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Article

A modified ancient Babylonian algorithm for nonlinear oscillators

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Abstract: This paper focuses on the frequency-amplitude relationship of nonlinear oscillators and proposes an improved ancient Babylonian algorithm. This algorithm constructs a solution framework based on linear and nonlinear operators in a unique iterative form, and cleverly selects the initial guess value and determines the frequency equation. Through in-depth exploration of several representative nonlinear oscillator examples (covering different forms of nonlinear terms and parameter settings), it fully demonstrates its specific operation and effectiveness verification process in the solution process. The results show that this algorithm performs well in weakly nonlinear oscillator problems, and the obtained results are highly consistent with existing methods or exact solutions. Moreover, it is equivalent to He's frequency formula under specific conditions, strongly supporting the effectiveness of the latter. At the same time, it clearly reveals the influence of the law of the nonlinear term coefficient and amplitude on the accuracy of the algorithm. However, in the case of strongly nonlinear systems, the algorithm has certain limitations. This study combines ancient numerical wisdom with modern nonlinear dynamics, providing a computationally simple and effective tool for oscillator engineering, while also indicating directions for improvement to enhance strong nonlinear performance.

Keywords: improved ancient Babylonian algorithm; nonlinear oscillator; frequency-amplitude relationship; He's frequency formula; ordinary differential equation

1. Introduction

Nonlinear oscillators play an important role in various fields such as mathematics, physics, and engineering. Their application in the field of energy harvesting has pioneered new approaches to addressing the energy crisis [1]. The significance of nonlinear oscillators lies not only in their extension from linear oscillators, but also in their ability to describe various complex phenomena observed in nature and scientific technology [2,3]. For instance, in mathematics, nonlinear oscillators provide models for chaos theory, facilitate the development of nonlinear wave equations, and discussions about the critical points of nonlinear oscillator potentials have advanced critical point theory. In physics, nonlinear oscillators are closely connected with classical mechanics, electricity, and quantum mechanics; they are essential tools for studying mechanical system oscillations, operations of circuits containing nonlinear elements, and behaviors of quantum systems under specific conditions. In engineering, nonlinear oscillators are extensively applied in mechanical dynamics; they can effectively analyze the dynamic characteristics of structures under dynamic loads, the stability of control systems, and the performance of mechanical devices such as engines and turbines [4].

With the rapid development of science and technology today, nonlinear vibration theory has become a core tool in fields such as three-dimensional (3D) printing technology [5], micro-electromechanical systems (MEMS) [6, 7], and flexible folding panels in intelligent buildings [7].

In the field of nonlinear oscillator research, frequency and amplitude are among its most fundamental characteristics, and they exhibit essential differences from linear oscillators. This relationship is also one of the key elements in understanding nonlinear dynamic behavior [7–9]. In engineering applications, the low-frequency characteristics of nonlinear vibration systems are frequently utilized; however, these systems often display large amplitudes in practical engineering situations, and this high-amplitude feature is of significant value in engineering applications [10, 11]. A profound understanding of the relationship between frequency and amplitude is crucial for exploring the deeper principles behind the motion patterns of nonlinear oscillatory systems. Such insight not only helps us clarify how energy flows and transfers within the system but also reveals why certain stability features or specific resonance effects occur during the operation of the system. In the field of microelectromechanical systems (MEMS), the interaction between frequency and amplitude is even more critical. It is precisely because of this close connection that we are able to effectively suppress and avoid the phenomenon of "pull-in" [12–14]—that is, the instability problem in which mechanical components lose balance and suddenly adhere to the electrode due to strong electrostatic forces.

The solution methods for nonlinear oscillators are generally divided into analytical methods and numerical methods. In recent years, in order to address the complexity of nonlinear oscillators, researchers have developed many modern methods in the field of analytical approaches. For example, the variational method [15], which solves problems by constructing an energy functional and finding the function that makes it extremal; nowadays, the critical point theory in variational methods has been widely applied to solve nonlinear partial differential equations. The variational iteration method [16], an improvement over the traditional variational method, uses an iterative approach to gradually approximate the exact solution. To handle more complex nonlinear terms, the Adomian Decomposition is introduced into the variational iteration process, resulting in the variational iteration method based on the Adomian Decomposition [17]; by using the characteristics of the Laplace transform to simplify the equation-solving process, one obtains the Laplace-based variational iteration method [18]. The dual Lagrange multiplier method [19], which constructs a dual problem and introduces Lagrange multipliers to handle constraints, can be seen as a generalization of the classical Lagrange multiplier method. The homotopy perturbation method [20, 21], a powerful tool combining homotopy theory and perturbation theory, transforms complex problems into simple ones through homotopy deformation. The point solution method [22] is mainly used to solve the high-precision slope value of the curve at a specific fixed point. The harmonic balance method is an analytical technique for determining the periodic steady-state response of nonlinear systems. It works by approximating the solution with a finite-term Fourier series. Substituting this series into the system equations and balancing the coefficients for each frequency component converts the differential equations into a set of nonlinear algebraic equations, which are then solved to obtain the periodic response. In damped vibration systems, both the homotopy perturbation method and the harmonic balance method have already been widely applied.

Although the variational iteration method and the homotopy perturbation method continue to play an important role in nonlinear vibration theory, their potential remains to be further explored and perfected [23–25]. In practical engineering applications, these two methods often require a large number of complicated calculations, and the process is relatively cumbersome. However, engineers tend to favor approaches that are both straightforward and efficient for quickly solving problems. It is in this context that He's frequency formula [26] has emerged. Characterized by an exceptionally concise ”one-step” procedure, the formula markedly reduces computational complexity and exhibits strong practicality, thereby offering a simple yet efficient means of determining system frequencies across a wide range of nonlinear oscillator problems. Owing to its simplicity and efficiency, the formula has been extensively adopted in various engineering disciplines. Specifically, He's frequency formula has been applied to the characteristic analysis of damped and forced nonlinear oscillators [27,28], effectively addressing the challenge of system response under external excitation. In fractal vibration systems [29], the formula can handle complex dynamic behaviors induced by spatial fractal dimensions. For nonlinear oscillators with generalized initial conditions [30], He's frequency formula extends its applicability by introducing parameterization of initial energy or phase angle. In strongly nonlinear vibration systems [31, 32], the formula demonstrates accuracy in large amplitude regimes by treating specific high-order nonlinear terms. Moreover, in time-delay vibration systems [33] and other complex systems with time-delay characteristics, He's frequency formula also exhibits strong adaptability. These applications collectively underscore the considerable value of He's frequency formula in theoretical research and practical engineering applications.

An ancient method related to He's frequency formula is the Babylonian algorithm, which is an ancient method for calculating square roots and is considered a remarkable representation of mathematical wisdom in ancient Western cultures. The history of the ancient Babylonian algorithm dates back to 1800–1600 BCE, with the famous YBC7289 clay tablet recording the Babylonians' calculation process for the square root of $\sqrt{2}$, the result of this algorithm differs very little from the modern value. Ji-Huan He proposed an innovative iterative method [34], drawing inspiration from the ancient Babylonian algorithm for calculating square roots. He used this ancient algorithm as the theoretical foundation and ingeniously combined it with the wisdom of ancient Chinese mathematics to construct a mathematical bridge across time and space. Ancient Western mathematics emphasized rigorous logic and adopted an axiomatic and deductive approach, whereby basic axioms were established to derive complex theorems. In contrast, ancient Chinese mathematics placed greater emphasis on practical experience and applications, summarizing patterns through extensive empirical observations and solving problems via intuitive calculations. Although there are obvious differences in their mathematical nature, they can complement each other in specific application contexts. The He Chengtian average, known as an ancient Chinese fractional interpolation algorithm, was originally devised to solve fractional approximation problems in astronomical calculations. Like the ancient Babylonian algorithm, it represents a valuable asset to human mathematical civilization. Given the striking similarities between the computational ideas of the old He Chengtian average and the iterative method used in the ancient Babylonian algorithm for calculating square roots and other values, both embodying the concept of numerical approximation, researchers have taken a keen interest in this connection. Ji-Huan He [34] elaborated on the relationship between the ancient Babylonian algorithm and the old He Chengtian average, and proposed an improved ancient Babylonian algorithm for addressing nonlinear oscillator problems. Several studies [35–37] have confirmed that this method is not only feasible but also efficient computationally. Furthermore, when applying the improved Babylonian algorithm to nonlinear oscillators, we found that under certain specific parameter conditions, this algorithm is equivalent to He's frequency formula.

In this paper, we will delve into the application of the improved ancient Babylonian algorithm in various specific nonlinear oscillator problems. Using this algorithm, we discuss the relationship between frequency and amplitude for these nonlinear oscillators. By comparing our results with existing methods, we demonstrate the practicality of the Babylonian algorithm in the study of nonlinear oscillators. At the same time, we also note the limitations of this method and briefly explain them. This research can be regarded as an extension of using the ancient Babylonian algorithm to study nonlinear oscillators, providing a new perspective for future studies of more complex nonlinear oscillator systems. It is hoped that this research will enhance the understanding of the frequency-amplitude relationship in nonlinear oscillators.

Our paper is organized as follows. In Section 2, we first introduce the iterative process of calculating square roots using the ancient Babylonian algorithm and provide a theoretical justification. Afterwards, we transition to the improved Babylonian algorithm, thereby leading to the abstract framework of the Babylonian algorithm for handling nonlinear oscillators. Further, in Section 3, we apply the improved ancient Babylonian algorithm to four specific examples of nonlinear oscillators, demonstrating the effectiveness and simplicity of our method. Finally, the conclusion remarks are described in Section 4.

2. Improved ancient Babylonian algorithm

Let us first briefly describe the process of the ancient Babylonian algorithm for calculating square roots. The essence of this algorithm is to use the arithmetic mean to continuously approximate the true square root. Specifically, to find the square root of a positive real number N, one first selects an initial guess x_0, and then, use the formula $x_{n+1} = (x_n + N/x_n)/2$ to get a new approximation x_{n+1} in each iteration, where the numbers x_n and N/x_n are two estimation values of the number $\sqrt{N}$. As the number of iterations increases, the value of x_{n+1} gets closer and closer to the true value $\sqrt{N}$, and it can easily be seen that it must always lie between x_n and N/x_n. Of course, this is an evaluation of the algorithm's reliability from the perspective of numerical computation. With the advancement of modern mathematics, we can rigorously prove the reasonableness of this iterative formula from the viewpoint of mathematical analysis. In fact, we can state it using the following theorem.

Theorem 1. *Let $N > 0$ and $a > 0$. Define the sequence $\{x_n\}$ by*

$$x_1 = a, x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right),$$

then the sequence $\{x_n\}$ convergence to $\sqrt{N}$.

Proof of Theorem 1. Notice that $x_1 = a > 0$, $N > 0$, by mathematical induction, we have $\forall n \in \mathrm{N}^+$, $x_n > 0$. Applying the inequality of arithmetic and geometric means, we obtain

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \geq \frac{1}{2} \cdot 2\sqrt{x_n \cdot \frac{N}{x_n}} = \sqrt{N}.$$

Therefore, $\sqrt{N}$ is a lower bound of the sequence.

Moreover, for $n \geq 2$

$$\frac{x_{n+1}}{x_n} = \frac{1}{2}\left(1 + \frac{N}{x_n^2}\right) \leq 1.$$

Thus, $\{x_n\}$ is a monotone sequence, then by the monotone convergence theorem, we know that sequence $\{x_n\}$ convergence. Let $\lim_{n\to\infty} x_n = A$. Applying the limit laws to the recurrence relation

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right),$$

and taking limits on both sides, we obtain

$$A = \frac{1}{2}\left(A + \frac{N}{A}\right).$$

Solving this equation yields $A = \pm\sqrt{N}$. Since $x_n \geq \sqrt{N}$ for all n, it follows that $\lim_{n\to\infty} x_n = \sqrt{N}$. □

This theorem demonstrates from the perspective of deductive reasoning that its validity can not only be tested through numerical calculation but also can be rigorously proven using the language of modern mathematical analysis. This iterative formula conforms to the rigorous proof system of ancient Western mathematics while also meeting the practical needs pursued in ancient Chinese mathematics.

In addition, we can also explain the iterative formula $x_{n+1} = (x_n + N/x_n)/2$ from a geometric perspective. Imagine a rectangle with an area of N, where one side length of x_n and the other side length of N/x_n, the actual square root $\sqrt{N}$ is the side length of the square with the same area as this rectangle. By continuously adjusting the side lengths of the rectangle to make it closer to a square, when the two sides x_n and N/x_n of the rectangle are equal, we can determine the side length of a square with an area of N, and then obtain the exact square root of $\sqrt{N}$.

Although several millennia have passed since the discovery of the ancient Babylonian algorithm, its effectiveness and simplicity remain undeniable. Through simple arithmetic operations (addition, division, averaging), it achieves quadratic convergence speed, reflecting the profound insight of ancient mathematicians. As a crystallization of wisdom from ancient Western mathematics, the ancient Babylonian algorithm is now used to handle some nonlinear oscillator problems. Similarly, the ancient Chinese He Chengtian average also plays an important role in solving nonlinear oscillator problems. The He Chengtian average uses a linear interpolation method to approximate the actual value, specifically by repeatedly iterating the weighted average of two boundary values to make the calculated approximate value approach the precise value. Next, we will introduce the iterative formula of the old He Chengtian average for quadratic equations.

For a general quadratic equation

$$x^2 = N. \tag{1}$$

The Old He Chengtian average [34] is

$$x_{n+1} = \frac{px_n + q\frac{N}{x_n}}{p+q} = \frac{p}{p+q}x_n + \left(1 - \frac{p}{p+q}\right)\frac{N}{x_n}, \tag{2}$$

where p and q are positive nature number, N is positive real number, x_n and $\frac{N}{x_n}$ are two approximate values. Through some simple calculations, we can find that

$$x_n \leq \sqrt{N} \leq \frac{N}{x_n}\text{or}\frac{N}{x_n} \leq \sqrt{N} \leq x_n. \tag{3}$$

We are denoting $\beta = p/p + q$, the Equation (2) can be rewritten as

$$x_{n+1} = \beta x_n + (1-\beta)\frac{N}{x_n}. \tag{4}$$

By using this iterative formula for multiple iterations, the approximate value x_{n+1} will get closer and closer to the true value $\sqrt{N}$ as the number of iterations increases. It is important to note that the initial guess must be different from $x_1 = N$. When $\beta = \frac{1}{2}$, the above Equation (4) is precisely the iteration formula of the ancient Babylonian algorithm; when $\beta = \frac{n-1}{n}$, the above Equation (4) then becomes the Newton iteration formula. Consequently, the ancient Babylon Algorithm can be seen as a special case of the old He Chengtian average method and Newton iteration method. In other words, both the ancient Babylonian algorithm and the old He Chengtian average method use weighted averaging techniques, allowing the estimated values on both sides of the true value to approach the true value through iterative formulas. When the estimated values on both sides are equal, we can obtain the precise value we need. We can expand the situation of the above quadratic equation to more general algebraic equations.

For more general algebraic equations

$$f(x) = 0, \tag{5}$$

we can observe that

$$x^2 = f(x) + x^2, \tag{6}$$

here, $f(x) + x^2$ replaces the position of N in the original quadratic Equation (1). Through Equation (4), we can obtain the corresponding ancient Babylonian iterative formula

$$x_{n+1} = \beta x_n + (1-\beta)\frac{f\left(x_n\right) + \left(x_n\right)^2}{x_n}. \tag{7}$$

The validity of this formula has already been verified in studies by He [26, 34]. Similar conclusions can also be drawn for more general differential equations. In this paper, we will focus on the abstract framework of the iterative algorithm of ancient Babylon in the differential equation, and apply it to the example in Section 3.

The iterative algorithm proposed by He [34] presents a novel approach for handling general differential equations of the form

$$L(u) + E(u) = 0, \tag{8}$$

where L is the linear differential operator, and E is the nonlinear operator. Equation (8) can be rewritten in the following form:

$$u^2 = L(u) + E(u) + u^2. \tag{9}$$

By using Equation (4) again, the structure of this iterative algorithm is as follows:

$$u_{n+1} = \beta u_n + (1-\beta)\frac{L\left(u_n\right) + E\left(u_n\right) + \left(u_n\right)^2}{u_n}, \tag{10}$$

where $0 < \beta < 1$ (weighting factor), the initial guess solution u_0 should satisfy all the boundary values and the initial conditions of Equation (8). A suitable choice of the initial solution plays an important role in effectively solving a nonlinear problem. For the general nonlinear oscillators, we can choose u_0 as follows:

$$u_0 = Acos(\omega t + \sigma), \tag{11}$$

where A is the amplitude, σ is the initial phase angle, and the frequency ω is determined by the following equation:

$$u_1\left(\bar{t}\right) = \beta u_0\left(\bar{t}\right) + (1-\beta)\frac{L\left(u_0\left(\bar{t}\right)\right) + E\left(u_0\left(\bar{t}\right)\right) + \left(u_0\left(\bar{t}\right)\right)^2}{u_0\left(\bar{t}\right)}, \tag{12}$$

where $\bar{t}$ is the location point. According to the steps above for solving the differential equation, we can obtain the connection between the ancient Babylonian algorithm and the He's frequency formula. For the nonlinear oscillator as follows:

$$u^{''} + f(u) = 0,\ u(0) = A, u^{'}(0) = 0. \tag{13}$$

Select the initial guess value

$$u_0(t) = Acos(\omega t). \tag{14}$$

After the previous discussion about the ancient Babylonian algorithm, we choose the weight factor $\beta = \frac{1}{2}$. Using Equation (10), we can obtain

$$u_1(t) = \frac{1}{2}u_0(t) + \frac{1}{2}\frac{u_0^{''}(t) + f\left(u_0(t)\right) + \left(u_0(t)\right)^2}{u_0(t)}. \tag{15}$$

Substituting Equation (14) into Equation (15) yields

$$u_1(t) = \frac{1}{2}Acos(\omega t) + \frac{1}{2}\frac{-A\omega^2 cos(\omega t) + f\left(u_0(t)\right) + A^2 cos^2(\omega t)}{Acos(\omega t)}. \tag{16}$$

The location point is chosen as

$$\omega \bar{t} = \frac{\pi}{6}. \tag{17}$$

It means that

$$u_1\left(\bar{t}\right) = \frac{\sqrt{3}}{2}A. \tag{18}$$

From Equation (12), we have

$$\omega = \sqrt{\frac{f\left(u_0\left(\bar{t}\right)\right)}{u_0\left(\bar{t}\right)}}. \tag{19}$$

Equation (19) is the He's frequency formula corresponding to Equation (13). This also shows that under the specific parameter setting, the ancient Babylonian algorithm has the same effect as the He's frequency formula, and inherits the characteristics of the He's frequency formula "one-step" method, which can greatly simplify the calculation process and reduce the complexity of the calculation. In the application of Section 3, we will continue to discuss the frequency-amplitude relationship of nonlinear oscillators of Equation (13).

The design inspiration for this algorithm originates from the fundamental concept of the ancient Babylonian iterative approximation method. It combines this traditional mathematical approach with modern differential equation solving techniques and has been specifically improved and optimized for the characteristics of nonlinear oscillators. In this way, the modified ancient Babylonian algorithm can be effectively applied to analyze and solve such complex dynamical systems. During the computation process, the algorithm uses iterative methods to gradually approximate the solution of the problem. As the number of iterations increases, the approximate solution obtained continuously converges to the true solution, thereby improving the accuracy of the results. In the iterative formula of this algorithm, both linear operators and nonlinear operators are introduced. The linear operator is mainly responsible for handling the linear terms in the differential equation, while the nonlinear operator is specifically used to process the nonlinear terms. This differentiated approach enables the algorithm to systematically address various components of the differential equation, thus enhancing its applicability and precision in solving a wide range of nonlinear oscillator problems.

In the selection of initial guess values and positioning points, the improved ancient Babylonian algorithm demonstrates extremely high flexibility. For nonlinear oscillators of the type represented by Equation (13), we chose specific parameters such that the frequency obtained using the ancient Babylonian algorithm matches the frequency derived from He's frequency formula. Furthermore, this algorithm allows researchers to independently adjust these key parameters according to the requirements of specific problems, enabling adaptation to different research scenarios. This high degree of adaptability is particularly important for solving various nonlinear oscillator problems encountered in practical work. Since the characteristics of nonlinear systems vary depending on the problem, it is often necessary to select different initial guess values and positioning points to achieve efficient and precise solutions. Overall, the improved ancient Babylonian algorithm not only enhances computational efficiency but also provides new perspectives for the development of analysis and solution techniques for nonlinear oscillators, serving as a transitional approach for handling more complex nonlinear oscillators.

3. Application to the nonlinear oscillators

This section selects four typical examples of nonlinear oscillators and analyzes and solves them using the improved ancient Babylonian algorithm. Each example represents different characteristics and behavioral patterns that nonlinear oscillators may encounter in practical engineering. Through the study of these examples, the aim is to provide a comprehensive perspective to evaluate the actual performance, advantages, and potential limitations of the algorithm under specific conditions.

In the study of this paper, we mainly focus on a special nonlinear oscillator that contains an odd nonlinear term, whose related characteristics have been elaborately discussed in reference [38]. The dynamical behavior of this type of oscillator has received considerable attention in studies, primarily because the presence of an odd nonlinear term often triggers asymmetric responses and high-dimensional chaotic behavior in oscillatory systems. Specifically, the introduction of the odd nonlinear term usually leads to a more complex phase space structure of the oscillator, thereby significantly affecting its frequency-amplitude relationship and overall steady-state stability. In this context, we introduce the Improved Babylonian Algorithm to conduct numerical analysis for this particular oscillator. Through the analysis of this example, our aim is to verify the effectiveness of the algorithm in handling asymmetric complex systems containing odd nonlinear terms. This verification not only helps to confirm the applicable scope of the algorithm itself, but also provides a theoretical basis for its potential applications in broader nonlinear oscillator models, including those containing mixtures of odd and even nonlinear terms.

The specific analysis process of these examples is as follows: First, the algorithm is introduced into the adaptation algorithm framework, and a mathematical model is established for the problem being addressed. We start by transforming the given differential equation into a form required by the framework, and rigorously set the initial guess values and key parameters in the algorithm based on theoretical and experimental experience. Next, we will record the iterative process of the algorithm in detail, focusing on observing the trend of the solution changes in each iteration. To objectively evaluate the performance of this algorithm in this specific scenario, we will compare the obtained numerical results with existing theoretical solutions or other recognized numerical methods. Through the above comparative analysis, we can not only quantitatively verify the accuracy and reliability of the algorithm but also further explore possible limitations of the algorithm when dealing with odd-order nonlinear terms, thus providing a theoretical basis and research direction for subsequent improvements.

When dealing with such old nonlinear oscillator problems, the increase in the degree of system nonlinearity significantly limits the applicability of the ancient Babylonian algorithm we adopted. Specifically, when the amplitude is large or the nonlinearity is extremely high, the convergence speed of the algorithm drops markedly, and even failures may occur. However, in specific situations where the amplitude is relatively small and the nonlinear effects are limited, the algorithm can still maintain good effectiveness. Thus, it is evident that the improved Babylonian algorithm still has limitations when facing strongly nonlinear oscillators, which points to a clear direction for future research: further optimization of the algorithm is needed to enhance its stability and applicability under conditions of strong nonlinearity.

The in-depth study of oscillator models containing odd nonlinear terms provides theoretical support for demonstrating the applicability and advantages of the improved ancient Babylonian algorithm in the analysis of nonlinear vibration systems. This research not only verifies the effectiveness of the algorithm in handling specific nonlinear oscillator problems, but also serves as an interesting point of penetration for further exploration of more complex and diverse nonlinear oscillator systems.

Example 1.

$$u^{''}\left(1+u^2\right)+u=0,\ u(0)=A,\ u^{'}(0)=0. \tag{20}$$

To rewrite Equation (20) in the form of Equation (10), we divide both sides by $\left(1+u^2\right)$*:*

$$u^{''}+\frac{u}{1+u^2}=0,\ u(0)=A,\ u^{'}(0)=0. \tag{21}$$

The initial guess value is chosen as:

$$u_0=Acos(\omega t). \tag{22}$$

Substituting $u_0\left(t\right)$ *into Equation (10), we obtain:*

$$\begin{gathered}u_1(t)=\beta u_0(t)+(1-\beta)\frac{u_0^{''}(t)+\frac{u_0(t)}{1+u_0^2(t)}+(u_0(t))^2}{u_0(t)}= \\ \beta Acos(\omega t)+(1-\beta)\frac{-A\omega^2cos(\omega t)+\frac{Acos(\omega t)}{1+A^2cos^2(\omega t)}+A^2cos^2(\omega t)}{Acos(\omega t)}.\end{gathered} \tag{23}$$

Here we typically set $\beta=\frac{1}{2}$*, and select the time point* $\bar{t}$ *such that* $\omega\bar{t}=\frac{\pi}{6}$ *[39]. This yields*

$$u_1\left(\bar{t}\right)=u_0\left(\bar{t}\right)=\frac{\sqrt{3}}{2}A. \tag{24}$$

From Equation (12), the frequency ω is given by:

$$\omega = \frac{1}{\sqrt{1+\frac{3}{4}A^2}}. \tag{25}$$

This result is consistent with that obtained using the homotopy perturbation method [38], demonstrating the validity of our approach.

Example 2. *We consider a common nonlinear oscillator [40],*

$$u^{''}u + 1 = 0, u(0) = A, u^{'}(0) = 0. \tag{26}$$

To make Equation (26) in the form of Equation (10), we rewrite Equation (26) as follows:

$$u^{''} + \frac{1}{u} = 0,\ u(0) = A,\ u^{'}(0) = 0. \tag{27}$$

First, select the initial guess value:

$$u_0 = Acos(\omega t). \tag{28}$$

By Equation (10), we have

$$\begin{gathered} u_1(t) = \beta u_0(t) + (1-\beta)\frac{u_0^{''}(t) + \frac{1}{u_0(t)} + (u_0(t))^2}{u_0(t)} = \\ \beta Acos(\omega t) + (1-\beta)\frac{-A\omega^2 cos(\omega t) + \frac{1}{Acos(\omega t)} + A^2cos^2(\omega t)}{Acos(\omega t)}. \end{gathered} \tag{29}$$

Then, we take $\beta = \frac{1}{2}$, the location point $\omega\bar{t} = \frac{\pi}{6}$, and so there is:

$$u_1(\bar{t}) = u_0(\bar{t}) = \frac{\sqrt{3}}{2}A. \tag{30}$$

According to Equation (12), we can easily get:

$$\omega = \frac{2}{\sqrt{3}A} = 1.15470A^{-1}. \tag{31}$$

Thus its approximate period is:

$$T = \frac{2\pi}{\omega} = \sqrt{3}\pi A = 5.44140A, \tag{32}$$

and its exact period [40] is:

$$T_{exact} = 2\sqrt{2\pi}A = 5.01325A. \tag{33}$$

The relative error of frequency is 7.86%, and such accuracy is achieved with only one iteration, indicating that this method is highly effective.

In the following, we consider a special nonlinear oscillator [41].

Example 3.

$$u^{''}+au+bu^{2n+1}=0, a\geq 0, b>0,\ n=1,2,3,\cdots,\ u(0)=A, u^{'}(0)=0. \quad (34)$$

Similarly, we assume the initial guess value:

$$u_0=Acos(\omega t). \quad (35)$$

Using Equation (10), we can obtain:

$$\begin{aligned} u_1(t) &= \beta u_0(t)+(1-\beta)\frac{u_0^{''}(t)+au_0(t)+bu_0^{2n+1}(t)+(u_0(t))^2}{u_0(t)} \\ &= \beta Acos(\omega t)+(1-\beta)\frac{-A\omega^2 cos(\omega t)+aAcos(\omega t)+bA^{2n+1}cos^{2n+1}(\omega t)+A^2cos^2(\omega t)}{Acos(\omega t)}. \end{aligned} \quad (36)$$

Similar to the first two examples, we choose $\beta=\frac{1}{2}$, and the location point $\omega\bar{t}=\frac{\pi}{6}$, yielding:

$$u_1\left(\bar{t}\right)=u_0\left(\bar{t}\right)=\frac{\sqrt{3}}{2}A. \quad (37)$$

Subsequently, based on Equation (12), we obtain:

$$\omega=\sqrt{a+b\left(\frac{3}{4}\right)^n A^{2n}}. \quad (38)$$

Here we provide the exact values of the frequency of this nonlinear oscillator [40]:

$$\omega_{exact}=\frac{2\pi}{4\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{a+\frac{b}{n+1}A^{2n}\left(1+\sin^2\theta+\sin^4\theta+\cdots+\sin^{2n}\theta\right)}}}. \quad (39)$$

It is easy to find that when $a=n=1$, Equation (34) became the famous Duffing equation:

$$u^{''}+u+bu^3=0,\ u(0)=A,\ u^{'}(0)=0. \quad (40)$$

At this point, Equation (38) gives the frequency of the equation as:

$$\omega=\sqrt{1+\frac{3}{4}bA^2}. \quad (41)$$

This result is consistent with that obtained in the study by He [34].

To further illustrate the validity of this approach for deriving frequencies, we consider the following special cases:

Case 1: $a=1, n=2$.

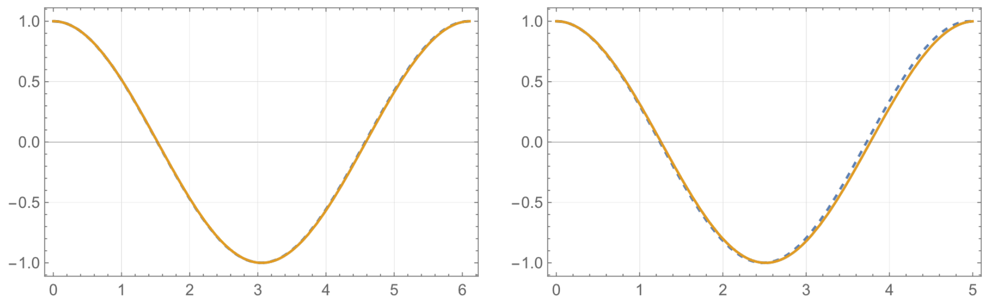
In this case, Equation (34) reduces to

$$u^{''}+u+bu^5=0,\ u(0)=A,\ u^{'}(0)=0. \quad (42)$$

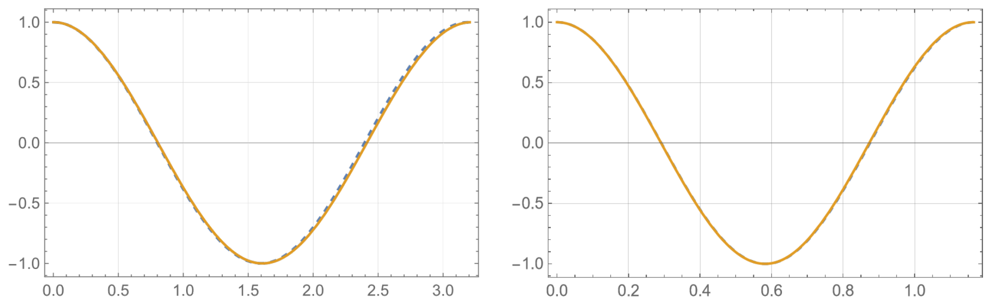
The approximation frequency can be obtained from Equation (38) as

$$\omega=\sqrt{1+\frac{9}{16}bA^4}. \quad (43)$$

Figure 1 shows excellent agreement between the approximate and the exact solution.



(a) $b = 0.1$, $\omega_{app} = 1.02774$. **(b)** $b = 1$, $\omega_{app} = 1.25$.



(c) $b = 5$, $\omega_{app} = 1.95256$. **(d)** $b = 50$, $\omega_{app} = 5.39676$.

Figure 1. Comparison of the approximate solution with the exact solution; dashed line: approximated solution and solid line: exact solution.

Case 2: $a = 10$, $n = 3$.

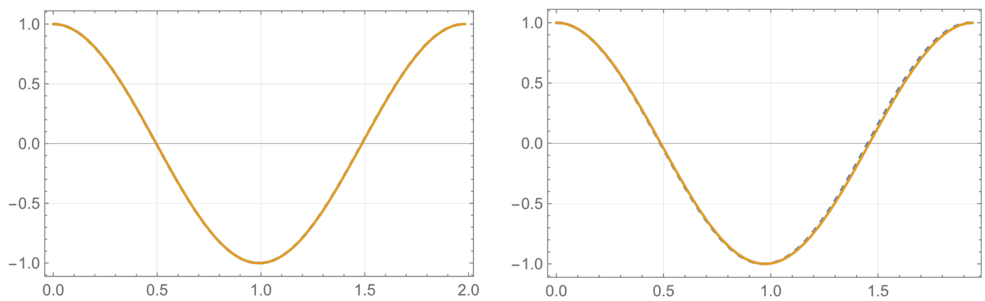
In this case, Equation (34) reduces to

$$u'' + 10u + bu^7 = 0,\ u(0) = A,\ u'(0) = 0. \tag{44}$$

The approximation frequency can be obtained by Equation (38)

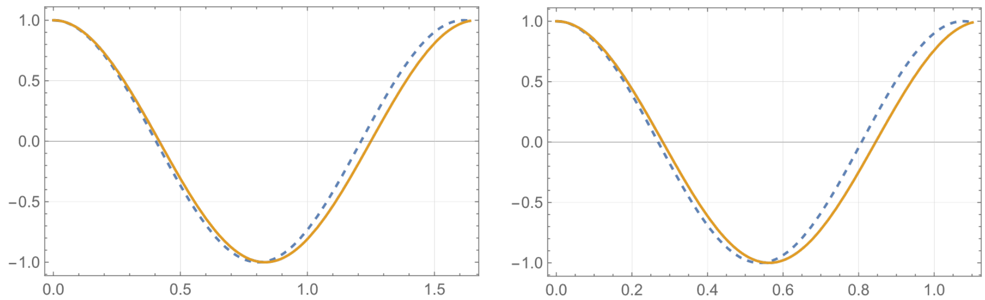
$$\omega = \sqrt{10 + \frac{27}{64} bA^6}. \tag{45}$$

Figure 2 shows that the approximate solution closely matches the exact solution.



(a) $b = 0.1$, $\omega_{app} = 3.16894$. **(b)** $b = 1$, $\omega_{app} = 3.22829$.

Figure 2. *Cont.*



(c) $b = 10$, $\omega_{app} = 3.77078$. **(d)** $b = 50$, $\omega_{app} = 5.57618$.

Figure 2. Comparison of the approximate solution with the exact solution.
Note: Dashed line: approximated solution and solid line: exact solution.

Case 3: $a = 10, b = 10, n = 3$.

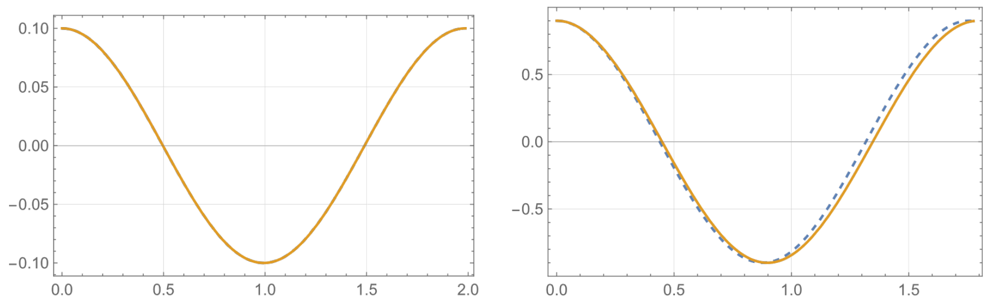
In this case, Equation (34) reduces to

$$u'' + 10u + 10u^7 = 0, u(0) = A, u'(0) = 0. \tag{46}$$

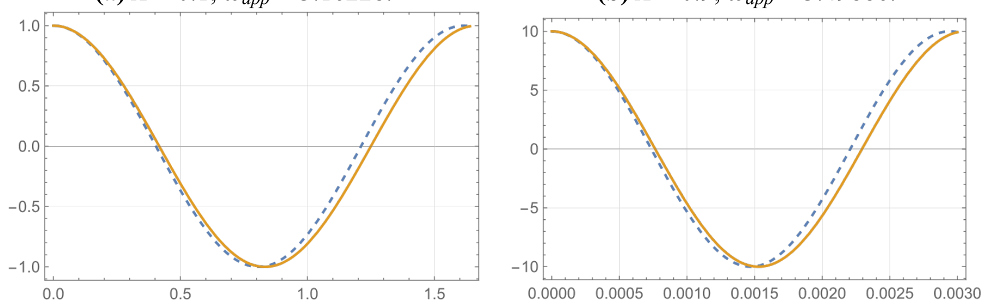
Its approximation frequency can be given by Equation (38)

$$\omega = \sqrt{10 + \frac{135}{32}A^6}. \tag{47}$$

Figure 3 shows excellent agreement of the obtained result with the exact one.



(a) $A = 0.1$, $\omega_{app} = 3.16228$. **(b)** $A = 0.9$, $\omega_{app} = 3.49886$.



(c) $A = 1$, $\omega_{app} = 3.77078$. **(d)** $A = 10$, $\omega_{app} = 2053.96$.

Figure 3. Comparison of the approximate solution with the exact solution.
Note: Dashed line: approximated solution and solid line: exact solution.

Figures 1 and **2** intuitively demonstrate the significant impact of changes in the nonlinear term coefficient on the approximate solutions of nonlinear oscillators. Specifically, when the coefficient of the nonlinear term is small (i.e., the system's behavior is close to linear oscillation), the approximate solution calculated by the improved Babylonian method almost coincides with the exact solution. This high-precision match occurs because, in cases where nonlinear effects have not yet become dominant, their disturbance to the system's dynamics is insufficient to cause significant deviations in the solving process. We select different fixed values for a and the degree of the nonlinear term, and discuss the comparison between the approximate solution and the exact solution when the amplitude is $A = 1$. We can observe that when the coefficient of the nonlinear term is $b \leq 1$, the exact solution and the approximate solution match closely. When $b \geq 1$, discrepancies begin to appear.

However, as the strength of the nonlinear term gradually increases, a discernible shift occurs. Discrepancies between the approximate and exact solutions start to surface. Interestingly, it is not a straightforward linear relationship where the error monotonically escalates with the growth of the nonlinear term coefficient. Instead, only when the nonlinear term becomes highly pronounced and exerts a strong influence does significant inaccuracy become evident. This implies that there exists a certain threshold or range within which the algorithm can still maintain a reasonable level of accuracy despite the presence of nonlinearities.

Figure 3 intuitively shows the impact of amplitude variation on the accuracy of approximate solutions for nonlinear oscillators, which helps us analyze the applicable range of approximation methods more deeply. Here, we select a fixed nonlinear coefficient $b = 10$ and observe the relationship between the approximate solution and the exact solution by changing the amplitude A. When $A \leq 0.1$, the approximate solution highly matches the exact solution. When A is greater than 0.1, differences begin to appear between the approximate solution and the exact solution. In regions with small amplitudes, the approximate solution closely matches the exact solution trajectory. This is because at low amplitudes, the system exhibits near-linear characteristics, the influence of nonlinear effects on the system dynamics is relatively weak, and the approximate solution is able to capture well the main behavior of the system. However, as the amplitude increases, the approximate solution gradually deviates from the exact solution. It is noteworthy that this deviation does not increase linearly but instead displays complex nonlinear features. Within certain critical amplitude ranges, the rate of deviation significantly slows or even tends to stabilize. This phenomenon indicates that the relationship between amplitude and error is not a simple linear or proportional one but is instead constrained by the multiple coupling effects of the system's internal nonlinear mechanisms.

In summary, the improved ancient Babylonian algorithm demonstrates high sensitivity to the coefficients of nonlinear terms and amplitude variations in the system. Specifically, these two factors are nonlinearly coupled during the numerical solution process, leading to significant differences between the resulting approximate and exact solutions. Nevertheless, based on the existing numerical experiment results, we still observe that the method maintains high numerical accuracy and stability under most parameter settings. This indicates that, although the algorithm's accuracy may be limited when dealing with extreme conditions such as very strong nonlinearity (e.g., extremely high powers) or large amplitude oscillations, the improved Babylonian algorithm still exhibits good applicability and potential value in the specific research field of nonlinear oscillator analysis.

Next, we apply the improved ancient Babylonian algorithm to a nonlinear oscillator system with damping characteristics.

Example 4. *In this example, we consider the following damped Duffing oscillator.*

$$u^{''} + au + bu^{'} + cu^3 = 0, u(0) = A, u^{'}(0) = 0. \tag{48}$$

When $b = 0$*, we can obtain the frequency easily, that is*

$$\omega = \sqrt{a + \frac{3}{4}cA^2}. \tag{49}$$

The calculation procedure for Equation (39) is analogous to the preceding three examples and will not be elaborated here. Notably, Equation (49) yields identical results to Equation (48) as obtained via the homotopy perturbation method [42]. Equation (48) can be linearized [42] as

$$u^{''} + \omega^2 u + bu^{'} = 0, u(0) = A,\ u^{'}(0) = 0. \tag{50}$$

Its solution is

$$u(t) = Ae^{\alpha t}\left[Acos(\Omega t) - \frac{\alpha A}{\Omega}sin(\Omega t)\right], \tag{51}$$

where $\alpha = -\frac{b}{2}$*,* $\Omega = \frac{\sqrt{4\omega^2 - b^2}}{2}$*. So the damped vibration systems can also be solved easily.*

4. Conclusion

This study proposes an improved scheme based on the ancient Babylonian algorithm for solving the frequency-amplitude relationship of nonlinear oscillators. By introducing a new iterative framework that systematically couples linear and nonlinear operators, this method overcomes the limitations of traditional approaches in handling complex nonlinear behaviors. First, we construct the frequency equation describing the nonlinear oscillator and design a targeted initial value strategy on this basis. Theoretical arithmetic shows that when dealing with weakly nonlinear oscillating systems, the algorithm can quickly obtain high-precision approximate solutions. Moreover, within specific parameter ranges, its numerical results are consistent with the theoretical exact solutions and the values calculated by the classical He's frequency formula, thereby verifying the effectiveness of the improved algorithm in terms of numerical stability and accuracy. For instance, in Example 1, the results of the ancient Babylonian algorithm are consistent with those of the homotopy perturbation method; in Example 2, the approximate solutions deviate by less than 7.86% from exact solutions. These results confirm the algorithm's reliability for weakly nonlinear systems.

It is noteworthy that the algorithm's adaptability is extended to damped nonlinear oscillators (Example 4), thereby demonstrating its versatility in managing real-world scenarios where energy dissipation is paramount. In contrast to conventional perturbation methods that depend on small nonlinearity parameters, this approach exhibits applicability to moderately strong nonlinearities, as demonstrated by consistent performance across a range of nonlinear stiffness coefficients. The method's efficacy stems from its iterative framework, which circumvents the necessity for intricate mathematical derivations while ensuring a high degree of accuracy. This characteristic positions it as a competitive alternative to established techniques, such as the homotopy perturbation method and the multi-scale method.

We must also acknowledge the limitations of the algorithm. In the case of strongly nonlinear systems, the approximate solutions obtained by the algorithm deviate significantly from the exact solutions. This is primarily due to the increasing complexity and non-linearity of the system, which the current form of the algorithm struggles to fully capture. Additionally, the amplitude of the oscillator also affects the accuracy of the algorithm. As the amplitude increases, the discrepancies between the approximate and exact solutions become more pronounced, although the relationship is not strictly monotonic.

This study bridges the gap between theoretical analysis and engineering demands by proposing an algorithm that combines computational efficiency with physical intuition. The algorithm has been successfully applied in the mechanical and electromechanical fields. It has been validated in micro-electro-mechanical systems (MEMS) resonators and vibration control devices, demonstrating its wide applicability. Through an in-depth investigation of nonlinear oscillator models, the application value is showcased when dealing with frequency-amplitude characteristics under different initial and boundary conditions.

The novel iterative framework is a system of interrelated components that work in concert to achieve a common objective. The enhanced algorithm provides a streamlined solution pathway for nonlinear oscillators with fractional derivatives [43–45] and differential equation-driven intelligent control [46].

This study reveals the practical application value of ancient mathematical concepts in the field of modern nonlinear dynamics, emphasizing the important role of interdisciplinary research methods in promoting the development of computational mechanics.

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