \varepsilon \nabla \boldsymbol{\phi}' + \varkappa \operatorname{div}(\nabla \mathbf{A}) - \alpha \mathbf{A}'. \end{aligned} \tag{19}$$

The proposed feedback control laws of Equation (19) are designed to act as an active damping mechanism.

Specifically, the velocity-dependent terms in Equation (19) mimic the effect of a virtual viscous damper, effectively dissipating the accumulated mechanical and electrical energy to suppress resonance and enhance the system's efficiency [24].

From a mathematical perspective, the control laws are motivated by the need to stabilize the system and regularize the magnetic potential. Equation (19) is derived using the Lyapunov direct method, ensuring that the time derivative of the energy is negative [25]. This guarantees that all solutions converge to the equilibrium state over time [26,27], and ensures that the system remains bounded and stable.

Unlike Bidouan et al. [22], we have chosen the steady state as the reference state and focus on the asymptotic reduction of a three-dimensional model to an effective two-dimensional formulation. The objective is not to re-derive the abstract evolution theory, but to justify the dimensional reduction, through a systematic asymptotic analysis, by using the compactness theory. We recognize that, in general, the magnetic field is considered negligible in the analysis of piezoelectric systems. We draw the scientific community's attention to situations in which the magnetic field may play an important role. For example, in the case of high-frequency electromagnetic oscillation, it can be utilized for stabilization.

Definition 2. *Let $T > 0$ be an arbitrary real number, $(\mathbf{v}^0, \mathbf{A}^0) \in (\mathbb{V}(\Omega), \mathbb{V}(\Omega))$ and $(\mathbf{v}^1, \mathbf{A}^1) \in (L^2(\Omega))^3 \times (L^2(\Omega))^3$, the triplet $(\mathbf{v}, \mathbf{A}, \boldsymbol{\phi})$ is said to be a weak solution of Equation (17) in $[0, T)$ if:*

1. $\mathbf{v} \in L^2(0, T; \mathbb{V}(\Omega)), \mathbf{v}' \in L^2(0, T; (L^2(\Omega))^3)$,
2. $\mathbf{A} \in L^2(0, T; \mathbb{V}(\Omega)), \mathbf{A}' \in L^2(0, T; (L^2(\Omega))^3)$,
3. $\boldsymbol{\phi} \in L^2(0, T; \Psi(\Omega))$,
4. *and* $\forall\, (\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}) \in (\mathbb{V}, \mathbb{V}, \Psi)$*, we have:*

$$\begin{cases} \text{(a)} & \displaystyle\int_\Omega \rho \mathbf{v}'' \cdot \widetilde{\mathbf{v}} + \int_\Omega Ce(\mathbf{v}) : e(\widetilde{\mathbf{v}}) + \int_\Omega P\mathbf{A}' : e(\widetilde{\mathbf{v}}) + \int_\Omega P\nabla\boldsymbol{\phi} : e(\widetilde{\mathbf{v}}) \\ & \displaystyle= -\int_{\Gamma+} \xi \mathbf{v}'' \cdot \widetilde{\mathbf{v}} - \int_{\Gamma+} \beta \mathbf{v}' \cdot \widetilde{\mathbf{v}}, \\ \text{(b)} & \displaystyle-\int_\Omega Pe(\mathbf{v}) \cdot \nabla\widetilde{\boldsymbol{\phi}} + \int_\Omega \epsilon \nabla\widetilde{\boldsymbol{\phi}} \cdot \nabla\boldsymbol{\phi} = 0, \\ \text{(c)} & \displaystyle\int_\Omega \epsilon \mathbf{A}'' \cdot \widetilde{\mathbf{A}} - \int_\Omega Pe(\mathbf{v}') \cdot \widetilde{\mathbf{A}} + \frac{1}{\mu_1}\int_\Omega \operatorname{curl}\mathbf{A} \cdot \operatorname{curl}\widetilde{\mathbf{A}} \\ & \displaystyle= -\varkappa \int_\Omega \nabla\mathbf{A} : \nabla\widetilde{\mathbf{A}} - \int_\Omega \alpha \mathbf{A}' \cdot \widetilde{\mathbf{A}} - \int_{\Gamma-} \vartheta \mathbf{A}'' \cdot \widetilde{\mathbf{A}} - \int_{\Gamma-} \gamma \mathbf{A}' \cdot \widetilde{\mathbf{A}}, \end{cases} \tag{20}$$

with the following initial conditions

$$\begin{cases} \text{(d)} & \displaystyle\left(\int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}\right)(0) = \int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^0, \quad \left(\int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}\right)(0) = \int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^0, \\ \text{(e)} & \displaystyle\left(\int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}'\right)(0) = \int_\Omega \widetilde{\mathbf{v}}^h \cdot \mathbf{v}^1, \quad \left(\int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}'\right)(0) = \int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^1. \end{cases}$$

Here, components (a)–(e) are collectively referred to as a single system denote by Equation (20)*.*

Lemma 1. *For some positive constants $\alpha, \beta, \gamma, \vartheta, \varkappa$ and ξ, let $\mathscr{E}(t)$ be defined as the energy of the piezoelectric system:*

$$\begin{aligned} \mathscr{E}(t) = \frac{1}{2}\Big(&\rho \int_\Omega |\mathbf{v}'|^2 + \int_\Omega \epsilon \mathbf{A}' \cdot \mathbf{A}' + \int_\Omega Ce(\mathbf{v}) : e(\mathbf{v}) + \int_\Omega \epsilon \nabla\boldsymbol{\phi} \cdot \nabla\boldsymbol{\phi} \\ &+ \frac{1}{\mu_1}\int_\Omega |\operatorname{curl}\mathbf{A}|^2 + \varkappa \int_\Omega |\nabla\mathbf{A}|^2 + \xi \int_{\Gamma+} |\mathbf{v}'|^2 + \vartheta \int_{\Gamma-} |\mathbf{A}'|^2\Big), \end{aligned} \tag{21}$$

then its time derivative is given by

$$\mathscr{E}'(t) = -\beta \int_{\Gamma+} |\mathbf{v}'|^2 - \alpha \int_\Omega |\mathbf{A}'|^2 - \vartheta \int_{\Gamma-} |\mathbf{A}'|^2. \tag{22}$$

Proof. Put $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}) = (\mathbf{v}', \mathbf{A}', \boldsymbol{\phi}')$ in Equation (20). It becomes

$$\begin{aligned}&\rho\int_\Omega \mathbf{v}''\cdot\mathbf{v}' + \int_\Omega Ce(\mathbf{v}) : e(\mathbf{v}') + \int_\Omega P\mathbf{A}' : e(\mathbf{v}') + \int_\Omega P\nabla\boldsymbol{\phi} : e(\mathbf{v}') - \int_\Omega Pe(\mathbf{v})\cdot\nabla\boldsymbol{\phi}' \\ &+ \int_\Omega \epsilon\nabla\boldsymbol{\phi}'\cdot\nabla\boldsymbol{\phi} + \int_\Omega \epsilon\mathbf{A}''\cdot\mathbf{A}' - \int_\Omega Pe(\mathbf{v}')\cdot\mathbf{A}' + \frac{1}{\mu_1}\int_\Omega \operatorname{curl}\mathbf{A}\cdot\operatorname{curl}\mathbf{A}' + \varkappa\int_\Omega \nabla\mathbf{A}' : \nabla\mathbf{A} \\ &= -\int_{\Gamma+}\xi\mathbf{v}''\cdot\mathbf{v}' - \int_{\Gamma+}\beta\mathbf{v}'\cdot\mathbf{v}' - \vartheta\int_{\Gamma-}\mathbf{A}''\cdot\mathbf{A} - \gamma\int_{\Gamma-}\mathbf{A}'\cdot\mathbf{A} - \alpha\int_\Omega \mathbf{A}'\cdot\mathbf{A},\end{aligned} \tag{23}$$

and consequently using integration by parts, it follows:

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\left(\rho\int_\Omega |\mathbf{v}'|^2 + \int_\Omega \epsilon\mathbf{A}'\cdot\mathbf{A}' + \int_\Omega Ce(\mathbf{v}) : e(\mathbf{v}) + \int_\Omega \epsilon\nabla\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi} + \frac{1}{\mu_1}\int_\Omega |\operatorname{curl}\mathbf{A}|^2\right.\right. \\ &\left.\left.+ \varkappa\int_\Omega |\nabla\mathbf{A}|^2 + \xi\int_{\Gamma+}|\mathbf{v}'|^2 + \vartheta\int_{\Gamma-}|\mathbf{A}'|^2\right)\right] = \int_\Omega P\,e(\mathbf{v})\cdot\nabla\boldsymbol{\phi}' - \int_\Omega P\nabla\boldsymbol{\phi} : e(\mathbf{v}') \\ &- \int_{\Gamma+}\beta|\mathbf{v}'|^2 - \int_\Omega \alpha|\mathbf{A}'|^2 - \int_{\Gamma-}\gamma|\mathbf{A}'|^2.\end{aligned} \tag{24}$$

Then, considering the identity,

$$\int_\Omega Pe(\mathbf{v}')\cdot\nabla\boldsymbol{\phi} = \frac{d}{dt}\left(\int_\Omega Pe(\mathbf{v})\cdot\nabla\boldsymbol{\phi}\right) - \int_\Omega Pe(\mathbf{v})\cdot\nabla\boldsymbol{\phi}', \tag{25}$$

and from Equation (20b) one can deduce that

$$\begin{cases} 2\displaystyle\int_\Omega P\,e(\mathbf{v})\cdot\nabla\boldsymbol{\phi}' = \frac{d}{dt}\int_\Omega \epsilon\nabla\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}, \\ \displaystyle\frac{d}{dt}\int_\Omega P\,e(\mathbf{v})\cdot\nabla\boldsymbol{\phi} = \frac{d}{dt}\int_\Omega \epsilon\nabla\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}, \end{cases} \tag{26}$$

then we obtain Equations (21) and (22). □

In the study by Bidouan et al. [22], the authors identified 11 types of stabilizable piezoelectric materials stated below.

$$\begin{aligned}
&P^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad
P^2 = \begin{pmatrix} P_{111} & P_{122} & P_{133} & 0 & 0 & 0 \\ P_{211} & P_{222} & P_{233} & 0 & 0 & 0 \\ P_{311} & P_{322} & P_{333} & 0 & 0 & 0 \end{pmatrix}, \\
&P^3 = \begin{pmatrix} P_{111} & 0 & 0 & 0 & P_{113} & P_{112} \\ P_{211} & 0 & 0 & 0 & P_{213} & P_{212} \\ P_{311} & 0 & 0 & 0 & P_{313} & P_{312} \end{pmatrix}, \quad
P^4 = \begin{pmatrix} 0 & 0 & P_{133} & P_{132} & P_{113} & 0 \\ 0 & 0 & P_{233} & P_{232} & P_{213} & 0 \\ 0 & 0 & P_{333} & P_{332} & P_{313} & 0 \end{pmatrix}, \\
&P^5 = \begin{pmatrix} 0 & P_{122} & 0 & P_{132} & P_{113} & 0 \\ 0 & P_{222} & 0 & P_{232} & P_{213} & 0 \\ 0 & P_{322} & 0 & P_{332} & P_{313} & 0 \end{pmatrix}, \quad
P^6 = \begin{pmatrix} 0 & 0 & P_{133} & P_{132} & 0 & P_{112} \\ 0 & 0 & P_{233} & P_{232} & 0 & P_{212} \\ 0 & 0 & P_{333} & P_{332} & 0 & P_{312} \end{pmatrix}, \\
&P^7 = \begin{pmatrix} 0 & P_{122} & 0 & 0 & P_{113} & P_{112} \\ 0 & P_{222} & 0 & 0 & P_{213} & P_{212} \\ 0 & P_{322} & 0 & 0 & P_{313} & P_{312} \end{pmatrix}, \quad
P^8 = \begin{pmatrix} 0 & 0 & P_{133} & 0 & P_{113} & P_{112} \\ 0 & 0 & P_{233} & 0 & P_{213} & P_{212} \\ 0 & 0 & P_{333} & 0 & P_{313} & P_{312} \end{pmatrix}, \\
&P^9 = \begin{pmatrix} 0 & P_{122} & 0 & P_{132} & 0 & P_{112} \\ 0 & P_{222} & 0 & P_{232} & 0 & P_{212} \\ 0 & P_{322} & 0 & P_{332} & 0 & P_{312} \end{pmatrix}, \quad
P^{10} = \begin{pmatrix} P_{111} & 0 & 0 & P_{132} & 0 & P_{112} \\ P_{211} & 0 & 0 & P_{232} & 0 & P_{212} \\ P_{311} & 0 & 0 & P_{332} & 0 & P_{312} \end{pmatrix}, \\
&\text{and } P^{11} = \begin{pmatrix} P_{111} & 0 & 0 & P_{132} & P_{113} & 0 \\ P_{211} & 0 & 0 & P_{232} & P_{213} & 0 \\ P_{311} & 0 & 0 & P_{332} & P_{313} & 0 \end{pmatrix}.
\end{aligned} \tag{27}$$

If one of them is considered, then if $\mathscr{E}'(t) = 0$ the strong stabilization of Equation (17) is achieved because **v** would be zero in a finite time. This is the situation, for instance, whenever there is an interval where

$$\beta \int_{\Gamma+} |\mathbf{v}'|^2 + \alpha \int_{\Omega} |\mathbf{A}'|^2 + \vartheta \int_{\Gamma-} |\mathbf{A}'|^2 = 0. \tag{28}$$

Let $\tilde{I} =]a, b[$ be the initial open interval of $\mathbb{R}^+$ where Equation (28) holds. Let $\tilde{t}$ be defined as the center of the interval

$$I =]a, \min(b, a+1)[. \tag{29}$$

Lemma 2. *Consider P such that $P_{ikl} = \delta_{kl}\delta_{ik}$, i.e., $P_{iii} = 1$. If I defined in Equation (29) exists, then $\mathbf{v} = 0$ in $I \times \Omega$.*

Proof. Assume that I exists, then

$$\beta \int_{\Gamma^+} |\mathbf{v}'|^2 + \alpha \int_{\Omega} |\mathbf{A}'|^2 + \vartheta \int_{\Gamma^-} |\mathbf{A}'|^2 = 0 \qquad \text{in} \quad I.$$

It follows that $\mathbf{A}' = \mathbf{A}'' = 0$ in $I \times \Omega$ and $\mathbf{v}' = 0$ on $I \times \Gamma^+$. Furthermore, from Equation (17c), we deduce:

$$Pe(\mathbf{v}'') = 0 \text{ in } I \times \Omega. \tag{30}$$

Therefore, $\partial_i v_i'' = 0, i = 1, 2, 3$. Since $\mathbf{v}' = 0$ on $I \times \Gamma^+$, we obtain the following system

$$\begin{cases} \text{(a)} \quad v_1'' = g_1(x_2, x_3) & \text{in} \quad I \times \Omega, \\ \text{(b)} \quad v_2'' = g_2(x_1, x_3) & \text{in} \quad I \times \Omega, \\ \text{(c)} \quad v_3'' = g_3(x_1, x_2) & \text{in} \quad I \times \Omega, \\ \text{(d)} \quad v_1 = 0 & \text{on} \quad I \times \Gamma^+, \\ \text{(e)} \quad v_2 = 0 & \text{on} \quad I \times \Gamma^+, \\ \text{(f)} \quad v_3 = 0 & \text{on} \quad I \times \Gamma^+, \end{cases} \tag{31}$$

for arbitrary functions g_1, g_2 and g_3. From Equation (31a,d), we deduce $v_1'' = 0$ in $I \times \Omega$. Similarly, Equation (31b,e) yields $v_2'' = 0$, while Equation (31c,f) results in $v_3'' = 0$. Employing the identities $\mathbf{v}'' = 0$ in $I \times \Omega$, $\quad \mathbf{v}' = 0$ on $I \times \Gamma^+$, and $\mathbf{A}' = 0$ in $I \times \Omega$, in Equation (20a), and substituting $\widetilde{\boldsymbol{\phi}} = \boldsymbol{\phi}$ in Equation (20b), we obtain:

$$\textstyle\int_\Omega Ce(\mathbf{v}) : e(\mathbf{v}) + \int_\Omega \epsilon \nabla \boldsymbol{\phi} \cdot \nabla \boldsymbol{\phi} = 0 \quad \text{in} \quad I. \tag{32}$$

Therefore, we have:

$$\textstyle\int_\Omega Ce(\mathbf{v}) : e(\mathbf{v}) = 0 \quad \text{in} \quad I. \tag{33}$$

Due to Korn's inequality, there exists a constant Υ such that:

$$\Upsilon \|\mathbf{v}\|^2 \leq \textstyle\int_\Omega Ce(\mathbf{v}) : e(\mathbf{v}) = 0 \quad \text{in} \quad I. \tag{34}$$

□

Lemma 3. *Consider P such that $P_{ijk} = 0$ if $j \neq k$ and*

$$\mathcal{M}_2 = \begin{pmatrix} P_{111} & P_{122} & P_{133} \\ P_{211} & P_{222} & P_{233} \\ P_{311} & P_{322} & P_{333} \end{pmatrix}$$

is nonsingular. If I defined in Equation (29) exists, then $\mathbf{v} = 0$ in $I \times \Omega$.

Proof. Assume that I exists, then

$$\beta \int_{\Gamma^+} |\mathbf{v}'|^2 + \alpha \int_{\Omega} |\mathbf{A}'|^2 + \vartheta \int_{\Gamma^-} |\mathbf{A}'|^2 = 0 \qquad \text{in} \quad I.$$

It follows that $\mathbf{A}' = \mathbf{A}'' = 0$ in $I \times \Omega$ and $\mathbf{v}' = 0$ on $I \times \Gamma^+$. Using a similar reasoning as in the proof of Lemma 2 we obtain the following:

$$Pe(\mathbf{v}'') = 0 \quad \text{in} \quad I \times \Omega,$$

with:

$$Pe(\mathbf{v}'') = \begin{pmatrix} P_{111} & P_{122} & P_{133} & 0 & 0 & 0 \\ P_{211} & P_{222} & P_{233} & 0 & 0 & 0 \\ P_{311} & P_{322} & P_{333} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_1 v_1'' \\ \partial_2 v_2'' \\ \partial_3 v_3'' \\ \dfrac{\partial_3 v_2'' + \partial_2 v_3''}{2} \\ \dfrac{\partial_1 v_3'' + \partial_3 v_1''}{2} \\ \dfrac{\partial_1 v_2'' + \partial_2 v_1''}{2} \end{pmatrix}. \tag{35}$$

Therefore, since $\mathcal{M}_2$ is not singular, it gives $\partial_j v_j'' = 0, j = 1, 2, 3$. Adopting the same procedure as in the proof of Lemma 2, we obtain $\mathbf{v} = 0$ in $I \times \Omega$. □

Lemma 4. *Consider P such that:*

$$P = \begin{pmatrix} P_{111} & 0 & 0 & 0 & P_{113} & P_{112} \\ P_{211} & 0 & 0 & 0 & P_{213} & P_{212} \\ P_{311} & 0 & 0 & 0 & P_{313} & P_{312} \end{pmatrix} \quad \text{and} \quad \mathcal{M}_3 = \begin{pmatrix} P_{111} & P_{112} & P_{113} \\ P_{211} & P_{212} & P_{213} \\ P_{311} & P_{312} & P_{313} \end{pmatrix} \tag{36}$$

is nonsingular. If I exists, then $\mathbf{v} = 0$ in $I \times \Omega$.

Proof. Assume that I exists, then

$$\beta \int_{\Gamma^+} |\mathbf{v}'|^2 + \alpha \int_{\Omega} |\mathbf{A}'|^2 + \vartheta \int_{\Gamma^-} |\mathbf{A}'|^2 = 0 \qquad \text{in} \quad I.$$

It follows that $\mathbf{A}' = \mathbf{A}'' = 0$ in $I \times \Omega$ and $\mathbf{v}' = 0$ on $I \times \Gamma^+$. Applying logic similar to that used in the proof of Lemma 2 we obtain the following:

$$Pe(\mathbf{v}'') = 0 \quad \text{in} \quad I \times \Omega.$$

Since $\mathcal{M}_3$ is nonsingular, we have:

$$\begin{cases} \text{(a) } \partial_1 v_1'' = 0 & \text{in } \Omega, \\ \text{(b) } \partial_1 v_2'' = -\,\partial_2 v_1'' & \text{in } \Omega, \\ \text{(c) } \partial_1 v_3'' = -\,\partial_3 v_1'' & \text{in } \Omega. \end{cases} \tag{37}$$

Identities $\mathbf{v}' = 0 \quad \text{on} \quad \Gamma^+$ and Equation (37a) give $v_1' = 0$ in Ω. Then, from Equation (37b) we deduce $\partial_1 v_2'' = -\partial_2 v_1'' = 0$ and $v_2'' = 0$ in Ω. Similarly, one proves that $v_3'' = 0$ in Ω using Equation (37c). Following the same procedure as in the proof of Lemma 2, we obtain $\mathbf{v} = 0$ in $I \times \Omega$. □

Lemma 5. *If Equation (28) holds in I, then $\mathscr{E} = 0$ in I.*

Proof. By substituting the control law q in Equation (20b) and using identity $\mathbf{v} = 0$ in $I \times \Omega$ and taking $\widetilde{\phi} = \phi$, we have the following:

$$\int_\Omega \varepsilon\, \nabla\phi \cdot \nabla\phi \;=\; 0. \tag{38}$$

Therefore, $\phi = 0$ in $I \times \Omega$. In addition, substituting the control laws $\mathbf{J}$ and $\mathbf{j}$ in Equation (20c), using identities $\mathbf{v} = 0$ and $\mathbf{A}' = 0$ in $I \times \Omega$, and taking $\widetilde{\mathbf{A}} = \mathbf{A}$, we have $\varkappa\|\nabla\mathbf{A}\|^2 + \|\mathrm{curl}\,\mathbf{A}\| = 0$ in I, consequently $\mathscr{E}(t) = 0$. □

This lemma ensures that the weak solution energy is decreasing or zero.

3.2. Existence of unique solution

Theorem 1. *Consider the feedback control laws in Equation (19) and one of the eleven types of stabilizable piezoelectric materials defined in Equation* (27)*. For arbitrarily chosen initial data* $(\mathbf{v}^0, \mathbf{A}^0) \in \mathbb{V}^2(\Omega)$, $\quad (\mathbf{v}^1, \mathbf{A}^1) \in L^2(\Omega) \times L^2(\Omega)$*, there exists a unique solution* $(\mathbf{v}, \mathbf{A}, \phi)$ *to Equation* (20) *in the sense of Definition 1. Furthermore, the energy $\mathscr{E}$ is decreasing. If the interval I defined in Equation* (29) *exists, then strong stabilization of Equation* (20) *is achieved, i.e.,* $\mathscr{E}(\tilde{t}) = 0$*.*

Proof. The solution of Equation (20) is built using the Galerkin's method.

Let $(vect\{v^n,\ n \in \mathbb{N}\})^3$ and $(vect\{\psi^n,\ n \in \mathbb{N}\})$ be the Hilbertian bases of $\mathbb{V}(\Omega)$ and $\Psi(\Omega)$, respectively; and define their finite-dimensional subspaces $\mathbb{V}_N$, and Ψ_N as follows:

$$\mathbb{V}_N = (vect\{v^n,\ 1 \le n \le N\})^3,$$

$$\Psi_N = (vect\{\psi^n,\ 1 \le n \le N\}).$$

Note that the finite-dimensional vector space sequence $\mathbb{V}_N \times \Psi_N$ converges to $\mathbb{V}(\Omega) \times \Psi(\Omega)$. Let $(\mathbf{v}^{N0})_{N\in\mathbb{N}} \subset \mathbb{V}_N$, $(\mathbf{v}^{N1})_{N\in\mathbb{N}} \subset \mathbb{V}_N$ be sequences that strongly converge to $\mathbf{v}^0 \in \mathbb{V}(\Omega)$, and $\mathbf{v}^1 \in (L^2(\Omega))^3$, respectively. Let $(\mathbf{A}^{N0})_{N\in\mathbb{N}} \subset \mathbb{V}_N$, $(\mathbf{A}^{N1})_{N\in\mathbb{N}} \subset \mathbb{V}_N$ be sequences that strongly converge to $\mathbf{A}^0 \in \mathbb{V}(\Omega)$, and $\mathbf{A}^1 \in (L^2(\Omega))^3$, respectively. Since **v** and **A** are in the same space , the same basis is used and only the coefficients differ. They can be written as follows.

$$\mathbf{v}^{N0}(x) = \sum_{n=1}^{N} \boldsymbol{\alpha}_0^n v^n(x), \quad \mathbf{v}^{N1}(x) = \sum_{n=1}^{N} \boldsymbol{\alpha}_1^n v^n(x),$$

$$\mathbf{A}^{N0}(x) = \sum_{n=1}^{N} \boldsymbol{\beta}_0^n v^n(x), \quad \mathbf{A}^{N1}(x) = \sum_{n=1}^{N} \boldsymbol{\beta}_1^n v^n(x),$$

and the unknowns $(\mathbf{v}^N, \mathbf{A}^N, \boldsymbol{\phi}^N)$ also follow similar expansions:

$$\mathbf{v}^N = \sum_{n=1}^{N} \mathbf{a}^n v^n, \quad \mathbf{A}^N = \sum_{n=1}^{N} \mathbf{b}^n v^n, \quad \boldsymbol{\phi}^N = \sum_{n=1}^{N} c^n \psi^n. \tag{39}$$

The discrete problem is then given by a system of ordinary differential equations whose unknowns are: $\mathbf{a}^n = (a_1^n, a_2^n, a_3^n)$, $\mathbf{b}^n = (b_1^n, b_2^n, b_3^n)$, c^n for $1 \leq n \leq N$, where a_i^n, b_i^n, and c^n are real functions.

The discrete variational problem is given by the following system: find $(\mathbf{v}^N, \mathbf{A}^N, \boldsymbol{\phi}^N) \in (\mathbb{V}_N \times \mathbb{V}_N \times \Psi_N)$ such that $\forall(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}) \in (\mathbb{V}_N \times \mathbb{V}_N \times \Psi_N)$:

$$\begin{cases} \text{(a)} \quad \rho \int_\Omega \mathbf{v}^{N''} \cdot \widetilde{\mathbf{v}} + \int_\Omega Ce(\mathbf{v}^N) : e(\widetilde{\mathbf{v}}) + \int_\Omega P\mathbf{A}^{N'} : e(\widetilde{\mathbf{v}}) + \int_\Omega P\nabla\boldsymbol{\phi}^N : e(\widetilde{\mathbf{v}}) = \int_{\Gamma+} \mathbf{s}^N \cdot \widetilde{\mathbf{v}}, \\ \text{(b)} \quad -\int_\Omega Pe(\mathbf{v}^N) \cdot \nabla\widetilde{\boldsymbol{\phi}} + \int_\Omega \epsilon\nabla\widetilde{\boldsymbol{\phi}} \cdot \nabla\boldsymbol{\phi}^N + \int_\Omega \epsilon\nabla\widetilde{\boldsymbol{\phi}} \cdot \mathbf{A}^{N'} = \int_\Omega q^N \cdot \widetilde{\boldsymbol{\phi}}, \\ \text{(c)} \quad \int_\Omega \epsilon\mathbf{A}^{N''} \cdot \widetilde{\mathbf{A}} + \int_\Omega \epsilon\nabla\boldsymbol{\phi}^{N'} \cdot \widetilde{\mathbf{A}} - \int_\Omega Pe(\mathbf{v}') \cdot \widetilde{\mathbf{A}} + \frac{1}{\mu_1}\int_\Omega \operatorname{curl}\mathbf{A}^N \cdot \operatorname{curl}\widetilde{\mathbf{A}} \\ \qquad = \int_\Omega \mathbf{J}^N \cdot \widetilde{\mathbf{A}} + \int_{\Gamma-} \mathbf{j}^N \cdot \widetilde{\mathbf{A}}, \end{cases} \tag{40}$$

with the following initial conditions

$$\begin{cases} \text{(d)} \quad \left(\int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^N\right)(0) = \int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^{N0}, \quad \left(\int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^N\right)(0) = \int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^{N0}, \\ \text{(e)} \quad \left(\int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^{N'}\right)(0) = \int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^{N1}, \quad \left(\int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^{N'}\right)(0) = \int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^{N1}. \end{cases}$$

Here, components (a)–(e) are collectively referred to as a single system denoted by Equation (40).

From it, the discrete control laws are given by

$$\begin{aligned} &q^N = -\operatorname{div}\left(\varepsilon\mathbf{A}^{N'}\right), \quad \mathbf{s}^N = -\left(\xi\mathbf{v}^{N''} + \beta\mathbf{v}^{N'}\right), \\ &\mathbf{j}^N = -\left(\vartheta\mathbf{A}^{N''} + \gamma\mathbf{A}^{N'} + \varkappa\nabla\mathbf{A}^N \cdot \mathbf{n}\right), \quad \mathbf{J}^N = \varepsilon\nabla\boldsymbol{\phi}^{N'} + \varkappa\operatorname{div}(\nabla\mathbf{A}^N) - \alpha\mathbf{A}^{N'}. \end{aligned} \tag{41}$$

The total energy $\mathscr{E}^N(t)$ of the discrete system to be estimated is:

$$\begin{aligned} \mathscr{E}^N(t) = \frac{1}{2}\Big(&\rho\int_\Omega |\mathbf{v}^{N'}|^2 + \int_\Omega \epsilon\mathbf{A}^{N'} \cdot \mathbf{A}^{N'} + \int_\Omega Ce(\mathbf{v}^N) : e(\mathbf{v}^N) + \int_\Omega \epsilon\nabla\boldsymbol{\phi}^N \cdot \nabla\boldsymbol{\phi}^N \\ &+ \frac{1}{\mu_1}\int_\Omega |\operatorname{curl}\mathbf{A}^N|^2 + \varkappa\int_\Omega |\nabla\mathbf{A}^N|^2 + \xi\int_{\Gamma+} |\mathbf{v}^{N'}|^2 + \vartheta\int_{\Gamma-} |\mathbf{A}^{N'}|^2\Big). \end{aligned} \tag{42}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the canonical basis of $\mathbb{R}^3$. Put $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}) = (\mathbf{e}_i v^k, \mathbf{e}_j v^k, \psi^k)$ in

Equation (40) and it follows:

$$
\begin{cases}
\text{(a)} \quad \rho \int_\Omega \mathbf{v}^{N''} \cdot \mathbf{e}_i v^k + \int_\Omega Ce(\mathbf{v}^N) : e(\mathbf{e}_i v^k) + \int_\Omega P\mathbf{A}^{N'} : e(\mathbf{e}_i v^k) + \int_\Omega P\nabla\boldsymbol{\phi}^N : e(\mathbf{e}_i v^k) \\
\qquad = \int_{\Gamma^-} \mathbf{s}^N \cdot \mathbf{e}_i v^k, \\
\text{(b)} \quad -\int_\Omega Pe(\mathbf{v}^N) \cdot \nabla\psi^k + \int_\Omega \varepsilon\nabla\psi^k \cdot \nabla\boldsymbol{\phi}^N + \int_\Omega \varepsilon\nabla\psi^k \cdot \mathbf{A}^{N'} = \int_\Omega q^N \psi^k, \\
\text{(c)} \quad \int_\Omega \varepsilon\mathbf{A}^{N''} \cdot \mathbf{e}_j v^k + \int_\Omega \varepsilon\nabla\boldsymbol{\phi}^{N'} \cdot \mathbf{e}_j v^k - \int_\Omega Pe(\mathbf{v}^N) \cdot \mathbf{e}_j v^k \\
\qquad + \frac{1}{\mu_1} \int_\Omega \operatorname{curl} \mathbf{A}^N \cdot \operatorname{curl} \mathbf{e}_j v^k = \int_\Omega \mathbf{J}^N \cdot \mathbf{e}_j v^k + \int_{\Gamma^-} \mathbf{j}^N \cdot \mathbf{e}_j v^k,
\end{cases} \tag{43}
$$

with the following initial conditions

$$
\begin{cases}
\text{(d)} \quad \left(\int_\Omega \mathbf{e}_i v^k \cdot \mathbf{v}^N\right)(0) = \int_\Omega \mathbf{e}_i v^k \cdot \mathbf{v}^{N0}, \quad \left(\int_\Omega \mathbf{e}_j v^k \cdot \mathbf{A}^N\right)(0) = \int_\Omega \mathbf{e}_j v^k \cdot \mathbf{A}^{N0}, \\
\text{(e)} \quad \left(\int_\Omega \mathbf{e}_i v^k \cdot \mathbf{v}^{N'}\right)(0) = \int_\Omega \mathbf{e}_i v^k \cdot \mathbf{v}^{N1}, \quad \left(\int_\Omega \mathbf{e}_j v^k \cdot \mathbf{A}^{N'}\right)(0) = \int_\Omega \mathbf{e}_j v^k \cdot \mathbf{A}^{N1},
\end{cases}
$$

$$1 \le k \le N, \quad 1 \le i \le 3.$$

Here, components (a)–(e) are collectively referred to as a single system denoted by Equation (43).

Considering the Equation (39) of the solution and the Equation (41) of the control laws in Equation (43), it becomes:

$$
\begin{cases}
\text{(a)} \quad \sum_{n=1}^{N} \frac{\partial^2 a_i^n}{\partial t^2} \left(\int_\Omega v^n v^k + \xi \int_{\Gamma^+} v^n v^k\right) + \sum_{n=1}^{N} a_j^n \int_\Omega Ce(\mathbf{e}_j v^n) : e(\mathbf{e}_i v^k) \\
\qquad + \sum_{n=1}^{N} \frac{\partial b_j^n}{\partial t} \int_\Omega P\mathbf{e}_j v^n : e(\mathbf{e}_i v^k) + \beta \sum_{n=1}^{N} \frac{\partial a_i^n}{\partial t} \int_{\Gamma^+} v^n v^k + \sum_{n=1}^{N} c^n \int_\Omega P\nabla\psi^n : e(\mathbf{e}_i v^k) = 0, \\
\text{(b)} \quad -\sum_{n=1}^{N} a_j^n \int_\Omega Pe(\mathbf{e}_j v^n) \cdot \nabla\psi^k + \sum_{n=1}^{N} c^n \left(\int_\Omega \varepsilon\nabla\psi^n \cdot \nabla\psi^k\right) = 0, \\
\text{(c)} \quad \sum_{n=1}^{N} \frac{\partial^2 b_i^n}{\partial t^2} \left(\int_\Omega \varepsilon(\mathbf{e}_i v^n) \cdot (\mathbf{e}_j v^k) + \vartheta \int_{\Gamma^-} v^n v^k\right) \\
\qquad - \sum_{n=1}^{N} \frac{\partial a_i^n}{\partial t} \int_\Omega Pe(\mathbf{e}_i v^n) \cdot \mathbf{e}_j v^k + \varkappa \sum_{n=1}^{N} b_i^n \int_\Omega \nabla v^n \cdot \nabla v^k \\
\qquad + \frac{1}{\mu_1} \sum_{n=1}^{N} b_i^n \int_\Omega \operatorname{curl} v^n \cdot \operatorname{curl} v^k + \sum_{n=1}^{N} \frac{\partial b_i^n}{\partial t} \alpha \int_\Omega v^n \cdot v^k + \sum_{n=1}^{N} \frac{\partial b_i^n}{\partial t} \gamma \int_{\Gamma^-} v^n \cdot v^k = 0,
\end{cases} \tag{44}
$$

with the following initial conditions

$$
\begin{cases}
\text{(d)} \quad \left(\sum_{n=1}^{N} a_i^n \int_\Omega v^k \cdot v^n\right)(0) = \int_\Omega \mathbf{e}_i v^k \cdot \mathbf{v}^{N0}, \quad \left(\int_\Omega v^k \cdot \mathbf{A}^N\right)(0) = \int_\Omega v^k \cdot \mathbf{A}^{N0}, \\
\text{(e)} \quad \left(\int_\Omega \mathbf{e}_i v^k \cdot \mathbf{v}^{N'}\right)(0) = \int_\Omega \mathbf{e}_i v^k \cdot \mathbf{v}^{N1}, \quad \left(\int_\Omega v^k \cdot \mathbf{A}^{N'}\right)(0) = \int_\Omega v^k \cdot \mathbf{A}^{N1}, \\
\quad 1 \le k \le N, \quad 1 \le i \le 3.
\end{cases}
$$

Here, components (a)–(e) are collectively referred to as a single system denoted by Equation (44).

Note that the matrices $\mathscr{M}_1 = \left(\int_\Omega \varepsilon\nabla\psi^n \cdot \nabla\psi^k\right)_{1\le n,k\le N}$, $\mathscr{M}_2 = \Big(\int_\Omega v^n v^k +$

$\xi \int_{\Gamma^-} v^n v^k \Big)_{1\leq n,k\leq N}$, and $\mathscr{M}_3 = \left(\int_\Omega \varepsilon(\mathbf{e}_i v^n)\cdot \mathbf{e}_j v^k + \vartheta \int_{\Gamma^-} v^n \cdot v^k \right)_{1\leq n,k\leq N}$ involved in the system of Equation (44) are positive definite, then from Equation (44b), we deduce that c^n can be expressed in terms of $\mathbf{a}^n$ and $\mathbf{b}^n$, and that Equation (44a,c) form a system of linear second-order ordinary differential equations which has a solution.

As for the energy estimation, let us put $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\phi}) = (\mathbf{v}^{N'}, \mathbf{A}^{N'}, \phi^{N'})$ in Equation (40), and consider the feedback control laws of Equation (41). It becomes

$$\begin{aligned}
&\rho\int_\Omega \mathbf{v}^{N''}\cdot\mathbf{v}^{N'} + \int_\Omega Ce(\mathbf{v}^N):e(\mathbf{v}^{N'}) + \int_\Omega P\mathbf{A}^{N'}:e(\mathbf{v}^{N'}) + \int_\Omega P\nabla\phi^N : e(\mathbf{v}^{N'})\\
&-\int_\Omega Pe(\mathbf{v}^N)\cdot\nabla\phi^{N'} + \int_\Omega \epsilon\nabla\phi^{N'}\cdot\nabla\phi^N + \int_\Omega \epsilon\mathbf{A}^{N''}\cdot\mathbf{A}^{N'}\\
&-\int_\Omega Pe(\mathbf{v}^{N'})\cdot\mathbf{A}^{N'} + \frac{1}{\mu_1}\int_\Omega \operatorname{curl}\mathbf{A}^N\cdot\operatorname{curl}\mathbf{A}^{N'} + \varkappa\int_\Omega \nabla\mathbf{A}^{N'}:\nabla\mathbf{A}^N \qquad (45)\\
&= -\int_{\Gamma+}\xi\mathbf{v}^{N''}\cdot\mathbf{v}^{N'} - \int_{\Gamma+}\beta\mathbf{v}^{N'}\cdot\mathbf{v}^{N'} - \vartheta\int_{\Gamma-}\mathbf{A}^{N''}\cdot\mathbf{A}^{N'}\\
&-\gamma\int_{\Gamma-}\mathbf{A}^{N'}\cdot\mathbf{A}^{N'} - \alpha\int_\Omega \mathbf{A}^{N'}\cdot\mathbf{A}^{N'},
\end{aligned}$$

and consequently, it follows that

$$\begin{aligned}
\mathscr{E}^{N'}(t) = &\int_\Omega P\,e(\mathbf{v}^N)\cdot\nabla\phi^{N'} - \int_\Omega P\nabla\phi^N : e(\mathbf{v}^{N'}) - \int_{\Gamma+}\beta|\mathbf{v}^{N'}|^2\\
&-\int_\Omega \alpha|\mathbf{A}^{N'}|^2 - \int_{\Gamma-}\gamma|\mathbf{A}^{N'}|^2.
\end{aligned} \qquad (46)$$

Then, considering the identity,

$$\int_\Omega Pe(\mathbf{v}^{N'})\cdot\nabla\phi^N = \frac{d}{dt}\left(\int_\Omega Pe(\mathbf{v}^N)\cdot\nabla\phi^N\right) - \int_\Omega Pe(\mathbf{v}^N)\cdot\nabla\phi^{N'}, \qquad (47)$$

and from the fact that from Equation (40b) one can deduce that

$$\begin{cases}
2\int_\Omega P\,e(\mathbf{v}^N)\cdot\nabla\phi^{N'} = \dfrac{d}{dt}\int_\Omega \epsilon\nabla\phi^N\cdot\nabla\phi^N,\\
\dfrac{d}{dt}\int_\Omega P\,e(\mathbf{v}^N)\cdot\nabla\phi^N = \dfrac{d}{dt}\int_\Omega \epsilon\nabla\phi^N\cdot\nabla\phi^N,
\end{cases} \qquad (48)$$

then the time derivative of the energy is

$$\mathscr{E}^{N'}(t) = -\int_{\Gamma+}\beta|\mathbf{v}^{N'}|^2 - \int_\Omega \alpha|\mathbf{A}^{N'}|^2 - \int_{\Gamma-}\gamma|\mathbf{A}^{N'}|^2. \qquad (49)$$

Thanks to Equation (49), the function $\mathscr{E}^N(t)$ is decreasing and for any N,

$$\mathscr{E}^N(t) \leq \mathscr{E}^N(0), \quad \forall t > 0. \qquad (50)$$

Since $\mathscr{E}^N(0)$ is bounded, $\mathscr{E}^N(t)$ is bounded in $L^\infty(0,T)$, then we exploit the compactness of the closed unit ball of reflexive Banach spaces for the weak topology. Hence, there exists $(\mathbf{v}, \mathbf{A}, \phi) \in (\mathbb{V}(\Omega)\times\mathbb{V}(\Omega)\times\Psi(\Omega))$ such that the following weak convergences hold for sub-sequences with the same notation.

$$\begin{aligned}
\mathbf{v}^{N'} &\rightharpoonup \mathbf{v}' \quad \text{in } L^2(0,T;L^2(\Omega)^3),\\
\mathbf{v}^{N'} &\rightharpoonup \mathbf{v}' \quad \text{in } L^2(0,T;L^2(\Gamma^+)^3),\\
e(\mathbf{v}^N) &\rightharpoonup e(\mathbf{v}) \quad \text{in } L^2(0,T;L^2(\Omega)^9),\\
\boldsymbol{\phi}^N &\rightharpoonup \boldsymbol{\phi} \quad \text{in } L^2(0,T;H^1(\Omega)),\\
\mathbf{A}^{N'} &\rightharpoonup \mathbf{A}' \quad \text{in } L^2(0,T;L^2(\Omega)^3),\\
\text{curl}\,\mathbf{A}^N &\rightharpoonup \text{curl}\,\mathbf{A} \quad \text{in } L^2(0,T;L^2(\Omega)^3),\\
\mathbf{A}^{N'} &\rightharpoonup \mathbf{A}' \quad \text{in } L^2(0,T;L^2(\Gamma^-)^3),\\
\nabla\mathbf{A}^N &\rightharpoonup \nabla\mathbf{A} \quad \text{in } L^2(0,T;L^2(\Omega)^9).
\end{aligned} \tag{51}$$

In order to obtain Equation (20), we take $(\widetilde{\mathbf{v}},\widetilde{\mathbf{A}},\widetilde{\boldsymbol{\phi}}) \in (\mathbb{V}_M \times \mathbb{V}_M \times \Psi_M)$ for some arbitrary $M \in \mathbb{N}$, consider the feedback control laws of Equation (41) and multiply Equation (40) by $\varphi(t) \in \mathscr{D}([0,T[)$, with $N > M$. Then, integrating by parts with respect to time, we obtain:

$$\begin{cases}
\text{(a)} \displaystyle\int_0^T\int_\Omega \rho\,\mathbf{v}^N\cdot\widetilde{\mathbf{v}}\,\varphi''(t) - \int_\Omega \rho\,\mathbf{v}^{N1}\cdot\widetilde{\mathbf{v}}\,\varphi(0) + \int_\Omega \rho\,\mathbf{v}^{N0}\cdot\widetilde{\mathbf{v}}\,\varphi'(0)\\
\quad + \displaystyle\int_0^T\int_\Omega C\,e(\mathbf{v}^N):e(\widetilde{\mathbf{v}})\,\varphi(t) + \int_0^T\int_\Omega P\mathbf{A}^{N'}:e(\widetilde{\mathbf{v}})\,\varphi(t) + \int_0^T\int_\Omega P\nabla\boldsymbol{\phi}^N:e(\widetilde{\mathbf{v}})\,\varphi(t)\\
\quad = -\beta\displaystyle\int_0^T\int_{\Gamma+}\mathbf{v}^{N'}\cdot\widetilde{\mathbf{v}}\,\varphi(t) + \xi\int_0^T\int_{\Gamma+}\mathbf{v}^{N'}\cdot\widetilde{\mathbf{v}}\,\varphi'(t) + \xi\int_{\Gamma+}\mathbf{v}^{N1}\cdot\widetilde{\mathbf{v}}\,\varphi(0),\\
\text{(b)} -\displaystyle\int_0^T\int_\Omega Pe(\mathbf{v}^N)\cdot\nabla\widetilde{\boldsymbol{\phi}}\,\varphi(t) + \int_0^T\int_\Omega \varepsilon\nabla\widetilde{\boldsymbol{\phi}}\cdot\nabla\boldsymbol{\phi}^N\,\varphi(t) = 0,\\
\text{(c)} \displaystyle\int_0^T\int_\Omega \varepsilon\mathbf{A}^N\cdot\widetilde{\mathbf{A}}\,\varphi''(t) - \int_\Omega \varepsilon\mathbf{A}^{N1}\cdot\widetilde{\mathbf{A}}\,\varphi(0) + \int_\Omega \varepsilon\mathbf{A}^{N0}\cdot\widetilde{\mathbf{A}}\,\varphi'(0)\\
\quad - \displaystyle\int_0^T\int_\Omega Pe(\mathbf{v}^{N'})\cdot\widetilde{\mathbf{A}}\,\varphi(t) + \frac{1}{\mu_1}\int_0^T\int_\Omega \text{curl}\,\mathbf{A}^N\cdot\text{curl}\,\widetilde{\mathbf{A}}\,\varphi(t)\\
\quad = -\alpha\displaystyle\int_0^T\int_\Omega \mathbf{A}^{N'}\cdot\widetilde{\mathbf{A}}\,\varphi(t) - \varkappa\int_0^T\int_\Omega \nabla\mathbf{A}^N:\nabla\widetilde{\mathbf{A}}\,\varphi(t)\\
\quad - \gamma\displaystyle\int_0^T\int_{\Gamma-}\mathbf{A}^{N'}\cdot\widetilde{\mathbf{A}}\,\varphi(t) + \vartheta\int_0^T\int_{\Gamma-}\mathbf{A}^{N'}\cdot\widetilde{\mathbf{A}}\,\varphi'(t) + \vartheta\int_{\Gamma-}\mathbf{A}^{N1}\cdot\widetilde{\mathbf{A}}\,\varphi(0).
\end{cases} \tag{52}$$

Finally, by letting N tend to ∞, the weak limit $(\mathbf{v},\mathbf{A},\boldsymbol{\phi})$ satisfies the following system:

$$\begin{cases}
\text{(a)} \displaystyle\int_0^T\int_\Omega \rho\,\mathbf{v}\cdot\widetilde{\mathbf{v}}\,\varphi''(t) - \int_\Omega \rho\,\mathbf{v}^1\cdot\widetilde{\mathbf{v}}\,\varphi(0) + \int_\Omega \rho\,\mathbf{v}^0\cdot\widetilde{\mathbf{v}}\,\varphi'(0)\\
\quad + \displaystyle\int_0^T\int_\Omega C\,e(\mathbf{v}):e(\widetilde{\mathbf{v}})\,\varphi(t) + \int_0^T\int_\Omega P\mathbf{A}':e(\widetilde{\mathbf{v}})\,\varphi(t) + \int_0^T\int_\Omega P\nabla\boldsymbol{\phi}:e(\widetilde{\mathbf{v}})\,\varphi(t)\\
\quad = -\beta\displaystyle\int_0^T\int_{\Gamma+}\mathbf{v}'\cdot\widetilde{\mathbf{v}}\,\varphi(t) + \xi\int_0^T\int_{\Gamma+}\mathbf{v}'\cdot\widetilde{\mathbf{v}}\,\varphi'(t) + \xi\int_{\Gamma+}\mathbf{v}^1\cdot\widetilde{\mathbf{v}}\,\varphi(0),\\
\text{(b)} -\displaystyle\int_0^T\int_\Omega Pe(\mathbf{v})\cdot\nabla\widetilde{\boldsymbol{\phi}}\,\varphi(t) + \int_0^T\int_\Omega \varepsilon\nabla\widetilde{\boldsymbol{\phi}}\cdot\nabla\boldsymbol{\phi}\,\varphi(t) = 0,\\
\text{(c)} \displaystyle\int_0^T\int_\Omega \varepsilon\mathbf{A}\cdot\widetilde{\mathbf{A}}\,\varphi''(t) - \int_\Omega \varepsilon\mathbf{A}^1\cdot\widetilde{\mathbf{A}}\,\varphi(0) + \int_\Omega \varepsilon\mathbf{A}^0\cdot\widetilde{\mathbf{A}}\,\varphi'(0)\\
\quad - \displaystyle\int_0^T\int_\Omega Pe(\mathbf{v}')\cdot\widetilde{\mathbf{A}}\,\varphi(t) + \frac{1}{\mu_1}\int_0^T\int_\Omega \text{curl}\,\mathbf{A}\cdot\text{curl}\,\widetilde{\mathbf{A}}\,\varphi(t)\\
\quad = -\alpha\displaystyle\int_0^T\int_\Omega \mathbf{A}'\cdot\widetilde{\mathbf{A}}\,\varphi(t) - \varkappa\int_0^T\int_\Omega \nabla\mathbf{A}:\nabla\widetilde{\mathbf{A}}\,\varphi(t)\\
\quad - \gamma\displaystyle\int_0^T\int_{\Gamma-}\mathbf{A}'\cdot\widetilde{\mathbf{A}}\,\varphi(t) + \vartheta\int_0^T\int_{\Gamma-}\mathbf{A}'\cdot\widetilde{\mathbf{A}}\,\varphi'(t) + \vartheta\int_{\Gamma-}\mathbf{A}^1\cdot\widetilde{\mathbf{A}}\,\varphi(0).
\end{cases} \tag{53}$$

Due to linearity and density arguments, Equation (53) holds for any $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}) \in (\mathbb{V}(\Omega) \times \mathbb{V}(\Omega) \times \Psi(\Omega))$. Assume that the system has two solutions that satisfy the same initial conditions. Since the system is linear, by integrating Equation (22) over $]0, t[$ we get:

$$\mathscr{E}(t) = -\int_0^t \left(\beta \int_{\Gamma^+} |\mathbf{v}'|^2 - \alpha \int_{\Omega} |\mathbf{A}'|^2 - \vartheta \int_{\Gamma^-} |\mathbf{A}'|^2 \right) ds, \tag{54}$$

assuming that the initial conditions are zero. For some positive constants $\alpha, \beta, \gamma, \vartheta, \varkappa$ and ξ, we deduce from Equation (21) that $\mathscr{E}(t) \geq 0$ and from Equation (54) that $\mathscr{E}(t) \leq 0$. This implies that the energy for the difference of two solutions is zero, thus uniqueness of the solution is proven. Let $(\mathbf{v}, \mathbf{A}, \boldsymbol{\phi})$ be a solution of Equation (53) and show that $(\mathbf{v}, \mathbf{A}, \boldsymbol{\phi})$ verifies identities in Equation (17). Let $\widetilde{\mathbf{A}} = 0, \widetilde{\mathbf{v}} = 0$ and $\widetilde{\boldsymbol{\phi}} = 0$ on Γ.

Using the Stokes' formula in Equation (53), we obtain:

$$\begin{cases} \rho \mathbf{v}'' - \operatorname{div}\big(Ce(\mathbf{v}) + P\mathbf{A}' + P\nabla\boldsymbol{\phi}\big) = 0 & \text{in } \Omega, \quad (55) \\ \operatorname{div}\big(Pe(v) - \varepsilon\nabla\boldsymbol{\phi} - \varepsilon\mathbf{A}'\big) = -\operatorname{div}(\varepsilon\mathbf{A}') & \text{in } \Omega, \quad (56) \\ (-Pe(\mathbf{v}) + \varepsilon\nabla\boldsymbol{\phi})' + \varepsilon\mathbf{A}'' + \dfrac{1}{\mu_1} \operatorname{curl}\operatorname{curl}\mathbf{A} = \varepsilon\nabla\boldsymbol{\phi}' + \varkappa \operatorname{div}(\nabla\mathbf{A}) - \alpha\mathbf{A}' & \text{in } \Omega, \quad (57) \end{cases}$$

in the sense of distribution because you can take the test functions to be C^∞ with compact support.

Furthermore, by using Stokes' formula once more after multiplying the Equations (55)–(57) by $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}) \in (\mathbb{V}(\Omega) \times \mathbb{V}(\Omega) \times \Psi(\Omega))$ and integrating over Ω, followed by multiplication by $\varphi(t) \in \mathscr{D}([0, T[)$, then integrating by parts with respect to time, we obtain:

$$\begin{cases} \text{(a) } \rho \displaystyle\int_0^T \int_\Omega \mathbf{v} \cdot \widetilde{\mathbf{v}} \varphi''(t) - \rho \int_\Omega \mathbf{v}'(0) \cdot \widetilde{\mathbf{v}} \varphi(0) + \rho \int_\Omega \mathbf{v}(0) \cdot \widetilde{\mathbf{v}} \varphi'(0) + \int_0^T \int_\Omega Ce(\mathbf{v}) : e(\widetilde{\mathbf{v}}) \varphi(t) \\ \quad + \displaystyle\int_0^T \int_\Omega P\mathbf{A}' : e(\widetilde{\mathbf{v}}) \varphi(t) + \int_0^T \int_\Omega P\nabla\phi : e(\widetilde{\mathbf{v}}) \varphi(t) \\ \quad = \displaystyle\int_0^T \int_{\Gamma_N} \big(Ce(\mathbf{v}) \cdot \mathbf{n} + P\mathbf{A}' \cdot \mathbf{n} + P\nabla\phi \cdot \mathbf{n}\big) \cdot \widetilde{\mathbf{v}} \varphi(t), \\ \\ \text{(b) } -\displaystyle\int_0^T \int_\Omega Pe(\mathbf{v}) \cdot \nabla\widetilde{\phi} \varphi(t) + \int_0^T \int_\Omega \varepsilon\nabla\widetilde{\phi} \cdot \nabla\phi \varphi(t) + \int_0^T \int_\Omega \varepsilon\nabla\widetilde{\phi} \cdot \mathbf{A}' \varphi(t) \\ \quad = -\displaystyle\int_0^T \int_\Omega \big(\operatorname{div}(\varepsilon\mathbf{A}')\big) \widetilde{\phi} \varphi(t), \\ \\ \text{(c) } \displaystyle\int_0^T \int_\Omega \varepsilon\mathbf{A} \cdot \widetilde{\mathbf{A}} \varphi''(t) - \int_\Omega \varepsilon\mathbf{A}'(0) \cdot \widetilde{\mathbf{A}} \varphi(0) + \int_\Omega \varepsilon\mathbf{A}(0) \cdot \widetilde{\mathbf{A}} \varphi'(0) - \int_0^T \int_\Omega Pe(\mathbf{v}') \cdot \widetilde{\mathbf{A}} \varphi(t) \\ \quad + \dfrac{1}{\mu_1} \displaystyle\int_0^T \int_\Omega \operatorname{curl}\mathbf{A} \cdot \operatorname{curl}\widetilde{\mathbf{A}} \varphi(t) = -\alpha \int_0^T \int_\Omega \mathbf{A}' \cdot \widetilde{\mathbf{A}} \varphi(t) - \varkappa \int_0^T \int_\Omega \nabla\mathbf{A} : \nabla\widetilde{\mathbf{A}} \varphi(t) \\ \quad + \varkappa \displaystyle\int_0^T \int_{\Gamma_N} (\nabla\mathbf{A} \cdot \mathbf{n}) \cdot \widetilde{\mathbf{A}} \varphi(t) + \dfrac{1}{\mu_1} \int_0^T \int_{\Gamma_N} (\operatorname{curl}\mathbf{A} \wedge \mathbf{n}) \cdot \widetilde{\mathbf{A}} \varphi(t). \end{cases} \tag{58}$$

Since the test functions $\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}$ and φ are arbitrarily chosen, comparing Equations (53) and (58) identify boundary and initial conditions. That is, we get Equation (17d–g,k,l).

For Dirichlet conditions of Equation (17h–j), they are obtained due to the definition of functional spaces.

Therefore, the weak limit $(\mathbf{v}, \mathbf{A}, \phi)$ is a unique solution of Equation (20) in the sense of Definition 1 and the four Lemmas 2–5 complete the proof. $\square$

4. Asymptotic analysis

Consider the Equation (17) of the plate with thickness $2h$, over the domain Ω^h which is a cylindrical domain of $\mathbb{R}^3$, and the middle surface ω, a domain of $\mathbb{R}^2$ as described in **Figure 2**.

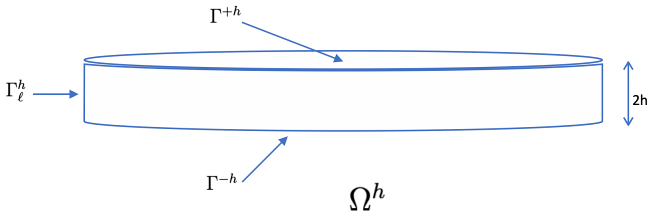

Figure 2. Piezoelectric plate.

Precisely, $\Omega^h = \omega \times]-h, h[$ with boundary $\Gamma^h = \Gamma^{+h} \cup \Gamma^{-h} \cup \Gamma^h_\ell$, where $\Gamma^{+h} = \omega \times \{+h\}$ is upper surface, and $\Gamma^{-h} = \omega \times \{-h\}$ is lower surface, and $\Gamma^h_\ell = \partial\omega \times]-h, h[$ is lateral surface, a reference domain is $\Omega = \omega \times]-1, 1[$. For $T > 0$ fixed, let $Q^h = [0, T] \times \Omega^h$. The Neumann boundary conditions are imposed on $\Gamma^h_N = \Gamma^{+h} \cup \Gamma^{-h}$. The Dirichlet boundary conditions are imposed on Γ^h_ℓ. Then, we have the piezoelectric system:

$$\begin{cases} \text{(a)} \quad \rho^h \mathbf{v}^{h''} - \operatorname{div}^h(Ce^h(\mathbf{v}^h) + P\nabla^h \phi^h(\mathbf{v}^h) + P\mathbf{A}^{h'}) = 0 & \text{in} \quad Q^h, \\ \text{(b)} \quad \operatorname{div}^h(Pe^h(\mathbf{v}^h) - \varepsilon \nabla^h \phi^h(\mathbf{v}^h) - \varepsilon \mathbf{A}^{h'}) = q^h & \text{in} \quad Q^h, \\ \text{(c)} \quad (-Pe^h(\mathbf{v}^h) + \varepsilon \nabla^h \phi^h(\mathbf{v}^h))' + \varepsilon \mathbf{A}^{h''} + \dfrac{1}{\mu_1} \operatorname{curl}^h \operatorname{curl}^h \mathbf{A}^h = \mathbf{J}^h & \text{in} \quad Q^h, \end{cases}$$

with boundary conditions

$$\begin{cases} \text{(d)} \quad (Ce^h(\mathbf{v}^h) + P\nabla^h \phi^h(\mathbf{v}^h) + P\mathbf{A}^{h'})\mathbf{n} = \mathbf{s}^h & \text{on} \quad \Gamma^{+h}, \\ \text{(e)} \quad (Ce^h(\mathbf{v}^h) + P\nabla^h \phi^h(\mathbf{v}^h) + P\mathbf{A}^{h'})\mathbf{n} = 0 & \text{on} \quad \Gamma^{-h}, \\ \text{(f)} \quad \operatorname{curl}^h \mathbf{A}^h \wedge \mathbf{n}/\mu_1 = -\mathbf{j}^h & \text{on} \quad \Gamma^{-h}, \\ \text{(g)} \quad \operatorname{curl}^h \mathbf{A}^h \wedge \mathbf{n}/\mu_1 = 0 & \text{on} \quad \Gamma^{+h}, \\ \text{(h)} \quad \mathbf{v}^h = 0 & \text{on} \quad \Gamma^h_\ell, \\ \text{(i)} \quad \phi^h = 0 & \text{on} \quad \Gamma^h, \\ \text{(j)} \quad \mathbf{A}^h = 0 & \text{on} \quad \Gamma^h_\ell, \end{cases} \tag{59}$$

and initial conditions

$$\begin{cases} \text{(k)} \quad \mathbf{v}^h(0) = \mathbf{v}^{0h}(x), \quad \mathbf{v}^{h'}(0) = \mathbf{v}^{1h}(x) & \text{on} \quad \Omega^h, \\ \text{(l)} \quad \mathbf{A}^h(0) = \mathbf{A}^{0h}(x), \quad \mathbf{A}^{h'}(0) = \mathbf{A}^{1h}(x) & \text{on} \quad \Omega^h. \end{cases}$$

Here, components (a)–(l) are collectively referred to as a single system denoted by Equation (59).

The weak formulation of Equation (59) is

$$\begin{cases} \text{(a)} \quad \int_{\Omega^h} \rho^h \mathbf{v}^{h''} \cdot \widetilde{\mathbf{v}}^h + \int_{\Omega^h} Ce^h(\mathbf{v}^h) : e^h(\widetilde{\mathbf{v}}^h) + \int_{\Omega^h} P\mathbf{A}^{h'} : e^h(\widetilde{\mathbf{v}}^h) \\ \qquad + \int_{\Omega^h} P\nabla^h \boldsymbol{\phi}^h(\mathbf{v}^h) : e^h(\widetilde{\mathbf{v}}^h) = \int_{\Gamma^{+h}} \mathbf{s}^h \cdot \widetilde{\mathbf{v}}^h, \\ \text{(b)} \quad -\int_{\Omega^h} Pe^h(\mathbf{v}^h) \cdot \nabla^h \widetilde{\boldsymbol{\phi}}^h(\widetilde{\mathbf{v}}^h) + \int_{\Omega^h} \varepsilon \nabla^h \widetilde{\boldsymbol{\phi}}^h(\widetilde{\mathbf{v}}^h) \cdot \nabla^h \boldsymbol{\phi}^h(\mathbf{v}^h) + \int_{\Omega^h} \varepsilon \nabla^h \widetilde{\boldsymbol{\phi}}^h(\widetilde{\mathbf{v}}^h) \cdot \mathbf{A}^{h'} \\ \qquad = \int_{\Omega^h} q^h \cdot \widetilde{\boldsymbol{\phi}}^h(\widetilde{\mathbf{v}}^h), \\ \text{(c)} \quad \int_{\Omega^h} \varepsilon \mathbf{A}^{h''} \cdot \widetilde{\mathbf{A}}^h + \int_{\Omega^h} \varepsilon \nabla^h \boldsymbol{\phi}^{h'}(\mathbf{v}^h) \cdot \widetilde{\mathbf{A}}^h - \int_{\Omega^h} Pe^h(\mathbf{v}^{h'}) \cdot \widetilde{\mathbf{A}}^h + \frac{1}{\mu_1} \int_{\Omega^h} \operatorname{curl}^h \mathbf{A}^h \cdot \operatorname{curl}^h \widetilde{\mathbf{A}}^h \\ \qquad = \int_{\Omega^h} \mathbf{J}^h \cdot \widetilde{\mathbf{A}}^h + \int_{\Gamma^{-h}} \mathbf{j}^h \cdot \widetilde{\mathbf{A}}^h, \end{cases} \tag{60}$$

with initial conditions

$$\begin{cases} \text{(d)} \quad \left(\int_{\Omega^h} \widetilde{\mathbf{v}}^h \cdot \mathbf{v}^h \right)(0) = \int_{\Omega^h} \widetilde{\mathbf{v}}^h \cdot \mathbf{v}^{0h}, \quad \left(\int_{\Omega^h} \widetilde{\mathbf{A}}^h \cdot \mathbf{A}^h \right)(0) = \int_{\Omega^h} \widetilde{\mathbf{A}}^h \cdot \mathbf{A}^{0h}, \\ \text{(e)} \quad \left(\int_{\Omega^h} \widetilde{\mathbf{v}}^h \cdot \mathbf{v}^h \right)(0) = \int_{\Omega^h} \widetilde{\mathbf{v}}^h \cdot \mathbf{v}^{1h}, \quad \left(\int_{\Omega^h} \widetilde{\mathbf{A}}^h \cdot \mathbf{A}^{h'} \right)(0) = \int_{\Omega^h} \widetilde{\mathbf{A}}^h \cdot \mathbf{A}^{1h}, \end{cases}$$

here, components (a)–(e) are collectively referred to as a single system denoted by Equation (60).

Among above,

$$\begin{aligned} & q^h = -\operatorname{div}^h \left(\varepsilon \mathbf{A}^{h'} \right), \quad \mathbf{s}^h = -h \left(\xi^h \mathbf{v}^{h''} + \beta^h \mathbf{v}^{h'} \right), \\ & \mathbf{j}^h = -h \left(\vartheta \mathbf{A}^{h''} + \gamma \mathbf{A}^{h'} + \varkappa \nabla^h \mathbf{A}^h \cdot \mathbf{n} \right), \quad \mathbf{J}^h = \varepsilon \nabla^h \boldsymbol{\phi}^{h'} + \varkappa \operatorname{div}^h (\nabla^h \mathbf{A}^h) - \alpha \mathbf{A}^{h'}. \end{aligned} \tag{61}$$

By asymptotic analysis, we obtain a simplified 2D model, which encapsulates the fundamental characteristics of the initial 3D piezoelectric model. The crucial link between the mechanical and electromagnetic fields is maintained, while this simplification drastically reduces computational complexity. The use of this 2D model is crucial for the design and analysis of thin piezoelectric devices, including sensors, actuators, and energy harvesters, particularly in tissue-engineering scaffolds and implantable devices.

Note that in the sequel, by default, the Latin indices take the values $1, 2, 3$, and Greek indices take the values $1, 2$.

4.1. Scaled variational formulation

We make a change of variables to bring the small parameter h on the operators only and no longer on the domain, to obtain convergence results in Hilbert spaces independent of h.

The rescaling choices are made to obtain the evolution of the Kirchhoff-Love model without losing the inertia term [4]. Alternative assumptions would correspond to different physical regimes, yielding different models.

We set $x = (x_1, x_2, x_3) \in \Omega$ and its associate $x^h = (x_1, x_2, hx_3) \in \Omega^h$, and make the following assumptions about the orders of magnitude of the solutions [4,5]:

$$\begin{aligned} & \mathbf{v}_1^h = h^2 \mathbf{v}_1(h)(x), \quad \mathbf{v}_2^h = h^2 \mathbf{v}_2(h)(x), \quad \mathbf{v}_3^h = h \mathbf{v}_3(h)(x), \\ & \boldsymbol{\phi}^h = h^3 \boldsymbol{\phi}(h)(x), \\ & \mathbf{A}_1^h = h^2 \mathbf{A}_1(h)(x), \quad \mathbf{A}_2^h = h^2 \mathbf{A}_2(h)(x), \quad \mathbf{A}_3^h = h^2 \mathbf{A}_3(h)(x). \end{aligned} \tag{62}$$

The assumed orders of magnitude are part of an asymptotic scaling strategy whose purpose is to isolate the dominant mechanical, electrical and dissipative effects in the thin-structure limit. These scalings are chosen to ensure a non-trivial balance between elastic, electrostatic and stabilization terms in the resulting 2D model.

Specifically, the scaling choices are justified by the principal of asymptotic consistency. The disparity between $\mathbf{v}_3^h$ ($O(h)$) and $\mathbf{v}_\alpha^h$ ($O(h^2)$) is the necessary condition for the 3D system to satisfy the Kirchhoff-Love constraint in the limit. Furthermore, scaling the electric and magnetic potentials ($\boldsymbol{\phi}^h$, $\mathbf{A}^h$) as $O(h^3)$ and $O(h^2)$ respectively ensures that the electromechanical coupling is preserved. This roadmap provides the rigorous link between the 3D physics and the derived 2D piezoelectric model. The initial conditions $\mathbf{v}^{0h}, \mathbf{A}^{0h}, \mathbf{v}^{1h}, \mathbf{A}^{1h}$ are transported in the same way and give rise to $\mathbf{v}^0(h), \mathbf{A}^0(h),\ \mathbf{v}^1(h),\ \mathbf{A}^1(h)$. In order to obtain the scaled variational formulation, operators can be scaled as follows.

For any $\mathbf{v} \in (H^1(\Omega)^3)$, let

$$\begin{aligned}
\kappa_{\alpha\beta}(h)(\mathbf{v}) &= \frac{1}{2}(\partial_\beta \mathbf{v}_\alpha + \partial_\alpha \mathbf{v}_\beta) = e^h_{\alpha\beta}(\mathbf{v}), \quad \alpha, \beta = 1, 2,\\
\kappa_{\alpha 3}(h)(\mathbf{v}) &= \frac{1}{2h}(\partial_3 \mathbf{v}_\alpha + \partial_\alpha \mathbf{v}_3) = \frac{1}{h} e^h_{\alpha 3}(\mathbf{v}), \quad \alpha = 1, 2,\\
\kappa_{33}(h)(\mathbf{v}) &= \frac{1}{2h^2}(\partial_3 \mathbf{v}_3 + \partial_3 \mathbf{v}_3) = \frac{1}{h^2}\partial_3 \mathbf{v}_3 = \frac{1}{h^2} e^h_{33}(\mathbf{v}).
\end{aligned}$$

For any $\mathbf{A} \in (L^2(\Omega)^3)$, let

$$\operatorname{curl}(h)\mathbf{A} = \left(\partial_2 \mathbf{A}_3 - \frac{1}{h}\partial_3 \mathbf{A}_2, \frac{1}{h}\partial_3 \mathbf{A}_1 - \partial_1 \mathbf{A}_3, \partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1\right),$$

and

$$\operatorname{div}(h)(\varepsilon \mathbf{A}) = \partial_1(\varepsilon_{1j}\mathbf{A}_j) + \partial_2(\varepsilon_{2j}\mathbf{A}_j) + \frac{1}{h}\partial_3(\varepsilon_{3j}\mathbf{A}_j),$$

furthermore, for any $\boldsymbol{\phi} \in H^1(\Omega)$, let

$$\nabla(h)\boldsymbol{\phi} = (h\partial_1\boldsymbol{\phi}, h\partial_2\boldsymbol{\phi}, \partial_3\boldsymbol{\phi}).$$

Using the scalings mentioned above, we have

$$\begin{aligned}
&e^h(\mathbf{v}^h) = h^2\kappa(h)(\mathbf{v}(h)), \quad \operatorname{curl}^h \mathbf{A}^h = h^2 \operatorname{curl}(h)\mathbf{A}(h), \quad \operatorname{div}^h(\varepsilon \mathbf{A}^h) = h^2 \operatorname{div}(h)(\varepsilon \mathbf{A}(h)),\\
&\nabla^h \boldsymbol{\phi}^h = h^2 \nabla(h)\boldsymbol{\phi}(h), \quad \nabla^h \mathbf{A}^h = h\nabla(h)\mathbf{A}(h).
\end{aligned}$$

All coefficients, including mechanical, piezoelectric, and electromagnetic, are independent of h, except the mass densities ρ^h, and the damping coefficients β^h, and ξ^h. Assume that there exist ρ, β and ξ such that

$$\rho^h = h^2\rho, \quad \beta^h = h^2\beta, \quad \xi^h = h^2\xi.$$

Note that the scaling $\rho^h = h^2\rho$ is the only choice that allows us to retrieve the vertical inertia term when h tends to zero. To establish the results of existence and uniqueness for the piezoelectric problem, we use the following spaces where the definitions and properties are presented in the studies by Pedregal [23] and Boukarou et al. [28]. Consider the following spaces:

$$\begin{aligned}
&\mathbb{V}(\Omega^h) = \left\{ \mathbf{v} \in (H^1(\Omega^h))^3 : \mathbf{v}|_{\Gamma_\ell^h} = 0 \right\}, \\
&\Psi(\Omega^h) = H_0^1(\Omega^h).
\end{aligned} \tag{63}$$

The spaces above, each endowed with the norms derived from $H^1(\Omega^h)$ are all Hilbert spaces [23].

Definition 3. *Let $T > 0$ be an arbitrary real number and consider the scalings of Equation (62), $(\mathbf{v}^0(h), \mathbf{A}^0(h)) \in (\mathbb{V}(\Omega))^2$ and $(\mathbf{v}^1(h), \mathbf{A}^1(h)) \in (L^2(\Omega))^3 \times (L^2(\Omega))^3$, the triplet $(\mathbf{v}(h), \mathbf{A}(h), \boldsymbol{\phi}(h))$ is a classical solution of the scaled-up Equation (59) on $[0, T)$ if:*

1. $\mathbf{v}(h) \in L^2(0, T; \mathbb{V}(\Omega)), \mathbf{v}'(h) \in L^2(0, T; (L^2(\Omega))^3)$,
2. $\mathbf{A}(h) \in L^2(0, T; \mathbb{V}(\Omega)), \mathbf{A}'(h) \in L^2(0, T; (L^2(\Omega))^3)$,
3. $\boldsymbol{\phi}(h) \in L^2(0, T; \Psi(\Omega))$,
4. *and the scaled-up Equation (59) holds in the sense of distributions.*

The triplet $(\mathbf{v}(h), \mathbf{A}(h), \boldsymbol{\phi}(h))$ is a weak solution of the scaled-up Equation (59) on $[0, T)$ if it satisfies conditions 1–3 above and $\forall(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}) \in (\mathbb{V}(\Omega) \times \mathbb{V}(\Omega) \times \Psi(\Omega))$, we have:

$$\begin{cases}
\text{(a)} & \displaystyle\int_\Omega \rho(h^2 \mathbf{v}_\alpha''(h)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}_3''(h)\widetilde{\mathbf{v}}_3) + \int_\Omega C\kappa(h)(\mathbf{v}(h)) : \kappa(h)(\widetilde{\mathbf{v}}) \\
& \displaystyle + \int_\Omega P\mathbf{A}'(h) : \kappa(h)(\widetilde{\mathbf{v}}) + \int_\Omega P\nabla(h)\boldsymbol{\phi}(\mathbf{v}(h)) : \kappa(h)(\widetilde{\mathbf{v}}) \\
& \displaystyle = -\int_{\Gamma+} \xi[h^2 \mathbf{v}_\alpha''(h)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}_3''(h)\widetilde{\mathbf{v}}_3] - \int_{\Gamma+} \beta[h^2 \mathbf{v}_\alpha'(h)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}_3'(h)\widetilde{\mathbf{v}}_3],
\end{cases}$$

$$\begin{cases}
\text{(b)} & \displaystyle -\int_\Omega P\kappa(h)(\mathbf{v}(h)) \cdot \nabla(h)\widetilde{\boldsymbol{\phi}} + \int_\Omega \varepsilon\nabla(h)\widetilde{\boldsymbol{\phi}} \cdot \nabla(h)\boldsymbol{\phi}(\mathbf{v}(h)) = 0,
\end{cases}$$

$$\begin{cases}
\text{(c)} & \displaystyle\int_\Omega \varepsilon\mathbf{A}''(h) \cdot \widetilde{\mathbf{A}} - \int_\Omega P\kappa(h)(\mathbf{v}'(h)) \cdot \widetilde{\mathbf{A}} + \frac{1}{\mu_1}\int_\Omega \operatorname{curl}(h)\,\mathbf{A}(h) \cdot \operatorname{curl}(h)\,\widetilde{\mathbf{A}} \\
& \displaystyle = -\varkappa\int_\Omega \frac{1}{h^2}\nabla(h)\mathbf{A}(h) : \nabla(h)\widetilde{\mathbf{A}} - \int_\Omega \alpha\mathbf{A}'(h) \cdot \widetilde{\mathbf{A}} \\
& \displaystyle - \int_{\Gamma-} (\vartheta\mathbf{A}''(h) + \gamma\mathbf{A}'(h)) \cdot \widetilde{\mathbf{A}},
\end{cases} \tag{64}$$

with initial conditions

$$\begin{cases}
\text{(d)} & \displaystyle\left(\int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}(h)\right)(0) = \int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^0(h), \quad \left(\int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}(h)\right)(0) = \int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^0(h), \\
\text{(e)} & \displaystyle\left(\int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}(h)'\right)(0) = \int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^1(h), \quad \left(\int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}(h)'\right)(0) = \int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^1(h).
\end{cases}$$

Here, components (a)–(e) are collectively referred to as a single system denoted by Equation (64)*.*

The energy of the Equation (64) is given by

$$\begin{aligned}
\mathscr{E}(h)(t) = \frac{1}{2}\Big(&\rho\int_\Omega (h^2\mathbf{v}_\alpha'(h)^2 + \mathbf{v}_3'(h)^2) + \int_\Omega \varepsilon\mathbf{A}'(h) \cdot \mathbf{A}'(h) + \int_\Omega C\kappa(h)(\mathbf{v}(h)) : \kappa(h)(\mathbf{v}(h)) \\
&+ \int_\Omega \varepsilon\nabla(h)\boldsymbol{\phi}(h)(\mathbf{v}(h)) \cdot \nabla(h)\boldsymbol{\phi}(h)(\mathbf{v}(h)) + \frac{1}{\mu_1}\int_\Omega |\operatorname{curl}(h)\,\mathbf{A}(h)|^2 \\
&+ \xi\int_{\Gamma+} [h^2\mathbf{v}_\alpha'(h)^2 + \mathbf{v}_3'(h)^2] + \vartheta\int_{\Gamma-} |\mathbf{A}'(h)|^2 + \varkappa\int_\Omega \frac{1}{h^2}|\nabla(h)\mathbf{A}(h)|^2\Big),
\end{aligned} \tag{65}$$

and following the same procedure as in the study by Bidouan et al. [22], we have

$$\mathscr{E}'(h)(t) = -\beta \int_{\Gamma^+} (h^2 \mathbf{v}'_\alpha(h)^2 + \mathbf{v}'_3(h)^2) - \alpha \int_\Omega |\mathbf{A}'(h)|^2 - \gamma \int_{\Gamma^-} |\mathbf{A}'(h)|^2. \quad (66)$$

The Equation (64) can be written in compact form as follows. Let $U(h) = (\mathbf{v}(h), \mathbf{A}(h), \boldsymbol{\phi}(\mathbf{v}(h)))$ and $V = (\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}})$, then

$$\begin{aligned} &\rho \int_\Omega (h^2 \mathbf{v}''_\alpha(h) \widetilde{\mathbf{v}}_\alpha + \mathbf{v}''_3(h)\widetilde{\mathbf{v}}_3) + \int_\Omega \varepsilon \mathbf{A}''(h) \cdot \widetilde{\mathbf{A}} + \int_{\Gamma^+} \xi [h^2 \mathbf{v}''_\alpha(h)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}''_3(h)\widetilde{\mathbf{v}}_3] \\ &+ \int_{\Gamma^+} \beta [h^2 \mathbf{v}'_\alpha(h)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}'_3(h)\widetilde{\mathbf{v}}_3] + \varkappa \int_\Omega \frac{1}{h^2} \nabla(h)\mathbf{A}(h) : \nabla \widetilde{\mathbf{A}} + \int_\Omega \alpha \mathbf{A}'(h) \cdot \widetilde{\mathbf{A}} \\ &+ \int_{\Gamma^-} (\vartheta \mathbf{A}''(h) + \gamma \mathbf{A}'(h)) \cdot \widetilde{\mathbf{A}} + a(U(h), V) + b(U(h), V) = 0, \end{aligned} \quad (67)$$

and

$$\begin{aligned} &\left(\int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}(h)\right)(0) = \int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^0(h), \quad \left(\int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}(h)\right)(0) = \int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^0(h), \\ &\left(\int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}(h)'\right)(0) = \int_\Omega \widetilde{\mathbf{v}} \cdot \mathbf{v}^1(h), \quad \left(\int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}(h)'\right)(0) = \int_\Omega \widetilde{\mathbf{A}} \cdot \mathbf{A}^1(h), \end{aligned}$$

where

$$\begin{aligned} a(h)(U,V) &= \int_\Omega C\kappa(h)(\mathbf{v}(h)) : \kappa(h)(\widetilde{\mathbf{v}}) + \int_\Omega \varepsilon \nabla(h)\widetilde{\boldsymbol{\phi}} \cdot \nabla(h)\boldsymbol{\phi}(\mathbf{v}(h)) \\ &\quad + \frac{1}{\mu_1} \int_\Omega \operatorname{curl}(h)\, \mathbf{A}(h) \cdot \operatorname{curl}(h)\, \widetilde{\mathbf{A}}, \\ b(h)(U,V) &= \int_\Omega P\mathbf{A}'(h) : \kappa(h)(\widetilde{\mathbf{v}}) + \int_\Omega P\nabla(h)\boldsymbol{\phi}(\mathbf{v}(h)) : \kappa(h)(\widetilde{\mathbf{v}}) \\ &\quad - \int_\Omega P\kappa(h)(\mathbf{v}(h)) \cdot \nabla(h)\widetilde{\boldsymbol{\phi}} - \int_\Omega P\kappa(h)(\mathbf{v}'(h)) \cdot \widetilde{\mathbf{A}}. \end{aligned}$$

Note that, using Equation (64b), one can express the electric potential $\boldsymbol{\phi}(h)$ in terms of $\mathbf{v}(h)$.

Let us state the following compactness theorem which is utilized in the sequel for establishing convergences.

Theorem 2. *Let E be a reflexive Banach space, and (x_n) be a bounded sequence in E. There exists a subsequence (x_{n_k}) of (x_n) which converges in the weak topology $\sigma(E, E^\star)$ [28].*

We make the following assumptions about the initial conditions:

Assumption 1. *The initial energy $\mathscr{E}(h)(0)$ is uniformly bounded.*

Assumption 2. *$(h(\mathbf{v}^1(h))_1, h(\mathbf{v}^1(h))_2, (\mathbf{v}^1(h))_3)$ converges to $(0, 0, (\mathbf{v}^1)_3)$ in $(L^2(\Omega))^3$.*

Hence, by applying Theorem 2, one can deduce weak convergence of subsequences of initial conditions.

4.2. Estimates and convergence results

Here, we prove the coercivity of the bilinear form $a(h)(U, V)$.

Lemma 6. *There exists a constant $\chi > 0$ such that for all $V = (\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}})$ and for all*

$0 < h < 1$

$$a(h)(V,V) \geq \chi \left(|\widetilde{\mathbf{v}}(h)|^2 + |\nabla(h)\widetilde{\phi}(h)|^2 + |\text{curl}(h)(\widetilde{\mathbf{A}})|^2 \right). \tag{68}$$

Proof. Using the properties of the elasticity tensor C, which is positive definite, we have:

$$\int_\Omega C\kappa(\widetilde{\mathbf{v}}) : \kappa(\widetilde{\mathbf{v}})\, dx \geq C_1 \|\kappa(\widetilde{\mathbf{v}})\|^2_{L^2(\Omega)},$$

for some constant $C_1 > 0$. Similarly, using the properties of the electric parameter ε:

$$\int_\Omega \varepsilon \nabla(h)\widetilde{\phi} \cdot \nabla(h)\widetilde{\phi}\, dx \geq C_2 \|\nabla(h)\widetilde{\phi}\|^2_{L^2(\Omega)},$$

for some constant $C_2 > 0$.

Therefore, putting $\chi = \min\left\{C_1, C_2, \dfrac{1}{\mu_1}\right\}$, it becomes Equation (68). Since $0 < h < 1$, due to Korn's inequality, there exists a positive constant $\mathscr{K}$ such that

$$\mathscr{K} \|\widetilde{\mathbf{v}}(h)\|_{\mathbb{V}} \leq \int_\Omega e(\widetilde{\mathbf{v}}(h)) : e(\widetilde{\mathbf{v}}(h))\, dx \leq \int_\Omega \kappa(h)(\widetilde{\mathbf{v}}(h)) : \kappa(h)(\widetilde{\mathbf{v}}(h))\, dx. \tag{69}$$

This proves the coercivity of the bilinear form $a(h)$. □

Let us now construct the spaces of the limit solutions. Recall that $\mathbf{v}$ is the Kirchhoff-Love displacement if it satisfies $e_{i3}(\mathbf{v}) = 0$ for all $i = 1, 2, 3$. By considering the boundary conditions, we define here the proper space for the limit displacement.

$$\mathbb{V}_{KL} = \{\mathbf{v} \in (H^1(\Omega))^3, \mathbf{v}|_{\Gamma_\ell} = 0, e_{i3}(\mathbf{v}) = 0, i = 1, 2, 3\}.$$

Equivalently, $\mathbb{V}_{KL}$ can be defined as in Raoult and Sène's study [5] by

$$\begin{aligned}\mathbb{V}_{KL} = \{&\widetilde{\mathbf{v}} \in (H^1(\Omega))^3, \exists(\eta_1, \eta_2) \in (H^1_0(\omega))^2, \eta_3 \in H^2_0(\omega),\\ &\widetilde{\mathbf{v}}_\alpha(x) = \eta_\alpha(x_1, x_2) - x_3\partial_\alpha\eta_3(x_1, x_2), \widetilde{\mathbf{v}}_3 = \eta_3(x_1, x_2)\}.\end{aligned}$$

The proper space of the limit vector potential is defined as

$$\mathbb{A}_l = \{\widetilde{\mathbf{A}} \in (L^2(\omega))^3, \partial_1\widetilde{\mathbf{A}}_2 - \partial_2\widetilde{\mathbf{A}}_1 \in L^2(\omega)\}.$$

The space for the limit scalar potential is defined as

$$\Psi_l = \{\widetilde{\phi} \in L^2(\Omega) : \partial_3\widetilde{\phi} \in L^2(\Omega)\}.$$

From Equation (66), we observe that the derivative of the energy $\mathscr{E}(h)(t)$ is negative, which implies that it is decreasing. Then, we deduce that

$$\mathscr{E}(h)(t) \leq \mathscr{E}(h)(0), \quad \forall\, t > 0. \tag{70}$$

Therefore, $\mathscr{E}(h)(t)$ is bounded in $L^\infty(0, T)$, and applying Theorem 2, we have the convergences stated in the following lemma.

Lemma 7. *There exist* $\mathbf{v} \in L^2(0,T;\mathbb{V}(\Omega))$*,* $\mathbf{A} \in L^2(0,T;H^1(\Omega)^3)$ *and* $\phi \in L^2(0,T;L^2(\Omega))$*, and subsequences with parameter* h *such that*

$$\mathbf{v}(h) \rightharpoonup \mathbf{v} \quad \text{in } L^2(0,T;H^1(\Omega)^3), \tag{71}$$

$$\mathbf{A}(h) \rightharpoonup \mathbf{A} \quad \text{in } L^2(0,T;H^1(\Omega)^3), \tag{72}$$

$$\phi(h)(\mathbf{v}(h)) \rightharpoonup \phi \quad \text{in } L^2(0,T;L^2(\Omega)), \tag{73}$$

and

$$(h\mathbf{v}_1'(h), h\mathbf{v}_2'(h), \mathbf{v}_3'(h)) \rightharpoonup (0,0,\mathbf{v}_3') \quad \text{in } L^2(0,T;L^2(\Omega)^3), \tag{74}$$

$$\mathbf{A}'(h) \rightharpoonup \mathbf{A}' \quad \text{in } L^2(0,T;L^2(\Omega)^3). \tag{75}$$

There exist κ *in* $L^2(0,T;L^2(\Omega)^9)$*,* $\mathcal{D}, \mathscr{B}^j$ *in* $L^2(0,T;L^2(\Omega)^3)$ *such that*

$$\kappa(h) \rightharpoonup \kappa \quad \text{in } L^2(0,T;L^2(\Omega)^9), \tag{76}$$

$$\text{curl}(h)\mathbf{A}(h) \rightharpoonup \mathcal{D} \quad \text{in } L^2(0,T;L^2(\Omega)^3), \tag{77}$$

$$\frac{1}{h}\nabla(h)\mathbf{A}_j(h) \rightharpoonup \mathscr{B}^j = (\partial_1\mathbf{A}_j, \partial_2\mathbf{A}_j, \mathscr{B}_3^j) \quad \text{in } L^2(0,T;L^2(\Omega)^3). \tag{78}$$

Furthermore,

- $\mathbf{v}$ *belongs to* $\mathbb{V}_{KL}$*, i.e.,* $e_{i3}(\mathbf{v}) = 0$*,*
- *the potential function satisfies*

$$\nabla(h)\phi(h)(\mathbf{v}(h)) \rightharpoonup (0,0,\partial_3\phi) \quad \text{in } L^2(0,T;L^2(\Omega)^3),$$

- $\mathbf{A}$ *is independent of* x_3 *and* $\mathcal{D}_3 = (\text{curl}\,\mathbf{A})_3$*.*

Proof. Throughout this proof, we use the fact that if a sequence (f_n) and the associated sequence of derivatives (f_n') converge to f and g respectively in the sense of distributions, then we can conclude that $f' = g$ in the sense of distributions and Theorem 2. Therefore, Equations (71)–(78) follow directly from the Equations (68) and (70). For the convergence of the potential function, one should only notice that, the Equation (70) implies that $\nabla(h)\phi(h)(\mathbf{v}(h))$ is bounded in $(L^2(\Omega))^3$, thus $\partial_3\phi(h)(\mathbf{v}(h))$ is bounded in $L^2(0,T;L^2(\Omega))$. Then, we prove that $\phi(h)(\mathbf{v}(h))$ is bounded in $L^2(0,T;L^2(\Omega))$ by using the Cauchy-Schwarz inequality and the fact that $\phi(h)$ is zero on the boundary.

As for the characterization of the limits, note that the convergence of $\kappa(h)$ implies that $\mathbf{v}$ is in $\mathbb{V}_{KL}$, and for the magnetic potential $\mathbf{A}(h)$, from Equation (70), it becomes

$$\begin{aligned}\text{curl}(h)\mathbf{A}(h) &= \left(\partial_2\mathbf{A}_3(h) - \frac{1}{h}\partial_3\mathbf{A}_2(h), \frac{1}{h}\partial_3\mathbf{A}_1(h) - \partial_1\mathbf{A}_3(h), \partial_1\mathbf{A}_2(h) - \partial_2\mathbf{A}_1(h)\right) \\ &\rightharpoonup (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3) \in L^2(0,T;L^2(\Omega)^3),\end{aligned} \tag{79}$$

where $\mathcal{D}_3$ is necessarily equal to $\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1$. Since $\frac{1}{h}\partial_3\mathbf{A}_j(h)$ converges to $\mathscr{B}_3^j$, due to Equation (78), we deduce that the limit $\mathbf{A}$ is independent of x_3. □

Figure 3 is the schematic representation of the controlled piezoelectric plate.

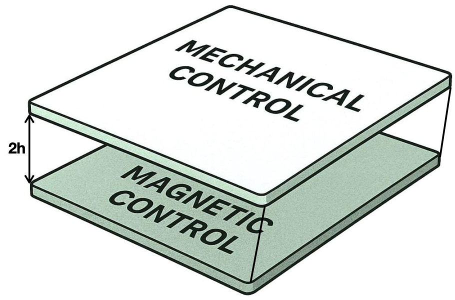

Figure 3. Controlled piezoelectric plate.

4.3. Identification of a 2D problem

This section aims to pass from the three-dimensional formulation to the bidimensional limit equations. In this transformation, the material constants are also transformed. We need to pass to the limit in the weak formulations of the scaled system. Consider the weak forms of Equation (64a–c) and pass to the limit as $h \longrightarrow 0$. Sobolev embeddings and compactness results (e.g., the Rellich-Kondrachov theorem) are utilized to extract weakly convergent subsequences from the sequences of approximate solutions $\mathbf{v}(h), \boldsymbol{\phi}(h), \mathbf{A}(h)$.

We use the obtained convergences to pass to the limit in the weak formulations of the equations to identify the limit problem. This typically involves proving that the weak limit satisfies the equations of the limit problem. By considering homogeneous and mechanically isotropic materials, the mechanical tensor is given by

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{80}$$

where λ and μ are Lamé moduli, and $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$. At this point, the constants related to the mechanical part are defined as follows:

$$c_{\alpha\beta\gamma\sigma} = \tfrac{2\mu\lambda}{\lambda+2\mu}\delta_{\alpha\beta}\delta_{\gamma\sigma} + 2\mu\delta_{\alpha\gamma}\delta_{\beta\sigma}. \tag{81}$$

The piezoelectric constants are transformed in the limit problem as

$$p_{i\alpha\beta} = P_{i\alpha\beta} - \tfrac{\lambda}{\lambda+2\mu}P_{i33}\delta_{\alpha\beta}, \tag{82}$$

which really combines Lamé moduli with piezoelectric constants. For any vector field $\mathbf{A} = (\mathbf{A}_i)$, we note

$$(p\mathbf{A})_{\alpha\beta} = p_{i\alpha\beta}\mathbf{A}_i. \tag{83}$$

Lastly, the electric coefficients ε_{ij} combined with the Lamé moduli and the piezoelectric constants produce

$$\widetilde{\varepsilon}_{ij} = \varepsilon_{ij} + \tfrac{1}{\mu}P_{i\alpha3}P_{j\alpha3} + \tfrac{1}{\lambda+2\mu}P_{i33}P_{j33}, \tag{84}$$

which can be written in the form

$$\widetilde{\varepsilon}_{ij} = \varepsilon_{ij} + \widetilde{p}_{ij}, \tag{85}$$

where

$$\widetilde{p}_{ij} = \tfrac{1}{\mu} P_{i\alpha 3} P_{j\alpha 3} + \tfrac{1}{\lambda+2\mu} P_{i33} P_{j33}. \tag{86}$$

Lemma 8. *The limits* $\mathbf{v}, \kappa$ *and* ϕ *in Lemma 7 are respectively in* $\mathbb{V}_{KL}, (L^2(\Omega))^9$ *and* Ψ *satisfy*

$$\begin{cases} \kappa_{\alpha\beta} = e_{\alpha\beta}(\mathbf{v}) & in \;\; L^2(\Omega), & (87)\\ \kappa_{\alpha 3} = -\dfrac{1}{2\mu}\left(P_{i\alpha 3}\mathbf{A}'_i + P_{3\alpha 3}\partial_3\phi\right) & in \;\; L^2(\Omega)\,, & (88)\\ \kappa_{33} = -\dfrac{1}{\lambda+2\mu}\left(\lambda e_{\gamma\gamma}(\mathbf{v}) + P_{i33}\mathbf{A}'_i + P_{333}\partial_3\phi\right) & in \;\; L^2(\Omega). & (89)\end{cases}$$

Proof. We have already proved that $\kappa_{i3}(h)$ is bounded in $L^2(\Omega)$, therefore $e_{\alpha 3}(\mathbf{v}(h))(= \;\; h\kappa_{\alpha 3}(h))(\mathbf{v}(h))$ and $e_{33}(\mathbf{v}(h))(= \;\; h^2\kappa_{33}(h))$ strongly converge to 0 in $L^2(\Omega)$ when h tends to 0. We have $e_{i3}(\mathbf{v}) = 0$, then $\mathbf{v} \in \mathbb{V}_{KL}$. Equation (87) is a consequence of Equation (71). Multiply Equation (64a) by h^2 and evaluate the limit when $h \to 0$, then using the convergences in Lemma 7 we obtain

$$\forall \widetilde{\mathbf{v}}_3 \in H_0^1(\Omega); \quad \textstyle\int_\Omega [(P\mathbf{A}')_{33} + (C\kappa)_{33} + P_{333}\partial_3\phi]\partial_3\widetilde{\mathbf{v}}_3 = 0.$$

Multiplying Equation (64a) by h and taking $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, 0)$, then using the convergences in Lemma 7 we obtain

$$\forall \widetilde{\mathbf{v}}_\alpha \in H_0^1(\Omega); \quad \int_\Omega [(P\mathbf{A}')_{\alpha 3} + (C\kappa)_{\alpha 3} + P_{3\alpha 3}\partial_3\phi]\partial_3\widetilde{\mathbf{v}}_\alpha = 0.$$

Therefore, $(P\mathbf{A}' + C\kappa)_{i3} + P_{3i3}\partial_3\phi = 0, i = 1,2,3.$ With the help of Equation (80), we solve for κ_{i3}. Using $\kappa_{\alpha\beta} = e_{\alpha\beta}(\mathbf{v})$, we obtain Equations (88) and (89). □

The form $\mathbf{v} = (\zeta_1 - x_3\partial_1\zeta_3, \zeta_2 - x_3\partial_2\zeta_3, \zeta_3)$ with $(\zeta_1, \zeta_2) \in (H_0^1(\omega))^2, \zeta_3 \in H_0^2(\omega)$, is used to write any $\mathbf{v}$ that is obtained in Lemma 8.

Theorem 3. *The second order polynomial in* x_3 *that represents the electric potential* ϕ *is expressed as*

$$\phi(x_1, x_2, x_3) = \sum_{n=0}^{2} \phi^n(x_1, x_2)x_3^n. \tag{90}$$

The coefficients $\phi^n(x_1, x_2)$ *are provided directly in relation to the displacement component* ζ_3*, as well as the provided boundary potential and piezoelectric constants.*

$$\begin{aligned} \phi^0 &= \tfrac{p_{3\alpha\beta}}{2\widetilde{\varepsilon}_{33}}\partial_{\alpha\beta}\zeta_3,\\ \phi^1 &= 0,\\ \phi^2 &= -\tfrac{p_{3\alpha\beta}}{2\widetilde{\varepsilon}_{33}}\partial_{\alpha\beta}\zeta_3. \end{aligned} \tag{91}$$

Proof. By passing to the limit in Equation (64b), one gets

$$\forall \widetilde{\phi} \in H_0^1(\Omega), \quad \int_\Omega \varepsilon_{33}\partial_3\phi\partial_3\widetilde{\phi} = \int_\Omega (P\kappa)_3\partial_3\widetilde{\phi},$$

or equivalently

$$\partial_3(\varepsilon_{33}\partial_3\boldsymbol{\phi} - (P\kappa)_3) = 0. \tag{92}$$

Using the fact that $\boldsymbol{\phi}$ satisfies Equation (92) and $\boldsymbol{\phi} = 0$ on Γ, and Lemma 8, we have

$$\begin{aligned} \varepsilon_{33}\partial_3^2\boldsymbol{\phi} &= \partial_3(P\kappa)_3 = \partial_3(P_{3kl}\kappa_{kl}) \\ &= P_{3\alpha\beta}\partial_3 e_{\alpha\beta}(\mathbf{v}) - P_{3\alpha 3}\partial_3\left[\frac{1}{\mu}\left(P_{i\alpha 3}\mathbf{A}'_i + P_{3\alpha 3}\partial_3\boldsymbol{\phi}\right)\right] \\ &\quad - P_{333}\partial_3\left[\frac{1}{\lambda+2\mu}\left(\lambda e_{\gamma\gamma}(\mathbf{v}) + P_{i33}\mathbf{A}'_i + P_{333}\partial_3\boldsymbol{\phi}\right)\right]. \end{aligned}$$

Hence, using identity $\mathbf{v} = (\zeta_1 - x_3\partial_1\zeta_3, \zeta_2 - x_3\partial_2\zeta_3, \zeta_3)$ and the fact that $\mathbf{A}$ is independent of x_3, it becomes

$$\varepsilon_{33}\partial_3^2\boldsymbol{\phi} = -P_{3\alpha\beta}\partial_{\alpha\beta}\zeta_3 - \frac{1}{\mu}P_{3\alpha 3}^2\partial_3^2\boldsymbol{\phi} + P_{333}\frac{\lambda}{\lambda+2\mu}\partial_{\gamma\gamma}\zeta_3 - \frac{P_{333}^2}{\lambda+2\mu}\partial_3^2\boldsymbol{\phi}. \tag{93}$$

From the integration of Equation (93) with respect to x_3, we deduce that there exists $\boldsymbol{\phi}^i$ $(i = 0, 1)$ such that

$$\boldsymbol{\phi} = \tfrac{1}{\tilde{\varepsilon}_{33}}\left[\tfrac{x_3^2}{2}\left(-P_{3\alpha\beta}\partial_{\alpha\beta}\zeta_3 + P_{333}\tfrac{\lambda}{\lambda+2\mu}\partial_{\gamma\gamma}\zeta_3\right) + x_3\boldsymbol{\phi}^1(x_1,x_2) + \boldsymbol{\phi}^0(x_1,x_2)\right], \tag{94}$$

since $\boldsymbol{\phi}$ is zero on the upper and lower faces of the plate Ω, we have $\boldsymbol{\phi}(x_1, x_2, -1) = \boldsymbol{\phi}(x_1, x_2, 1) = 0$. Therefore

$$\boldsymbol{\phi}^0(x_1,x_2) = -\frac{1}{2}\left(-P_{3\alpha\beta}\partial_{\alpha\beta}\zeta_3 + P_{333}\frac{\lambda}{\lambda+2\mu}\partial_{\gamma\gamma}\zeta_3\right), \quad \boldsymbol{\phi}^1(x_1,x_2) = 0,$$

which ends the proof by replacing $\partial_{\gamma\gamma}\zeta_3$ by $\delta_{\alpha\beta}\partial_{\alpha\beta}\zeta_3$. □

To pass to the limit and deduce the limit problem, we multiply Equations (64a–c) by $\varphi(t) \in \mathscr{D}([0,T[)$, take $(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}, \widetilde{\boldsymbol{\phi}}) \in (\mathbb{V}(\Omega) \times \mathbb{V}(\Omega) \times \Psi(\Omega))$, then integrate by parts with respect to time t to obtain:

$$\begin{cases} \text{(a)} \quad -\displaystyle\int_0^T\int_\Omega \rho(h^2\mathbf{v}'_\alpha(h)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}'_3(h)\widetilde{\mathbf{v}}_3)\varphi'(t) + \int_0^T\int_\Omega C\kappa(\mathbf{v}(h)) : \kappa(\widetilde{\mathbf{v}})\varphi(t) \\ \quad + \displaystyle\int_0^T\int_\Omega P\mathbf{A}'(h) : \kappa(h)(\widetilde{\mathbf{v}})\varphi(t) + \int_0^T\int_\Omega P\nabla(h)\boldsymbol{\phi}(h)(\mathbf{v}(h)) : \kappa(h)(\widetilde{\mathbf{v}})\varphi(t) \\ \quad - \displaystyle\int_\Omega \rho(h^2\mathbf{v}'_\alpha(h)(0)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}'_3(h)(0)\widetilde{\mathbf{v}}_3)\varphi(0) = \int_0^T\int_{\Gamma+} \xi\left(h^2\mathbf{v}'_\alpha(h)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}'_3(h)\widetilde{\mathbf{v}}_3\right)\varphi'(t) \\ \quad + \displaystyle\int_{\Gamma+} \xi\left(h^2\mathbf{v}'_\alpha(h)(0)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}'_3(h)(0)\widetilde{\mathbf{v}}_3)\right)\varphi(0) - \int_0^T\int_{\Gamma+} \beta\left(h^2\mathbf{v}'_\alpha(h)\widetilde{\mathbf{v}}_\alpha + \mathbf{v}'_3(h)\widetilde{\mathbf{v}}_3\right)\varphi(t), \\ \text{(b)} \quad -\displaystyle\int_0^T\int_\Omega P\kappa(h)(\mathbf{v}(h)) \cdot \nabla(h)\widetilde{\boldsymbol{\phi}}\varphi(t) + \int_0^T\int_\Omega \varepsilon\nabla(h)\widetilde{\boldsymbol{\phi}} \cdot \nabla(h)\boldsymbol{\phi}(h)(\mathbf{v}(h))\varphi(t) = 0, \\ \text{(c)} \quad -\displaystyle\int_0^T\int_\Omega \varepsilon\mathbf{A}'(h) \cdot \widetilde{\mathbf{A}}\varphi'(t) + \frac{1}{\mu_1}\int_0^T\int_\Omega \operatorname{curl}(h)\mathbf{A}(h) \cdot \operatorname{curl}(h)\widetilde{\mathbf{A}}\varphi(t) \\ \quad - \displaystyle\int_0^T\int_\Omega P\kappa(h)(\mathbf{v}'(h)) \cdot \widetilde{\mathbf{A}}\varphi(t) = -\varkappa\int_0^T\int_\Omega \frac{1}{h^2}\nabla\mathbf{A}(h) : \nabla\widetilde{\mathbf{A}}\varphi(t) \\ \quad - \alpha\displaystyle\int_0^T\int_\Omega \mathbf{A}'(h) \cdot \widetilde{\mathbf{A}}\varphi(t) + \vartheta\int_0^T\int_{\Gamma-} \mathbf{A}'(h) \cdot \widetilde{\mathbf{A}}\varphi'(t) + \int_{\Gamma-} \vartheta\mathbf{A}'(h)(0) \cdot \widetilde{\mathbf{A}}\varphi(0) \\ \quad - \displaystyle\int_0^T\int_{\Gamma-} \gamma\mathbf{A}(h) \cdot \widetilde{\mathbf{A}}\varphi'(t) + \int_\Omega \varepsilon\mathbf{A}'(h)(0) \cdot \widetilde{\mathbf{A}}\varphi(0). \end{cases} \tag{95}$$

Lemma 9. *Any weak limit* $\mathcal{D}$ *for* $\mathrm{curl}(h)\mathbf{A}(h)$ *obtained in Lemma 7 satisfies* $\mathcal{D}_\alpha = 0, \alpha = 1,2$. *Therefore,* $\mathcal{D} = (0, 0, \partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1)$.

Proof. Multiply Equation (95c) by h. Then, passing to the limit when $h \to 0$ we deduce that

$$\forall\widetilde{\mathbf{A}}_1 \in H^1(\omega), \quad \int_\omega \mathcal{D}_2\partial_2\widetilde{\mathbf{A}}_1 = 0, \quad \forall\widetilde{\mathbf{A}}_2 \in H^1(\omega), \quad \int_\omega \mathcal{D}_1\partial_1\widetilde{\mathbf{A}}_2 = 0. \tag{96}$$

This shows that $\mathcal{D}_\alpha = 0$ since there is no boundary condition on $\widetilde{\mathbf{A}}$. □

Let

$$d_{\alpha\beta\gamma\delta} = \frac{p_{3\alpha\beta}p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}, \tag{97}$$

and d the associated tensor.

Lemma 10. *The limit* $(\mathbf{v}, \mathbf{A}, \boldsymbol{\phi})$ *of scaled solution given in Lemma 7 is the unique solution of the problem: Find* $(\mathbf{v}, \mathbf{A}, \boldsymbol{\phi}) \in (\mathbb{V}_{KL}(\Omega) \times \mathbb{A}_l(\Omega) \times \Psi_l(\Omega))$ *such that* $\forall\,(\widetilde{\mathbf{v}}, \widetilde{\mathbf{A}}) \in (\mathbb{V}_{KL}(\Omega) \times \mathbb{A}_l(\Omega))$,

$$\begin{aligned}
&\int_\omega \rho\zeta_3''\eta_3 + \int_\omega c e_{\alpha\beta}(\zeta_1,\zeta_2)\cdot e_{\alpha\beta}(\eta_1,\eta_2) + \int_\omega p_{i\alpha\beta}e_{\alpha\beta}(\eta_1,\eta_2)\mathbf{A}_i' \\
&+ \frac{1}{3}\int_\omega p_{3\alpha\beta}\frac{p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}\partial_{\gamma\delta}(\zeta_3)\partial_{\alpha\beta}\eta_3 + \int_\omega (\xi\zeta_3'' + \beta\zeta_3')\eta_3 \\
&- \frac{1}{3}\int_\omega p_{3\alpha\beta}\frac{p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}\partial_{\alpha\beta}(\zeta_3)\partial_{\gamma\delta}\eta_3 + \frac{1}{3}\int_\omega \varepsilon_{33}\frac{p_{3\alpha\beta}}{\widetilde{\varepsilon}_{33}}\frac{p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}\partial_{\gamma\delta}(\zeta_3)\partial_{\alpha\beta}\eta_3 \\
&+ \int_\omega \varepsilon\mathbf{A}''\cdot\widetilde{\mathbf{A}} - \int_\omega p_{i\alpha\beta}e_{\alpha\beta}(\zeta_1',\zeta_2')\widetilde{\mathbf{A}}_i + \frac{1}{\mu_1}\int_\omega(\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1)(\partial_1\widetilde{\mathbf{A}}_2 - \partial_2\widetilde{\mathbf{A}}_1) \\
&+ \varkappa\int_\omega \partial_\alpha\mathbf{A}_j\partial_\alpha\widetilde{\mathbf{A}}_j + \int_\omega \alpha\mathbf{A}'\cdot\widetilde{\mathbf{A}} + \int_\omega(\vartheta\mathbf{A}'' + \gamma\mathbf{A}')\cdot\widetilde{\mathbf{A}} = 0,
\end{aligned} \tag{98}$$

with the initial conditions

$$\begin{cases} \zeta(0) = \zeta^0, \quad \zeta'(0) = \zeta^1, \\ \mathbf{A}(0) = \mathbf{A}^0, \quad \mathbf{A}'(0) = \mathbf{A}^1. \end{cases}$$

Proof. Let $\mathbf{v} \in \mathbb{V}_{KL}, \boldsymbol{\phi} \in \Psi_l$ and $\mathbf{A} \in \mathbb{A}_l$. Multiply Equation (67) by $\varphi(t) \in \mathscr{D}([0,T[)$ and integrate by parts with respect to time t. Evaluate the limit when h tends to 0 and use the weak convergences in Lemmas 7 and 8, we deduce

$$\begin{aligned}
&\int_\Omega \rho\mathbf{v}_3''\widetilde{\mathbf{v}}_3 + \int_\Omega (c\kappa(\mathbf{v}))_{\alpha\beta}\cdot\kappa_{\alpha\beta}(\widetilde{\mathbf{v}}) + \int_\Omega p_{i\alpha\beta}\kappa_{\alpha\beta}(\widetilde{\mathbf{v}})\mathbf{A}_i' + \int_\Omega p_{3\alpha\beta}\kappa_{\alpha\beta}(\widetilde{\mathbf{v}})\partial_3\phi \\
&- \int_\Omega p_{3\alpha\beta}\kappa_{\alpha\beta}(\mathbf{v})\partial_3\widetilde{\phi} + \int_\Omega \varepsilon_{33}\partial_3\widetilde{\phi}\cdot\partial_3\phi + \int_\Omega \varepsilon\mathbf{A}''\cdot\widetilde{\mathbf{A}} - \int_\Omega p_{i\alpha\beta}\kappa_{\alpha\beta}(\mathbf{v}')\widetilde{\mathbf{A}}_i \\
&+ \frac{1}{\mu_1}\int_\Omega(\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1)(\partial_1\widetilde{\mathbf{A}}_2 - \partial_2\widetilde{\mathbf{A}}_1) + \int_{\Gamma^+}(\xi\mathbf{v}_3'' + \beta\mathbf{v}_3')\widetilde{\mathbf{v}}_3 \\
&+ \varkappa\int_\Omega \partial_\alpha\mathbf{A}_j\partial_\alpha\widetilde{\mathbf{A}}_j + \int_\Omega \alpha\mathbf{A}'\cdot\widetilde{\mathbf{A}} + \int_{\Gamma^-}(\vartheta\mathbf{A}'' + \gamma\mathbf{A}')\cdot\widetilde{\mathbf{A}} = 0,
\end{aligned} \tag{99}$$

with the initial conditions

$$\begin{cases} \mathbf{v}_3(0) = \mathbf{v}_3^0, & \mathbf{v}_3'(0) = \mathbf{v}_3^1, \\ \mathbf{A}(0) = \mathbf{A}^0, & \mathbf{A}'(0) = \mathbf{A}^1. \end{cases}$$

For $\mathbf{v} = (\zeta_1 - x_3\partial_1\zeta_3, \zeta_2 - x_3\partial_2\zeta_3, \zeta_3)$, $\widetilde{\mathbf{v}} = (\eta_1 - x_3\partial_1\eta_3, \eta_2 - x_3\partial_2\eta_3, \eta_3) \in \mathbb{V}_{KL}$, expression of ϕ in Equations (90) and (91), for $\mathbf{A}, \widetilde{\mathbf{A}}$ in $\mathbb{A}_l(\Omega)$, then we get Equation (98).

Regarding the uniqueness of the solution, since we are dealing with a linear system, assume that the initial conditions are zero, and it suffices to apply Lemma 11 proven below. ☐

Lemma 11. *The energy $\mathscr{E}(t)$ of the limit problem of Equation (98) is given by*

$$\begin{aligned} \mathscr{E}(t) = \frac{1}{2}\Big(&\int_\omega \rho\,|\zeta_3'|^2 + \int_\omega c e_{\alpha\beta}(\zeta_1,\zeta_2)\cdot e_{\alpha\beta}(\zeta_1,\zeta_2) + \frac{1}{3}\int_\omega \varepsilon_{33}\left|\frac{p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}\partial_{\gamma\delta}(\zeta_3)\right|^2 + \int_\omega \varepsilon\mathbf{A}'\cdot\mathbf{A}' \\ &+ \frac{1}{\mu_1}\int_\omega |\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1|^2 + \int_\omega \xi|\zeta_3'|^2 + \vartheta\int_\omega |\mathbf{A}'|^2 + \varkappa\int_\omega |(\partial_1\mathbf{A}_j, \partial_2\mathbf{A}_j, \mathscr{B}_3^j)|^2\Big), \end{aligned} \tag{100}$$

and its derivative by

$$\mathscr{E}'(t) = -\beta\int_\omega |\zeta_3'|^2 - \alpha\int_\omega |\mathbf{A}'|^2 - \gamma\int_\omega |\mathbf{A}'|^2. \tag{101}$$

Proof. Putting $(\eta, \widetilde{\mathbf{A}}, \widetilde{\phi}) = (\zeta', \mathbf{A}', \phi')$ in Equation (98), yields

$$\begin{aligned} &\int_\omega \rho\zeta_3''\zeta_3' + \int_\omega e_{\alpha\beta}(\zeta_1,\zeta_2)\cdot e_{\alpha\beta}(\zeta_1',\zeta_2') + \int_\omega p_{i\alpha\beta}e_{\alpha\beta}(\zeta_1',\zeta_2')\mathbf{A}_i' + \frac{1}{3}\int_\omega p_{3\alpha\beta}\frac{p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}\partial_{\gamma\delta}(\zeta_3)\partial_{\alpha\beta}\zeta_3' \\ &+ \int_\omega (\xi\zeta_3'' + \beta\zeta_3')\zeta_3' - \frac{1}{3}\int_\omega p_{3\alpha\beta}\frac{p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}\partial_{\alpha\beta}(\zeta_3)\partial_{\gamma\delta}\zeta_3' + \frac{1}{3}\int_\omega \varepsilon_{33}\frac{p_{3\alpha\beta}}{\widetilde{\varepsilon}_{33}}\frac{p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}\partial_{\gamma\delta}(\zeta_3)\partial_{\alpha\beta}\zeta_3' \\ &+ \int_\omega \varepsilon\mathbf{A}''\cdot\mathbf{A}' - \int_\omega p_{i\alpha\beta}e_{\alpha\beta}(\zeta_1',\zeta_2')\mathbf{A}_i' + \frac{1}{\mu_1}\int_\omega (\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1)(\partial_1\mathbf{A}_2' - \partial_2\mathbf{A}_1') + \varkappa\int_\omega \partial_\alpha\mathbf{A}_j\partial_\alpha\mathbf{A}_j' \\ &+ \int_\omega \alpha\mathbf{A}'\cdot\mathbf{A}' + \int_\omega (\vartheta\mathbf{A}'' + \gamma\mathbf{A}')\cdot\mathbf{A}' = 0. \end{aligned}$$

This gives,

$$\begin{aligned} &\int_\omega \rho\zeta_3''\zeta_3' + \int_\omega c e_{\alpha\beta}(\zeta_1,\zeta_2)\cdot e_{\alpha\beta}(\zeta_1',\zeta_2') + \int_\omega (\xi\zeta_3'' + \beta\zeta_3')\zeta_3' + \frac{1}{3}\int_\omega \varepsilon_{33}\frac{p_{3\alpha\beta}}{\widetilde{\varepsilon}_{33}}\frac{p_{3\gamma\delta}}{\widetilde{\varepsilon}_{33}}\partial_{\gamma\delta}(\zeta_3)\partial_{\alpha\beta}\zeta_3' \\ &+ \int_\omega \varepsilon\mathbf{A}''\cdot\mathbf{A}' + \frac{1}{\mu_1}\int_\omega (\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1)(\partial_1\mathbf{A}_2' - \partial_2\mathbf{A}_1') + \varkappa\int_\omega \partial_\alpha\mathbf{A}_j\partial_\alpha\mathbf{A}_j' \\ &+ \int_\omega \alpha\mathbf{A}'\cdot\mathbf{A}' + \int_\omega (\vartheta\mathbf{A}'' + \gamma\mathbf{A}')\cdot\mathbf{A}' = 0. \end{aligned}$$

Integration by parts concludes the proof. ☐

4.4. The limit boundary value problem

Theorem 4. *The set of limit partial differential equations consists of*

1. *a bending equation for the vertical displacement ζ_3: for an arbitrary η_3, let*

$$m_{\alpha\beta}(\eta_3) = \frac{4\mu}{3}\left(\partial_{\alpha\beta}\eta_3 + \frac{\lambda}{\lambda+2\mu}\delta_{\alpha\beta}\Delta\eta_3\right) + \frac{2}{3}d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}\eta_3. \tag{102}$$

Then ζ_3 *solves*

$$\begin{cases} 2\rho\zeta_3'' + \partial_{\alpha\beta} m_{\alpha\beta}(\zeta_3) = -(\xi\zeta_3'' + \beta\zeta_3') \quad \text{in } \omega \times (0,T), & (103)\\ \zeta_3 = \partial_{\mathbf{n}}\zeta_3 = 0 \quad \text{on } \partial\omega, & (104)\\ \zeta_3(0) = \zeta_3^0, \quad \zeta_3'(0) = \zeta_3^1; & (105)\end{cases}$$

2. *a coupled system in the horizontal displacement components* ζ_1, ζ_2 *and the three components of the potential* $\mathbf{A}$*. For arbitrary* η_1, η_2*, let*

$$\widetilde{n}_{\alpha\beta}(\eta_1,\eta_2) = 4\mu\left(e_{\alpha\beta}(\eta_1,\eta_2) + \frac{\lambda}{\lambda+2\mu}\delta_{\alpha\beta}e_{\rho\rho}(\eta_1,\eta_2)\right). \tag{106}$$

Then $\zeta_1, \zeta_2, \mathbf{A}_1, \mathbf{A}_2$ *and* $\mathbf{A}_3$ *solve*

$$\begin{cases} \partial_\beta \widetilde{n}_{\alpha\beta}(\zeta_1,\zeta_2) + 2p_{i\alpha\beta}\partial_\beta \mathbf{A}_i' = 0, \ \alpha = 1,2 & \text{in } \omega \times (0,T), \quad (107)\\ (\zeta_1,\zeta_2) = 0 & \text{on } \partial\omega \times (0,T), \quad (108)\\ \zeta_\alpha(0) = \zeta_\alpha^0, \quad \zeta_\alpha'(0) = \zeta_\alpha^1 & \text{in } \omega, \quad (109)\end{cases}$$

$$\begin{cases} (\widetilde{\varepsilon}\mathbf{A}'')_1 - p_{1\alpha\beta}e_{\alpha\beta}(\zeta_1',\zeta_2') + \dfrac{1}{\mu_1}\partial_2(\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1) & \\ \quad = \varkappa\partial_{\alpha\alpha}\mathbf{A}_1 - \alpha\mathbf{A}_1' - \vartheta\mathbf{A}_1'' - \gamma\mathbf{A}_1' & \text{in } \omega \times (0,T), \quad (110)\\ (\widetilde{\varepsilon}\mathbf{A}'')_2 - p_{2\alpha\beta}e_{\alpha\beta}(\zeta_1',\zeta_2') - \dfrac{1}{\mu_1}\partial_1(\partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1) & \\ \quad = \varkappa\partial_{\alpha\alpha}\mathbf{A}_2 - \alpha\mathbf{A}_2' - \vartheta\mathbf{A}_2'' - \gamma\mathbf{A}_2' & \text{in } \omega \times (0,T), \quad (111)\\ (\widetilde{\varepsilon}\mathbf{A}'')_3 - p_{3\alpha\beta}e_{\alpha\beta}(\zeta_1',\zeta_2') & \\ \quad = \varkappa\partial_{\alpha\alpha}\mathbf{A}_3 - \alpha\mathbf{A}_3' - \vartheta\mathbf{A}_3'' - \gamma\mathbf{A}_3' & \text{in } \omega \times (0,T), \quad (112)\end{cases}$$

$$\begin{cases} \partial_1\mathbf{A}_2 - \partial_2\mathbf{A}_1 = 0 & \text{on } \partial\omega \times (0,T), \quad (113)\\ \mathbf{A} = 0 & \text{on } \partial\omega \times I, \quad (114)\\ \mathbf{A}(0) = \mathbf{A}^0, \quad \mathbf{A}'(0) = \mathbf{A}^1 & \text{in } \omega; \quad (115)\end{cases}$$

3. *its energy decays.*

Proof. In Equation (98), take $\mathscr{W} = (\widetilde{\mathbf{v}}, 0)$ with $\widetilde{\mathbf{v}} = (-x_3\partial_1\eta_3, -x_3\partial_2\eta_3, \eta_3)$, $\eta_3 \in H_0^2(\omega)$ and $\widetilde{\mathbf{A}} = 0$ to get

$$\begin{aligned} &\int_\omega 2\rho\zeta_3''\eta_3\,d\omega + \frac{2}{3}\int_\omega c_{\gamma\iota\alpha\beta}\partial_{\gamma\iota}\zeta_3\partial_{\alpha\beta}\eta_3\,d\omega + \frac{2}{3}\int_\omega d_{\alpha\beta\gamma\iota}\partial_{\gamma\iota}\zeta_3\partial_{\alpha\beta}\eta_3\,d\omega \\ &= -\int_\omega (\xi\zeta_3'' - \beta\zeta_3')\eta_3. \end{aligned} \tag{116}$$

Let us take $\eta_3 \in \mathscr{D}(\omega)$ and multiply the Equation (116) by $\varphi(t) \in \mathscr{D}(]0,T[)$, then integrate with respect to t and deduce Equation (103). Apply Green's formula to Equation (103) and take into account the singularities to get Equation (104) and we have the system in ζ_3. Let $\widetilde{\mathbf{v}} = 0$, $\widetilde{\mathbf{A}} = (0, 0, \widetilde{\mathbf{A}}_3)$ in Equation (98), then

$$\int_\omega (\widetilde{\varepsilon}\mathbf{A}'')_3 \widetilde{\mathbf{A}}_3 - \int_\omega p_{3\alpha\beta} e_{\alpha\beta}(\zeta_1', \zeta_2') \widetilde{\mathbf{A}}_3 - \varkappa \int_\omega \partial_{\alpha\alpha} \mathbf{A}_3 \widetilde{\mathbf{A}}_3 + \alpha \int_\omega \mathbf{A}_3' \widetilde{\mathbf{A}}_3 \quad r$$
$$+ \int_\omega (\vartheta \mathbf{A}_3'' + \gamma \mathbf{A}_3') \widetilde{\mathbf{A}}_3 = 0,$$

and we deduce Equation (112). Let $\widetilde{\mathbf{v}} = (\eta_1, 0, 0), \widetilde{\mathbf{A}} = 0$ in Equation (98) to get

$$-\partial_\beta \widetilde{n}_{1\beta}(\zeta_1, \zeta_2) - 2p_{i1\beta} \partial_\beta \mathbf{A}_i' = 0, \tag{117}$$

which proves Equation (107) for $\alpha = 1$. Let $\widetilde{\mathbf{v}} = (0, \eta_2, 0), \widetilde{\mathbf{A}} = 0$ in Equation (98) to get

$$-\partial_\beta \widetilde{n}_{2\beta}(\zeta_1, \zeta_2) - 2p_{i2\beta} \partial_\beta \mathbf{A}_i' = 0, \tag{118}$$

which proves Equation (107) for $\alpha = 2$. Let $\widetilde{\mathbf{v}} = 0$, $\widetilde{\mathbf{A}} = (\widetilde{\mathbf{A}}_1, 0, 0)$ in Equation (98) to get

$$\int_\omega (\widetilde{\varepsilon}\mathbf{A}'')_1 \widetilde{\mathbf{A}}_1 - \int_\omega p_{1\alpha\beta} e_{\alpha\beta}(\zeta_1', \zeta_2') \widetilde{\mathbf{A}}_1 - \frac{1}{\mu_1} \int_\omega (\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1) \partial_2 \widetilde{\mathbf{A}}_1 + \int_\omega \varkappa \partial_\alpha \mathbf{A}_1 \partial_\alpha \widetilde{\mathbf{A}}_1$$
$$+ \int_\omega \alpha \mathbf{A}_1' \widetilde{\mathbf{A}}_1 + \int_\omega \vartheta \mathbf{A}_1'' \widetilde{\mathbf{A}}_1 + \int_\omega \gamma \mathbf{A}_1' \widetilde{\mathbf{A}}_1 = 0.$$

This proves Equation (110). Let $\widetilde{\mathbf{v}} = 0$, $\widetilde{\mathbf{A}} = (0, \widetilde{\mathbf{A}}_2, 0)$ in Equation (98) to get

$$\int_\omega (\widetilde{\varepsilon}\mathbf{A}'')_2 \widetilde{\mathbf{A}}_2 - \int_\omega p_{2\alpha\beta} e_{\alpha\beta}(\zeta_1', \zeta_2') \widetilde{\mathbf{A}}_2 + \frac{1}{\mu_1} \int_\omega (\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1) \partial_2 \widetilde{\mathbf{A}}_2 + \int_\omega \varkappa \partial_\alpha \mathbf{A}_2 \partial_\alpha \widetilde{\mathbf{A}}_2$$
$$+ \int_\omega \alpha \mathbf{A}_2' \widetilde{\mathbf{A}}_2 + \int_\omega \vartheta \mathbf{A}_2'' \widetilde{\mathbf{A}}_2 + \int_\omega \gamma \mathbf{A}_2' \widetilde{\mathbf{A}}_2 = 0.$$

This proves Equation (111).

Now we show that $\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = 0$ on $\partial\omega$. To do this, it suffices to apply the procedure used in Theorem 4.3.2 of the study by Sène [16], where two identities are established: $\forall\, \varphi \in \mathscr{D}(]0, T[)$, $(\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2) \in H^1(\omega)^2$, we have:

$$\int_{\partial\omega} \left(\mathbf{n}_2 \int_0^T (\mathrm{curl}\, \mathbf{A})_3 \varphi \, dt \right) \widetilde{\mathbf{A}}_1 \, d\partial\omega = \int_{\partial\omega} \left(\mathbf{n}_1 \int_0^T (\mathrm{curl}\, \mathbf{A})_3 \varphi \, dt \right) \widetilde{\mathbf{A}}_2 \, d\partial\omega = 0. \tag{119}$$

Thus $\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = 0$ on $\partial\omega$.

Finally, the decay of the limit energy defined in Equation (100) is obtained thanks to Equation (101). □

5. Concluding remarks

The main contribution of this paper is the reduction of the 3D piezoelectric model to an effective 2D model. This 2D model is rigorously justified by asymptotic analysis when the thickness parameter $h \to 0$ which retains the correct limit behavior of the original system. Importantly, the analysis shows that the linear control laws scale with the thickness and they enter the limiting equations, thereby preserving well-posedness and energy dissipation of the reduced model. From a modeling and computational perspective, the resulting 2D model significantly reduces the dimensional complexity.

This enables efficient simulation, analysis, and implementation of linear feedback controllers that would be computationally prohibitive in a full 3D setting. Moreover, it explicitly exerts the influence of control parameters on bending and plane responses, providing a clearer insight into stability and performance properties. Here are a few remarks that describe the change in behavior of a stabilized piezoelectric plate and take advantage of its small thickness $2h$.

1. The limit equations reveal that magnetic field-based control affects the average horizontal displacement more than the vertical displacement, whereas the electric potential is closely related to the latter.
2. Observing the rate of decrease in the limit energy given in Equation (101), we note that in the case of thin plates, the internal magnetic control and the boundary magnetic control related to the variation of the magnetic field with respect to time play the same role. Therefore, when the thickness $2h$ is very small, in Equation (61), the internal control given by the variation of the magnetic field with respect to time $\alpha\mathbf{A}^{h'}$ can be relaxed. Moreover, mechanical boundary controls related to horizontal displacements can be relaxed. Then, since the magnitude of the horizontal displacement velocity is h times that of the vertical displacement velocity, it suffices to control with the latter.
3. One of the most important observations is that the rate of energy decay is composed of terms given by the derivatives of the displacement vector and the magnetic potential with respect to time, it is obvious that if these functions are constant over a time interval, there is no longer any stabilization because the energy is constant in that interval. This issue was addressed by Bidouan et al. [22], where the authors prove that for a set of 11 types of piezoelectric materials, in such a case, the energy $\mathscr{E}$ is not only constant but equal to zero, which means that strong stabilization is achieved. Apart from these 11 types, the problem remains open. The following comments reveal that such a problem vanishes as the thickness of the plate tends to zero.
4. As for horizontal displacements, let us consider Equations (107)–(109) in the time interval I where the time derivative of the magnetic potential is zero:

$$\begin{cases} \partial_\beta \widetilde{n}_{\alpha\beta}(\zeta_1,\zeta_2) = 0, \ \alpha = 1,2, & \text{in } \omega \times I, \quad (120) \\ (\zeta_1,\zeta_2) = 0 & \text{on } \partial\omega \times I, \quad (121) \\ -p_{1\alpha\beta} e_{\alpha\beta}(\zeta_1',\zeta_2') + \dfrac{1}{\mu_1}\partial_2(\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1) = \varkappa \partial_{\alpha\alpha}\mathbf{A}_1 & \text{in } \omega \times I, \quad (122) \\ -p_{2\alpha\beta} e_{\alpha\beta}(\zeta_1',\zeta_2') - \dfrac{1}{\mu_1}\partial_1(\partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1) = \varkappa \partial_{\alpha\alpha}\mathbf{A}_2 & \text{in } \omega \times I, \quad (123) \\ -p_{3\alpha\beta} e_{\alpha\beta}(\zeta_1',\zeta_2') = \varkappa \partial_{\alpha\alpha}\mathbf{A}_3 & \text{in } \omega \times I, \quad (124) \\ \partial_1 \mathbf{A}_2 - \partial_2 \mathbf{A}_1 = 0 & \text{on } \partial\omega \times I, \quad (125) \\ \mathbf{A} = 0 & \text{on } \partial\omega \times (0,T), \quad (126) \\ \mathbf{A}(0) = \mathbf{A}^0, \quad \mathbf{A}'(0) = 0 & \text{in } \omega, \quad (127) \\ \zeta_\alpha(0) = \zeta_\alpha^0, \quad \zeta_\alpha'(0) = \zeta_\alpha^1 & \text{in } \omega. \quad (128) \end{cases}$$

Thus, Equations (120) and (121) implies that $(\zeta_1, \zeta_2) = 0$ in $\omega \times I$, then we deduce from Equations (122)–(125) that $\mathbf{A} = 0$. This can be proven by using the Lax-Milgram Theorem.

5. In the present work, if we consider the issue of the energy decay rate being zero, the most significant finding of this work is that, in that case, the energy is approximately zero for any type of piezoelectric material, when the thickness of the plate is sufficiently small. To argue this conclusion, let us consider the limit system of Equations (103) and (104) when the first term of the energy decay rate ζ_3' is zero in a time interval I:

$$\begin{cases} \partial_{\alpha\beta} m_{\alpha\beta}(\zeta_3) = 0 \quad \text{in } \omega \times I, & (129) \\ \zeta_3 = \partial_{\mathbf{n}} \zeta_3 = 0 \quad \text{on } \partial\omega \times I. & (130) \end{cases}$$

Therefore, the vertical component of the displacement vector is zero, because the Equations (129) and (130) have a unique solution which is zero.
Multiply Equation (129) by ζ_3 and integrate over ω:

$$\int_\omega \partial_{\alpha\beta} m_{\alpha\beta}(\zeta_3) \zeta_3 \, d\omega = 0.$$

Integrate by parts twice and use the boundary Equation (130) to get,

$$\int_\omega m_{\alpha\beta}(\zeta_3) \partial_{\alpha\beta} \zeta_3 \, d\omega = 0.$$

This integral vanishes only if $\partial_{\alpha\beta}\zeta_3 = 0$ in ω. This implies that ζ_3 is at most an affine function. Together with the boundary condition $\zeta_3 = 0$ and $\partial_{\mathbf{n}}\zeta_3 = 0$, we deduce $\zeta_3 = 0$.

In conclusion, if the limit energy of the system is constant in a time interval I, then it is necessarily zero. Hence, we achieve strong stabilization. And what the current asymptotic analysis reveals is that when the thickness of a piezoelectric plate is very small, we no longer need the restrictions imposed by Lemmas 2–5 on the characteristics of the piezoelectric materials to obtain stabilization or strong stabilization of its perturbations.

Looking ahead, we intend to apply the same procedure to non-linear deformations, as our study is currently only applicable to small deformations. Another interesting avenue is to extend this work to mechanically anisotropic materials.

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