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Article

Bipolar fuzzy dominance rough WASPAS approach for AI-based radar evaluation

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Abstract: The selection of an AI-based radar system to detect drones is a multi-criteria decision-making (MCDM) problem with many conflicting criteria. Exchanges between positive and negative aspects of each radar system in uncertain and incomplete information should be assessed by decision-makers. The traditional MCDM models are usually not effective in dealing with such complexities, especially when both positive and negative aspects are involved, and comparative reasoning is needed (dominance). In order to address these shortcomings, this article suggests an advanced model using the bipolar fuzzy dominance rough set (BFDRS) approach. The suggested method combines fuzzy logic to deal with uncertainty, dominance-based rough sets to model preferences, and bipolar fuzzy sets to manage dual-natured assessments. In order to operationalize the framework, we propose two new aggregation operators, namely bipolar fuzzy dominance rough dombi averaging (BFDRDA) and bipolar fuzzy dominance rough dombi geometric (BFDRDG), to combine expert opinions in the context of multiple criteria successfully. After that, we develop an MCDM methodology, which is the WASPAS method, within the framework of BFDRS, to prioritize AI radar alternatives in the presence of uncertainty. An extensive case study proves the relevance of the suggested model, and a comparative analysis with the currently existing ones proves its strength and higher decision-support abilities in complex and contradictory environments.

Keywords: artificial intelligence; radar system; multi-criteria decision making (MCDM); bipolar fuzzy dominance rough set approach

1. Introduction

AI-assisted radar technologies have become a standard component of modern investigation and defense, with the ability to process vast amounts of data in real time. The technologies make use of machine learning algorithms to enhance the detection, classification, and tracking of targets, especially in high-clutter or low signal-to-noise environments. Modern airspace security faces challenges. While drones offer advantages to commercial, industrial, agricultural, and defense sectors, they have also presented severe security threats. Drones may be used to conduct unlawful surveillance and disrupt essential infrastructure, and can even be deployed as weapons in terrorism or military operations. Radar technologies have developed in response to these new challenges. Traditional radar systems are resistant to typical aerial threats but fail in general to detect, track, and identify drones due to their size, their ability to fly at low altitude, stealth mode, and their erratic trajectory. Such vulnerability has offered a chance to design and implement AI-driven radar systems that bring the aspects of intelligence, responsiveness, and efficiency into modern air defense and surveillance systems. AI-radar systems affect state-of-the-art machine learning and deep learning algorithms to process large amounts of radar data in real time. In contrast to conventional radar, which is primarily based on fixed rules, AI-based systems have the capability to learn and adapt through past detections. This learning capability enables such systems to enhance recognition, minimize false alarms, and even identify even the most undetectable airborne threats in varying environmental settings. Through pattern, anomaly, and trajectory analysis, AI may enable the separation of drones from other radar cross-section objects, like birds, which is normally hard to do through conventional systems. Dynamic thresholding and adaptive filters are possible with AI. providing the best performance in even the highest-clutter or lowest-signal-to-noise-ratio cases, like in a battlefield or the city. Multi-sensor data fusion techniques are aided by AI to show rich situational awareness images. The radar systems have an ability to learn from previous detections to enhance prediction, tracking, and allocation of resources. SRC Silent Archer is one of the cognitive radar systems using AI to prioritize and make decisions about threats. AI real-time processing will decrease the operator workload and response time to a complex situation. The history of detections can be learned by these radar systems in order to improve predictive tracking and resource utilization. Thales SMART-L and SRC Silent Archer are cognitive radar systems using AI to provide threat assessment and real-time decision support. Quick AI processing can alleviate the workload on the operator and respond faster in dynamic environments. AI puts the detection of anomalies at a high level and opens the prospects of self-diagnostic abilities, which increases the reliability and lightness of maintenance of the system. As unmanned aerial threats continue to pose a persistent threat, AI-driven radar systems will continue to play a crucial role in providing intelligent and resilient airspace security. Also, AI can be used in conjunction with radar technology to introduce autonomous surveillance features, where machines are able to work independently with little or no human intervention. The AI procedures continuously readjust the levels of detection based on environmental knowledge and threat behavior, and limit threats of false positives and false negatives. In military applications, artificial intelligence radar can be used to conduct electronic warfare in the name of monitoring enemy signals and countermeasures through dynamic evolution. Civilian applications of artificial intelligence radar technology also exist, especially in the scenario of airport security. where AI and radar work together to avoid runways and ensure the detection of unauthorized drones. Radars that are powered by AI are increasingly being applied in maritime surveillance, border control, and disaster tracking due to their ability to be more versatile. The transition to edge AI allows the radar systems to work on the ground, improving the speed and safety of the information. The latest technologies that are trending include the combination of AI radar and 5G communication infrastructure to facilitate smart and networked defensive systems. Since the threats are ever-changing, the technology of space warfare surveillance must also be made nimble, intelligent, and highly responsive at all times. Further advancement and implementation of AI-powered radar systems will create a drastic change in the perception, interpretation, and counteraction of threats.

1.1. Research problem and motivation

Dominance-based Rough Set (DRS) theory is an extension of classical Rough Set (RS) theory, which has relied on equivalence relations to classify and analyze crisp data within information systems. Although RS is good at separating well-defined subsets from data, it cannot handle order, preference, or ranking factors that are quite essential in most real-world applications. To overcome this deficiency, DRS utilizes dominance relations to facilitate comparison among the objects in terms of their relative performance. This extension is useful for more organized analysis in determining preferred choices through dominance-based classification. However, DRS fails to deal with fuzzy or imprecise information, which often occurs under complex decision-making scenarios. To avoid this, the Fuzzy Dominance-based Rough Set Approach (FDRSA) was presented by combining the Fuzzy Set (FS) theory with the DRS. This combined model provides an adaptable and robust means of handling uncertainty, incompleteness, and preference-based data simultaneously. FDRSA is particularly useful in situations where there are fuzzy edges and rough approximations combined with dominance relations within datasets. However, it has one strong limitation: it is not able to properly show bipolar situations when data involves both positive and negative elements. Bipolarity is found in actual life, like in health care decision-making, where one intervention can yield high effectiveness in the control of disease (favorable attribute) but at the expense of serious side effects or cost (unfavorable attributes). This illustrates a double-natured quality of both merits and demerits. Although Bipolar Fuzzy Rough Sets (BFRS) can represent such bipolar information with dual sides inside a rough set paradigm, they do not have the mechanism of including dominance relations. Accordingly, neither FDRSA nor BFRS itself can yet tackle problems of the combination of uncertainty, vagueness, dominance, and bipolarity. This deficiency underscores the need for an integrated, holistic model of decision-making that can work on all of these at the same time, an urgent one in intricate data worlds like those existing in computer science and related fields.

For the selection of an AI-based radar system, a MCDM process is used based on several criteria in uncertain and dynamic situations. The BFDRS model happens to be best suited for the task owing to its capacity for processing simultaneously positive and negative ratings at the same time, known as bipolarity. This implies it can capture the interest of characteristics (e.g., more detection distance) and the undesirability of others (e.g., high cost to maintain). Also, the fuzzy aspect enables the model to handle imprecise or fuzzy expert opinions, typical of military and defense-related evaluations. The dominance-based framework enables the comparison of radar options on a relative basis across all characteristics instead of individual rankings. Lastly, rough set theory provides a means to handle imprecise classification limits, supporting more flexible and realistic decision-making with incomplete information. Combining these strengths, BFDRS gives a strong, intelligent tool to aid strategic decisions in AI radar system deployment for detecting drones. **Figure 1** shows how the bipolar fuzzy dominance rough set approach is used in the selection of an AI-based radar system.

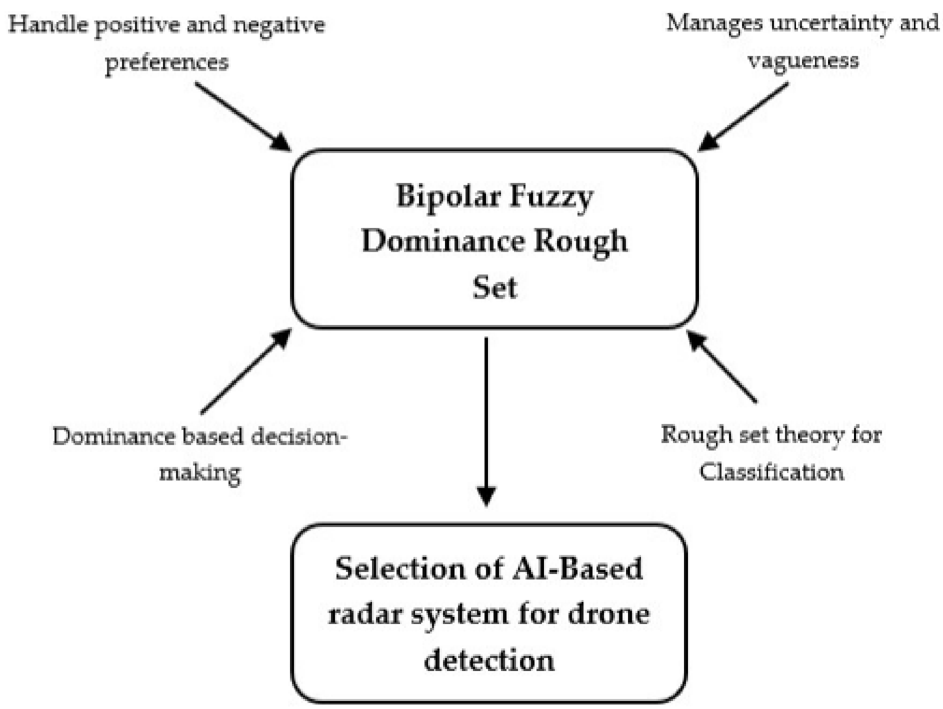

Figure 1. BFDRS is used in the selection of an AI-based radar system.

1.2. Contribution

The methodology of the current study develops the Dombi AOs via bipolar fuzzy dominance rough techniques. This study presents two fundamental operators: the bipolar fuzzy dominance rough Dombi averaging (BFDRDA) and the bipolar fuzzy dominance rough Dombi geometric (BFDRDG) operators. The unified framework of the advanced operators allows them to handle fuzzy measures of criteria and their positive and negative sides, dominance, and lower and upper approximations for better evaluation abilities. Our approach combines the developed operators to form a new MCDM framework that copes with decision issues involving bipolarity, dominance, and roughness in addition to uncertain information structures. An in-depth case study illustrates the real-world application of our suggested framework through the analysis and shortlisting of AI-based radar systems. We prove the efficiency and usefulness of our method by considering a comparative analysis with other available theoretical frameworks. Our suggested work showcases its worthiness with the comparative assessment that establishes its capability of dealing with assessment decision-making scenarios. The contribution of this article is shown in **Figure 2**.

1.3. Novelty of the research

This research is unique in the sense that a complete bipolar fuzzy dominance rough WASPAS (BFDR-WASPAS) decision-making model for the evaluation of AI radar systems has never been conducted previously, in which no prior methodology has incorporated bipolar fuzzy modelling, dominance rough set reasoning, and WASPAS aggregation. The literature that remains usually views bipolar uncertainty, dominance relations, and weighted aggregation as a unique perspective of analysis. Instead, we take these elements and interrelate them to form a consistent approach, which in turn explains positive and negative beliefs, dominance-inspired approximations, and additive multiplicative scoring. This hybrid gives much more realistic models of dual-attitude uncertainty, retains coarse boundary information, and enhances the stability of rankings. In such a way, the proposed BFDRWASPAS model is a mathematically viable and practically powerful instrument that significantly extends the current MCDM techniques employed to conduct intelligent radar evaluation.

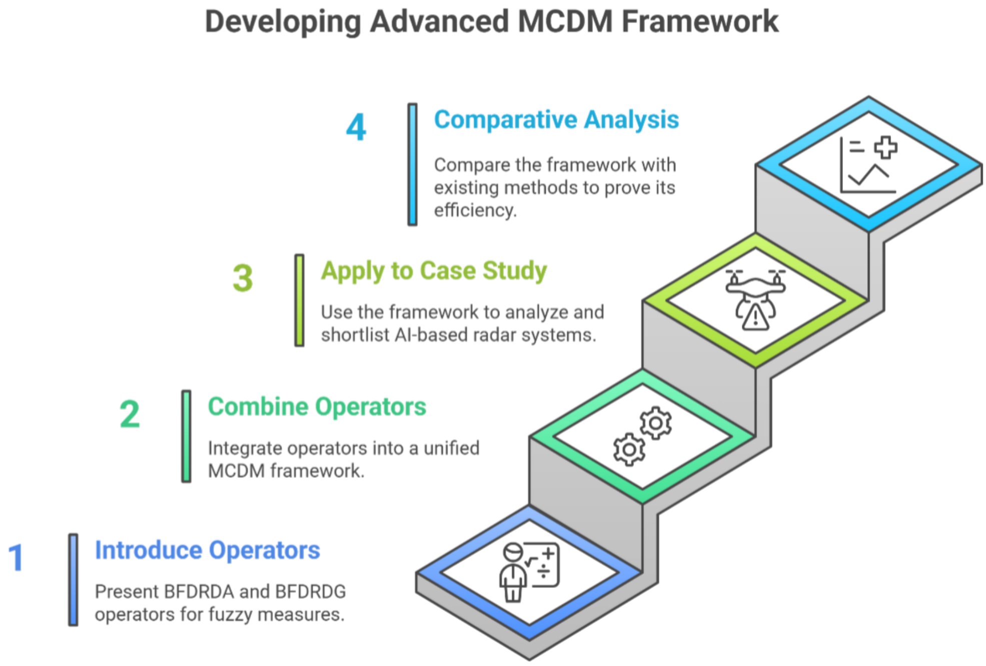

Figure 2. The flowchart of the contribution.

- Preliminary integration of the BFDR figures with the WASPAS strategy, which will create a novel decision-making framework not available in the past literature.
- The introduction of bipolar dominance provides rough approximations to depict both positive and negative uncertainty simultaneously, which makes radar evaluation easier to read.
- Most fuzzy and bipolar MCDM models ignore a mathematically structured model that preserves the boundary-region information of rough sets of dominance data.
- Expansion to real AI radar systems, in which the model can be demonstrated capable of dealing with complex, competing, and inaccurate assessment criteria.

1.4. Layout of the paper

Section 2 consists of a review of the relevant literature. Section 3, together with essential preliminaries, includes definitions of RS FS, DRS, FDRS, and Bipolar Fuzzy Rough Sets (BFRS). In Section 4, the Bipolar Fuzzy Dominance-based Rough Set (BFDRS) framework is introduced, where the bipolar fuzzy dominance relation is formally defined and demonstrated. Subsection 4.1 outlines the operational laws—union, intersection, and complement for Bipolar Fuzzy Dominance-based Rough Set Approximation (BFDRSA). Section 5 develops Dombi weighted arithmetic and Dombi weighted geometric aggregation operators within the Bipolar Fuzzy Dominance-based Rough Approximation (BFDRA) framework, providing formal definitions for the BFDRDWA and BFDRDWG operators. Section 6 presents the

WASPAS approach applied to BFDRNs. In Section 7, the case study is explained, including the application related to the bipolar fuzzy dominance rough approach. Section 8 offers a comparative analysis of several theoretical models, including RSs, FSs, BFRs, DRSs, Fuzzy Dominance-based Rough Sets (FDRs), Bipolar Fuzzy Rough Sets (BFRs), and Bipolar Fuzzy Dominance-based Rough Sets (BFDRs). Lastly, Section 9 concludes the paper by summarizing the key findings.

2. Literature review

Various scholars write about the use of radar systems in various areas. The survey of radar systems in medical applications was discussed by Pisa et al. [1]. Long et al. [2] give an in-depth summary of entomological radar systems, their work giving prominence to the possibility of advanced radar architectures, such as phased array and multi-frequency systems, in ecological surveillance and early pest warning. Rohling and Moller [3] presented three types of continuous radar waveforms that were used in automotive radar sensors. All the objects within the estimated range are sensed by this sensor. Orman et al. [4] explained how a complex radar system involving three tasks, search, track, and weapon guidance, can be modeled to control it. Wellig et al. [5] highlight the trend in the world to highly developed radar technologies, such as cognitive and multi-static systems, in developing effective counter solutions. Millimeter wave radar was proposed by Ślesicka et al. [6] with the assistance of an AI-based method, which assists in identifying concealed objects. More recent literature has discussed various methods of drone detection, with a special focus on radar technology and intelligent decision-making systems. Musa et al. [7] carried out an in-depth investigation of AI-powered frequency-modulated continuous wave (FMCW) radars that could be used to identify drones in the vicinity of runways on airport grounds. Their interest is in the increased use of artificial intelligence in enhancing radar signal processing to enable target identification and tracking under cluttered conditions. A previous, introductory review of radar-based detection of copter drones was published by Singh et al. [8], which states the advantages and disadvantages of the various types of radar systems to detect low-flying drones. They differentiate radar systems according to frequency bands and configuration and test their performance in various environments. Their results show that conventional radar systems will have a hard time differentiating between drones and birds or other airborne obstructions, and require algorithmic enhancements and hybrid detection techniques. Volkov et al. [9] re-oriented the focus towards the application of fuzzy mathematical algorithms in autonomous systems such as drones. Their article introduces fuzzy logic as a great enabler of dealing with uncertainty and inaccurate information in the process of making real-time decisions. Although the paper puts the primary emphasis on autonomous vehicle navigation, it offers comparative examples to UAVs, indicating that fuzzy systems may be used to improve the control of drones, the ability to detect obstacles, and the ability to identify dynamic targets. Sharjeel [10] continues the thread of fuzzy systems with their proposal of a decision support system (DSS) of air defense fire control using a fuzzy network of ranking and chasing several aerial targets. Their study shows how DSS using fuzzy logic can be implemented in the defense systems of a military to improve the response time and accuracy in the process of detecting and neutralizing threats such as enemy drones. The strategy is consistent with recent trends in network-centric warfare, where situational awareness and timely decision-making are of utmost importance. According to the literature, although radar is an embedded sensing technique, smart algorithms to handle it can enhance its efficiency significantly. Future studies should be about embedded systems, which are radar-fuzzy logic-based AI, and provide real-time and reliable drone detection in both civilian and military drone applications.

2.1. Fuzzy backgrounds

In 1982, Pawlak [11] defined a classical rough set (RS) that is comprised of the upper approximation (UA) and lower approximation (LA) of a set, with respect to an equivalence relation. RS is a functional approach to handling uncertain, vague, and imprecise information. Pawlak [12] proposed the RS theory. Qi et al. [13] generalized the rough set theory and tested the radar anti-jamming capability. Their approach manipulates large, unfinished, and duplicate data. The strategy does a good job of maximizing evaluation indices with the entropy-based attribute reduction and is tested in a real-world radar setup. Wu et al. [14] proposed an improved radial basis function (RBF) neural network to recognize a radar emitter that uses rough sets to enhance its performance. Greco et al. [15] established the principles of Dominance-Based Rough Set Theory (DRST). DRST is a generalization of the classical rough set theory needed to process preference-ordered data used in the analysis of decisions. DRST uses dominance relations between attributes and decision outcomes based on rough sets, depending on indiscernibility relations. This can be applied to multi-criteria decision making, where rankings and preferences play a central role. DRST can be used in the fields in different ways, such as risk assessment, resource allocation, and classification. The model was further developed by Błaszczyński et al. [16], who applied it in decision support systems. Azar et al. [17] addressed dominance in rough set-based classification systems that are improved. Chakhar et al. [18] suggest a method to facilitate group multicriteria classification, based on the dominance-based rough set approach.

The theory of fuzzy sets (FS) was proposed by Zadeh in 1965 [19], and it provides a mathematical framework to deal with uncertain and imprecise data. FS is the extension of crisp theory whereby each element has a membership grade; its membership degree has a range of 0 to 1 to indicate the extent to which an element is a member of a set. FS is the most central in the modeling of real-world ambiguity. FS theory has numerous applications in real life. According to Mamdani and Assilian [20], the use of FS has proliferated in control systems. Maries and Sherif [21] present the application of FS theory with special focus on fuzzy industrial controllers. Bellman and Zadeh [22] outlined fuzzy set theory decision-making. Steimann [23] provides the application of fuzzy set theory and artificial intelligence in the medical area. Solonska and Zhyrnov [24] explain how the fuzzy set theory can be used in analyzing radar data. Another key concept that goes beyond simple RS is Fuzzy RS (FRS), initially introduced by Dubois and Prade [25]. FRS offers a more adaptable way of processing ambiguous and complicated data by combining the power of FS theory with RST. Other fuzzy approximations that are also introduced with FRS include a fuzzy equivalence relation, as well as lower and upper approximations. Moreover, Radzikowska and Kerre proposed a comparison of FRS [26]. Yeung et al. [27] introduce certain upper and lower approximation operators of fuzzy sets based on arbitrary fuzzy relations and discuss their relations. Along with FRS, there is another significant development, the Bipolar Fuzzy Rough Set (BFRS), introduced by Yang et al. [28]. Unlike traditional approaches, which are founded on one domain of membership, BFRS speaks of uncertainty in bipolarity, positive and negative membership levels. BFRS provides a more detailed and expressive representation of uncertainty with the combination of roughness and the values of uncertainty that are bipolar. Han et al. [29] also extended their study on bipolar-valued rough fuzzy sets and demonstrated the use of rough sets in decision information systems. It is a two-point of view model of human reasoning that enhances the modeling of human reasoning in complex settings. Greco et al. [30] introduced the fuzzy dominance-based rough set theory, which is based on the classical rough set theory, but with the addition of the fuzzy set theory and dominance relations, and is to be applied to multi-attribute decision-making issues, where the information is imprecise, vague, or fuzzy. Within this method, dominance relations used to rank objects on various criteria are generalized using fuzzy logic, which allows the partial domination degree instead of the strict domination. The implication of this is that an object can be ranked higher than the other as a result of a combination of values of attributes, each of which is ranked to some extent of preference. The rough set aspect of the model then uses these fuzzy dominance relationships to develop lower and upper approximations of classes of decisions, which indicates the uncertainty in the classification. The technique finds application in most useful in problems of decision that have subjective judgment, ambiguity, and gradation, such as human-based evaluations, risk analysis, and preference analysis. Sang et al. [31] explain the fuzzy dominance rough feature selection of active anti-noise. Wang and Jiang [32] explore the concept of fuzzy dominance and its use in evolutionary many-objective optimization. Mahmood and Rehman [33] generalized complex fuzzy theory to the usage of positive-negative information: bipolar complex fuzzy sets, and used them to develop generalized similarity measures when uncertainty is present. Their performance in the laboratory also enhanced the power of expression of fuzzy modeling in bipolar and complex environments. Albaity et al. [34] developed a hesitant fuzzy rough MCDM model of data source optimization in data science fusion based on this. Greco et al. [35] introduced the dominance-based rough set approach to handle preference relations in decision-making, while Zhang [36] proposed bipolar fuzzy sets to concurrently model positive and negative information under uncertainty.

In the literature reviewed above, we discover that rough set and fuzzy set theories that were developed by Pawlak [11, 12] and Zadeh [19], respectively, have been extensively employed to study uncertainty modeling. Classical rough sets can be used to work with indiscernibility, but are restricted in their ability to work with graded and preference-based data. Even though Greco dominance-based rough set theory (DRST) [15] tackled the issue of processing and ranking preferences, it nonetheless relies on crisp relations [15–18]. Nevertheless, fuzzy rough sets and fuzzy set extensions facilitate greater flexibility when it comes to imprecision [25–27], but they cannot process bipolar data. Bipolar fuzzy rough sets (BFRS) [28,29] enhance expressiveness by combining positive and negative membership, but are not effective in dominance-based decision structures. Even though a part of this problem is addressed by fuzzy dominance rough set models [30], the current methods still do not fully incorporate both bipolarity and dominance relations, and this shows that there is a gap in addressing the complex problem of decision making.

2.2. WASPAS approach

The WASPAS method is a strong multi-criteria decision-making (MCDM) technique that combines the strengths of two classical methods: the Weighted Sum Model and the Weighted Product Model. It is particularly useful for solving complex DM problems involving multiple conflicting attributes. In the WASPAS approach, each alternative is evaluated by first normalizing the decision matrix and then calculating two performance scores for each option, one based on the additive utility model (WSM) and the other on the multiplicative utility model (WPM). These scores are then combined using a linear combination controlled by a coefficient (usually 0.5) to produce a final ranking value for each alternative. This hybrid scoring mechanism increases accuracy, reduces sensitivity to scale differences, and ensures greater flexibility in modeling decision-maker preferences. The WASPAS methodology is particularly useful in decision-making situations such as supplier selection, project appraisal, technology ranking, or any field where a balanced assessment of both multiplicative and collective effects of criteria is needed. Alinezhad and Khalili [37] systematically introduced the WASPAS method and highlighted its theoretical aspect and practical applicability in MADM problems. They described how the union of additive and multiplicative utility models enhances the precision and trustworthiness of decisions, particularly when faced with incongruent or ambiguous criteria. This initial work laid the foundation for future developments and applications in different fields. Radomska-Zalas [38] further applied WASPAS to a chosen technological process, demonstrating its effectiveness in industrial decision-making under real-world circumstances. Through the use of WASPAS for the selection of the best technological operations configuration, the research verified the ability of the method to process technical information and facilitate process optimization efficiently. In the context of computational intelligence, Rehman and Mahmood [39] combined WASPAS with the Binary Coded Feature-WASPAS (BCF-WASPAS) approach and used Einstein operators to improve feature selection in software defect prediction. Their research identified the power of the method in handling high-dimensional data, providing accurate attribute ranking, and assisting in more accurate predictive modeling in software development. Mahmood et al. [40] also used a sophisticated Pythagorean fuzzy rough WASPAS technique to classify business engineering problem potential solutions. Their work provided a new decision-making model able to handle imprecise and fuzzy information, proving the high flexibility and versatility of WASPAS when combined with superior fuzzy logic and rough set theories. Further development of WASPAS into the fuzzy realm was done by Tufail and Shabir [41], who proposed a WASPAS technique for measuring roughness in bipolar fuzzy sets. Their paper emphasized the significance of bipolar fuzzy covering and rough approximations to improve the evaluation of uncertainty and polarity in decision spaces.

Jaleel [42] used the WASPAS technique in agricultural robotics, leveraging bipolar complex fuzzy soft sets and Dombi aggregation operators. The paper illustrated the usability of the technique for intelligent systems and robotics, particularly in situations where uncertain and dynamic input conditions require flexible but structured decision models. Taken together, the research highlights the increasing visibility and versatility of the WASPAS approach across various domains, from manufacturing and business engineering to robotics and computational modeling. Its integration with fuzzy, rough, and complex set theories demonstrates its development toward solving real-world problems with multiple facets, making WASPAS a central technique in current MADM approaches.

3. Preliminaries

We revised certain fundamental ideas that were already established before presenting the new concepts, which include RS, FS, DRS, FDRS, and BFRS related to the proposed work.

Definition 1 ([11])**.** *Let ϑ be the fixed set and $\breve{\mathfrak{Y}}$ it is an equivalence relation. Then, the pair (ϑ, $\breve{\mathfrak{Y}}$) is an approximation space (AS). The LA and UA of $\wp \subset \vartheta$ w.r.t (ϑ, $\breve{\mathfrak{Y}}$) are indicated and described as:*

$$\underline{\breve{\mathfrak{Y}}}(\wp) = \left\{ \left(\widetilde{\mathfrak{d}} \in \vartheta : \left[\widetilde{\mathfrak{d}}\right]_{\breve{\mathfrak{Y}}} \subseteq \wp \right) \right\},$$

$$\overline{\breve{\mathfrak{Y}}}(\wp) = \left\{ \left(\widetilde{\mathfrak{d}} \in \vartheta : \left[\widetilde{\mathfrak{d}}\right]_{\breve{\mathfrak{Y}}} \cap \wp \neq \varnothing \right) \right\}.$$

Then, the pair $\left(\underline{\breve{\mathfrak{Y}}}(\wp),\ \overline{\breve{\mathfrak{Y}}}(\wp)\right)$ is called RS, with $\underline{\breve{\mathfrak{Y}}}(\wp) \neq \overline{\breve{\mathfrak{Y}}}(\wp)$.

Definition 2 ([19])**.** *A FS $\mathcal{F}$ is defined on the fix set ϑ and denoted as:*

$$\mathcal{F} = \left\{ \left(\widetilde{\mathfrak{d}}, \mathcal{Z}\left(\widetilde{\mathfrak{d}}\right)\right) \mid \widetilde{\mathfrak{d}} \in \vartheta \right\},$$

where $\mathcal{Z}\left(\widetilde{\mathfrak{d}}\right) \in [0,1]$.

Definition 3 ([35])**.** *Let, $\mathbb{D} = (\vartheta,\ \mathbb{A}\mathfrak{t} \bigcup \check{D})$ be a fuzzy decision system, $\wp \subset \mathbb{A}\mathfrak{t}$, for $\widetilde{\mathfrak{d}}$, $\widetilde{\mathfrak{q}} \in \vartheta$ the dominance relation determined by the fuzzy attribute set $\wp$.*

$$\check{D}_{\wp} = \left\{ \left(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}\right) \in \vartheta \times \vartheta \mid \mathcal{Z}\left(\widetilde{\mathfrak{d}}, \check{\mathfrak{p}}\right) \geq \mathcal{Z}\left(\widetilde{\mathfrak{q}}, \check{\mathfrak{p}}\right),\ \forall\, \check{\mathfrak{p}} \in \wp \right\},$$

which means the value of $\widetilde{\mathfrak{d}}$ on $\check{\mathfrak{p}}$ is superior to the value of $\widetilde{\mathfrak{q}}$ on $\check{\mathfrak{p}}$.

i. $\left[\widetilde{\mathfrak{d}}\right]^{\succcurlyeq}_{\check{D}_{\wp}} = \left\{ \widetilde{\mathfrak{q}} \in \vartheta \mid \left(\widetilde{\mathfrak{q}},\ \widetilde{\mathfrak{d}}\right) \in \check{D}_{\wp} \right\}$;

ii. $\left[\widetilde{\mathfrak{d}}\right]^{\preccurlyeq}_{\check{D}_{\wp}} = \left\{ \widetilde{\mathfrak{q}} \in \vartheta \mid \left(\widetilde{\mathfrak{d}}, \widetilde{\mathfrak{q}}\right) \in \check{D}_{\wp} \right\}$.

$\left[\widetilde{\mathfrak{d}}\right]^{\succcurlyeq}_{\check{D}_{\wp}}$ It denotes the set of objects that dominate $\widetilde{\mathfrak{d}}$ and $\left[\widetilde{\mathfrak{d}}\right]^{\preccurlyeq}_{\check{D}_{\wp}}$ denote the set of objects that are dominated by $\widetilde{\mathfrak{d}}$.

Definition 4 ([30])**.** *For $\mathbb{D} = (\vartheta,\ \mathbb{A}\mathfrak{t} \bigcup \check{D})$ $\wp \subset \mathbb{A}\mathfrak{t}$, for $\widetilde{\mathfrak{d}}$, $\widetilde{\mathfrak{q}} \in \vartheta$, then for all $\mathscr{o} \in N$, the LA and UA of $\mathcal{Z}^{\succcurlyeq}_{\mathscr{o}}$ with FDR are denoted by $\underline{\breve{\mathfrak{Y}}}\left(\mathcal{Z}^{\succcurlyeq}_{\mathscr{o}}\right)$ and $\overline{\breve{\mathfrak{Y}}}\left(\mathcal{Z}^{\succcurlyeq}_{\mathscr{o}}\right)$ respectively, and* *are represented and defined as:*

$$\underline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\succcurlyeq}\right)=\left\{\left(\widetilde{\mathfrak{d}}, \mathcal{Z}_{\underline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\succcurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)\right):\ \widetilde{\mathfrak{d}}\in\vartheta\right\},$$

$$\overline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\succcurlyeq}\right)=\left\{\left(\widetilde{\mathfrak{d}}, \mathcal{Z}_{\overline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\succcurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)\right):\ \widetilde{\mathfrak{d}}\in\vartheta\right\}.$$

Where the membership of each element $\widetilde{\mathfrak{d}}\in\vartheta$ is defined as:

$$\mathcal{Z}_{\underline{\mathcal{Y}}\left(\mathcal{Z}_{e}^{\succcurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)=\bigwedge\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left(\mathcal{Z}_{\left(\mathcal{Z}_{e}^{\succcurlyeq}\right)}\left(\widetilde{\mathfrak{q}}\right)\bigvee\left(1-\check{D}_{\wp}\left(\widetilde{\mathfrak{q}},\widetilde{\mathfrak{d}}\right)\right)\right),$$

$$\mathcal{Z}_{\overline{\mathcal{Y}}\left(\mathcal{Z}_{e}^{\succcurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)=\bigvee\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left(\mathcal{Z}_{\left(\mathcal{Z}_{e}^{\succcurlyeq}\right)}\left(\widetilde{\mathfrak{q}}\right)\bigwedge\left(\check{D}_{\wp}\left(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}\right)\right)\right).$$

Similarly, the LA and UA of $\mathcal{Z}_{e}^{\preccurlyeq}$ with FDRS denoted by $\underline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)$ and $\overline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)$ respectively, are represented and defined as:

$$\underline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)=\left\{\left(\widetilde{\mathfrak{d}}, \mathcal{Z}_{\underline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)\right):\ \widetilde{\mathfrak{d}}\in\vartheta\right\},$$

$$\overline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)=\left\{\left(\widetilde{\mathfrak{d}}, \mathcal{Z}_{\overline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)\right):\ \widetilde{\mathfrak{d}}\in\vartheta\right\}.$$

Where the membership of each element $\widetilde{\mathfrak{d}}\in\vartheta$ is defined as:

$$\mathcal{Z}_{\underline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)=\bigwedge\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left(\mathcal{Z}_{\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{q}}\right)\bigvee\left(1-\check{D}_{\wp}\left(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}\right)\right)\right),$$

$$\mathcal{Z}_{\overline{\breve{\mathcal{Y}}}\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)=\bigvee\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left(\mathcal{Z}_{\left(\mathcal{Z}_{e}^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{q}}\right)\bigwedge\left(\check{D}_{\wp}\left(\widetilde{\mathfrak{q}},\widetilde{\mathfrak{d}}\right)\right)\right).$$

Definition 5 ([36])**.** *A BFS ϑ is defined and denoted as:*

$$\vartheta=\left\{<\widetilde{\mathfrak{d}},\ \left(\mathcal{Z}^{+}(\widetilde{\mathfrak{d}}),\mathcal{Z}^{-}\left(\widetilde{\mathfrak{d}}\right)\right)>|\ \widetilde{\mathfrak{d}}\in\vartheta\right\},$$

where $\mathcal{Z}^{+}(\widetilde{\mathfrak{d}})\ \in[0,\ 1]$ is a PMD and $\mathcal{Z}^{-}\left(\widetilde{\mathfrak{d}}\right)\ \in[-1,\ 0]$ is an NMD for each $\widetilde{\mathfrak{d}}\in\vartheta$.

Definition 6 ([28])**.** *Let, ϑ be the fixed set, $\breve{\mathcal{Y}}$ be a bipolar fuzzy equivalence relation, and $\wp=\{(\widetilde{\mathfrak{d}},\ (\mathcal{Z}_{\wp}^{+}(\widetilde{\mathfrak{d}}),\mathcal{Z}_{\wp}^{-}(\widetilde{\mathfrak{d}})))|\widetilde{\mathfrak{d}}\in\vartheta\}\ \in BFS(\vartheta)$. Then, the pair ($\vartheta$, $\breve{\mathcal{Y}}$) is a bipolar fuzzy approximation space (BFAS). Then, the LA and UA with $\wp$ respect to (ϑ, $\breve{\mathcal{Y}}$) are indicated and described as:*

$$\underline{\breve{\mathcal{Y}}}(\wp)=\left\{\left(\widetilde{\mathfrak{d}},\left(\chi^{+}{}_{\breve{\mathcal{Y}}}\left(\widetilde{\mathfrak{d}}\right),\chi^{-}{}_{\underline{\breve{\mathcal{Y}}}}\left(\widetilde{\mathfrak{d}}\right)\right)\right)|\ \widetilde{\mathfrak{d}}\in\vartheta\right\},$$

$$\overline{\breve{\mathcal{Y}}}(\wp)=\left\{\left(\widetilde{\mathfrak{d}},\left(\chi^{+}{}_{\overline{\breve{\mathcal{Y}}}}\left(\widetilde{\mathfrak{d}}\right),\chi^{-}{}_{\overline{\breve{\mathcal{Y}}}}\left(\widetilde{\mathfrak{d}}\right)\right)\right)|\ \widetilde{\mathfrak{d}}\in\vartheta\right\}.$$

Where,

$$\chi^{+}{}_{\underline{\breve{\mathcal{Y}}}}\left(\widetilde{\mathfrak{d}}\right)=\bigwedge\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left[\mathcal{Z}_{\breve{\mathcal{Y}}}^{+}\left(\widetilde{\mathfrak{d}},\widetilde{\mathfrak{q}}\right)\ \bigvee\ \mathcal{Z}_{\wp}^{+}\left(\widetilde{\mathfrak{q}}\right)\right]=\underline{\chi\omega^{+}\left(\widetilde{\mathfrak{d}}\right)},$$

$$\chi^{-}{}_{\underline{\breve{\mathcal{Y}}}}(\widetilde{\mathfrak{d}}) = \bigvee_{\widetilde{\mathfrak{q}}\in\vartheta}\left[\mathcal{Z}^{-}_{\breve{\mathcal{Y}}}(\widetilde{\mathfrak{d}},\widetilde{\mathfrak{q}}) \bigwedge \mathcal{Z}^{-}_{\wp}(\widetilde{\mathfrak{q}})\right] = \underline{\chi^{-}(\widetilde{\mathfrak{d}})},$$

$$\chi^{+}{}_{\overline{\breve{\mathcal{Y}}}}(\widetilde{\mathfrak{d}}) = \bigvee_{\widetilde{\mathfrak{q}}\in\vartheta}\left[\mathcal{Z}^{+}_{\breve{\mathcal{Y}}}(\widetilde{\mathfrak{d}},\widetilde{\mathfrak{q}}) \bigwedge \mathcal{Z}^{+}_{\wp}(\widetilde{\mathfrak{q}})\right] = \overline{\chi^{+}(\widetilde{\mathfrak{d}})},$$

$$\chi^{-}{}_{\overline{\breve{\mathcal{Y}}}}(\widetilde{\mathfrak{d}}) = \bigwedge_{\widetilde{\mathfrak{q}}\in\vartheta}\left[\mathcal{Z}^{-}_{\breve{\mathcal{Y}}}(\widetilde{\mathfrak{d}},\widetilde{\mathfrak{q}}) \bigvee \mathcal{Z}^{-}_{\wp}(\widetilde{\mathfrak{q}})\right] = \overline{\chi^{-}(\widetilde{\mathfrak{d}})}.$$

Then, the pair $\left(\underline{\breve{\mathcal{Y}}}(\wp),\ \overline{\breve{\mathcal{Y}}}(\wp)\right)$ is called BFRS, with $\underline{\breve{\mathcal{Y}}}(\wp) \neq \overline{\breve{\mathcal{Y}}}(\wp)$ for simplicity, $\Delta_{BFRS} = \left\{\left(\widetilde{\mathfrak{d}},\ \left(\left(\underline{\chi^{+}(\widetilde{\mathfrak{d}})}, \underline{\chi^{-}(\widetilde{\mathfrak{d}})}\right), \left(\overline{\chi^{+}(\widetilde{\mathfrak{d}})}, \overline{\chi^{-}(\widetilde{\mathfrak{d}})}\right)\right)\right) \mid \widetilde{\mathfrak{d}} \in \vartheta\right\}$ denotes the BFRS.

4. Bipolar fuzzy dominance rough set

Here, we devise the notion of bipolar fuzzy dominance rough set and related results.

Definition 7. *Consider the universe ϑ and for all $R \in BFR(\vartheta \times \vartheta)$, and $R = \{\langle(\widetilde{\mathfrak{d}},\widetilde{\mathfrak{q}}),\ M^{+}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}),\ M^{-}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}})\rangle \mid M^{+} : \vartheta_1 \times \vartheta_2 \to [0,\ 1],\ M^{-} : \vartheta_1 \times \vartheta_2 \to [-1,\ 0]\}$. The bipolar fuzzy dominance-based relation can show the reliability of positive MG and negative MG of the dominance principle between objects. For the decision system $\mathbb{D} = (\vartheta,\ AT \cup \{d\})$, and for $a \in AT$, R is a BFD relation concerning attribute a, for AT and every $(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{d}}) \in \vartheta \times \vartheta$, it is expressed by:*

$$\mathrm{R_{AT}}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}) = \left\{\left(\chi^{+}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}),\ \chi^{-}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}})\right)\right\} = \left\{\left(\wedge\left\{\chi^{+}_{\mathrm{R_a}}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}) :\ \mathrm{a} \in \mathrm{AT}\right\},\ \wedge\left\{\chi^{-}_{\mathrm{R_a}}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}) :\ \mathrm{a} \in \mathrm{AT}\right\}\right)\right\}.$$

Definition 8. *For the decision system $\mathbb{D} = (\vartheta,\ AT \cup \{d\})$, $A \subseteq AT$, bipolar fuzzy dominance-based relation R_{AT} concerning A, for all $n \in N$, and then the LA and UA of the class $\mathcal{Z}^{\succcurlyeq}_{n}$ concerningR_{AT} by using the implicator and T-norms are expressed as:*

$$\underline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{\mathrm{n}}) = \left\{\widetilde{\mathfrak{d}},\ \chi^{+}_{\underline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{\mathrm{n}})}(\widetilde{\mathfrak{d}}),\ \chi^{-}_{\underline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{\mathrm{n}})}(\widetilde{\mathfrak{d}})\right\},$$

$$\overline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{\mathrm{n}}) = \left\{\widetilde{\mathfrak{d}},\ \chi^{+}_{\overline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{\mathrm{n}})}(\widetilde{\mathfrak{d}}),\ \chi^{-}_{\overline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{\mathrm{n}})}(\widetilde{\mathfrak{d}})\right\},$$

where, $\chi^{+}_{\underline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{o})}(\widetilde{\mathfrak{d}})$ denotes the positive lower membership grade of the class $\mathcal{Z}^{\succcurlyeq}_{o}$, $\chi^{-}_{\underline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{o})}(\widetilde{\mathfrak{d}})$ denotes the negative lower membership grade of the class $\mathcal{Z}^{\succcurlyeq}_{o}$.

$$\chi^{+}_{\underline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{o})}(\widetilde{\mathfrak{d}}) = \bigwedge_{\widetilde{\mathfrak{q}}\in\vartheta}\left[1 - \mathrm{R}^{+}_{\mathrm{A}}(\widetilde{\mathfrak{q}},\widetilde{\mathfrak{d}}) \bigvee \mathcal{Z}^{\succcurlyeq +}_{o}(\widetilde{\mathfrak{q}})\right],\ \chi^{-}_{\underline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{o})}(\widetilde{\mathfrak{d}}) = \bigwedge_{\widetilde{\mathfrak{q}}\in\vartheta}\left[-1 - \mathrm{R}^{-}_{\mathrm{A}}(\widetilde{\mathfrak{q}},\widetilde{\mathfrak{d}}) \bigvee \mathcal{Z}^{\succcurlyeq -}_{o}(\widetilde{\mathfrak{q}})\right],$$

here, $\chi^{+}_{\overline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{o})}(\widetilde{\mathfrak{d}})$ denotes the positive upper membership grade of the class $\mathcal{Z}^{\succcurlyeq}_{o}$, $\chi^{-}_{\overline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{o})}(\widetilde{\mathfrak{d}})$ denotes the negative upper membership grade of the class $\mathcal{Z}^{\succcurlyeq}_{o}$.

$$\chi^{+}_{\overline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{o})}(\widetilde{\mathfrak{d}}) = \bigvee_{\widetilde{\mathfrak{q}}\in\vartheta}\left[\mathrm{R}^{+}_{\mathrm{A}}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}) \bigwedge \mathcal{Z}^{\succcurlyeq +}_{o}(\widetilde{\mathfrak{q}})\right],\ \chi^{-}_{\overline{\mathcal{Y}}(\mathcal{Z}^{\succcurlyeq}_{o})}(\widetilde{\mathfrak{d}}) = \bigvee_{\widetilde{\mathfrak{q}}\in\vartheta}\left[\mathrm{R}^{-}_{\mathrm{A}}(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}) \bigwedge \mathcal{Z}^{\succcurlyeq -}_{o}(\widetilde{\mathfrak{q}})\right],$$

the upper approximation (UA) and lower approximation (LA) of the class $\mathcal{Z}^{\preccurlyeq}_{o}$ concerning

R_{AT} are expressed by:

$$\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)=\left\{\widetilde{\mathfrak{d}},\ \chi^{+}_{\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right),\ \chi^{-}_{\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)\right\},$$

$$\overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)=\left\{\widetilde{\mathfrak{d}},\ \chi^{+}_{\overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right),\ \chi^{-}_{\overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)\right\},$$

where, $\chi^{+}_{\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)$ denotes the positive lower membership grade of the class $\mathcal{Z}_o^{\preccurlyeq}$, $\chi^{-}_{\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)$ denotes the negative lower membership grade of the class $\mathcal{Z}_o^{\preccurlyeq}$.

$$\chi^{+}_{\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)=\bigwedge\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left[1-\mathrm{R}_{\mathrm{A}}^{+}\left(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}\right)\bigvee\mathcal{Z}_o^{\preccurlyeq +}(\widetilde{\mathfrak{q}})\right],\ \chi^{-}_{\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)=\bigwedge\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left[-1-\mathrm{R}_{\mathrm{A}}^{-}\left(\widetilde{\mathfrak{d}},\ \widetilde{\mathfrak{q}}\right)\bigvee\mathcal{Z}_o^{\preccurlyeq -}(\widetilde{\mathfrak{q}})\right],$$

here, $\chi^{+}_{\overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)$ denotes the positive upper membership grade of the class $\mathcal{Z}_o^{\preccurlyeq}$, $\chi^{-}_{\overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)$ denotes the negative upper membership grade of the class $\mathcal{Z}_o^{\preccurlyeq}$.

$$\chi^{+}_{\overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)=\bigvee\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left[\mathrm{R}_{\mathrm{A}}^{+}\left(\widetilde{\mathfrak{q}},\widetilde{\mathfrak{d}}\right)\bigwedge\mathcal{Z}_o^{\preccurlyeq +}(\widetilde{\mathfrak{q}})\right],\chi^{-}_{\overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)}\left(\widetilde{\mathfrak{d}}\right)=\bigvee\nolimits_{\widetilde{\mathfrak{q}}\in\vartheta}\left[\mathrm{R}_{\mathrm{A}}^{-}\left(\widetilde{\mathfrak{q}},\widetilde{\mathfrak{d}}\right)\bigwedge\mathcal{Z}_o^{\preccurlyeq -}(\widetilde{\mathfrak{q}})\right].$$

Then the framework ($\underline{\mathcal{Y}}$ ($\mathcal{Z}_o^{\succcurlyeq}$), $\overline{\mathcal{Y}}$ ($\mathcal{Z}_o^{\succcurlyeq}$), $\underline{\mathcal{Y}}$ ($\mathcal{Z}_o^{\preccurlyeq}$), $\overline{\mathcal{Y}}$ ($\mathcal{Z}_o^{\preccurlyeq}$)), it is called bipolar fuzzy dominance-based rough set (BFDRS) concerning (ϑ, $\mathcal{Y}$) and $\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\succcurlyeq}\right), \overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\succcurlyeq}\right)$, $\underline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right), \overline{\mathcal{Y}}\left(\mathcal{Z}_o^{\preccurlyeq}\right)$: BFD ($\vartheta$) $\rightarrow$ BFD (ϑ) are referred to as LA and UA operators. Here, $\Psi_{\mathrm{BFDN}}=\left(\left(\underline{\beth}^{+}_{\mathcal{Z}_o^{\succcurlyeq}},\ \underline{\beth}^{-}_{\mathcal{Z}_o^{\succcurlyeq}},\ \overline{\beth}^{+}_{\mathcal{Z}_o^{\succcurlyeq}},\ \overline{\beth}^{-}_{\mathcal{Z}_o^{\succcurlyeq}}\right),\left(\underline{\beth}^{+}_{\mathcal{Z}_o^{\preccurlyeq}},\ \underline{\beth}^{-}_{\mathcal{Z}_o^{\preccurlyeq}},\ \overline{\beth}^{+}_{\mathcal{Z}_o^{\preccurlyeq}},\ \overline{\beth}^{-}_{\mathcal{Z}_o^{\preccurlyeq}}\right)\right)$, it is a bipolar fuzzy dominance rough number (BFDRN).

Definition 9. *The mathematical representation of the score value is explained as:*

$$\mathrm{S}_{\Psi}=\frac{1}{8}\left(4+\underline{\beth_o^{+\leq}}+\underline{\beth_o^{-\leq}}+\overline{\beth_o^{+\leq}}+\overline{\beth_o^{-\leq}}+\underline{\beth_o^{+\geq}}+\underline{\beth_o^{-\geq}}+\overline{\beth_o^{+\geq}}+\overline{\beth_o^{-\geq}}\right),$$

where $\mathrm{S}_{\Psi}\in[0,\ 1]$.

Definition 10. *The mathematical representation of the accuracy value is explained as:*

$$\mathrm{A}_{\Psi}=\left(\frac{\underline{\beth_o^{+\leq}}+\underline{\beth_o^{-\leq}}+\overline{\beth_o^{+\leq}}+\overline{\beth_o^{-\leq}}+\underline{\beth_o^{+\geq}}+\underline{\beth_o^{-\geq}}+\overline{\beth_o^{+\geq}}+\overline{\beth_o^{-\geq}}}{8}\right),$$

where $\mathrm{A}_{\Psi}\in[0,\ 1]$.

Definition 11. *Let* $\Psi_1=\left(\left(\underline{\beth_{1o}^{+\preccurlyeq}},\ \underline{\beth_{1o}^{-\preccurlyeq}},\overline{\beth_{1o}^{+\preccurlyeq}},\ \overline{\beth_{1o}^{-\preccurlyeq}}\right),\ \left(\underline{\beth_{1o}^{+\succcurlyeq}},\ \underline{\beth_{1o}^{-\succcurlyeq}},\ \overline{\beth_{1o}^{+\succcurlyeq}},\ \overline{\beth_{1o}^{-\succcurlyeq}}\right)\right)$ *and* $\Psi_2=\left(\left(\underline{\beth_{2o}^{+\preccurlyeq}},\ \underline{\beth_{2o}^{-\preccurlyeq}},\ \overline{\beth_{2o}^{+\preccurlyeq}},\ \overline{\beth_{2o}^{-\preccurlyeq}}\right),\ \left(\underline{\beth_{2o}^{+\succcurlyeq}},\ \underline{\beth_{2o}^{-\succcurlyeq}},\ \overline{\beth_{2o}^{+\succcurlyeq}},\ \overline{\beth_{2o}^{-\succcurlyeq}}\right)\right)$ *be any two BFDRNs for* $\aleph\in N,\ then$ *Operational laws are given by:*

1. $$\Psi_1 \oplus \Psi_2 = \left(\begin{array}{c}
\left(\begin{array}{c}
\left(1-\frac{1}{1+\left\{\left(\frac{\underline{\beth_{1_o}^{+\preccurlyeq}}}{1-\underline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}+\left(\frac{\underline{\beth_{2_o}^{+\preccurlyeq}}}{1-\underline{\beth_{2_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\left(\frac{1+\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}+\left(\frac{1+\left|\underline{\beth_{2_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(1-\frac{1}{1+\left\{\left(\frac{\overline{\beth_{1_o}^{+\preccurlyeq}}}{1-\overline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}+\left(\frac{\overline{\beth_{2_o}^{+\preccurlyeq}}}{1-\overline{\beth_{2_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\left(\frac{1+\left|\overline{\beth_{1_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{1_o}^{+\preccurlyeq}}\right|}\right)^{\aleph}+\left(\frac{1+\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right), \\
\left(\begin{array}{c}
\left(1-\frac{1}{1+\left\{\left(\frac{\underline{\beth_{1_o}^{+\succcurlyeq}}}{1-\underline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}+\left(\frac{\underline{\beth_{2_o}^{+\succcurlyeq}}}{1-\underline{\beth_{2_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\left(\frac{1+\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}+\left(\frac{1+\left|\underline{\beth_{2_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(1-\frac{1}{1+\left\{\left(\frac{\overline{\beth_{1_o}^{+\succcurlyeq}}}{1-\overline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}+\left(\frac{\overline{\beth_{2_o}^{+\succcurlyeq}}}{1-\overline{\beth_{2_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\left(\frac{1+\left|\overline{\beth_{1_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{1_o}^{+\succcurlyeq}}\right|}\right)^{\aleph}+\left(\frac{1+\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right)
\end{array}\right)$$

2. $$\Psi_1 \otimes \Psi_2 = \left(\begin{array}{c}
\left(\begin{array}{c}
\left(\frac{1}{1+\left\{\left(\frac{1-\underline{\beth_{1_o}^{+\preccurlyeq}}}{\underline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}+\left(\frac{1-\underline{\beth_{2_o}^{+\preccurlyeq}}}{\underline{\beth_{2_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\left(\frac{\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}+\left(\frac{\left|\underline{\beth_{2_o}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\frac{1}{1+\left\{\left(\frac{1-\overline{\beth_{1_o}^{+\preccurlyeq}}}{\overline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}+\left(\frac{1-\overline{\beth_{2_o}^{+\preccurlyeq}}}{\overline{\beth_{2_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\left(\frac{\left|\overline{\beth_{1_o}^{+\preccurlyeq}}\right|}{1+\left|\overline{\beth_{1_o}^{+\preccurlyeq}}\right|}\right)^{\aleph}+\left(\frac{\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right), \\
\left(\begin{array}{c}
\left(\frac{1}{1+\left\{\left(\frac{1-\underline{\beth_{1_o}^{+\succcurlyeq}}}{\underline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}+\left(\frac{1-\underline{\beth_{2_o}^{+\succcurlyeq}}}{\underline{\beth_{2_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\left(\frac{\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}+\left(\frac{\left|\underline{\beth_{2_o}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\frac{1}{1+\left\{\left(\frac{1-\overline{\beth_{1_o}^{+\succcurlyeq}}}{\overline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}+\left(\frac{1-\overline{\beth_{2_o}^{+\succcurlyeq}}}{\overline{\beth_{2_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\left(\frac{\left|\overline{\beth_{1_o}^{+\succcurlyeq}}\right|}{1+\left|\overline{\beth_{1_o}^{+\succcurlyeq}}\right|}\right)^{\aleph}+\left(\frac{\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}{1+\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right)
\end{array}\right)$$

3. $$\lambda\Psi_1 = \left(\begin{array}{c}
\left(\left(1-\frac{1}{1+\left\{\lambda\left(\frac{\underline{\beth_{1_o}^{+\preccurlyeq}}}{1-\underline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, \frac{-1}{1+\left\{\lambda\left(\frac{1+\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \left(1-\frac{1}{1+\left\{\lambda\left(\frac{\overline{\beth_{1_o}^{+\preccurlyeq}}}{1-\overline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, \frac{-1}{1+\left\{\lambda\left(\frac{1+\left|\overline{\beth_{1_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right), \\
\left(\left(1-\frac{1}{1+\left\{\lambda\left(\frac{\underline{\beth_{1_o}^{+\succcurlyeq}}}{1-\underline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, \frac{-1}{1+\left\{\lambda\left(\frac{1+\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \left(1-\frac{1}{1+\left\{\lambda\left(\frac{\overline{\beth_{1_o}^{+\succcurlyeq}}}{1-\overline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, \frac{-1}{1+\left\{\lambda\left(\frac{1+\left|\overline{\beth_{1_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right)
\end{array}\right)$$

4. $$\Psi_1{}^{\lambda}=\left(\begin{array}{c}\left(\left(\frac{1}{1+\left\{\lambda\left(\frac{1-\underline{\beth_{1_o}^{+\preccurlyeq}}}{\underline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},-1+\frac{1}{1+\left\{\left(\frac{\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\left(\frac{1}{1+\left\{\lambda\left(\frac{1-\overline{\beth_{1_o}^{+\preccurlyeq}}}{\overline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},-1+\frac{1}{1+\left\{\lambda\left(\frac{\left|\overline{\beth_{1_o}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right),\\ \left(\left(\frac{1}{1+\left\{\lambda\left(\frac{1-\underline{\beth_{1_o}^{+\succcurlyeq}}}{\underline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},-1+\frac{1}{1+\left\{\lambda\left(\frac{\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\left(\frac{1}{1+\left\{\lambda\left(\frac{1-\overline{\beth_{1_o}^{+\succcurlyeq}}}{\overline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},-1+\frac{1}{1+\left\{\lambda\left(\frac{\left|\overline{\beth_{1_o}^{-\succcurlyeq}}\right|}{1+\left|\overline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right)\end{array}\right)$$

5. Bipolar fuzzy dominance rough Dombi aggregation operators

In this section of the article, we will demonstrate Dombi arithmetic AOs and Dombi geometric AOs with BFDRNs.

Definition 12. *Let,* $\Psi_{j}=\left(\left(\underline{\beth_{jm}^{+\preccurlyeq}},\underline{\beth_{jm}^{-\preccurlyeq}},\overline{\beth_{jm}^{+\preccurlyeq}},\overline{\beth_{jm}^{-\preccurlyeq}}\right),\left(\underline{\beth_{jm}^{+\succcurlyeq}},\underline{\beth_{jm}^{-\succcurlyeq}},\overline{\beth_{jm}^{+\succcurlyeq}},\overline{\beth_{jm}^{-\succcurlyeq}}\right)\right)$ *be the family of BFDRNs, where* $j=1,2,3,\ldots,m$. *Next, the following defines the concept of the BFDRWA operator:*

$$\text{BFDRDWA}\left(\Psi_1,\Psi_2,\Psi_3,\ldots,\Psi_{\mathrm{m}}\right)=\bigoplus_{j=1}^{\mathrm{m}}\left(\omega_{j}\Psi_{j}\right)=\omega_1\Psi_1\oplus\omega_2\Psi_2\oplus\ldots\oplus\omega_{\mathrm{m}}\Psi_{\mathrm{m}}. \tag{1}$$

Where $\omega=\{\omega_1,\omega_2,\ldots,\omega_{\mathrm{m}}\}$ represent the weight vectors such that $\sum_{j=1}^{\mathrm{m}}\omega_{j}=1$.

Theorem 1. *Let,* $\Psi_{j}=\left(\left(\underline{\beth_{jm}^{+\preccurlyeq}},\underline{\beth_{jm}^{-\preccurlyeq}},\overline{\beth_{jm}^{+\preccurlyeq}},\overline{\beth_{jm}^{-\preccurlyeq}}\right),\left(\underline{\beth_{jm}^{+\succcurlyeq}},\underline{\beth_{jm}^{-\succcurlyeq}},\overline{\beth_{jm}^{+\succcurlyeq}},\overline{\beth_{jm}^{-\succcurlyeq}}\right)\right)$ *be the family of BFDRNs, where* $j=1,2,3,\ldots,m$. *Now, we have obtained the aggregated result, and in the results, we can see that we get a BFDRN again.*

$$\text{BFDRDWA}\left(\Psi_1,\Psi_2,\Psi_3,\ldots,\Psi_{\mathrm{m}}\right)=$$

$$\left(\begin{array}{c}\left(\left(1-\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_{j}\left(\frac{\underline{\beth_{j_o}^{+\preccurlyeq}}}{1-\underline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\frac{-1}{1+\left\{\sum_{j=1}^{m}\omega_{j}\left(\frac{1+\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\left(1-\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_{j}\left(\frac{\overline{\beth_{j_o}^{+\preccurlyeq}}}{1-\overline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\frac{-1}{1+\left\{\sum_{j=1}^{m}\omega_{j}\left(\frac{1+\left|\overline{\beth_{j_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right),\\ \left(\left(1-\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_{j}\left(\frac{\underline{\beth_{j_o}^{+\succcurlyeq}}}{1-\underline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\frac{-1}{1+\left\{\sum_{j=1}^{m}\omega_{j}\left(\frac{1+\left|\underline{\beth_{j_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\left(1-\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_{j}\left(\frac{\overline{\beth_{j_o}^{+\succcurlyeq}}}{1-\overline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\frac{-1}{1+\left\{\sum_{j=1}^{m}\omega_{j}\left(\frac{1+\left|\overline{\beth_{j_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{j_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right)\end{array}\right) \tag{2}$$

$\omega=\{\omega_1,\omega_2,\ldots,\omega_{\mathrm{m}}\}^{T}$ represent the weight vectors such that $\sum_{j=1}^{\mathrm{m}}\omega_{j}=1$.

Proof. We prove the theorem by using mathematical induction. First, we assume for

$\boldsymbol{j} = 2.$

$$\omega_1\ \Psi_1 = \left(\begin{array}{l} \left(\begin{array}{l} \left(1-\frac{1}{1+\left\{\omega_1\left(\frac{\underline{\beth_{1_o}^{+\preccurlyeq}}}{1-\underline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(1-\frac{1}{1+\left\{\omega_1\left(\frac{\overline{\beth_{1_o}^{+\preccurlyeq}}}{1-\overline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\overline{\beth_{1_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right) \end{array} \right), \\ \left(\begin{array}{l} \left(1-\frac{1}{1+\left\{\omega_1\left(\frac{\underline{\beth_{1_o}^{+\succcurlyeq}}}{1-\underline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(1-\frac{1}{1+\left\{\omega_1\left(\frac{\overline{\beth_{1_o}^{+\succcurlyeq}}}{1-\overline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\overline{\beth_{1_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right) \end{array} \right) \end{array} \right)$$

$$\omega_2\ \Psi_2 = \left(\begin{array}{l} \left(\begin{array}{l} \left(1-\frac{1}{1+\left\{\omega_2\left(\frac{\underline{\beth_{2_o}^{+\preccurlyeq}}}{1-\underline{\beth_{2_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_2\left(\frac{1+\left|\underline{\beth_{2_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(1-\frac{1}{1+\left\{\omega_2\left(\frac{\overline{\beth_{2_o}^{+\preccurlyeq}}}{1-\overline{\beth_{2_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right) \end{array} \right), \\ \left(\begin{array}{l} \left(1-\frac{1}{1+\left\{\omega_2\left(\frac{\underline{\beth_{2_o}^{+\succcurlyeq}}}{1-\underline{\beth_{2_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_2\left(\frac{1+\left|\underline{\beth_{2_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(1-\frac{1}{1+\left\{\omega_2\left(\frac{\overline{\beth_{2_o}^{+\succcurlyeq}}}{1-\overline{\beth_{2_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_2\left(\frac{1+\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right) \end{array} \right) \end{array} \right)$$

$$\mathrm{BFDRDWA}\left(\Psi_1,\Psi_2\right)=\ \omega_1\ \Psi_1\oplus\omega_2\ \Psi_2=$$

$$\left(\begin{array}{c}\left(\begin{array}{c}\left(1-\frac{1}{1+\left\{\omega_1\left(\frac{\underline{\beth_{1_o}^{+\preccurlyeq}}}{1-\underline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}+\omega_2\left(\frac{\underline{\beth_{2_o}^{+\preccurlyeq}}}{1-\underline{\beth_{2_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{1_o}^{-\preccurlyeq}}\right|}\right)^{o}+\omega_2\left(\frac{1+\left|\underline{\beth_{2_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\ \left(1-\frac{1}{1+\left\{\omega_1\left(\frac{\overline{\beth_{1_o}^{+\preccurlyeq}}}{1-\overline{\beth_{1_o}^{+\preccurlyeq}}}\right)^{\aleph}+\omega_2\left(\frac{\overline{\beth_{2_o}^{+\preccurlyeq}}}{1-\overline{\beth_{2_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\overline{\beth_{1_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{1_o}^{+\preccurlyeq}}\right|}\right)^{\aleph}+\omega_2\left(\frac{1+\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\frac{1}{1+\left\{\omega_1\left(\frac{\underline{\beth_{1_o}^{+\succcurlyeq}}}{1-\underline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}+\omega_2\left(\frac{\underline{\beth_{2_o}^{+\succcurlyeq}}}{1-\underline{\beth_{2_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{1_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}+\omega_2\left(\frac{1+\left|\underline{\beth_{2_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\ \left(1-\frac{1}{1+\left\{\omega_1\left(\frac{\overline{\beth_{1_o}^{+\succcurlyeq}}}{1-\overline{\beth_{1_o}^{+\succcurlyeq}}}\right)^{\aleph}+\omega_2\left(\frac{\overline{\beth_{2_o}^{+\succcurlyeq}}}{1-\overline{\beth_{2_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_1\left(\frac{1+\left|\overline{\beth_{1_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{1_o}^{+\succcurlyeq}}\right|}\right)^{\aleph}+\omega_2\left(\frac{1+\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right)\end{array}\right)$$

$$\mathrm{BFDRDWA}\left(\Psi_1,\Psi_2\right)=\left(\begin{array}{c}\left(\begin{array}{c}\left(1-\frac{1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{\underline{\beth_{j_o}^{+\preccurlyeq}}}{1-\underline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{1+\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\ \left(1-\frac{1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{\overline{\beth_{j_o}^{+\preccurlyeq}}}{1-\overline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{1+\left|\overline{\beth_{j_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(1-\frac{1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{\underline{\beth_{j_o}^{+\succcurlyeq}}}{1-\underline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{1+\left|\underline{\beth_{j_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\ \left(1-\frac{1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{\overline{\beth_{j_o}^{+\succcurlyeq}}}{1-\overline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{2}\omega_j\left(\frac{1+\left|\overline{\beth_{j_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right)\end{array}\right)$$

This proves that (2) is hold for $m=2$. Now assume that (2) is true for $m=k$, for $\mathrm{k}\geq 2$

$$\mathrm{BFDRDWA}_{(\Psi_1,\Psi_2,\Psi_3,\,\ldots,\,\Psi_\mathrm{k})} = \left(\begin{matrix}\left(\begin{matrix}\left(1-\frac{1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{\beth_{j_e}^{+\preccurlyeq}}{1-\beth_{j_e}^{+\preccurlyeq}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{1+\left|\beth_{j_e}^{-\preccurlyeq}\right|}{\left|\beth_{j_e}^{-\preccurlyeq}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(1-\frac{1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{\overline{\beth_{j_e}^{+\preccurlyeq}}}{1-\overline{\beth_{j_e}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{1+\left|\overline{\beth_{j_e}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{2_e}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{matrix}\right), \\ \left(\begin{matrix}\left(1-\frac{1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{\beth_{j_e}^{+\succcurlyeq}}{1-\beth_{j_e}^{+\succcurlyeq}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{1+\left|\beth_{j_e}^{-\succcurlyeq}\right|}{\left|\beth_{j_e}^{-\succcurlyeq}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(1-\frac{1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{\overline{\beth_{j_e}^{+\succcurlyeq}}}{1-\overline{\beth_{j_e}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{1+\left|\overline{\beth_{j_e}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{2_e}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{matrix}\right)\end{matrix}\right)$$

Next, we will show that (2) is true for $m = k + 1$.

$$\mathrm{BFDRDWA}\,(\Psi_1,\Psi_2,\Psi_3,\ldots,\Psi_{\mathrm{k}+1}) = \bigoplus_{j=1}^{\mathrm{k}+1}(\omega_j\ \Psi_j) \oplus \omega_{\mathrm{k}+1}\ \Psi_{\mathrm{k}+1} =$$

$$\left(\begin{matrix}\left(\begin{matrix}\left(1-\frac{1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{\beth_{j_e}^{+\preccurlyeq}}{1-\beth_{j_e}^{+\preccurlyeq}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{1+\left|\beth_{j_e}^{-\preccurlyeq}\right|}{\left|\beth_{j_e}^{-\preccurlyeq}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(1-\frac{1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{\overline{\beth_{j_e}^{+\preccurlyeq}}}{1-\overline{\beth_{j_e}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{1+\left|\overline{\beth_{j_e}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{2_e}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{matrix}\right), \\ \left(\begin{matrix}\left(1-\frac{1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{\beth_{j_e}^{+\succcurlyeq}}{1-\beth_{j_e}^{+\succcurlyeq}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{1+\left|\beth_{j_e}^{-\succcurlyeq}\right|}{\left|\beth_{j_e}^{-\preccurlyeq}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(1-\frac{1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{\overline{\beth_{j_e}^{+\succcurlyeq}}}{1-\overline{\beth_{j_e}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k}\omega_j\left(\frac{1+\left|\overline{\beth_{j_e}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{2_e}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{matrix}\right)\end{matrix}\right) \oplus$$

$$\left(\left(\left(1-\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\beth^{+\preccurlyeq}_{(k+1)_e}}{1-\beth^{+\preccurlyeq}_{(k+1)_e}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_{k+1}\left(\frac{1+\left|\beth^{-\preccurlyeq}_{(k+1)_e}\right|}{\left|\beth^{-\preccurlyeq}_{(k+1)_e}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\left(1-\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\overline{\beth^{+\preccurlyeq}_{(k+1)_e}}}{1-\overline{\beth^{+\preccurlyeq}_{(k+1)_e}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_{k+1}\left(\frac{1+\left|\overline{\beth^{-\preccurlyeq}_{(k+1)_e}}\right|}{\left|\overline{\beth^{-\preccurlyeq}_{(k+1)_e}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right),\left(\left(1-\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\beth^{+\succcurlyeq}_{(k+1)_e}}{1-\beth^{+\succcurlyeq}_{(k+1)_e}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_{k+1}\left(\frac{1+\left|\beth^{-\succcurlyeq}_{(k+1)_e}\right|}{\left|\beth^{-\succcurlyeq}_{(k+1)_e}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\left(1-\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\overline{\beth^{+\succcurlyeq}_{(k+1)_e}}}{1-\overline{\beth^{+\succcurlyeq}_{(k+1)_e}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\omega_{k+1}\left(\frac{1+\left|\overline{\beth^{-\succcurlyeq}_{(k+1)_e}}\right|}{\left|\overline{\beth^{-\succcurlyeq}_{(k+1)_e}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right)\right)$$

$$\mathrm{BFDRDWA}\left(\Psi_1,\Psi_2,\Psi_3,\ \ldots,\ \Psi_{k+1}\right)=$$

$$\left(\left(\left(1-\frac{1}{1+\left\{\sum_{j=1}^{k+1}\omega_j\left(\frac{\beth^{+\preccurlyeq}_{j_e}}{1-\beth^{+\preccurlyeq}_{j_e}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k+1}\omega_j\left(\frac{1+\left|\beth^{-\preccurlyeq}_{j_e}\right|}{\left|\beth^{-\preccurlyeq}_{j_e}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\left(1-\frac{1}{1+\left\{\sum_{j=1}^{k+1}\omega_j\left(\frac{\overline{\beth^{+\preccurlyeq}_{j_e}}}{1-\overline{\beth^{+\preccurlyeq}_{j_e}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k+1}\omega_j\left(\frac{1+\left|\overline{\beth^{-\preccurlyeq}_{j_e}}\right|}{\left|\overline{\beth^{-\preccurlyeq}_{2_e}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right),\left(\left(1-\frac{1}{1+\left\{\sum_{j=1}^{k+1}\omega_j\left(\frac{\beth^{+\succcurlyeq}_{j_e}}{1-\beth^{+\succcurlyeq}_{j_e}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k+1}\omega_j\left(\frac{1+\left|\beth^{-\succcurlyeq}_{j_e}\right|}{\left|\beth^{-\preccurlyeq}_{j_e}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\left(1-\frac{1}{1+\left\{\sum_{j=1}^{k+1}\omega_j\left(\frac{\overline{\beth^{+\succcurlyeq}_{j_e}}}{1-\overline{\beth^{+\succcurlyeq}_{j_e}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ \frac{-1}{1+\left\{\sum_{j=1}^{k+1}\omega_j\left(\frac{1+\left|\overline{\beth^{-\succcurlyeq}_{j_e}}\right|}{\left|\overline{\beth^{-\succcurlyeq}_{2_e}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\right)\right)$$

This shows that equation (2) is held for $m=k+1$.

Definition 13. *Let,* $\Psi_{\boldsymbol{j}}=\left(\left(\underline{\beth^{+\preccurlyeq}_{jm}},\ \underline{\beth^{-\preccurlyeq}_{jm}},\ \overline{\beth^{+\preccurlyeq}_{jm}},\ \overline{\beth^{-\preccurlyeq}_{jm}}\right),\ \left(\underline{\beth^{+\succcurlyeq}_{jm}},\ \underline{\beth^{-\succcurlyeq}_{jm}},\ \overline{\beth^{+\succcurlyeq}_{jm}},\ \overline{\beth^{-\succcurlyeq}_{jm}}\right)\right)$ *be the family of BFDRNs, where* $\boldsymbol{j}=1,\ 2,\ 3,\ \ldots,m$. *Then the BFDRDWG operator is a mapping* $\Psi^{e}\to\Psi$: *such that*

$$\mathrm{BFDRDWG}\left(\Psi_1,\Psi_2,\Psi_3,\ \ldots,\ \Psi_{\mathrm{m}}\right)=\bigotimes_{\boldsymbol{j}=1}^{\mathrm{m}}\left(\Psi_{\boldsymbol{j}}\right)^{\omega_{\boldsymbol{j}}}. \tag{3}$$

Where $\omega=\{\omega_1,\ \omega_2,\ \ldots,\ \omega_{\mathrm{m}}\}$ represent the weight vectors such that $\sum_{\boldsymbol{j}=1}^{\mathrm{m}}\omega_{\boldsymbol{j}}=1$.

Theorem 2. *Let,* $\Psi_{\boldsymbol{j}}=\left(\left(\underline{\beth^{+\preccurlyeq}_{jm}},\ \underline{\beth^{-\preccurlyeq}_{jm}},\ \overline{\beth^{+\preccurlyeq}_{jm}},\ \overline{\beth^{-\preccurlyeq}_{jm}}\right),\ \left(\underline{\beth^{+\succcurlyeq}_{jm}},\ \underline{\beth^{-\succcurlyeq}_{jm}},\ \overline{\beth^{+\succcurlyeq}_{jm}},\ \overline{\beth^{-\succcurlyeq}_{jm}}\right)\right)$ *be the* *family of BFDRNs, where $\boldsymbol{j} = 1,\ 2,\ 3,\ \ldots, m$. Now, we have obtained the accumulated result, and in the results, we can see that we get a BFDRN again.*

$$
\mathrm{BFDRDWG}\left(\Psi_1, \Psi_2, \Psi_3,\ \ldots,\ \Psi_{\mathrm{m}}\right) = \underset{\boldsymbol{j}=1}{\overset{\mathrm{m}}{\otimes}} \left(\Psi_{\boldsymbol{j}}\right)^{\omega_{\boldsymbol{j}}} =
$$

$$
\left(\begin{array}{l}
\left(\begin{array}{l}
\left(\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{m}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\gimel_{\mathrm{j}_{\wp}}^{+\preccurlyeq}}}{\underline{\gimel_{\mathrm{j}_{\wp}}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{m}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\gimel_{\mathrm{j}_{\wp}}^{-\preccurlyeq}}\right|}{1+\left|\underline{\gimel_{\mathrm{j}_{\wp}}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{m}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\gimel_{\mathrm{j}_{\wp}}^{+\preccurlyeq}}}{\overline{\gimel_{\mathrm{j}_{\wp}}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{m}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\gimel_{\mathrm{j}_{\wp}}^{-\preccurlyeq}}\right|}{1+\left|\overline{\gimel_{\mathrm{j}_{\wp}}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right), \\
\left(\begin{array}{l}
\left(\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{m}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\gimel_{\mathrm{j}_{\wp}}^{+\succcurlyeq}}}{\underline{\gimel_{\mathrm{j}_{\wp}}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{m}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\gimel_{\mathrm{j}_{\wp}}^{-\succcurlyeq}}\right|}{1+\left|\underline{\gimel_{\mathrm{j}_{\wp}}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{m}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\gimel_{\mathrm{j}_{\wp}}^{+\succcurlyeq}}}{\overline{\gimel_{\mathrm{j}_{\wp}}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{m}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\gimel_{\mathrm{j}_{\wp}}^{+\succcurlyeq}}\right|}{1+\left|\overline{\gimel_{\mathrm{j}_{\wp}}^{+\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right)
\end{array}\right) \tag{4}
$$

Proof. We prove by using mathematical induction.

Consider $m = 2$, then for the Equation (4),

$$
\mathrm{BFDRDWG}\left(\Psi_1, \Psi_2\right) = \Psi_1 \otimes \Psi_2
$$

$$
= \left(\begin{array}{l}
\left(\begin{array}{l}
\left(\dfrac{1}{1+\left\{\omega_1\left(\frac{1-\underline{\gimel_{1_{\wp}}^{+\preccurlyeq}}}{\underline{\gimel_{1_{\wp}}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\dfrac{1}{1+\left\{\omega_1\left(\frac{\left|\underline{\gimel_{1_{\wp}}^{-\preccurlyeq}}\right|}{1+\left|\underline{\gimel_{1_{\wp}}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\dfrac{1}{1+\left\{\omega_1\left(\frac{1-\overline{\gimel_{1_{\wp}}^{+\preccurlyeq}}}{\overline{\gimel_{1_{\wp}}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\dfrac{1}{1+\left\{\omega_1\left(\frac{\left|\overline{\gimel_{1_{\wp}}^{-\preccurlyeq}}\right|}{1+\left|\overline{\gimel_{1_{\wp}}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right), \\
\left(\begin{array}{l}
\left(\dfrac{1}{1+\left\{\omega_1\left(\frac{1-\underline{\gimel_{1_{\wp}}^{+\succcurlyeq}}}{\underline{\gimel_{1_{\wp}}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\dfrac{1}{1+\left\{\omega_1\left(\frac{\left|\underline{\gimel_{1_{\wp}}^{-\succcurlyeq}}\right|}{1+\left|\underline{\gimel_{1_{\wp}}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\dfrac{1}{1+\left\{\omega_1\left(\frac{1-\overline{\gimel_{1_{\wp}}^{+\succcurlyeq}}}{\overline{\gimel_{1_{\wp}}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\dfrac{1}{1+\left\{\omega_1\left(\frac{\left|\overline{\gimel_{1_{\wp}}^{-\succcurlyeq}}\right|}{1+\left|\overline{\gimel_{1_{\wp}}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right)
\end{array}\right) \otimes
$$

$$\left(\begin{array}{l}\left(\begin{array}{l}\left(\frac{1}{1+\left\{\omega_2\left(\frac{1-\underline{\beth_{2_e}^{+\preccurlyeq}}}{\underline{\beth_{2_e}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_2\left(\frac{\left|\underline{\beth_{2_e}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{2_e}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(\frac{1}{1+\left\{\omega_2\left(\frac{1-\overline{\beth_{2_e}^{+\preccurlyeq}}}{\overline{\beth_{2_e}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_2\left(\frac{\left|\overline{\beth_{2_e}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{2_e}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right), \\ \left(\begin{array}{l}\left(\frac{1}{1+\left\{\omega_2\left(\frac{1-\underline{\beth_{1_e}^{+\succcurlyeq}}}{\underline{\beth_{1_e}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_2\left(\frac{\left|\underline{\beth_{2_e}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{2_e}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(\frac{1}{1+\left\{\omega_2\left(\frac{1-\overline{\beth_{2_e}^{+\succcurlyeq}}}{\overline{\beth_{2_e}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_2\left(\frac{\left|\overline{\beth_{2_e}^{-\succcurlyeq}}\right|}{1+\left|\overline{\beth_{2_e}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right)\end{array}\right)$$

$$\mathrm{BFDRDWG}\left(\Psi_1, \Psi_2\right) =$$

$$\left(\begin{array}{l}\left(\begin{array}{l}\left(\frac{1}{1+\left\{\omega_1\left(\frac{1-\underline{\beth_{1_e}^{+\preccurlyeq}}}{\underline{\beth_{1_e}^{+\preccurlyeq}}}\right)^{\aleph}+\omega_2\left(\frac{1-\underline{\beth_{2_e}^{+\preccurlyeq}}}{\underline{\beth_{2_e}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_1\left(\frac{\left|\underline{\beth_{1_e}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{1_e}^{-\preccurlyeq}}\right|}\right)^{\aleph}+\omega_2\left(\frac{\left|\underline{\beth_{2_e}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{2_e}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(\frac{1}{1+\left\{\omega_1\left(\frac{1-\overline{\beth_{1_e}^{+\preccurlyeq}}}{\overline{\beth_{1_e}^{+\preccurlyeq}}}\right)^{\aleph}+\omega_2\left(\frac{1-\overline{\beth_{2_e}^{+\preccurlyeq}}}{\overline{\beth_{2_e}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_1\left(\frac{\left|\overline{\beth_{1_e}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{1_e}^{-\preccurlyeq}}\right|}\right)^{\aleph}+\omega_2\left(\frac{\left|\overline{\beth_{2_e}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{2_e}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right), \\ \left(\begin{array}{l}\left(\frac{1}{1+\left\{\omega_1\left(\frac{1-\underline{\beth_{1_e}^{+\succcurlyeq}}}{\underline{\beth_{1_e}^{+\succcurlyeq}}}\right)^{\aleph}+\omega_2\left(\frac{1-\underline{\beth_{1_e}^{+\succcurlyeq}}}{\underline{\beth_{1_e}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_1\left(\frac{\left|\underline{\beth_{1_e}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{1_e}^{-\succcurlyeq}}\right|}\right)^{\aleph}+\omega_2\left(\frac{\left|\underline{\beth_{2_e}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{2_e}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(\frac{1}{1+\left\{\omega_1\left(\frac{1-\overline{\beth_{1_e}^{+\succcurlyeq}}}{\overline{\beth_{1_e}^{+\succcurlyeq}}}\right)^{\aleph}+\omega_2\left(\frac{1-\overline{\beth_{2_e}^{+\succcurlyeq}}}{\overline{\beth_{2_e}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_1\left(\frac{\left|\overline{\beth_{1_e}^{-\succcurlyeq}}\right|}{1+\left|\overline{\beth_{1_e}^{-\succcurlyeq}}\right|}\right)^{\aleph}+\omega_2\left(\frac{\left|\overline{\beth_{2_e}^{-\succcurlyeq}}\right|}{1+\left|\overline{\beth_{2_e}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right)\end{array}\right)$$

$$\mathrm{BFDRDWG}\left(\vartheta_1,\vartheta_2\right)=\bigotimes_{\boldsymbol{j}=1}^{2}\left(\Psi_{\boldsymbol{j}}\right)^{\omega_{\boldsymbol{j}}}=$$

$$\left(\begin{array}{c}\left(\begin{array}{c}\left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{2}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\beth_{j\wp}^{+\preccurlyeq}}}{\underline{\beth_{j\wp}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{2}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\beth_{j\wp}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{j\wp}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\ \left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{2}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\beth_{j\wp}^{+\preccurlyeq}}}{\overline{\beth_{j\wp}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{2}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\beth_{j\wp}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{j\wp}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{2}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\beth_{j\wp}^{+\succcurlyeq}}}{\underline{\beth_{j\wp}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{2}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\beth_{j\wp}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{j\wp}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\ \left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{2}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\beth_{j\wp}^{+\succcurlyeq}}}{\overline{\beth_{j\wp}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{2}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\beth_{j\wp}^{+\succcurlyeq}}\right|}{1+\left|\overline{\beth_{j\wp}^{+\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right)\end{array}\right)$$

This shows that Equation (2), is held for $m=2$. Now, assume that (4) also holds for $m=k$.

$$\mathrm{BFDRDWG}\left(\Psi_1,\Psi_2,\Psi_3,\ \ldots,\ \Psi_{\mathrm{k}}\right)=\bigotimes_{\boldsymbol{j}=1}^{\mathrm{k}}\left(\Psi_{\boldsymbol{j}}\right)^{\omega_{\boldsymbol{j}}}=$$

$$\left(\begin{array}{c}\left(\begin{array}{c}\left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\beth_{j\wp}^{+\preccurlyeq}}}{\underline{\beth_{j\wp}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\beth_{j\wp}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{j\wp}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\ \left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\beth_{j\wp}^{+\preccurlyeq}}}{\overline{\beth_{j\wp}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\beth_{j\wp}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{j\wp}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right),\\ \left(\begin{array}{c}\left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\beth_{j\wp}^{+\succcurlyeq}}}{\underline{\beth_{j\wp}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\beth_{j\wp}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{j\wp}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\ \left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\beth_{j\wp}^{+\succcurlyeq}}}{\overline{\beth_{j\wp}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}},\ -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\beth_{j\wp}^{+\succcurlyeq}}\right|}{1+\left|\overline{\beth_{j\wp}^{+\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right)\end{array}\right)$$

Now, we will show that (4) is held for $m=k+1$,

$$\mathrm{BFDRDWG}\left(\Psi_1, \Psi_2, \Psi_3, \ldots, \Psi_{\mathrm{k}+1}\right) = \bigotimes_{\boldsymbol{j}=1}^{\mathrm{k}+1} \left(\Psi_{\boldsymbol{j}}\right)^{\omega_{\boldsymbol{j}}} =$$

$$\left(\begin{array}{c}
\left(\begin{array}{c}
\left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\beth_{j_o}^{+\preccurlyeq}}}{\underline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\beth_{j_o}^{+\preccurlyeq}}}{\overline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\beth_{j_o}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right), \\
\left(\begin{array}{c}
\left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\beth_{j_o}^{+\succcurlyeq}}}{\underline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\beth_{j_o}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{j_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\beth_{j_o}^{+\succcurlyeq}}}{\overline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\beth_{j_o}^{+\succcurlyeq}}\right|}{1+\left|\overline{\beth_{j_o}^{+\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right)
\end{array}\right) \otimes$$

$$\left(\begin{array}{c}
\left(\begin{array}{c}
\left(\frac{1}{1+\left\{\omega_{k+1}\left(\frac{1-\underline{\beth_{(k+1)_o}^{+\preccurlyeq}}}{\underline{\beth_{(k+1)_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\left|\underline{\beth_{(k+1)_o}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{(k+1)_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\frac{1}{1+\left\{\omega_{k+1}\left(\frac{1-\overline{\beth_{(k+1)_o}^{+\preccurlyeq}}}{\overline{\beth_{(k+1)_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\left|\overline{\beth_{(k+1)_o}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{(k+1)_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right), \\
\left(\begin{array}{c}
\left(\frac{1}{1+\left\{\omega_{k+1}\left(\frac{1-\underline{\beth_{(k+1)_o}^{+\succcurlyeq}}}{\underline{\beth_{(k+1)_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\left|\underline{\beth_{(k+1)_o}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{(k+1)_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\
\left(\frac{1}{1+\left\{\omega_{k+1}\left(\frac{1-\overline{\beth_{(k+1)_o}^{+\succcurlyeq}}}{\overline{\beth_{(k+1)_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\omega_{k+1}\left(\frac{\left|\overline{\beth_{(k+1)_o}^{-\succcurlyeq}}\right|}{1+\left|\overline{\beth_{(k+1)_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right)
\end{array}\right)$$

$$\mathrm{BFDRDWG}\left(\Psi_1, \Psi_2, \Psi_3, \ldots, \Psi_{\mathrm{k}+1}\right) = \underset{\boldsymbol{j}=1}{\overset{\mathrm{k}+1}{\otimes}} \left(\Psi_{\boldsymbol{j}}\right)^{\omega_{\boldsymbol{j}}} =$$

$$\left(\begin{array}{c}\left(\begin{array}{c}\left(\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k+1}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\beth_{\mathrm{j}\mathscr{o}}^{+\preccurlyeq}}}{\underline{\beth_{\mathrm{j}\mathscr{o}}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k+1}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\beth_{\mathrm{j}\mathscr{o}}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{\mathrm{j}\mathscr{o}}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k+1}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\beth_{\mathrm{j}\mathscr{o}}^{+\preccurlyeq}}}{\overline{\beth_{\mathrm{j}\mathscr{o}}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k+1}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\beth_{\mathrm{j}\mathscr{o}}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{\mathrm{j}\mathscr{o}}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right), \\ \left(\begin{array}{c}\left(\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k+1}\omega_{\boldsymbol{j}}\left(\frac{1-\underline{\beth_{\mathrm{j}\mathscr{o}}^{+\succcurlyeq}}}{\underline{\beth_{\mathrm{j}\mathscr{o}}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k+1}\omega_{\boldsymbol{j}}\left(\frac{\left|\underline{\beth_{\mathrm{j}\mathscr{o}}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{\mathrm{j}\mathscr{o}}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right), \\ \left(\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k+1}\omega_{\boldsymbol{j}}\left(\frac{1-\overline{\beth_{\mathrm{j}\mathscr{o}}^{+\succcurlyeq}}}{\overline{\beth_{\mathrm{j}\mathscr{o}}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\dfrac{1}{1+\left\{\sum_{\boldsymbol{j}=1}^{k+1}\omega_{\boldsymbol{j}}\left(\frac{\left|\overline{\beth_{\mathrm{j}\mathscr{o}}^{+\succcurlyeq}}\right|}{1+\left|\overline{\beth_{\mathrm{j}\mathscr{o}}^{+\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)\end{array}\right)\end{array}\right)$$

Hence, prove that Equation (4) is held for $m = k + 1$, this implies that it is true for every m.

6. Bipolar fuzzy dominance rough WASPAS approach

To handle the MCDM techniques, many DM tools have been developed. In this section, we have defined the WASPAS technique.

Assume that $\mathit{\Lambda}_L = \{\mathit{\Lambda}_1, \mathit{\Lambda}_2, \mathit{\Lambda}_3, \ldots, \mathit{\Lambda}_m\}$ denote the set of 'm' alternatives and $\mathit{\Lambda}_T = \{\mathit{\Lambda}_{T1}, \mathit{\Lambda}_{T2}, \mathit{\Lambda}_{T3}, \ldots, \mathit{\Lambda}_{T\mathscr{o}}\}$ denote the set of'$\mathscr{o}$' attributes. Let $\{\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, \ldots, \mathrm{e}_\hbar\}$ be the set of $\hbar$ experts for each alternative $\mathit{\Lambda}_\mathrm{i}$ ($i = 1, 2, 3, \ldots, m$) against attributes $\mathit{\Lambda}_{T\boldsymbol{j}}$ ($\boldsymbol{j} =$ 1, 2, 3, .., $\mathscr{o}$). Let $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_\mathscr{o})^\mathrm{T}$ be the WVs for attributes $\mathit{\Lambda}_{T\boldsymbol{j}}$ and $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3, \ldots, \upsilon_\hbar)^\mathrm{T}$ be the weight vectors for experts e_z ($z = 1, 2, 3, \ldots, \hbar$) such that $\sum_{\boldsymbol{j}=1}^{\mathscr{o}} \omega = 1$ and $\sum_{z=1}^{\hbar} \upsilon_z = 1$. The algorithm of the WASPAS is given by :

Step 1: Assume that the decision analyst provides their assessment in the form of BFDRNs corresponding to each attribute.

$$M_{r\times\mathscr{o}} = \left[\begin{array}{cccc}
\left(\begin{array}{c}\left(\underline{\beth_{(11)\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(11)\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(11)\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(11)\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(11)\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(11)\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(11)\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(11)\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right) & \left(\begin{array}{c}\left(\underline{\beth_{(12)\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(12)\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(12)\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(12)\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(12)\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(12)\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(12)\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(12)\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right) & \cdots & \left(\begin{array}{c}\left(\underline{\beth_{(1\mathscr{o})\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(1\mathscr{o})\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(1\mathscr{o})\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(1\mathscr{o})\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(1\mathscr{o})\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(1\mathscr{o})\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(1\mathscr{o})\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(1\mathscr{o})\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right) \\
\left(\begin{array}{c}\left(\underline{\beth_{(21)\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(21)\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(21)\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(21)\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(21)\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(21)\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(21)\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(21)\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right) & \left(\begin{array}{c}\left(\underline{\beth_{(22)\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(22)\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(22)\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(22)\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(22)\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(22)\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(22)\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(22)\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right) & \cdots & \left(\begin{array}{c}\left(\underline{\beth_{(2\mathscr{o})\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(2\mathscr{o})\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(2\mathscr{o})\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(2\mathscr{o})\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(2\mathscr{o})\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(2\mathscr{o})\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(2\mathscr{o})\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(2\mathscr{o})\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\left(\begin{array}{c}\left(\underline{\beth_{(r1)\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(r1)\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(r1)\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(r1)\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(r1)\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(r1)\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(r1)\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(r1)\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right) & \left(\begin{array}{c}\left(\underline{\beth_{(r2)\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(r2)\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(r2)\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(r2)\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(r2)\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(r2)\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(r2)\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(r2)\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right) & \cdots & \left(\begin{array}{c}\left(\underline{\beth_{(r\mathscr{o})\mathrm{m}}^{+\preccurlyeq}}, \underline{\beth_{(r\mathscr{o})\mathrm{m}}^{-\preccurlyeq}}, \overline{\beth_{(r\mathscr{o})\mathrm{m}}^{+\preccurlyeq}}, \overline{\beth_{(r\mathscr{o})\mathrm{m}}^{-\preccurlyeq}}\right), \\ \left(\underline{\beth_{(r\mathscr{o})\mathrm{m}}^{+\succcurlyeq}}, \underline{\beth_{(r\mathscr{o})\mathrm{m}}^{-\succcurlyeq}}, \overline{\beth_{(r\mathscr{o})\mathrm{m}}^{+\succcurlyeq}}, \overline{\beth_{(r\mathscr{o})\mathrm{m}}^{-\succcurlyeq}}\right)\end{array}\right)
\end{array}\right].$$

Step 2: Normalize the data;

Step 3: Compute the weighted sum (WS) by using the formula given below:

$$
\mu_i^{WS} = \left(\begin{array}{c}
\left(\begin{array}{c}
\left(1-\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{\underline{\beth_{j_o}^{+\preccurlyeq}}}{1-\underline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, \frac{-1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{1+\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}{\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\
\left(1-\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{\overline{\beth_{j_o}^{+\preccurlyeq}}}{1-\overline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, \frac{-1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{1+\left|\overline{\beth_{j_o}^{-\preccurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right),\\
\left(\begin{array}{c}
\left(1-\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{\underline{\beth_{j_o}^{+\succcurlyeq}}}{1-\underline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, \frac{-1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{1+\left|\underline{\beth_{j_o}^{-\succcurlyeq}}\right|}{\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\
\left(1-\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{\overline{\beth_{j_o}^{+\succcurlyeq}}}{1-\overline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, \frac{-1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{1+\left|\overline{\beth_{j_o}^{-\succcurlyeq}}\right|}{\left|\overline{\beth_{2_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right)
\end{array}\right)
$$

Step 4: Computing the weighted product (WP) by using the formula given below:

$$
\mu_i^{WP} = \left(\begin{array}{c}
\left(\begin{array}{c}
\left(\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{1-\underline{\beth_{j_o}^{+\preccurlyeq}}}{\underline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}{1+\left|\underline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\
\left(\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{1-\overline{\beth_{j_o}^{+\preccurlyeq}}}{\overline{\beth_{j_o}^{+\preccurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{\left|\overline{\beth_{j_o}^{-\preccurlyeq}}\right|}{1+\left|\overline{\beth_{j_o}^{-\preccurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right),\\
\left(\begin{array}{c}
\left(\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{1-\underline{\beth_{j_o}^{+\succcurlyeq}}}{\underline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{\left|\underline{\beth_{j_o}^{-\succcurlyeq}}\right|}{1+\left|\underline{\beth_{j_o}^{-\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right),\\
\left(\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{1-\overline{\beth_{j_o}^{+\succcurlyeq}}}{\overline{\beth_{j_o}^{+\succcurlyeq}}}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}, -1+\frac{1}{1+\left\{\sum_{j=1}^{m}\omega_j\left(\frac{\left|\overline{\beth_{j_o}^{+\succcurlyeq}}\right|}{1+\left|\overline{\beth_{j_o}^{+\succcurlyeq}}\right|}\right)^{\aleph}\right\}^{\frac{1}{\aleph}}}\right)
\end{array}\right)
\end{array}\right)
$$

Step 5: Finally, use WASPAS techniques for ordering alternatives as given by

$$
\mathrm{B}_i = \frac{\mathrm{B}_i^{WS} + \mathrm{B}_i^{WP}}{2};
$$

Step 6: Utilize the definition of score value to get score values to assess the best alternative.

All of the above algorithms are given in **Figure 3**.

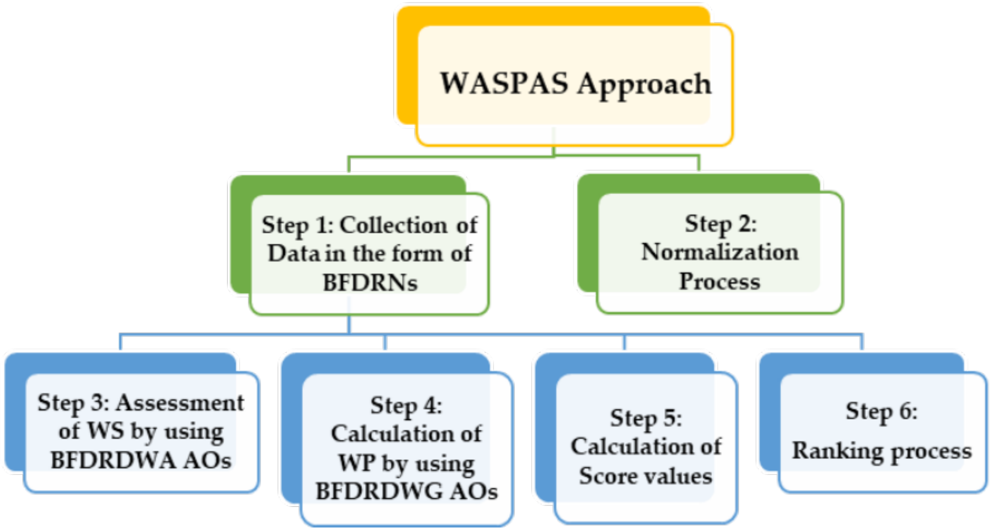

Figure 3. Flow diagram of the WASPAS technique.

7. Case study

To cope with increasing security challenges and the increase in air threats, the country has embarked upon an overall modernization of its defense infrastructure. A prime element of this modernization drive is enhancing air surveillance and threat detection capabilities. In pursuit of this goal, the government seeks to introduce an AI-based radar system that will be able to effectively detect, identify, and follow different airborne targets, such as planes, drones, and missiles, in real time. Such radar systems are required to demonstrate effective performance under different operational environments, including weather and day/night conditions, using signal processing methods and artificial intelligence to improve detection and identification quality.

Suppose we have four radar systems given by

1) Radar System Λ_1;
2) Radar System Λ_2;
3) Radar System Λ_3;
4) Radar System Λ_4.

The information about these radar systems is given in **Table 1**.

Table 1. Description of different radar systems.

Radar system	Description
Λ_1	Radar System Λ_1 is a state-of-the-art, AI-driven signal-free radar system, hence almost invisible and highly beneficial for stealth surveillance. The system uses ambient electromagnetic radiation such as FM radio, DAB, and DVB-T signals to detect and track airborne targets, including low-observable drones. AI and advanced signal processing algorithms are used to process reflections and filter out real targets from clutter. TwinVis provides real-time situational awareness in 3D tracking. Its passive design makes it possible to operate in electromagnetic-sensitive environments without producing interference. Several drones can be detected in parallel by the radar, including in mountainous or urban environments. The radar is modular and scalable in design, suitable for fixed and mobile installations. It is an economical, stealth system for persistent drone and airspace surveillance.

Table 1. *Cont.*

Radar system	Description
Λ_2	Radar System Λ_2 is an active electronically scanned array (AESA) radar for long-range air and missile defense. Passive radars are not comparable to it, as they don't send out signals; instead, they actively send out signals and can identify small, high-speed targets like micro-drones at ranges over 2000 km for ballistic missiles and a few hundred kilometers for air targets. It employs AI-driven adaptive beamforming and in-real-time classification of signals to precisely track multiple threats. SMART-L MMR can, at the same time, perform volume search, high-altitude scanning, and detection of low-altitude drones. It is designed for optimization on naval and ground platforms and can be easily integrated into defense networks. The software-defined architecture of the radar facilitates AI updates in real-time and mission-specific adaptation. Its ability to track thousands of objects in real-time makes it highly effective in responding to attacks by coordinated swarms of drones.
Λ_3	Radar System Λ_3 is unique with its solid-state platform, small size, and hemispheric coverage, optimized for detecting short- to medium-range threats, such as small drones. Compared to long-range or passive systems, MHR is optimized for tactical mobility and is installed on a vehicle or forms part of mobile defense units. It uses AESA technology with AI-based algorithms for real-time 360-degree surveillance, target classification, and threat prioritization. It operates at a high update rate and low latency for detecting low-altitude, high-speed drones in dynamic battle scenarios. It provides cueing for electro-optic sensors and countermeasure systems. RADA radars are modular, enabling multi-radar configurations that enhance tracking accuracy and target discrimination. Its AI-based filtering enables it to discriminate between drones, birds, and ground clutter with high reliability, even in cluttered environments.
Λ_4	Radar System Λ_4 is an end-to-end, AI-driven solution that detects, tracks, classifies, identifies, and neutralizes hostile unmanned aerial systems, including drone swarms. Not compliant with any of the traditional radar system architectures, Silent Archer integrates multiple sensor modalities, air surveillance radar, electronic warfare (EW) systems, direction-finding units, and electro-optical/infrared (EO/IR) cameras into a fused, modular architecture. Sensor fusion enables the system to provide real-time situational awareness and threat assessment. Its open, sensor-agnostic architecture facilitates seamless integration with a range of command and control systems and kinetic weapon platforms to achieve multiple-layered defense. Highlight-worthy is the fact that Silent Archer enables flexible deployment configurations, such as fixed-site installation, mobile expeditionary units, and fly-away kits, to provide flexibility in diverse operational environments.

Based on this information, **Table 2 s**hows that systems have the following four characteristics as attributes.

Table 2. Characteristics as attributes of each radar system.

Symbols	Criteria	Explanation
Λ_{T_1}	Range	Range is the highest distance a radar system can detect and resolve a target. Range is based on transmitter power, antenna size, frequency, and environmental conditions. A greater range enables early detection and better preparation against impending threats. Intelligent algorithms in AI-based radar optimize signal processing to increase effective range. High-range systems are most appropriate for defense and surveillance.
Λ_{T_2}	Resolution	Resolution is the ability of the radar to distinguish between two closely located objects. It could be range resolution (distance between them) or angular resolution (angle between them). Higher resolution provides improved, higher-definition target imagery, which is important in dense environments. AI enhances resolution with adaptive filtering and advanced signal interpretation. This generates more precise target discrimination and fewer false alarms.
Λ_{T_3}	Accuracy	Accuracy is the extent to which the measured data of the radar (e.g., position, speed, or direction) coincides with actual values. High accuracy is needed in tracking rapidly moving or tiny targets, such as drones. Accuracy is determined by system calibration, signal purity, and algorithms for processing. AI contributes to enhancing accuracy through learning from previous mistakes and adjusting to noise or interference. Precise radars are essential in precise targeting or navigation applications.

Table 2. *Cont.*

Symbols	Criteria	Explanation
Λ_{T_4}	Detection Capability	Detection ability is the radar's effectiveness in detecting and identifying different objects in adverse conditions. High detection ability indicates the radar's ability to work effectively in adverse weather, low visibility, or electromagnetic interference. AI methods improve detection by learning patterns, filtering noise, and adapting sensitivity in real-time. This allows the radar to detect low-signature or stealthy objects such as small drones. High detection ability provides a robust and stable threat surveillance system.

Suppose that these attributes have a weight vector (0.13, 0.28, 0.23, 0.36). N ow we use the step-wise algorithm. The overall discussion is given by :

Step 1: In **Table 3**, collect data from experts in the model of BFDRN.

Table 3. Expert assessment values in the model of BFDRNs.

	Λ_{T_1}	Λ_{T_2}	Λ_{T_3}	Λ_{T_4}
Λ_1	$\left(\begin{pmatrix}(0.38,\ -0.25),\\(0.58,\ -0.85)\end{pmatrix},\begin{pmatrix}(0.56,\ -0.11),\\(0.44,\ -0.24)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.45,\ -0.41),\\(0.74,\ -0.45)\end{pmatrix},\begin{pmatrix}(0.75,\ -0.23),\\(0.85,\ -0.77)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.12,\ -0.32),\\(0.52,\ -0.16)\end{pmatrix},\begin{pmatrix}(0.56,\ -0.42),\\(0.22,\ -0.74)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.32,\ -0.65),\\(0.29,\ -0.56)\end{pmatrix},\begin{pmatrix}(0.49,\ -0.11),\\(0.44,\ -0.24)\end{pmatrix}\right)$
Λ_2	$\left(\begin{pmatrix}(0.22,\ -0.52),\\(0.57,\ -0.45)\end{pmatrix},\begin{pmatrix}(0.66,\ -0.31),\\(0.48,\ -0.27)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.39,\ -0.64),\\(0.73,\ -0.37)\end{pmatrix},\begin{pmatrix}(0.26,\ -0.82),\\(0.12,\ -0.46)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.42,\ -0.23),\\(0.75,\ -0.41)\end{pmatrix},\begin{pmatrix}(0.30,\ -0.11),\\(0.32,\ -0.12)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.55,\ -0.52),\\(0.73,\ -0.85)\end{pmatrix},\begin{pmatrix}(0.65,\ -0.11),\\(0.21,\ -0.64)\end{pmatrix}\right)$
Λ_3	$\left(\begin{pmatrix}(0.18,\ -0.33),\\(0.19,\ -0.74)\end{pmatrix},\begin{pmatrix}(0.16,\ -0.41),\\(0.23,\ -0.51)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.33,\ -0.37),\\(0.25,\ -0.19)\end{pmatrix},\begin{pmatrix}(0.36,\ -0.91),\\(0.47,\ -0.73)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.38,\ -0.62),\\(0.98,\ -0.85)\end{pmatrix},\begin{pmatrix}(0.99,\ -0.94),\\(0.97,\ -0.75)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.67,\ -0.31),\\(0.69,\ -0.52)\end{pmatrix},\begin{pmatrix}(0.51,\ -0.19),\\(0.96,\ -0.84)\end{pmatrix}\right)$
Λ_4	$\left(\begin{pmatrix}(0.21,\ -0.22),\\(0.54,\ -0.90)\end{pmatrix},\begin{pmatrix}(0.78,\ -0.43),\\(0.95,\ -0.20)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.58,\ -0.76),\\(0.69,\ -0.75)\end{pmatrix},\begin{pmatrix}(0.74,\ -0.84),\\(0.41,\ -0.95)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.87,\ -0.52),\\(0.88,\ -0.45)\end{pmatrix},\begin{pmatrix}(0.86,\ -0.75),\\(0.85,\ -0.52)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.82,\ -0.61),\\(0.96,\ -0.21)\end{pmatrix},\begin{pmatrix}(0.34,\ -0.31),\\(0.69,\ -0.64)\end{pmatrix}\right)$

Step 2: All the data is benefit type, so there is no need to normalize the data.

Step 3: Calculation of weighted sum values B_i^{WS} is given in **Table 4**.

Table 4. Weighted Sum (*WS*) values for each Λ_{T_i}.

	Λ_{T_1}	Λ_{T_2}	Λ_{T_3}	Λ_{T_4}
BFDRDWA	$\left(\begin{pmatrix}(0.36,\ -0.36),\\(0.62,\ -0.34)\end{pmatrix},\begin{pmatrix}(0.64,\ -0.21),\\(0.75,\ -0.34)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.46,\ -0.37),\\(0.72,\ -0.39)\end{pmatrix},\begin{pmatrix}(0.57,\ -0.21),\\(0.30,\ -0.30)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.56,\ -0.35),\\(0.95,\ -0.35)\end{pmatrix},\begin{pmatrix}(0.97,\ -0.34),\\(0.95,\ -0.45)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.81,\ -0.40),\\(0.93,\ -0.34)\end{pmatrix},\begin{pmatrix}(0.78,\ -0.39),\\(0.88,\ -0.40)\end{pmatrix}\right)$

Step 4: Evaluation of weighted product values B_i^{WP} is given in **Table 5**.

Table 5. Weighted Product (*WP*) values for each Λ_{T_i}.

	Λ_{T_1}	Λ_{T_2}	Λ_{T_3}	Λ_{T_4}
BFDRDWG	$\left(\begin{pmatrix}(0.20,\ -0.13),\\(0.38,\ -0.13)\end{pmatrix},\begin{pmatrix}(0.55,\ -0.07),\\(0.33,\ -0.13)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.36,\ -0.13),\\(0.69,\ -0.14)\end{pmatrix},\begin{pmatrix}(0.34,\ -0.09),\\(0.21,\ -0.12)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.31,\ -0.12),\\(0.30,\ -0.14)\end{pmatrix},\begin{pmatrix}(0.31,\ -0.14),\\(0.44,\ -0.17)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.41,\ -0.15),\\(0.71,\ -0.13)\end{pmatrix},\begin{pmatrix}(0.45,\ -0.14),\\(0.64,\ -0.15)\end{pmatrix}\right)$

Step 5: Finally, we use the WASPAS method and find the unique bipolar fuzzy dominance rough value of each alternative, and the values are shown in **Table 6**:

$$\mathrm{B}_i = \frac{\mathrm{B}_i{}^{WS} + \mathrm{B}_i{}^{WP}}{2}.$$

Table 6. Bipolar fuzzy dominance rough values for each alternative Λ_{T_i}.

	$\mathbf{B}_i(\mathbf{\Lambda}_{T_1})$	$\mathbf{B}_i(\mathbf{\Lambda}_{T_2})$	$\mathbf{B}_i(\mathbf{\Lambda}_{T_3})$	$\mathbf{B}_i(\mathbf{\Lambda}_{T_4})$
$\mathbf{B}_i$	$\left(\left\langle\begin{matrix}(0.0093,\ -0.0111),\\(0.3981,\ -0.0110)\end{matrix}\right\rangle \left\langle\begin{matrix}(0.6087,\ -0.0044),\\(0.8512,\ -0.0114)\end{matrix}\right\rangle\right)$	$\left(\left\langle\begin{matrix}(0.0722,\ -0.0118),\\(0.9046,\ -0.0133)\end{matrix}\right\rangle \left\langle\begin{matrix}(0.2131,\ -0.0062),\\(0.0045,\ -0.0096)\end{matrix}\right\rangle\right)$	$\left(\left\langle\begin{matrix}(0.1784,\ -0.0099),\\(0.9999,\ -0.0121)\end{matrix}\right\rangle \left\langle\begin{matrix}(0.9999,\ -0.0125),\\(0.9999,\ -0.0173)\end{matrix}\right\rangle\right)$	$\left(\left\langle\begin{matrix}(0.9567,\ -0.0139),\\(0.9996,\ -0.0114)\end{matrix}\right\rangle \left\langle\begin{matrix}(0.9180,\ -0.0134),\\(0.9954,\ -0.0147)\end{matrix}\right\rangle\right)$

Step 6: By using Definition (11), we calculate the score values and assess the best alternative.

$$\mathrm{S}\left(\mathrm{B}\left(\Lambda_1\right)\right) = \frac{1}{8}(4 + 0.0093 - 0.0111 + 0.3981 - 0.0110 + 0.6087 - 0.0044 + 0.8512 - 0.0114) = 0.7286;$$

$$\mathrm{S}\left(\mathrm{B}\left(\Lambda_2\right)\right) = \frac{1}{8}(4 + 0.0722 - 0.0118 + 0.9046 - 0.0133 + 0.2131 - 0.0062 + 0.0045 - 0.0096) = 0.6441;$$

$$\mathrm{S}\left(\mathrm{B}\left(\Lambda_3\right)\right) = \frac{1}{8}(4 + 0.1784 - 0.0099 + 0.9999 - 0.0121 + 0.9999 - 0.0125 + 0.9999 - 0.0173) = 0.8907;$$

$$\mathrm{S}\left(\mathrm{B}\left(\Lambda_4\right)\right) = \frac{1}{8}(4 + 0.9567 - 0.0139 + 0.9996 - 0.0114 + 0.9180 - 0.0134 + 0.9954 - 0.0147) = 0.9770.$$

Score values and ranking of each alternative are shown in **Table 7**.

Table 7. Score values and ranking.

$\mathbf{S}\left(\mathbf{B}\left(\Lambda_i\right)\right)$	**Score values**
$\mathbf{S}\left(\mathbf{B}\left(\Lambda_1\right)\right)$	0.7286
$\mathbf{S}\left(\mathbf{B}\left(\Lambda_2\right)\right)$	0.6441
$\mathbf{S}\left(\mathbf{B}\left(\Lambda_3\right)\right)$	0.8907
$\mathbf{S}\left(\mathbf{B}\left(\Lambda_4\right)\right)$	0.9770
Ranking	$\mathrm{S}\left(\mathrm{B}\left(\Lambda_4\right)\right) > \mathrm{S}\left(\mathrm{B}\left(\Lambda_3\right)\right) > \mathrm{S}\left(\mathrm{B}\left(\Lambda_1\right)\right) > \mathrm{S}\left(\mathrm{B}\left(\Lambda_2\right)\right)$

Results show that alternative Λ_4 has the highest score, 0.9770, and is ranked first because the alternative has a higher and stable score on such critical factors as reliability and technical efficiency that are fundamental in defense systems. Alternative Λ_3 scored 0.8907 and ranks second and is strong in terms of its operational ability, but is not as well balanced as Λ_4. Alternative $\Lambda_1 = 0.7286$ is ranked the third option and has moderate performance and a fair trade-off between cost and effectiveness. On the other hand, $\Lambda_2 = 0.6441$ is the lowest ranking because of poor performance in major attributes. In general, the ranking shows the importance of trade-offs among several criteria and outlines practical trade-offs, which is useful for making informed decisions in defense applications.

Sensitivity analysis

Sensitivity analysis plays an important role in validating the stability of the proposed decision-making framework. In this study, a sensitivity analysis examines the impact of variations in the criteria weights on the final ranking of AI radar systems obtained using the Bipolar Fuzzy Dominance Rough WASPAS methodology. Different sets of weight combinations are considered by systematically altering the importance of each criterion while keeping the overall structure of the model unchanged. The results of the analysis demonstrate that although minor changes occur in the ranking positions under varying weight scenarios, the overall ranking pattern remains consistent. This indicates that the proposed method is robust and less sensitive to fluctuations in criteria weights. Moreover, the most preferred alternative maintains its top position across the majority of test cases, which confirms the stability and effectiveness of the model in handling uncertainty and bipolar fuzzy information. Let the new set of criteria weights be $w_s = (0.2,\ 0.31,\ 0.26,\ 0.23)$. In **Table 8,** bipolar fuzzy dominance rough values for each alterative $\varLambda_{T_i}$ are calculated.

Table 8. Bipolar fuzzy dominance rough values of alterative $\varLambda_{T_i}$ for new weights.

	$\mathbf{B}_i(\mathbf{\Lambda}_{T_1})$	$\mathbf{B}_i(\mathbf{\Lambda}_{T_2})$	$\mathbf{B}_i(\mathbf{\Lambda}_{T_3})$	$\mathbf{B}_i(\mathbf{\Lambda}_{T_4})$
$\mathbf{B}_i$	$\left(\begin{pmatrix}(0.009,\ -0.010),\\(0.475,\ -0.010)\end{pmatrix}\begin{pmatrix}(0.654,\ -0.004),\\(0.875,\ -0.012)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.045,\ -0.011),\\(0.887,\ -0.012)\end{pmatrix}\begin{pmatrix}(0.180,\ -0.007),\\(0.006,\ -0.008)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.094,\ -0.010),\\(0.999,\ -0.012)\end{pmatrix}\begin{pmatrix}(0.999,\ -0.014),\\(0.999,\ -0.016)\end{pmatrix}\right)$	$\left(\begin{pmatrix}(0.949,\ -0.013),\\(0.998,\ -0.013)\end{pmatrix}\begin{pmatrix}(0.941,\ -0.014),\\(0.997,\ -0.013)\end{pmatrix}\right)$

Step 6: By using Definition (11), we calculate the score values and assess the best alternative.

$$\mathrm{S}\left(\mathrm{B}\left(\varLambda_1\right)\right) = \frac{1}{8}(4 + 0.009 - 0.010 + 0.475 - 0.010 + 0.654 - 0.004 + 0.875 - 0.012) = 0.7472;$$

$$\mathrm{S}\left(\mathrm{B}\left(\varLambda_2\right)\right) = \frac{1}{8}(4 + 0.045 - 0.011 + 0.887 - 0.012 + 0.180 - 0.007 + 0.006 - 0.008) = 0.6351;$$

$$\mathrm{S}\left(\mathrm{B}\left(\varLambda_3\right)\right) = \frac{1}{8}(4 + 0.094 - 0.010 + 0.999 - 0.012 + 0.999 - 0.014 + 0.999 - 0.016) = 0.8801;$$

$$\mathrm{S}\left(\mathrm{B}\left(\varLambda_4\right)\right) = \frac{1}{8}(4 + 0.949 - 0.013 + 0.998 - 0.013 + 0.941 - 0.014 + 0.997 - 0.013) = 0.9791.$$

Table 9 shows the score values and ranking of each alternative.

Table 9. Score values and ranking using weights $\varLambda_i$.

S(B($\varLambda_i$))	**Score values**
S(B($\varLambda_1$))	0.7472
S(B($\varLambda_2$))	0.6351
S(B($\varLambda_3$))	0.8801
S(B($\varLambda_4$))	0.9791
Ranking	S(B($\varLambda_4$)) $>$ S(B($\varLambda_3$)) $>$ S(B($\varLambda_1$)) $>$ S(B($\varLambda_2$))

Hence, the sensitivity analysis verifies that the proposed bipolar fuzzy dominance rough WASPAS approach provides reliable and consistent results, making it a suitable tool for the evaluation and assessment of AI radar systems in uncertain decision environments.

8. Comparative analysis

In this section, we compare the proposed bipolar fuzzy dominance rough WASPAS (BFDR-WASPAS) method with existing fuzzy WASPAS-based approaches to highlight its effectiveness and robustness. To prove the effectiveness and excellence of the suggested bipolar fuzzy dominance rough set BFDRS-WASPAS framework in uncertainty and decision-making complexity management, it is compared with several existing studies. In the work by Li et al. [43], a spherical cubic fuzzy WASPAS method is proposed that makes use of spherical cubic fuzzy data to reflect the level of uncertainty. Dominance relationships and crude estimates are not taken into account, however. On the other hand, the suggested BFDRS–WASPAS model combines dominance-based rough sets with bipolar fuzzy data to provide better alternative discrimination. Cheng [44] deals with uncertainty in decision-making by generalizing the WASPAS model with spherical fuzzy sets. This is an effective way to model vagueness, but it cannot simultaneously model positive as well as negative desires. Even though effective in depicting vagueness, this method does not have the ability to reflect both positive and negative preferences at the same time. The proposed model overcomes this limitation by integrating dominance rough sets with bipolar fuzzy structures to generate more comprehensive and precise assessments. To achieve sustainable selection of suppliers, Singh et al. [45] use dominance-based rough set analysis, which is effective in capturing preference ordering. However, the ability of their model to deal with two aspects of uncertainty is constrained by the lack of bipolar fuzzy information. This can be improved with the proposed BFDRSWASP approach, which integrates dominance rough sets and bipolar fuzzy representation, resulting in stronger and more adaptable decision results. Shanthi and Preethi [46] are using a bipolar fuzzy WASPAS methodology to examine electrochemical machining processes. Bipolar fuzzy rough sets (BFRS) are an important advancement of FS, FRS, and BFS, incorporating negative membership values as well as positive membership values with the notion of lower and upper approximations. This model is particularly good at dealing with the uncertainty and gaps that are commonly present in rough data, and providing an analysis that is much deeper and more refined than can be obtained with BFS alone. Despite these advantages, BFRS is not good enough at managing the data related to dominance relations, which is essential in most real-world decision-making problems. To address this, our solution is based on the premises of BFRS but with the ability to model dominance-based situations. This not only makes our structure more flexible, but also more powerful. It is interesting to note that, when we leave out the dominance aspect in our model, our model will automatically reduce to the conventional BFRS, and this makes our model general and inclusive. Tufail and Shabir [41] develop a WASPAS method that uses rough approximations to partially address uncertainty based on bipolar fuzzy covering rough sets. Their approach, however, does not provide dominance-based analysis, which is a key to structured preference modeling. The proposed BFDRS-WASPAS architecture integrates dominance relations, which allow prioritizing alternatives to a greater extent and achieve more coherent outcomes of decision-making. All in all, the suggested model is superior to current methods and provides more reliable decision-making results under complicated circumstances due to the ability to impose bipolar ambiguity, dominance relations, and approximations. The comparative framework is given in **Table 10**.

Table 10. Comparative analysis of overall work.

Theories-WASPAS	$S(B(\Lambda_1))$	$S(B(\Lambda_2))$	$S(B(\Lambda_3))$	$S(B(\Lambda_4))$	Ranking
FS-WASPAS [43,44]	×	×	×	×	×
DRS-WASPAS [45]	×	×	×	×	×
BFS-WASPAS [46]	×	×	×	×	×
BFRS-WASPAS [41]	×	×	×	×	×
BFDRDWA-WASPAS	0.6397	0.5975	0.7428	0.7332	$\Lambda_3 > \Lambda_4 > \Lambda_1 > \Lambda_2$
BFDRDWG-WASPAS	0.6260	0.6397	0.6005	0.7041	$\Lambda_4 > \Lambda_2 > \Lambda_1 > \Lambda_3$
BFDR-WASPAS	0.9770	0.8907	0.7286	0.6441	$\Lambda_4 > \Lambda_3 > \Lambda_1 > \Lambda_2$

Figure 4 shows the graphical representations of our comparison.

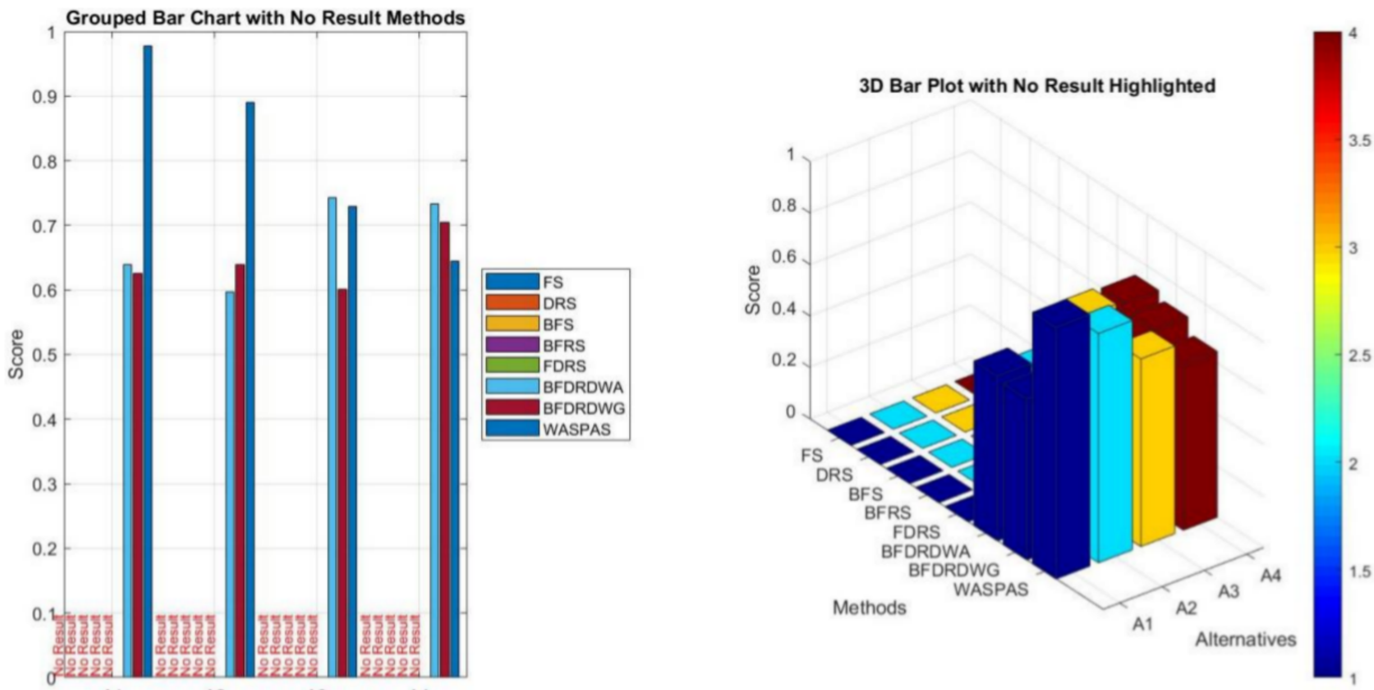

Figure 4. The graphical representation of the ranking of different frameworks.

9. Conclusion

A new model of bipolar fuzzy dominance rough dombi WASPAS (BFDRDWA and BFDRDWGA) was made in this work to evaluate and select AI-based radar systems. The suggested model successfully incorporates the use of bipolar fuzzy information and dominance rough approximations and Dombi aggregation, allowing it to treat uncertainty and conflicting criteria more thoroughly. The findings also show that in the BFDRDWA operator, alternative Λ_3 is the best choice, and in the BFDRDWGA operator, alternative Λ_4 is the best choice, which proves the consistency and strength of the suggested approach. Further, comparative and sensitivity studies demonstrate that the model offers consistent rankings and enhanced discrimination ability in comparison to the current fuzzy-WASPAS models. Although these benefits exist, the proposed model has some drawbacks, such as greater complexity of the computations since it involves combining several theoretical elements and relies on expert-selected weights and parameters that can affect the ultimate decision results.

For future research, the proposed framework can be generalized to more sophisticated settings, e.g., bipolar fuzzy dominance rough sets. Moreover, a combination of the bipolar fuzzy dominance rough structure with other multi-criteria decision-making models, including the VIKOR method and TOPSIS, could be more flexible and comparative decision-supporting systems.

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Abbreviations

Notations	Meaning	Notations	Meaning
$\wp$	Any fixed subset	$\chi^{-}{}_{\underline{\breve{\mathcal{Y}}}}(\widetilde{\mathfrak{d}})$	Negative membership of bipolar fuzzy rough number
$\underline{\breve{\mathcal{Y}}}(\wp)$	Lower Approximation of $\wp$	$\underline{\mathcal{Y}}(\mathcal{Z}_o)$	Lower Approximation of class $\mathcal{Z}_o$
$\overline{\breve{\mathcal{Y}}}(\wp)$	Upper Approximation of $\wp$	$\overline{\mathcal{Y}}(\mathcal{Z}_o)$	Upper approximation of class $\mathcal{Z}_o$
$\mathbb{D}$	Decision System	$\underline{\gimel}^{+}_{\mathcal{Z}_o^{\preccurlyeq}}$	Lower positive bipolar fuzzy dominance rough number of class $\mathcal{Z}_o^{\preccurlyeq}$
$\mathbb{A}\mathrm{t}$	Set of Attributes	$\underline{\gimel}^{-}_{\mathcal{Z}_o^{\preccurlyeq}}$	Lower negative bipolar fuzzy dominance rough number of class $\mathcal{Z}_o^{\preccurlyeq}$
$\mathcal{Z}_o^{\succcurlyeq}$	Dominated Class	$\overline{\gimel}^{+}_{\mathcal{Z}_o^{\preccurlyeq}}$	Upper positive bipolar fuzzy dominance rough number of class $\mathcal{Z}_o^{\preccurlyeq}$
$\mathcal{Z}_o^{\preccurlyeq}$	Dominating Class	$\overline{\gimel}^{-}_{\mathcal{Z}_o^{\preccurlyeq}}$	Upper negative bipolar fuzzy dominance rough number of class $\mathcal{Z}_o^{\preccurlyeq}$
$\chi^{+}{}_{\underline{\breve{\mathcal{Y}}}}(\widetilde{\mathfrak{d}})$	Positive membership of bipolar fuzzy rough number	Ψ_{BFDN}	Bipolar fuzzy dominance rough number

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