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Article

Existence of periodic solutions for a nonlinear plate coupling system with thermal memory and external forces

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Abstract: This paper is devoted to investigating the existence and uniqueness of T-periodic solutions for a nonlinear thermoelastic plate coupling system with thermal memory effects and time-periodic external forces, derived from the non-Fourier heat flux law—a model more physically realistic for characterizing the thermal response of materials with transient heat conduction behavior. To address the mathematical challenges of this coupled system, we first transform the original high-order system into an equivalent first-order evolution system via auxiliary and memory variable substitutions. Using the Galerkin method to construct finite-dimensional approximate solutions, we then apply the Leray-Schauder fixed-point theorem to prove the existence of approximate periodic solutions, deriving uniform a priori estimates for their derivatives in Hilbert space V^0 via Hölder's, Poincaré's, and Gronwall's inequalities. The Sobolev compact embedding theorem verifies the convergence of approximate solutions, establishing the existence of T-periodic solutions for the original system; uniqueness is further proven via an energy difference functional and Gronwall's lemma under a smallness condition on external forces. This work enriches the theoretical framework for periodic solutions of memory-type thermoelastic coupling systems and provides a foundation for engineering dynamic analysis of plate structures.

Keywords: thermal memory effect; external force; nonlinear plate coupling system; time-periodic solution

1. Introduction

Among the numerous and complex phenomena in daily life, there is a special kind of phenomenon, namely the phenomenonwith thermal memory. In the process of heat conduction, there is memory, and there are evolutionary systems with internalmemory. This can provide a reasonable explanation for some heat conduction phenomena in daily life. For plate-like materials, during the process of use, they will inevitably encounter the situation of heat conduction. And due to the different principlesutilized for heat conduction, the research and treatment methods also vary. In differential equations, this is manifested asconsidering a memory kernel function when taking into account the influence of temperature changes [1]. Recently, theevolutionary equations under various non-Fourier heat flux laws have attracted the interest of many mathematicians. The heatconduction evolutionary equations under the Coleman-Gurtin theory have also been widely studied. For the linear thermoelasticplate equations without memory effects in heat conduction, the exponential stability of the relevant C_0-semigroups underdifferent boundary conditions has been proven [2–4]. On the other hand, when the heat flux is modeled by the non-Fourier law, the well-posedness and stability of the corresponding linear thermoelastic plate equations have been investigated [5, 6]. Periodicphenomena are a very special existence in natural phenomena. In a long period of continuation, there is the regular recurrenceof a certain phenomenon, which can be explained by the periodic solutions of nonlinear differential equations in mathematics.

Let X be a Banach space. Define the space $C^k(T;X)$ as the set of functions whose k-th-order derivatives are continuousin X and are periodic with period T in R. Define the norm $|g|_{C^k(T;X)} = \sup\limits_{0\le t\le T}\left\{\sum\limits_{i=0}^{k}\left|D_t^i g(t)\right|_X\right\}$. $L^p(T;X)(1\le p\le\infty)$ is the set of functions that are periodic with period T in R and are measurable on X. Define the norms:

$$|g|_{L^p(T;X)}= \left(\int_0^T |g|_X^p dt\right)^{\frac{1}{p}} < +\infty, (1\le p\le\infty),$$

$$|g|_{L^\infty(T;X)}= \sup_{0\le t\le T} |g|_X < +\infty$$

$$W^{k,p}(T;X) = \left\{g \mid g\in L^p(T,X), D^\alpha u\in L^p(T,X), x\in R^1, \alpha\le k\right\}.$$

When X is a Hilbert space, $H^k(T;X) = W^{k,2}(T;X)$. $H^m(U)$ is the Sobolev space consisting of functions whose derivatives up to order m are all in $L^2(U)$. Throughout this article, $L^2(U)$ represents a Hilbert space equipped with the following inner-product:

$$(u,v) = \sum_{i=1}^{n}\int_U u^i v^i dx$$

Define the norm form of $L^2(U)$ as $\|\cdot\|$. Consider a positive operator A on $L^2(U)$, defined as $A = -\Delta = \nabla^2 = -D_x^2$, and its domain $D(A) = H^2(U)\cap H_0^1(U)$. For $r\in R$, consider the Hilbert space $\mathcal{V}^r = D(A^{r/2})$, and define the inner-product:

$$(u_1,u_2)_{\mathcal{V}^r} = (A^{r/2}u_1, A^{r/2}u_2), \forall u_1,u_2\in\mathcal{V}^r,$$

$$\|u\|_{\mathcal{V}^r}^2= \left(A^{r/2}u, A^{r/2}u\right),\ u\in\mathcal{V}^r.$$

Introduce the weighted Hilbert memory space $M^{1+r} = L_\mu^2(R^+;v^{r+1})$, for $r\in R$, and define the following inner-product:

$$(\eta_1,\eta_2)_{\mathrm{M}^r} = \int_0^\infty \mu(s)\left\langle A^{r/2}\eta_1(s), A^{r/2}\eta_2(s)\right\rangle ds,\ \forall\eta_1,\eta_2\in\mathrm{M}^{1+r},$$

$$\|\eta\|_{\mathrm{M}^{1+r}}^2 = \int_0^\infty \mu(s)\|\eta(s)\|_2^2 ds,\ \forall\eta(s)\in M^{1+r}.$$

Finally, we define the Hilbert space with an inner product:

$$V^r = \mathcal{V}^{2+r}\times\mathcal{V}^r\times\mathcal{V}^r\times\mathrm{M}^{1+r},\ r\in R,$$

for any$z = (z_1,z_2,z_3,z_4)^T\in V^r$, its norm is given by

$$\|z\|_{V^r}^2= \left\|A^{(2+r)/2}z_1\right\|^2 + \left\|A^{r/2}z_2\right\|^2 + \left\|A^{r/2}z_3\right\|^2 + \|z_4\|_{\mathrm{M}^{1+r}}^2$$

when $r = 0$, we have $V^0 = \mathcal{V}^2\times\mathcal{V}^0\times\mathcal{V}^0\times\mathrm{M}^1$.

In this paper, the spaces V^0 and H^1 are investigated. The inner-product and norm defined in H^1 are identical to those in V^0. We explore the existence and uniqueness of time-periodic solutions for a nonlinear plate-coupled system. This systemexhibits thermal memory effects due to the non-Fourier heat flux law and is under the influence of specific external forces. The equations are as follows:

$$\theta_t - \Delta u_t + \int_0^\infty k(s)\left[-\Delta\theta(t-s)\right] ds + g(x,t) = 0, \tag{1}$$

$$u_{tt} - \Delta u_t + \Delta\left(\Delta u + \theta\right) + f(u) = 0, \tag{2}$$

$$u(t) = \Delta u(t) = 0, \tag{3}$$

$$\theta(t) = 0, \tag{4}$$

$$\theta(-s) = \Psi(s). \tag{5}$$

Among them, for $(t,x) \in R^+ \times \Omega,\ u = u(t,x)$ is a real-valued function on $[0,+\infty) \times U$ is the temperature change from the self-equilibrium reference value, u is the vertical displacement of the plate. The function $\mathit{\Psi} : R^+ \times U \to R$ is called the initial history of temperature. It is assumed that the memory kernel function $k : R^+ \to R$ is a positive, bounded and convex function that vanishes at infinity. $f(u) \in C^2(R,R)$ is a sufficiently smooth real-valued function. The given external force $g(x,t)$ has a periodic solution with period T in time t, $g(x,t) \in H^1(T,V^0)$ and

$$\sup_{0\le t\le T} \|g(x,t)\|_{L^2} \equiv K.$$

In addition, to ensure the uniform boundedness of approximate solutions and the validity of subsequent higher-order derivative estimates, we impose a smallness condition on the external force: $\sup_{0\le t\le T} \|g(x,t)\|_{L^2} \equiv K \ll 1$, where K is a sufficiently small positive constant independent of the truncation order N and the parameter $\lambda \in (0,1)$ in the Leray-Schauder fixed-point theorem.

Comprehensive and meticulous research work has been carried out on the thermoelastic plate equations. In recent decades, the long-time behavior of the solutions of plate equations has been studied by many scholars. Woinowsky-Krieger [7] in 1950 and Berger [8] in 1955 established the elastic vibration equations, from which the plate equations were derived. In these studies, we note that in recent decades, the long-time behavior of the solutions of plate equations with thermal memory effects caused by Fourier's law has been studied by many scholars. Dridi [9] integrated thermal damping and viscoelastic damping, extended the theoretical framework to dynamic boundary conditions, proved the exponential or polynomial decay of solutions, overcame the limitation of static boundary conditions in the original literature, and thus established a theoretical basis applicable to the stability analysis of elastically constrained plates in engineering practice. In 1997, Rivera and Shibata [10] considered the initial-boundary value problem.

$$\begin{cases} u_{tt} - h\triangle u_{tt} + \triangle^2 u + \alpha\triangle\theta = 0, \\ \theta_t - \beta\triangle\theta - \alpha\triangle u_t = 0. \end{cases}$$

And it was proved that as time increases, the energy related to the system decays exponentially to zero. When $h > 0$, the system belongs to the weakly hyperbolic type. The estimation of the second- (or third-) order derivatives of the solution on the boundary during this process has been studied in detail [11, 12] for the work in control theory [13,14]. For $\gamma = 0$, in 2006, Kim [15] explored the thermoelastic plate equation with Dirichlet boundary conditions proposed by him in 1992 and proved the exponential decay of the solution. In 2001, Giorgi and Pata [4] considered an abstract linear thermoelastic system with memory and gave the results of existence, uniqueness, and continuous dependence.

$$\begin{cases} \partial_{tt}u + \Delta\left[\Delta u + \vartheta\right] = 0, \\ \partial_t\vartheta + \vartheta - \Delta u_t + \int_0^\infty \left[\beta(\sigma)\vartheta(t-\sigma) - k(\sigma)\Delta\vartheta(t-\sigma)\right] d\sigma = 0. \end{cases}$$

In the presence of memory, the system is shown tobe exponentially stable. These thin plates are affected by thermal deformation and the laws of heat conduction. In 2005, Grasselli et al. [16] considered the evolution problem in the two-dimensional theory of linear hereditary thermoelasticity.

$$\begin{cases} u_{tt}(t) + \Delta\left(\Delta u(t) + \vartheta(t)\right) = 0, \\ \vartheta_t(t) + \int_0^\infty k(s)\left[c\vartheta(t-s) - \Delta\vartheta(t-s)\right] ds - \Delta u_t(t) = 0. \end{cases}$$

This problem simulates a Kirchhoff thermoelastic thin plate with heat-conduction memory effects. In the same year, the Kirchhoff plate equations with storage conditions at the boundary [17] and the stability of the solution of the von Kármán system were studied:

$$\begin{cases} \frac{\partial u}{\partial v} + \int_0^t g_1(t-s)\left(\mathrm{B}_1 u(s) + \rho_1 \frac{\partial u}{\partial v}(s)\right) ds = 0, \\ u - \int_0^t g_2(t-s)\left(\mathrm{B}_2 u(s) - \rho_2 u(s)\right) ds = 0. \end{cases}$$

In 2008, Wu [18] considered a nonlinear plate equation with thermal memory effects due to the non - Fourier heat-flux law:

$$\begin{cases} \vartheta_t - \Delta u_t + \int_0^\infty k(s)\left[-\Delta\vartheta(t-s)\right] ds = 0, \\ u_{tt} - \Delta u_t + \Delta\left(\Delta u + \vartheta\right) + f(u) = 0, \end{cases}$$

where the notations have the same meanings as in the study by Grasselli et al. [16]. The existence and uniqueness of the global solution, as well as the existence and uniqueness of the global attractor, were proved. By using appropriate Łojasiewicz–Simon inequalities, under the assumption that the nonlinear term f is real-analytic, as time tends to infinity, the convergence of the entire solution to a single steady-state was proved. In addition, an estimate of the convergence rate was provided. In 2014, Barbosa and Ma [19] studied the long-time dynamics of the solutions of extendable plate equations with thermal memory. This problem corresponds to a thermoelastic model based on the non-Fourier heat-flux theory. The authors of this paper studied the long-time behavior of a class of nonlinear problems and simulated a thermoelastic extensible plate using the non-Fourier heat-conduction law. In 2023, BK Kakumani and SP Yadav considered a viscoelastic plate equation with a logarithmic nonlinearity in the presence of a nonlinear frictional damping term. They used the Faedo-Galerkin method to prove the existence of solutions to the problem and also proved some general decay-rate results [20]. In 2024, Wang and Liu studied the asymptotic behavior of the nonlinear MGT plate equation in the unbounded domain. By using the energy method in the Fourier space, they proved the optimal decay-estimate results for the non-critical case. The optimality was analyzed by considering the asymptotic expansion of the eigenvalues [21]. In 2024, Ma, Chen, and Deng derived a geometrically nonlinear plate model based on the Kirchhoff hypothesis and the large-deflection hypothesis and provided an error analysis of the corresponding element-free Galerkin method. Numerical experiments, involving clamped square and circular plates with uniform and non-uniform nodes, as well as a plate model with a corner singularity, were presented to validate the theoretical results [22].

This paper mainly aims to study the existence of periodic solutions for a nonlinear plate-coupled system with thermal memory and external force terms. A brief overview of the research status of periodic solutions in the references cited in this paper is presented. In 1997, Hisako Kato [23] proved the existence of periodic solutions for the Navier-Stokes equations by applying the Galerkin method and using the Leray-Schauder fixed-point theorem. In 2001, Academician Guo and Professor Du [24] also proved the existence of periodic solutions for the weakly damped Schrödinger-Boussinesq equations by applying the Galerkin method and using the Leray-Schauder fixed-point theorem. In 2019, Luo and Du [25] used the Galerkin method and the Leray-Schauder fixed-point theorem to prove the existence of time-periodic solutions for the 3-D viscous primitive equations of large-scale dry atmosphere.

Wang and Ma first investigated the existence of pullback attractors for the plate equation with variable delay on the unbounded domain $\mathbb{R}^n$ by means of the theory of multivalued dynamical systems [26]. In 2023, Sun et al. considered the 2D magnetohydrodynamics equations with horizontal dissipation and horizontal magnetic diffusion. The classical solution in $H - k$ $(k \geq 2)$ has been obtained; due to partial dissipation and strong nonlinearity, the global well-posedness of the weak solution in H^1 is still unknown. By combining the classic Galerkin's method with Brouwer's fixed-point theorem, the existence of time-periodic solutions in H^1 with small initial values is obtained [27]. In 2024, Chai and Zhou were concerned with numerical approximations to the one-dimensional Cahn-Hilliard equation with time-periodic solutions [28]. Vasilyeva et al. focused on the core mechanisms of heat and mass transfer and established the coupled governing equations for phase transition processes in porous media [29].

However, it should be emphasized that the nonlinear thermoelastic plate system with memory considered in this paper, described by the evolution equation $(u, v, \theta, \eta)(t + T) = (u, v, \theta, \eta)(t) + f(u) + g(x, t)$, has obvious innovations and extensions compared with the existing research results. On the one hand, from the perspective of system modeling, most of the existing thermoelastic plate systems adopt the Fourier heat conduction law, which assumes that the heat flux propagates at an infinite speed and cannot accurately describe the thermal response of materials with fast transient processes or microscale characteristics. In contrast, our system is derived based on the non-Fourier heat conduction law, which introduces the thermal memory effect through the internal variable η, making the model more in line with the actual physical properties of materials such as polymers and composites under complex thermal environments. On the other hand, compared with the existing literature that either ignores the external force term or only considers constant external forces, this paper incorporates the time-dependent external force $g(x,t)$ into the coupled system. This extension makes the system more applicable to practical engineering scenarios (such as plates subjected to time-periodic loads or thermal disturbances), but also brings new mathematical challenges due to the coupling of time-periodic external excitation, nonlinearity, and memory effect.

The main contributions of this paper are explicitly summarized as follows:

First, we establish a more general and physically realistic nonlinear thermoelastic plate system model with thermal memory and time-dependent external force, which fills the gap that existing studies rarely consider the coupling effect of non-Fourier thermal memory and time-periodic external excitation.

Second, we propose a rigorous analytical framework for proving the existence and uniqueness of T-periodic solutions for the proposed system: by combining the Galerkin method with the Leray-Schauder fixed-point theorem, we first construct the approximate periodic solutions and verify their boundedness, then prove the convergence of the approximate solutions in the appropriate functional space, and finally obtain the existence of T-periodic solutions; on this basis, we further prove the uniqueness of the periodic solution by using the energy estimation method and the Gronwall inequality.

Third, the research results of this paper not only enrich the theoretical system of periodic solutions for nonlinear thermoelastic coupling systems with memory but also provide a reliable theoretical basis for the dynamic analysis and stability control of engineering structures such as plates under complex thermal and mechanical loads.

Based on this, it is proved that there exists a unique time-periodic solution for the nonlinear plate coupled system with thermal memory effects and external force terms, and the period is T, as follows

$$(u, v, \theta, \eta)(t+T) = (u, v, \theta, \eta)(t). \tag{6}$$

2. Basic definition theorem

In this section, we list several classical inequalities that will be used in the subsequent proof process. These inequalities are standard results in functional analysis, and their specific roles in this paper are briefly explained as follows:

Hölder's inequality: Used to estimate the integral of the product of functions, especially in the process of estimating the inner product of nonlinear terms and solution functions.

Lemma 1 (Hölder's inequality [30]). *Let* $1 \le p \le \infty$, *and* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $u \in L^p(\Omega)$

and $v \in L^q(\Omega)$*, then* $uv \in L^1(\Omega)$*, and*

$$\int_\Omega |u \cdot v| dx \leq \left(\int_\Omega |u| dx \right)^{\frac{1}{p}} \left(\int_\Omega |v| dx \right)^{\frac{1}{q}},$$

or the more general Hölder's inequality:

$$\left| \int_\Omega \prod_{i=1}^{s} u_i dx \right| \leq \prod_{i=1}^{s} \left(\int_\Omega |u_i|^{\lambda_i} dx \right)^{\frac{1}{\lambda_i}},$$

where $\lambda_i \geq 1$*,* $\sum_{i=1}^{s} \frac{1}{\lambda_i} = 1$*. For the special case of* $s = 2$*, we have*

$$\int_\Omega |u \cdot v| dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Young's inequality: Applied to handle the product terms in energy estimation and to obtain the boundedness of related terms by splitting the product into the sum of two terms.

Lemma 2 (Young's inequality [30])**.** *(i)For any* $a, b > 0$*, there is* $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$*, where* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. When* $p = q = 2$*, it is the Cauchy inequality.*
(ii) For any $a, b \in R$*, there is* $|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q$*.*

Cauchy's inequality with ε: A flexible tool for adjusting the coefficients of terms in energy inequality, which is crucial for obtaining the dissipative structure of the energy functional.

Lemma 3 (Cauchy's inequality with ε [30])**.**

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2.$$

It holds for any $\varepsilon > 0$ *and any non-negative* a *and* b*.*

Gronwall's inequality: The core tool for proving the boundedness of solutions and the uniqueness of periodic solutions, which can convert differential inequalities into integral inequalities to obtain the estimation of solution norms.

Lemma 4 (Gronwall's inequality [31])**.** *Let* $u(t)$*,* $h(t)$ *and* $g(t)$ *be three locally integrable functions on* $[t_0, +\infty]$*. If they satisfy the differential equation,*

$$\frac{du}{dt} + \gamma u(t) \leq h(t),$$

then for $\gamma \geq 0$*, we have*

$$u(t) \leq \exp(-\gamma(t_1 - t_0))u(t_0) + \int_0^x \exp(-\gamma(t - s))h(s)ds.$$

Poincaré's inequality: Used to establish the equivalence between the norm of the function and the norm of its gradient in the Sobolev space, which simplifies the estimation of the solution's higher-order derivatives.

Lemma 5 (Poincaré inequality [30])**.**

$$\|u\|_{L^2}^2 \leq C_p^2 \|\nabla u\|_{L^2}^2,$$

where C_p is a positive constant that only depends on Ω.

Lemma 6. *For $m < \varepsilon < l$, $g = 0, 1$, $g = 0, 1$, $\forall B_i \in V^0$, $i = 1, 2$, then we have*

$$(a)\ -\varepsilon(Au, B_1) \geq -\frac{\alpha\varepsilon}{4}\|u\|^2 - \frac{\varepsilon}{\alpha}\|B_1\|^2$$

$$(b)\ -\int_U B_1 Au \,\mathrm{d}x \geq -\frac{\alpha}{4}\|B_1\|^2 - \frac{1}{\alpha}\|u\|_2^2$$

$$(c)\ |\pm(A^g B_1, B_2)| \leq \|A^g B_1\| \|B_2\| \leq \frac{1}{\alpha}\|A^g B_1\|^2 + \frac{\alpha}{4}\|B_2\|^2$$

$$(d)\ \left|\pm\int_0^\infty \varepsilon u(s) < A^g B_1, \eta^t(s) > \mathrm{d}s\right| \leq \frac{\varepsilon\alpha k_0}{4}\|A^{g/2} B_1\|^2 + \frac{\varepsilon}{\alpha D_1^{g/2}}\|\eta^t\|_{M^1}^2$$

Proof. Lemma 6(a) and Lemma 6(b) can be obtained by taking $\varepsilon = \frac{\alpha}{2}$ in Lemma 3. For Lemma 6(c) and Lemma 6(d), first make estimations and then use Lemma 3 with $\varepsilon = \frac{\alpha}{2}$.

□

Lemma 7 (Leray-Schauder Fixed-Point Theorem [31])**.** *Let $A : E \to E$ be completely continuous. If the set*

$$\{x \mid x \in E,\ x = \lambda Ax, 0 < \lambda < 1\}$$

is bounded, then A must have a fixed-point in the closed ball T of E, where

$$T = \{x \mid x \in E,\ \|x\| \leq R\},$$

$$R = \sup\{\|x\| \mid x = \lambda Ax, 0 < \lambda < 1\}.$$

3. Equivalent transformation of the system

The memory kernel function $\mu(s)$ and the nonlinear term $f(u)$ are required to satisfy the following conditions:

(I) For any $s \in R^+$, there exists $\sigma_1 > \frac{8C_{21}+C_5}{8(1-C_1)}$ $(0 < C_1 < 1)$, such that

$$\mu(s) \in C^1(R^+) \cap L^1(R^+),\ \mu(s) \geq 0,\ \mu'(s) + \sigma_1\mu(s) \leq 0;$$

Physical background: This condition ensures that the memory effect decays exponentially over time, which is consistent with the physical fact that the influence of historical temperature on the current heat conduction gradually weakens.

(II) Assume that $k(s)$ decays to zero as $s \to \infty$ (i.e., k is zero at infinity). Additionally, define the memory kernel

$$\mu(s) = -k'(s).$$

In recent years, evolutionary equations governed by various non-Fourier heat flux laws have attracted significant interest from mathematicians. Let q denote the heat flux vector; according to Coti et al.'s study [6], the linear constitutive relation for q is given by

$$q(t) = -\int_0^\infty k(s)\nabla\theta(t-s)ds,$$

where $k(s)$ is the thermal conductivity relaxation function. If $k(s)$ is the Dirac delta function at zero (corresponding to instantaneous heat propagation), the above equation reduces to the classical Fourier's law $q = -\nabla\theta$. A typical form of $k(s)$ satisfying the decay property is

$$k(s) = \frac{1}{\sigma}e^{-\frac{s}{\sigma}}, \quad \sigma > 0, \quad k_0 = \frac{1}{\sigma}.$$

Physical Background of the Decay Condition for $\mu(s)$: The memory kernel $\mu(s) = -k'(s)$ is directly derived from the non-Fourier heat flux law $q(t) = -\int_0^\infty k(s)\nabla\theta(t-s)ds$. The decay condition $\mu'(s) + \sigma_1\mu(s) \leq 0$ (with $\sigma_1 > 0$) enforces the exponential decay of the thermal memory effect over time, which aligns with physical reality: the influence of historical temperature variations on the current heat conduction of materials (e.g., polymers, composite plates) diminishes gradually as the time lag s increases. This is an intrinsic physical property of memory-dependent heat conduction, distinguishing the non-Fourier model from the classical Fourier model.

Mathematical Necessity of the Decay Condition: This decay condition is a critical prerequisite for constructing the dissipative structure of the energy functional in our analysis:

(1) By virtue of $\mu'(s) + \sigma_1\mu(s) \leq 0$, we can rigorously estimate the memory integral term and derive the positive dissipative term $\sigma_1\|\eta\|^2_{\mathrm{M}^1}$ in the energy inequality, where η is the internal variable characterizing thermal memory effects.

(2) This dissipative term is indispensable for proving the uniform boundedness of Galerkin approximate solutions (Section 4). It further enables the application of the Leray-Schauder fixed-point theorem and compactness arguments to establish the existence of T-periodic solutions.

(3) Without this decay condition, the memory term would not contribute to energy dissipation, making it impossible to obtain a priori estimates of the solution norm—thus invalidating the subsequent proofs of existence and uniqueness of periodic solutions.

(III) The nonlinear function $f(u) \in C^2(R, R)$ satisfies

$$\liminf_{|s|\to+\infty} \frac{f(s)}{s} > -\frac{1}{C_U},\ f(0) = 0,\ \forall s \in R,$$

$$\liminf_{|s|\to\infty} \frac{sf(s) - C_2F(s)}{s^2} \geq 0,\ \forall s \in R.$$

Physical example: The Duffing-type nonlinearity $f(u) = au + bu^3$ ($a > -\frac{1}{C_U}, b \in R$) satisfies the above conditions. When $|u|\to\infty$, $\frac{f(u)}{u} = a + bu^2 \to +\infty$ (if $b > 0$) or tends to a (if $b = 0$), both of which are greater than $-\frac{1}{C_U}$. At the same time, $sf(s) - C_2F(s) = s(au + bu^3) - C_2(\frac{1}{2}au^2 + \frac{1}{4}bu^4)$, and its limit as $|u|\to\infty$ is non-negative when C_2 is appropriately chosen.

The memory kernel function $\mu(s)$ and the nonlinear term $f(u)$ satisfy the conditions (I)–(III) as before. Introduce $\eta_t + \eta_s = \theta$ to deal with the memory integral $\int_0^\infty k(s)[-\Delta\theta(t-s)]ds$, and Equation (1) can be equivalently transformed into

$$\theta_t - \Delta u_t - \int_0^\infty \mu(s)\Delta\eta(s)ds + g(x,t) = 0, \tag{7}$$

where $g(x,t) \in H^1(T, V^0)$.

Next, for convenience, introduce the transformation $u_t + \varepsilon u = v$, where ε is a positive constant within the following range:

$$\frac{\alpha\iota}{k_0\alpha - 1} < \varepsilon < \min\{\rho_{41}, \rho_{42}, \rho_{43}, \rho_{44}\}.$$

The transformation $u_t + \varepsilon u = v$ ($\varepsilon > 0$) is a key auxiliary variable transformation. Physically, it combines the plate's transverse displacement u and its vibration velocity u_t into v, where εu simulates linear viscous damping to match the actual dynamic behavior of elastic materials without changing the system's core characteristics. Mathematically, it reduces the original high-order coupled Partial Differential Equations (PDE) system to a first-order one by decoupling u_{tt}, introduces a positive dissipative term to ensure the boundedness of Galerkin approximate solutions (a prerequisite for the Leray-Schauder fixed-point theorem), and unifies the periodicity conditions of u and v to simplify the proof of T-periodic solution existence and uniqueness.

Explanation of the transformation: This transformation can convert the second-order time derivative term u_{tt} in Equation (2) into a first-order derivative term of v, which is convenient for constructing the energy functional and analyzing the dissipative properties of the system. By choosing an appropriate range of ε, we can ensure that the lower bound of the interval is less than the upper bound: since $\alpha > \frac{1}{k_0}$ (from the subsequent parameter constraints), we have $k_0\alpha - 1 > 0$, so $\frac{\alpha l}{k_0\alpha - 1}$ is a finite positive number; at the same time, $\rho_{41}, \rho_{42}, \rho_{43}, \rho_{44}$ are all positive constants determined by the system parameters, thus the interval for ε is non-empty.

After the transformation, Equations (1) and (2) can be transformed into

$$\begin{cases} \theta_t - \Delta(v - \varepsilon u) - \int_0^\infty \mu(s)\Delta\eta(s)ds + g(x,t) = 0, \\ v_t - \varepsilon v + \varepsilon^2 u - \Delta(v - \varepsilon u) + \Delta(\Delta u + \theta) + f(u) = 0, \end{cases}$$

initial values $\theta(t) = 0, u(t) = \Delta u(t) = 0,\ x \in \partial U$.

Combining the above analysis, Equations (1)–(5) can be equivalently transformed into

$$\theta_t - \Delta(v - \varepsilon u) - \int_0^\infty \mu(s)\Delta\eta(s)ds + g(x,t) = 0, \tag{8}$$

$$v_t - \varepsilon v + \varepsilon^2 u - \Delta(v - \varepsilon u) + \Delta(\Delta u + \theta) + f(u) = 0, \tag{9}$$

$$u_t + \varepsilon u = v, \tag{10}$$

$$\eta_t + \eta_s = \theta, \tag{11}$$

$$(u, v, \theta, \eta)(t + T) = (u, v, \theta, \eta)(t), \tag{12}$$

$$\theta(t) = 0, u(t) = \Delta u(t) = 0, x \in \partial U. \tag{13}$$

Prove the existence and uniqueness of the periodic solution with period T in the space $V^0 = \mathcal{V}^2 \times \mathcal{V}^0 \times \mathcal{V}^0 \times \mathrm{M}^1$.

4. Approximate solutions and convergence

Based on the equivalent transformation of the original nonlinear plate coupling system completed in the previous section, which converts the system with thermal memory and external force terms into a more analyzable evolution system as in Equations (8)–(13), this section focuses on the construction of approximate solutions, their a priori estimates, and the verification of convergence. The existence of time-periodic solutions of the original system cannot be directly proved due to the complexity of nonlinear terms and memory integral terms; therefore, we first construct approximate solutions using the Galerkin method, then obtain their uniform boundedness through a series of a priori estimates of high-order derivatives and time derivatives, and finally rely on the compact embedding theorem of Sobolev spaces to verify the convergence of the approximate solution sequence, laying a solid foundation for the subsequent proof of the existence and uniqueness of time-periodic solutions.

4.1. Existence of approximate solutions

In this part, we will prove the existence of approximate solutions to Equations (8)–(12). Let ω_j $(j = 1, 2, \dots)$ be a complete orthonormal system in V^0, which is composed of the eigen-functions of the Stokes operator A.

The approximate solution $z_N = (u_N, v_N, \theta_N, \eta_N)$ of Equations (8)–(12) is expressed as follows:

$$u_N(t) = \sum_{j=1}^{N} a_{jN}(t)\omega_j, \quad \theta_N(t) = \sum_{j=1}^{N} c_{jN}(t)\omega_j,$$

$$v_N(t) = \sum_{j=1}^{N} b_{jN}(t)\omega_j, \quad \eta_N(t) = \sum_{j=1}^{N} d_{jN}(t)\omega_j.$$

Corresponding to $Y_N = (\tau_N, q_N, \varphi_N, p_N) \in C^1(T, W_N')$ in the following text, we have

$$\tau_N(t) = \sum_{j=1}^{N} a'_{jN}(t)\omega_j, \quad \varphi_N(t) = \sum_{j=1}^{N} c'_{jN}(t)\omega_j,$$

$$q_N(t)=\sum_{j=1}^{N} b'_{jN}(t)\omega_j, \quad p_N(t)=\sum_{j=1}^{N} d'_{jN}(t)\omega_j.$$

Among them, $a_{jN}(t),\ a'_{jN}(t),\ c_{jN}(t),\ c'_{jN}(t),\ b_{jN}(t),\ b'_{jN}(t),\ d_{jN}(t),\ d'_{jN}(t)$ $(j=1,2,\cdots)$ are coefficient functions of the variable $t\in R^+$. Based on the Galerkin method, the coefficients $a_{jN}(t),c_{jN}(t),b_{jN}(t),d_{jN}(t)$ satisfy the following nonlinear ordinary differential equations:

$$\left(\theta_{Nt}-\Delta(v_N-\varepsilon u_N)-\int_0^\infty \mu(s)\Delta\eta_N(s)ds+g(x,t),\omega_j\right)=0, \tag{14}$$

$$\left(v_{Nt}-\varepsilon v_N+\varepsilon^2 u_N-\Delta(v_N-\varepsilon u_N)+\Delta(\Delta u_N+\theta_N)+f(u_N),\omega_j\right)=0, \tag{15}$$

$$(u_{Nt}+\varepsilon u_N,\omega_j)=(v_N,\omega_j), \tag{16}$$

$$(\eta_{Nt}+\eta_{Ns},\omega_j)=(\theta_N,\omega_j), \tag{17}$$

$$(u_N,v_N,\theta_N,\eta_N)(t+T)=(u_N,v_N,\theta_N,\eta_N)(t). \tag{18}$$

Let W'_N be a subspace of V^0 composed of $\omega_1,\omega_2,\cdots,\omega_N$. It is easy to know that for any $Y_N=(\tau_N,q_N,\varphi_N,p_N)\in C^1(T,W'_N)$, for the system of linear equations:

$$(\theta_{Nt},\omega_j)=\left(\Delta(q_N-\varepsilon\tau_N)+\int_0^\infty \mu(s)\Delta p_N(s)ds-g(x,t),\omega_j\right), \tag{19}$$

$$(v_{Nt}-\varepsilon v_N+\varepsilon^2 u_N,\omega_j)=(\Delta(q_N-\varepsilon\tau_N)-\Delta(\Delta\tau_N+\varphi_N)-f(\tau_N),\omega_j), \tag{20}$$

$$(u_{Nt}+\varepsilon u_N,\omega_j)=(v_N,\omega_j), \tag{21}$$

$$(\eta_{Nt}+\eta_{Ns},\omega_j)=(\theta_N,\omega_j), \tag{22}$$

There exists a unique time-periodic solution $z_N=(u_N,v_N,\theta_N,\eta_N)\in C^1(T,W'_N)$. Therefore, a continuous and compact function $C^1(T,W'_N)$ is constructed in $F: Y_N\to z_N$. Then, by applying the Leray-Schauder fixed-point theorem, the existence of approximate solutions to Equations (14)–(18) can be obtained. When applying the fixed-point theorem, we only need to prove the following. After replacing the nonlinear terms $\lambda\int_0^\infty \mu(s)\Delta\eta_N(s)ds$ and $\lambda f(u_N)$ with $\int_0^\infty \mu(s)\Delta\eta_N(s)ds$ and $f(u_N)$ respectively, prove that

$$\left(\theta_{Nt}-\Delta(v_N-\varepsilon u_N)-\lambda\int_0^\infty \mu(s)\Delta\eta_N(s)ds+g(x,t),\omega_j\right)=0, \tag{23}$$

$$\left(v_{Nt}-\varepsilon v_N+\varepsilon^2 u_N-\Delta(v_N-\varepsilon u_N)+\Delta(\Delta u_N+\theta_N)+\lambda f(u_N),\omega_j\right)=0. \tag{24}$$

For all possible solutions, they are bounded. That is, prove $\sup\limits_{0\le t\le T}\|z_N(t)\|\le C$, where C is a constant independent of $\lambda(0<\lambda<1)$. Since we have simplified the system of Equations (1)–(5) to (14)–(18), we only need to prove that the solutions of the system of Equations (14)–(18) are periodic with period T.

Theorem 1. *We have z_N as the approximate solution of Equations (8)–(13), and we* *only need*

$$\sup_{0\le t\le T}\|z_N(t)\|\le C,$$

where $C>0$ *is a constant independent of* λ *and* N.

Proof. First, perform the following multiplications, summations, and then take the inner products for Equations (23)–(24) and (16)–(17) respectively, and make estimations. Firstly, multiply Equation (23) by $c_{jN}(t)$ to get $c_{jN}(t)\omega_j$, and then sum over $c_{jN}(t)\omega_j$ $(j=1,2,3,\dots)$. Then take the inner-product of $\theta_N(t)$ and Equation (23), that is

$$(\theta_{Nt}-\Delta(v_N-\varepsilon u_N),\theta_N)=\left(\lambda\int_0^\infty\mu(s)\Delta\eta_N(s)ds-g(x,t),\theta_N\right),$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta_N\|^2=&\int_U\Delta v_N\theta_N dx-\varepsilon\int_U\Delta u_N\theta_N dx\\&-\lambda\int_0^\infty\mu(s)\langle A\eta_N(s),\theta_N\rangle ds-\int_U g(x,t)\theta_N dx.\end{aligned}$$

Secondly, multiply Equation (24) by $b_{jN}(t)$ to get $b_{jN}(t)\omega_j$, and then sum over $b_{jN}(t)\omega_j$ $(j=1,2,3,\dots)$. Then take the inner-product of $v_N(t)$ and Equation (24), we have

$$(v_{Nt}-\varepsilon v_N+\varepsilon^2u_N+Av_N,v_N)=(-\varepsilon\Delta u_N-\Delta(\Delta u_N+\theta_N)-\lambda f(u_N),v_N),$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|v_N\|^2-\varepsilon\|v_N\|^2+\varepsilon^2\int_U u_Nv_N dx+\|A^{1/2}v_N\|^2\\&=-\varepsilon\int_U\Delta u_Nv_N dx-\int_U\Delta^2u_Nv_N dx-\int_U\Delta\theta_Nv_N dx-\int_U\lambda f(u_N)v_N dx,\end{aligned}$$

next, multiply Equation (16) by $a_{jN}(t)$ to get $a_{jN}(t)\omega_j$, and then sum over $a_{jN}(t)\omega_j$ $(j=1,2,3,\dots)$. Then take the inner-product of $u_N(t)$ and Equation (16), that is

$$\begin{aligned}(u_{N_t}+\varepsilon u_N,u_N)_{\mathcal{V}^2}&=(v_N,u_N)_{\mathcal{V}^2},\\\frac{1}{2}\frac{d}{dt}\|u_N\|_2^2+\varepsilon\|u_N\|_2^2&=\int_U\Delta^2v_Nu_N dx.\end{aligned}$$

Finally, multiply Equation (17) by $d_{jN}(t)$ to get $d_{jN}(t)\omega_j$, and then sum over $d_{jN}(t)\omega_j$ $(j=1,2,3,\dots)$. Then take the inner-product of $\eta_N(t)$ and Equation (17), that is

$$\begin{aligned}&(\eta_{N_t}+\eta_{Ns},\eta_N)_{\mathcal{M}_1}=(\theta_N,\eta_N)_{\mathcal{M}_1},\\&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_N\|^2_{\mathcal{M}^1}-\int_0^\infty\mu'(s)\left\langle A^{1/2}\eta_N(s),A^{1/2}\eta_N\right\rangle\mathrm{d}s\\&\quad=\lambda\int_0^\infty\mu(s)\left\langle A^{1/2}\eta_N(s),A^{1/2}\theta_N\right\rangle\mathrm{d}s.\end{aligned}$$

In summary, after organizing and combining like-terms, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|z_N\|^2_{\mathcal{V}^0}+\varepsilon\|u_N\|^2_2-\varepsilon\|v_N\|^2+\|A^{1/2}v_N\|^2\\&-\int_0^\infty u'(s)\|A^{1/2}\eta_N\|^2ds+\varepsilon^2\int_U u_Nv_Ndx\\&=-\varepsilon\int_U\Delta u_N\theta_Ndx-\varepsilon\int_U\Delta u_Nv_Ndx\\&-\int_U\lambda f(u_N)v_Ndx-\int_U g(x,t)\theta_Ndx.\end{aligned}\tag{25}$$

Let $J_N(t)=-\int_0^\infty\mu(s)\langle\theta_N(t),\eta_N(s)\rangle ds$, and take its derivative.

By combining the energy equality with the derivative of $J_N(t)$ and using the classical inequalities in Section 2, we estimate each term in the equation. The key steps are as follows: Use Cauchy's inequality with ε to adjust the coefficients of the cross terms, such as $\int_U\Delta u_N\theta_Ndx$ and $\int_U f(u_N)v_Ndx$. Use the condition $\mu'(s)+\sigma_1\mu(s)\le 0$ to estimate the memory term, obtaining the dissipative term $\sigma_1|\eta_N|^2_{\mathrm{M}^1}$. Use the smallness condition $K\ll 1$ of the external force to control the term $\int_U g(x,t)\theta_Ndx$.

After arranging and combining the terms, we obtain the following differential inequality:

$$\begin{aligned}&\frac{d}{dt}J_N(t)+\frac{k_0}{2}\|\theta_N\|^2\\&\le\frac{k_0}{2}\|\nabla v_N\|^2+\frac{\varepsilon^2k_0}{2}\|\nabla u_N\|^2+C_2\|\eta_N\|^2_{\mathrm{M}^1}\\&-C_1\int_0^\infty\mu'(s)\|A^{1/2}\eta_N\|^2ds+\frac{k_0}{2}\|g(x,t)\|^2,\end{aligned}\tag{26}$$

where $C_2=1+k_0>0$. By using the spectral representation $A=\int_\delta^\infty\lambda dE_\lambda$ ($\delta>0$, which is the minimum eigenvalue of A), the following inequality holds

$$\|A^\alpha u_N\|\le\delta^{\alpha-\beta}\|A^\beta u_N\|,\quad(0\le\alpha\le\beta).$$

Next, in order to apply Gronwall's inequality, we perform the following transformation, Equation $(26)\times 2\varepsilon+$Equation (25), and by using the above-mentioned inequalities, we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|z_N\|^2_{\mathcal{V}^0}+2\varepsilon J_N(t)\right)+\varepsilon k_0\|\theta_N\|^2+\varepsilon\|u_N\|^2_2-\varepsilon^3\|\nabla u_N\|^2-\varepsilon\|v_N\|^2\\&-\varepsilon\|A^{1/2}v_N\|^2+\|A^{1/2}v_N\|^2-2\varepsilon C_2\|\eta_N\|^2_{\mathcal{M}^1}-\int_0^\infty\mu'(s)\|A^{1/2}\eta_N\|^2ds\\&+2\varepsilon C_1\int_0^\infty\mu'(s)\|A^{1/2}\eta_N\|^2ds+\varepsilon^2\int_U u_Nv_Ndx\\&\le-\varepsilon\int_U\Delta u_N\theta_Ndx-\varepsilon\int_U\Delta u_Nv_Ndx-\lambda\int_U f(u_N)v_Ndx-\int_U g(x,t)\theta_Ndx.\end{aligned}$$

Since there exists $\frac{\alpha}{4}<m<1$ and $N_1=N_1(m)>0$, such that $f(s)s\ge-\frac{1-m}{C_U}s^2$ for all $|s|\ge N_1$. Next, for the convenience of calculation, we use the inequalities in the lemma for estimation. After simplifying and combining like terms, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\Psi_N(t)\right)+\varepsilon k_0\|\theta_N\|^2+\varepsilon\|u_N\|_2^2-\varepsilon^3 k_0\|\nabla u_N\|^2-\varepsilon\|v_N\|^2+\|A^{1/2}v_N\|^2\\
&-\varepsilon k_0\|A^{1/2}v_N\|^2+\left(\sigma_1(1-2\varepsilon C_1)-2\varepsilon C_2\right)\|\eta_N\|_{\mathcal{M}^1}^2\\
&\le\frac{\varepsilon\alpha}{4}\|u_N\|_2^2+\frac{\varepsilon}{\alpha}\|\theta_N\|^2+\varepsilon(1-m)\|Au_N\|^2-\varepsilon C(|U|,f)+\frac{\varepsilon^3}{\delta^2}\|u_N\|_2^2\\
&+\frac{\varepsilon^2}{\delta}\|u_N\|_2^2+\left(\varpi+\frac{k_0}{2}\right)\|g(x,t)\|^2+\iota\|\theta_N\|^2.
\end{aligned}$$

Among them, ϖ and ι are arbitrary positive constants. Let

$$\Psi_N(t)=\frac{1}{2}\|Z_N(t)\|_{\mathcal{V}^0}^2+2\varepsilon J_N(t)+\int_U F(u_N)dx+\frac{\varepsilon^2}{2}\|u_N\|^2+\frac{\varepsilon}{2}\|\nabla u_N\|^2,$$

the above-mentioned formula becomes

$$\begin{aligned}
&\frac{d}{dt}\Psi_N(t)+\left(\varepsilon k_0-\frac{\varepsilon}{\alpha}-\iota\right)\|\theta_N\|^2+\left(\varepsilon m-\frac{\varepsilon^3}{\delta^2}-\frac{\varepsilon^3 k_0+\varepsilon^2}{\delta}-\frac{\varepsilon\alpha}{4}\right)\|u_N\|_2^2\\
&+\left(\delta-\varepsilon k_0\delta-\varepsilon\right)\|v_N\|^2+\left(\sigma_1-2\sigma_1\varepsilon C_1-2\varepsilon C_2\right)\|\eta_N\|_{\mathcal{M}^1}^2\\
&\le-\varepsilon C(|U|,f)+\left(\varpi+\frac{k_0}{2}\right)\|g(x,t)\|^2=\left(\varpi+\frac{k_0}{2}\right)K^2-\varepsilon C(|U|,f),
\end{aligned}\tag{27}$$

where $C(|U|,f)=|U|\min_{|s|\le N_1}f(s)s$. For $\Psi_N(t)$, we can have

$$C_3\|Z_N(t)\|_{\mathcal{V}^0}^2-C_4\le\Psi_N(t)\le\|Z_N(t)\|_{\mathcal{V}^0}^2+\Phi(\|u_N\|_{v^2}),$$

where $\Phi(\|u_N\|_{v^2})=|U|\max_{|y|\le\|u_N\|_{v^2}}|F(y)|$. For the sake of simplicity, we take

$$\rho=\min(\rho_1,\rho_2,\rho_3,\rho_4),$$

where

$$\begin{aligned}
\rho_1&=\varepsilon m-\frac{\varepsilon^3}{\delta^2}-\frac{\varepsilon^3 k_0+\varepsilon^2}{\delta}-\frac{\varepsilon\alpha}{4},\\
\rho_2&=\delta-\varepsilon k_0\delta-\varepsilon,\\
\rho_3&=\varepsilon k_0-\frac{\varepsilon}{\alpha}-\iota,\\
\rho_4&=\sigma_1-2\sigma_1\varepsilon C_1-2\varepsilon C_2.
\end{aligned}$$

To ensure that ρ is positive, here we take $\frac{1}{k_0-2C_1 t}<\alpha<4m$ and $C_1<\frac{k_0}{2\iota}$. Meanwhile, let

$$\rho_{41}=\min\left\{\frac{\delta}{\delta k_0+1},\frac{\sigma_1}{2\sigma_1 C_1+2C_2}\right\},$$

then Equation (27) is equivalent to the following formula:

$$\frac{d}{dt}\|Z_N(t)\|_{\mathcal{V}^0}^2+\frac{\rho}{C_3}\|Z_N(t)\|_{\mathcal{V}^0}^2\le\frac{\left(\varpi+\frac{k_0}{2}\right)K^2-\varepsilon C(|U|,f)}{C_3}.\tag{28}$$

Integrating Equation (28), we can obtain

$$\int_0^T \|Z_N(t)\|_{\mathcal{V}^0}^2 dt \leq \frac{\left(\varpi + \frac{k_0}{2}\right) K^2 T - \varepsilon C(|U|, f) T}{\rho}. \tag{29}$$

By applying the First Mean Value Theorem for integrals to Equation (29), there exists $t^* \in [0, T]$ such that

$$\|Z_N(t^*)\|_{\mathcal{V}^0}^2 \leq \frac{\left(\varpi + \frac{k_0}{2}\right) K^2 - \varepsilon C(|U|, f)}{\rho}. \tag{30}$$

Combined with Equation (28), integrating from t^* to $t + T$ $(t \in [0, T])$, we have

$$\sup_{0 \leq t \leq T} \|Z_N(t)\|_{\mathcal{V}^0}^2 \leq \frac{\left(\varpi + \frac{k_0}{2}\right) K^2 - \varepsilon C(|U|, f)}{\rho} + \frac{2\left(\varpi + \frac{k_0}{2}\right) K^2 T - 2\varepsilon C(|U|, f) T}{C_3} = C^2,$$

where C^2 is independent of λ and N. In conclusion, it can be proven that the approximate solution $Z_N(t)$ exists for Equations (8)–(13). □

4.2. A priori estimates

In this section, to demonstrate the convergence of the approximate solutions obtained in the previous subsection, we will derive estimates for higher-order and lower-order derivatives as well as some integral terms.

4.2.1. Estimation of high-order derivatives

Before starting to prove the lemma about the uniform boundedness of $A^r z_N(t)$, we note that we can choose the basis functions $\{\omega_j; j = 1, 2, \dots\}$, where ω_j is an eigenfunction of A and also an eigenfunction of A^r. So we have $A\omega_j = \mu_j \omega_j$, and then $A^r \omega_j = \mu_j^r \omega_j$, where μ_j is an eigenvalue of A. Next, we prove the following lemma.

Lemma 8. *Let $z_N(t)$ be the solution of Equations (8)–(13). Then*

$$\|A^r z_N(t)\| \leq C_0, \quad t \in (-\infty, +\infty),$$

where $C_0 > 0$ is a constant independent of N.

Proof. First, perform the following multiplications, summations, and inner-product operations on Equations (23)–(24) and (16)–(17) respectively for estimation. Multiply Equation (23) by $A^{2r} c_{jN}(t)$ to get $A^{2r} c_{jN}(t) \omega_j$, and then sum $A^{2r} c_{jN}(t) \omega_j$ $(j = 1, 2, 3, \dots)$ to obtain $A^{2r} \theta_N(t)$.

Take the inner-product of $A^{2r}\theta_N(t)$ and Equation (23), that is

$$\left(\theta_{N_t}, A^{2r}\theta_N\right) = \left(\Delta(v_N - \varepsilon u_N) + \lambda \int_0^\infty \mu(s)\Delta\eta_N(s)\,\mathrm{d}s - g(x,t), A^{2r}\theta_N\right),$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^r\theta_N\|^2 = -\int_U A^{2r+1}v_N\theta_N\,\mathrm{d}x + \varepsilon\int_U A^{2r+1}u_N\theta_N\,\mathrm{d}x$$
$$-\lambda\int_0^\infty \mu(s)\left\langle A^{r+1/2}\eta_N(s), A^{r+1/2}\theta_N\right\rangle \mathrm{d}s - \int_U g(x,t)A^{2r}\theta_N\,\mathrm{d}x.$$

Secondly, multiply Equation (24) by $A^{2r}b_{jN}(t)$ to get $A^{2r}b_{jN}(t)\omega_j$, and then sum $A^{2r}b_{jN}(t)\omega_j$ $(j = 1,2,3,\dots)$ to obtain $A^{2r}v_N(t)$. Take the inner-product of $v_N(t)$ and Equation (24), we have

$$\left(v_{N_t} - \varepsilon v_N + \varepsilon^2 u_N + Av_N, A^{2r}v_N\right)$$
$$= \left(-\varepsilon\Delta u_N - \Delta(\Delta u_N + \theta_N) - \lambda f(u_N), A^{2r}v_N\right),$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^r v_N\|^2 - \varepsilon\|A^r v_N\|^2 + \varepsilon^2\int_U A^{2r}u_N v_N\,\mathrm{d}x + \|A^{r+1/2}v_N\|^2$$
$$= \varepsilon\int_U A^{2r+1/2}u_N v_N\,\mathrm{d}x - \int_U A^{2r+2}u_N v_N\,\mathrm{d}x$$
$$+ \int_U A^{2r+1}\theta_N v_N\,\mathrm{d}x - \lambda\int_U f(u_N)A^{2r}v_N\,\mathrm{d}x.$$

Then, multiply Equation (16) by $A^{2r}a_{jN}(t)$ to get $A^{2r}a_{jN}(t)\omega_j$, and then sum $A^{2r}a_{jN}(t)\omega_j$ $(j = 1,2,3,\dots)$ to obtain $A^{2r}u_N(t)$. Take the inner-product of $A^{2r}u_N(t)$ and Equation (16), that is

$$\left(u_{N_t} + \varepsilon u_N, A^{2r}u_N\right)_{\mathcal{V}^2} = \left(v_N, A^{2r}u_N\right)_{\mathcal{V}^2},$$

$$\frac{1}{2}\frac{d}{dt}\|A^r u_N\|_2^2 + \varepsilon\|A^r u_N\|_2^2 = \int_U A^{2r+2}v_N u_N dx.$$

Finally, multiply Equation (17) by $A^{2r}d_{jN}(t)$ to get $A^{2r}d_{jN}(t)\omega_j$, and then sum $A^{2r}d_{jN}(t)\omega_j$ $(j = 1,2,3,\dots)$ to obtain $A^{2r}\eta_N(t)$. Take the inner-product of $A^{2r}\eta_N(t)$ and Equation (17), that is

$$\left(\eta_{Nt} + \eta_{Ns}, A^{2r}\eta_N\right)_{\mathrm{M}_1} = \left(\theta_N, A^{2r}\eta_N\right)_{\mathrm{M}_1},$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^r\eta_N\|_{\mathrm{M}^1}^2 - \int_0^\infty \mu'(s)\|A^{r+1/2}\eta_N\|^2\,\mathrm{d}s$$
$$= \lambda\int_0^\infty \mu(s)\left\langle A^{r+1/2}\eta_N(s), A^{r+1/2}\theta_N\right\rangle \mathrm{d}s.$$

In summary, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^r z_N\|_{V^0}^2 + \varepsilon\|A^r u_N\|_2^2 - \varepsilon\|A^r v_N\|^2 + \|A^{r+1/2}v_N\|^2 - \int_0^\infty \mu'(s)\|A^{r+1/2}\eta_N\|^2\,\mathrm{d}s\\&+\varepsilon^2\int_U A^{2r}u_N v_N\,\mathrm{d}x = \varepsilon\int_U A^{2r+1}u_N\theta_N\,\mathrm{d}x + \varepsilon\int_U A^{2r+1}u_N v_N\,\mathrm{d}x\\&-\lambda\int_\Omega f(u_N)A^{2r}v_N\,\mathrm{d}x - \int_U g(x,t)A^{2r}\theta_N\,\mathrm{d}x.\end{aligned} \tag{31}$$

Let $A^{2r}J_N(t) := -\int_0^\infty \mu(s)\left\langle A^r\theta_N(t), A^r\eta_N(s)\right\rangle \mathrm{d}s$, then

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}A^{2r}J_N(t)+\frac{k_0}{2}\|A^r\theta_N\|^2\\
&\quad\le \frac{k_0}{2}\|A^{r+1/2}v_N\|^2+\frac{\varepsilon^2k_0}{2}\|A^{r+1/2}u_N\|^2+C_2\|A^r\eta_N\|^2_{\mathrm{M}^1}\\
&\quad - C_1\int_0^\infty \mu'(s)\|A^{r+1/2}\eta_N\|^2\,\mathrm{d}s+\frac{D_1k_0}{2}\|g(x,t)\|^2,
\end{aligned} \tag{32}$$

where $D_1 = \|A^r\|^2$. To apply Gronwall's inequality, we perform the following transformation, Equation $(32)\times 2\varepsilon+$ Equation (31).

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|A^rz_N(t)\|^2_{V^0}+2\varepsilon A^{2r}J_N(t)\right)+\varepsilon k_0\|A^r\theta_N\|^2+\varepsilon\|A^ru_N\|_2^2\\
&\quad-\varepsilon^3\|A^{r+1/2}u_N\|^2-\varepsilon\|A^rv_N\|^2-\varepsilon\|A^{r+1/2}v_N\|^2+\|A^{r+1/2}v_N\|^2\\
&\quad-2\varepsilon C_2\|A^r\eta_N\|^2_{\mathrm{M}^1}-\int_0^\infty \mu'(s)\|A^{r+1/2}\eta_N\|^2\,\mathrm{d}s\\
&\quad+2\varepsilon C_1\int_0^\infty \mu(s)\|A^{r+1/2}\eta_N\|^2\,\mathrm{d}s+\varepsilon^2\int_\Omega A^{2r}u_Nv_N\,\mathrm{d}x\\
&\le -\varepsilon\int_\Omega A^{2r}\Delta u_N\theta_N\,\mathrm{d}x-\varepsilon\int_\Omega A^{2r}\Delta u_Nv_N\,\mathrm{d}x\\
&\quad-\lambda\int_\Omega A^{2r}f(u_N)v_N\,\mathrm{d}x-\int_\Omega A^{2r}g(x,t)\theta_N\,\mathrm{d}x
\end{aligned}$$

After simplifying and combining like terms, we obtain

$$\frac{d}{dt}\left(\Psi_N(t)\right)+\rho^*\|z_N(t)\|^2\le\left(\varpi+\frac{D_1k_0}{2}\right)K^2-\varepsilon C(|U|,f), \tag{33}$$

where

$$\begin{aligned}
&\Psi_N(t)=\frac{1}{2}\|A^rz_N(t)\|^2_{V^0}+2\varepsilon A^{2r}J_N(t)-\left|\int_U A^{2r}F(u_N)\,\mathrm{d}x\right|\\
&\quad+\frac{\varepsilon^2}{2}\|A^ru_N\|^2+\frac{\varepsilon}{2}\|\nabla A^ru_N\|^2,\\
&\rho^*=\min\{\rho_1^*,\rho_2,\rho_3,\rho_4\},\\
&\rho_1^*=\varepsilon-\frac{\varepsilon^3}{\delta^2}-\frac{\varepsilon^3k_0+\varepsilon^2}{\delta}-\frac{\varepsilon\alpha}{4}-\varepsilon(1-m)\delta^r,\\
&\rho_2=\delta-\varepsilon k_0\delta-\varepsilon,\\
&\rho_3=\varepsilon k_0-\frac{\varepsilon}{\alpha}-\iota,\\
&\rho_4=\sigma_1-2\sigma_1\varepsilon C_1-2\varepsilon C_2.
\end{aligned}$$

The range of parameters has been restricted above to make ρ^* positive, where $\frac{4-\alpha-4\delta^r}{4\delta^r}<m<1$. By estimating $\Psi_N(t)$, we can obtain

$$C_{31}\|A^rz_N(t)\|^2_{V^0}-C_{41}\le\Psi_N(t)\le\|A^rz_N(t)\|^2_{V^0}+\Phi\left(\|A^ru\|_{\mathcal{V}^2}\right).$$

In summary, we have

$$\|A^r z_N(t^*)\|_{V^0} \le \|Z_N(t^*)\|_{V^0} \le \frac{\left(\varpi + \frac{D_1 k_0}{2}\right)^{1/2} K - \varepsilon^{1/2} \left(C(|U|, f)\right)^{1/2}}{\rho^{1/2}},$$

when we take $t = t^*, K < 1$, we can get the following inequality:

$$\|A^r z_N(t)\|_{V^0} \le \frac{\left(\varpi + \frac{D_1 k_0}{2}\right)^{1/2} K^{1/2} - \varepsilon^{1/2} \left(C(|U|, f)\right)^{1/2}}{\rho^{1/2}}.$$

Let

$$T^* = \sup\left\{T \mid \|A^r z_N(t)\| \le \frac{\left(\varpi + \frac{D_1 k_0}{2}\right)^{1/2} K^{1/2} - \varepsilon^{1/2} \left(C(|U|, f)\right)^{1/2}}{\rho^{1/2}}, \forall t \in [t^*, T)\right\},$$

from the value of K, we can get $T^* = \infty$. Actually, if T^* $(t^* < T^*)$ is finite, the following conclusions should hold

$$\|A^r z_N(t)\|_{V^0} \le \frac{\left(\varpi + \frac{D_1 k_0}{2}\right)^{1/2} K^{1/2} - \varepsilon^{1/2} \left(C(|U|, f)\right)^{1/2}}{\rho^{1/2}}, \forall t \in [t^*, T^*),$$

$$\|A^r z_N(T^*)\|_{V^0} = \frac{\left(\varpi + \frac{D_1 k_0}{2}\right)^{1/2} K^{1/2} - \varepsilon^{1/2} \left(C(|U|, f)\right)^{1/2}}{\rho^{1/2}}.$$

Then Equation (33) can be transformed into

$$\frac{d}{dt} C_3 \|A^r z_N(t)\|_{V^0}^2 + \rho^* \|A^r z_N\|^2 \le \left(\varpi + \frac{D_1 k_0}{2}\right) K^2 - \varepsilon C(|U|, f).$$

When $t = T^*$, we can get $\frac{d}{dt}\|A^r z_N(t)\|_{V^0}^2 < 0$. Therefore, for any $t \in [T^*, T^* + \varsigma)$, we have

$$\|A^r z_N(t)\|_{V^0} \le \frac{\left(\varpi + \frac{D_1 k_0}{2}\right)^{1/2} K^{1/2} - \varepsilon^{1/2} \left(C(|U|, f)\right)^{1/2}}{\rho^{*1/2}}.$$

Then, according to the assumption, $T^* = \infty$ holds. In addition, due to the periodicity of $z_N(t)$, for any $t \in (-\infty, \infty)$, we have

$$\|A^r z_N(t)\|_{V^0} \le \frac{\left(\varpi + \frac{D_1 k_0}{2}\right)^{1/2} K^{1/2} - \varepsilon^{1/2} \left(C(|U|, f)\right)^{1/2}}{\rho^{1/2}} = C_0.$$

Next, to show the convergence of the approximate solutions, we will derive the estimation of high-order derivatives. First, we recall Lemma 8. That is, if K is small enough, the approximate solutions satisfy

$$\sup_t \|A^r z_N(t)\|_{V^0} \le C(K).$$

$C(K)$ represents a constant that depends on K, but is independent of N. Moreover, it is easy to see that if K is small enough, then $C(K) < C$ holds for any positive constant C. In the subsequent proofs, we will take this conclusion as a fact. □

4.2.2. H^1 estimation of $Z_N(t)$ and V^0 estimation of Z_{Nt}

Lemma 9. *Let $z_N(t)$ be the approximate solution of Equations (8)–(13). Then we have*

$$\sup_t \|\nabla z_N(t)\|_{V^0} \leq C(K_0, K),$$

where $K_0 = \left(\int_0^T \|g\|^2 dt\right)^{1/2}$, and $C(K_0, K)$ is a constant independent of N.

Proof. Multiply Equations (19)–(22) by $(Ac_{jN}, Ab_{jN}, Aa_{jN}, Ad_{jN})$, sum them up, and then take the inner-product (similar to Lemma 8), we get

$$(\theta_{Nt}, A\theta_N) = \left(\Delta(q_N - \varepsilon\tau_N) + \int_0^\infty \mu(s)\Delta p_N(s)\,\mathrm{d}s - g(x,t), A\theta_N\right), \tag{34}$$

$$\left(v_{Nt} - \varepsilon v_N + \varepsilon^2 u_N, Av_N\right) = \left(\Delta(q_N - \varepsilon\tau_N) - \Delta(\Delta\tau_N + \varphi_N) - f(\tau_N), Av_N\right), \tag{35}$$

$$\left(u_{Nt} + \varepsilon u_N, Au_N\right) = \left(v_N, Au_N\right), \tag{36}$$

$$\left(\eta_{Nt} + \eta_{Ns}, A\eta_N\right) = \left(\theta_N, A\eta_N\right). \tag{37}$$

By repeating the process of taking the inner product and then estimating as in Lemma 8, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla z_N\|_{V^0}^2 + \varepsilon\|\nabla u_N\|_2^2 - \varepsilon\|\nabla v_N\|^2 + \|Av_N\|^2 \\
&- \int_0^\infty \mu'(s)\|A\eta_N\|^2\,\mathrm{d}s + \varepsilon^2\int_U Au_N v_N\,\mathrm{d}x \\
&= \varepsilon\int_U A^{1+1}u_N\theta_N\,\mathrm{d}x + \varepsilon\int_U A^{1+1}u_N v_N\,\mathrm{d}x \\
&- \lambda\int_U f(u_N)Av_N\,\mathrm{d}x - \int_U g(x,t)A\theta_N\,\mathrm{d}x.
\end{aligned} \tag{38}$$

Similar to taking the corresponding $AJ_N(t)$ in Lemma 8, we have

$$\begin{aligned}
&\frac{d}{dt}AJ_N(t) + \frac{k_0}{2}\|\nabla\theta_N\|^2 \\
&\leq \frac{k_0}{2}\|Av_N\|^2 + \frac{\varepsilon^2 k_0}{2}\|Au_N\|^2 + C_{21}\|\nabla\eta_N\|_{\mathrm{M}^1}^2 \\
&- C_1\int_0^\infty \mu'(s)\|A\eta_N\|^2 ds + k_0\|g(x,t)\|\|A\eta_N\|_{\mathrm{M}^1},
\end{aligned}$$

where $\Psi_{N_2}(t) = \frac{1}{2}\|\nabla z_N(t)\|_{V^0}^2 + 2\varepsilon AJ_N(t) + \frac{\varepsilon^2}{2}\|\nabla u_N\|^2 + \frac{\varepsilon}{2}\|Au_N\|^2$, $C_{21} = 1/2 + k_0 > 0$. Here we can obtain

$$\begin{aligned}
&\tfrac{d}{dt}C_{32}\|\nabla z_N(t)\|_{V^0}^2 + \rho^{**}\|Az_N\|_{V^0}^2 \\
&\leq \|g\|\|A\theta_N\| + k_0\|g(x,t)\|\|A\eta_N\|_{\mathrm{M}_1} + \|f\|\|Av_N\|, \\
&\leq e\|g\|\|Az_N\|_{V^0} + \|f\|\|Az_N\|_{V^0}
\end{aligned} \tag{39}$$

where $e = \max\{1, k_0\}$, $\rho^{**} = \min\{\rho_1^{**}, \rho_2, \rho_3^*, \rho_4^*\}$,

$$\begin{aligned}
\rho_1^{**} &= \varepsilon - \frac{\varepsilon^3}{\delta^2} - \frac{\varepsilon^3 k_0 + \varepsilon^2}{\delta} - \frac{\varepsilon\alpha}{4}, \\
\rho_2 &= \delta - \varepsilon k_0 \delta - \varepsilon, \\
\rho_3^* &= \varepsilon k_0 - \frac{\varepsilon}{\alpha}, \\
\rho_4^* &= \sigma_1 - 2\sigma_1 \varepsilon C_1 - 2\varepsilon C_{21}.
\end{aligned}$$

The range of parameters has been restricted above to make ρ^{**} positive, where $\frac{1}{k_0} < \alpha < 4$. Meanwhile, let

$$\rho_{42} = \min\left\{\frac{\sigma_1}{2\sigma_1 C_1 + 2C_{21}}, \frac{-\delta + \delta\sqrt{G}}{2\delta k_0 + 2}\right\}, \quad G = 1 + (1 + \delta k_0)(4 - \alpha).$$

According to the assumption of the nonlinear term f, we have

$$\|f\| \leq \frac{1-m}{C_U}\|u_N\| \leq \frac{1-m}{C_U}\|z_N\|_{V^0} \leq \frac{1-m}{\delta C_U}\|Az_N\|_{V^0},$$

where $m \in \left(\frac{\alpha}{4}, 1\right)$, $|u| \geq N$ and $N = N(m) > 0$.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} C_{32}\|\nabla z_N(t)\|_{V^0}^2 + \rho^{**}\|Az_N\|_{V^0}^2 \\
&\leq e\|g\|\|A\theta_N\| + \|f\|\|Av_N\| \\
&\leq e\|g\|\|Az_N\|_{V^0} + \frac{1-m}{\delta C_U}\|Az_N\|_{V^0}^2
\end{aligned} \tag{40}$$

Integrating the above-mentioned formula, it can be transformed into

$$\rho^{**}\|Az_N\|_{V^0} \leq eK_0 + \frac{1-m}{\delta C_U}\|Az_N\|_{V^0}. \tag{41}$$

By moving terms and performing combined operations, we get

$$\|Az_N\|_{V^0}^2 \leq \frac{e^2 K_0^2}{\left(\rho^{**} - \frac{1-m}{\delta C_U}\right)^2} = E_0, \tag{42}$$

Also, since

$$\|\nabla z_N(t^*)\|_{V^0}^2 \leq \delta^{-1}\|Az_N(t^*)\|_{V^0}^2 \leq \delta^{-1}E_0, \quad t^* \in [0, T], \tag{43}$$

integrating from t^* to $t + T$ $(t \in [0, T])$ combined with Equation (40), we obtain

$$C_{32}\|\nabla z_N(t)\| - C_{32}\|\nabla z_N(t^*)\| \leq \left(eK_0 - \rho^{**} - \frac{1-m}{\delta C_U}\right) 2TE_0. \tag{44}$$

Combining Equations (43) and (44), we have

$$\sup_t \|\nabla z_N(t)\|_{V^0}^2 \leq \delta^{-1}E_0 + \frac{\left(eK_0 - \rho^{**} - \frac{1-m}{\delta C_U}\right) 2TE_0}{C_{32}} = C(K_0),$$

where $C(K_0)$ is independent of N. □

Lemma 10. *Let z_N be the approximate solution of Equations (8)–(13). Then we have*

$$\sup_t \|z_{Nt}(t)\|_{V^0} \leq C(K_0, K_1, K),$$

where $K_0 = \left(\int_0^T \|g\|^2 dt\right)^{1/2}$, $K_1 = \left(\int_0^T \|g_t\|^2 dt\right)^{1/2}$, *and* $C(K_0, K_1, K)$ *is a constant independent of* N.

Proof. Multiply Equations (19)–(22) by $(c_{jNt}, b_{jNt}, a_{jNt}, d_{jNt})$ respectively, sum them up, and then take the inner-product (similar to Lemma 8), we get

$$(\theta_{Nt}, \theta_N) = \left(\Delta(q_N - \varepsilon\tau_N) + \int_0^\infty \mu(s)\Delta p_N(s)ds - g(x,t), \theta_{Nt}\right), \tag{45}$$

$$\left(v_{Nt} - \varepsilon v_N + \varepsilon^2 u_N, v_{Nt}\right) = \left(\Delta(q_N - \varepsilon\tau_N) - \Delta(\Delta\tau_N + \varphi_N) - f(\tau_N), v_{Nt}\right), \tag{46}$$

$$(u_{Nt} + \varepsilon u_N, u_{Nt}) = (v_N, u_{Nt}), \tag{47}$$

$$(\eta_{Nt} + \eta_{Ns}, \eta_{Nt}) = (\theta_N, \eta_{Nt}). \tag{48}$$

Taking the inner-product of the above equations respectively and then summing up the four equations, we obtain

$$\begin{aligned}
&\|\nabla z_N\|_{V^0}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(-\int_0^\infty \mu'(s)\|A^{1/2}\eta_N\|^2\,\mathrm{d}s \\
&\quad + \varepsilon\|u_N\|_2^2 - \varepsilon\|v_N\|^2 + \|A^{1/2}v_N\|^2\bigg) + \varepsilon^2\int_U u_N v_{Nt}\,\mathrm{d}x \\
&= \lambda\int_0^\infty \mu(s)\left\langle A^{1/2}\eta_{Nt}(s), A^{1/2}\theta_N\right\rangle \mathrm{d}s \\
&\quad - \lambda\int_0^\infty \mu(s)\left\langle A\eta_N(s), \theta_{Nt}\right\rangle \mathrm{d}s + \int_U \Delta^2 v_N u_{Nt}\,\mathrm{d}x - \int_U \Delta^2 u_N v_{Nt}\,\mathrm{d}x \\
&\quad + \int_U \Delta v_N\theta_{Nt}\,\mathrm{d}x - \int_U \Delta\theta_N v_{Nt}\,\mathrm{d}x - \varepsilon\int_U \Delta u_N v_{Nt}\,\mathrm{d}x - \varepsilon\int_U \Delta u_N\theta_{Nt}\,\mathrm{d}x \\
&\quad - \int_U \lambda f(u_N)v_{Nt}\,\mathrm{d}x - \int_U g(x,t)\theta_{Nt}\,\mathrm{d}x \\
&= \|A^{1/2}\theta_N\|^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\lambda\int_0^\infty \mu(s)\|A^{1/2}\eta_N\|\,\mathrm{d}s}{\|A^{1/2}\theta_N\|} + \|v_N\|^2\frac{\mathrm{d}}{\mathrm{d}t}\int_U \frac{\Delta^2 u_N + \Delta\theta_N}{v_N}\,\mathrm{d}x \\
&\quad - \varepsilon\int_U \Delta u_N v_{Nt}\,\mathrm{d}x - \varepsilon\int_U \Delta u_N\theta_{Nt}\,\mathrm{d}x - \int_U \lambda f(u_N)v_{Nt}\,\mathrm{d}x - \int_U g(x,t)\theta_{Nt}\,\mathrm{d}x.
\end{aligned}$$

Estimating the above-mentioned formula using inequalities, we have

$$\begin{aligned}
&\|z_{Nt}\|_{V^0}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigg(-\int_0^\infty \mu'(s)\|A^{1/2}\eta_N\|^2\,\mathrm{d}s \\
&\quad + \varepsilon\|u_N\|_2^2 - \varepsilon\|v_N\|^2 + \|A^{1/2}v_N\|^2\bigg) + \varepsilon^2\int_U u_N v_{Nt}\,\mathrm{d}x \\
&\leq \|A^{1/2}\theta_N\|^2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\int_0^\infty \mu(s)\|A^{1/2}\eta_N\|\,\mathrm{d}s}{\|A^{1/2}\theta_N\|} \\
&\quad + \|v_N\|^2\frac{\mathrm{d}}{\mathrm{d}t}\int_U \frac{\Delta^2 u_N + \Delta\theta_N}{v_N}\,\mathrm{d}x + \varepsilon\|u_N\|_2^2 + \frac{\varepsilon}{2}\|v_{Nt}\|^2 \\
&\quad + \frac{\varepsilon}{2}\|\theta_{Nt}\|^2 + \left|\int_U f(u_N)v_{Nt}\,\mathrm{d}x\right| + 2\|g\|^2 + \frac{1}{8}\|\theta_{Nt}\|^2.
\end{aligned}$$

After transformation and simplification, we get

$$\begin{aligned}
&\|z_{Nt}\|_{V^0}^2+\frac{1}{2}\frac{d}{dt}\left(-\int_0^\infty \mu'(s)\|A^{1/2}\eta_N\|^2 ds+\varepsilon\|u_N\|_2^2-\varepsilon\|v_N\|^2+\|A^{1/2}v_N\|^2\right)\\
&\le \frac{1}{2}\|u_{Nt}\|_2^2+\left(\frac{1+k_0D_{11}+\varepsilon}{2}+\frac{1}{8}\right)\|\theta_{Nt}\|^2+\frac{1}{2}\|\eta_{Nt}\|_{\mathrm{M}^1}^2\\
&+D_{12}\|v_N\|^2+\frac{1+k_0D_{11}}{2}\|\theta_N\|^2\\
&+\frac{1}{2}\|\eta_N\|_{\mathrm{M}^1}^2+\left(\frac{\varepsilon^2}{2\delta^2}+\varepsilon+\frac{1}{2}\right)\|u_N\|_2^2+\left(\frac{\varepsilon^2+\varepsilon}{2}+D_{12}\right)\|v_{Nt}\|^2+\\
&\left|\int_U f(u_N)v_{Nt}dx\right|+2\|g\|^2,
\end{aligned}$$

that is

$$\begin{aligned}
&\|z_{Nt}\|_{V^0}^2+\frac{1}{2}\frac{d}{dt}\left(\sigma_1\|\eta_N\|_{\mathrm{M}^1}^2+\varepsilon\|u_N\|_2^2+(\delta-\varepsilon)\|v_N\|^2+\frac{\varepsilon}{4}\|\theta_N\|^2\right)\\
&\le L\|z_{Nt}\|_{V^0}^2+R\|z_{Nt}\|_{V^0}^2+\frac{1-m}{\delta C_U}\|z_N\|_{V^0}^2+2\|g\|^2,
\end{aligned}$$

among them

$$\begin{aligned}
&L=\max\left\{\frac{\varepsilon^2+\varepsilon}{2}+D_{12},\frac{1+k_0D_{11}+\varepsilon}{2}+\frac{1+\varepsilon}{8}\right\},\\
&R=\max\left\{\frac{\varepsilon^2}{2\delta^2}+\varepsilon+\frac{1}{2},D_{12},\frac{1+k_0D_{11}}{2}+\frac{\varepsilon}{8}\right\},\\
&Q=\min\left\{\sigma,\delta-\varepsilon,\frac{\varepsilon}{4}\right\}.
\end{aligned}$$

By restricting the parameters and choosing an appropriate α, we have $1-L>0$. Meanwhile, let

$$\rho_{43}=\min\left\{-\frac{1}{2}+\Delta_1,\frac{3-4k_0D_{11}}{5}\right\},\quad \Delta_1=\sqrt{\frac{9}{4}-2D_{12}},$$

after moving terms and combining like-terms, the following inequality is obtained

$$\int_0^t\|z_{Nt}\|_{V^0}^2dt\le\frac{T\left(R+\frac{1-m}{\delta C_U}\right)}{1-L}C(K)+\frac{2K_0^2}{1-L}=C(K,K_0). \tag{49}$$

In addition, to distinguish Equations (14)–(17) with respect to t, multiply by the derivatives $(c_{jNt},b_{jNt},a_{jNt},d_{jNt})$ and sum over j, and we get

$$(\theta_{Ntt},\theta_{Nt})=\left(\Delta(q_{Nt}-\varepsilon\tau_{Nt})+\lambda\int_0^\infty\mu(s)\Delta p_{Nt}(s)ds-g_t(x,t),\theta_{Nt}\right), \tag{50}$$

$$\begin{aligned}(v_{Ntt}-\varepsilon v_{Nt}+\varepsilon^2u_{Nt},v_{Nt})=(&\Delta(q_{Nt}-\varepsilon\tau_{Nt})-\Delta(\Delta\tau_{Nt}+\varphi_{Nt})\\&-\lambda f_t(\tau_N)\tau_{Nt},v_{Nt}),\end{aligned} \tag{51}$$

$$(u_{Ntt}+\varepsilon u_{Nt},u_{Nt})=(v_{Nt},u_{Nt}), \tag{52}$$

$$(\eta_{Ntt}+\eta_{Nst},\eta_{Nt})=(\theta_{Nt},\eta_{Nt}). \tag{53}$$

by repeating the process of estimation after taking the inner product in Lemma 8, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\|z_{Nt}\|_{V^0}^2+\bigg(-\int_0^\infty \mu'(s)\|A^{1/2}\eta_{Nt}\|^2\,\mathrm{d}s\\
&\quad+\varepsilon\|u_{Nt}\|_2^2-\varepsilon\|v_{Nt}\|^2+\|A^{1/2}v_{Nt}\|^2\bigg)\Bigg)+\varepsilon^2\int_U u_{Nt}v_{Nt}\,\mathrm{d}x\\
&=\lambda\int_0^\infty \mu(s)\left\langle A^{1/2}\eta_{Nt}(s),A^{1/2}\theta_{Nt}\right\rangle\,\mathrm{d}s\\
&\quad-\lambda\int_0^\infty \mu(s)\left\langle A\eta_{Nt}(s),\theta_N\right\rangle\,\mathrm{d}s+\int_U \Delta^2 v_{Nt}u_{Nt}\,\mathrm{d}x-\int_U \Delta^2 u_{Nt}v_{Nt}\,\mathrm{d}x\\
&\quad+\int_U \Delta v_{Nt}\theta_{Nt}\,\mathrm{d}x-\int_U \Delta\theta_{Nt}v_{Nt}\,\mathrm{d}x-\varepsilon\int_U \Delta u_{Nt}v_{Nt}\,\mathrm{d}x\\
&\quad-\varepsilon\int_U \Delta u_{Nt}\theta_{Nt}\,\mathrm{d}x-\int_U \lambda f_t u_{Nt}v_{Nt}\,\mathrm{d}x-\int_U g_t(x,t)\theta_{Nt}\,\mathrm{d}x.
\end{aligned}$$

After arranging and combining like-terms, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_{Nt}\|_{V^0}^2+\sigma_1\|\eta_{Nt}\|_{\mathrm{M}^1}^2+\varepsilon\|u_N\|_2^2+\left(\frac{1}{C_p^2}-\varepsilon\right)\|v_{Nt}\|^2\\
&\leq-\varepsilon\int_U \Delta u_{Nt}v_{Nt}\,\mathrm{d}x-\varepsilon\int_U \Delta u_{Nt}\theta_{Nt}\,\mathrm{d}x-\varepsilon^2\int_U u_{Nt}v_{Nt}\,\mathrm{d}x\\
&\quad-\int_U \lambda f_t u_{Nt}v_{Nt}\,\mathrm{d}x-\int_U g_t(x,t)\theta_{Nt}\,\mathrm{d}x\\
&\leq\left(\frac{\varepsilon^2}{2\delta^2}+\frac{2\varepsilon}{\alpha}\right)\|u_{Nt}\|_2^2+\left(\frac{\varepsilon^2}{2}+\frac{\alpha\varepsilon}{4}\right)\|v_{Nt}\|^2\\
&\quad+\left(\frac{\alpha\varepsilon}{4}+\frac{1}{8}\right)\|\theta_{Nt}\|^2+\left|\int_U f_t u_{Nt}v_{Nt}\,\mathrm{d}x\right|+2\|g_t\|^2.
\end{aligned}\tag{54}$$

Based on the assumption of the nonlinear term f, after taking derivatives, we can obtain the inequality $\|f_t\|\leq\frac{1-m}{C_U}$. Then we have

$$\begin{aligned}
&\frac{d}{dt}J_{Nt}(t)+\frac{k_0}{2}\|\theta_{Nt}\|^2\leq\frac{k_0D_{11}}{2}\|v_{Nt}\|^2+\frac{\varepsilon^2k_0}{2D_{11}}\|u_{Nt}\|_2^2\\
&\quad+\left(C_{21}+C_1\sigma_1+\frac{C_5}{8}\right)\|\eta_{Nt}\|_{\mathrm{M}^1}^2+2k_0\|g_t(x,t)\|^2,
\end{aligned}\tag{55}$$

where $D_{11}=\|A^{1/2}\|^2$, $D_{12}=\|A\|^2<\frac{9}{8}$. Combining Equations (54) and (55), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|z_{Nt}\|_{V^0}^2+J_{Nt}(t)\right)+\sigma_1\|\eta_{Nt}\|_{\mathrm{M}^1}^2+\varepsilon\|u_{Nt}\|_2^2\\
&\quad+\frac{k_0}{2}\|\theta_{Nt}\|^2+\left(\frac{1}{C_p^2}-\varepsilon\right)\|v_{Nt}\|^2\\
&\leq\left(\frac{\varepsilon^2}{2\delta^2}+\frac{2\varepsilon}{\alpha}+\frac{1}{\delta^2}+\frac{\varepsilon^2k_0}{2D_{11}}\right)\|u_{Nt}\|_2^2\\
&\quad+\left(\frac{\varepsilon^2}{2}+\frac{\alpha\varepsilon}{4}+\frac{1-m}{2C_U}+\frac{k_0D_{11}}{2}\right)\|v_{Nt}\|^2\\
&\quad+\left(\frac{\alpha\varepsilon}{4}+\frac{1}{8}\right)\|\theta_{Nt}\|^2+\left(C_{21}+C_1\sigma_1+\frac{C_5}{8}\right)\|\eta_{Nt}\|_{\mathrm{M}^1}^2\\
&\quad+(2+2k_0)\|g_t\|^2.
\end{aligned}$$

Since $C_6\|z_{Nt}\|_{V^0}^2 - C_7 \le \frac{1}{2}\|z_{Nt}\|_{V^0}^2 + J_{Nt}(t)$, after scaling the above-mentioned formula, we have

$$\frac{d}{dt}C_6\|z_{Nt}\|_{V^0}^2 + L_1\|z_{Nt}\|_{V^0}^2 \le R_1\|z_{Nt}\|_{V^0}^2 + (2+2k_0)\|g_t\|^2, \tag{56}$$

where

$$L_1 = \min\{\varepsilon,\ \frac{1}{C_p^2} - \varepsilon,\ \frac{k_0}{2},\ \sigma_1\},$$

$$R_1 = \max\Big\{\frac{\varepsilon^2}{2\delta^2} + \frac{2\varepsilon}{\alpha} + \frac{1}{\delta^2} + \frac{\varepsilon^2 k_0}{2D_{11}},\ \frac{\varepsilon^2}{2} + \frac{\alpha\varepsilon}{4} + \frac{1-m}{2C_U} + \frac{k_0 D_{11}}{2},\ \frac{\alpha\varepsilon}{4} + \frac{1}{8},\ C_{21} + C_1\sigma_1 + \frac{C_5}{8}\Big\}.$$

By restricting the parameters, we have $L_1 - R_1 > 0$. When $m > \frac{C_p^2 - 2C_U + C_U C_p^2 k_0 D_{11}}{C_p^2}$, let

$$\rho_{44} = \min\left\{\frac{\delta^2 D_{11}(-2+\alpha+\alpha\sqrt{I_1})}{\alpha(D_{11}+k_0\delta^2)}, -1 - \frac{\alpha}{4} + \sqrt{I_2}, \frac{4k_0 - 1}{2\alpha}\right\},$$

where $I_1 = 1 - \frac{4}{\alpha} + \frac{4}{\alpha^2} - \frac{2}{\delta^4} - \frac{2k_0}{\delta^2 D_{11}}$, $I_2 = 1 + \frac{\alpha}{2} + \frac{\alpha^2}{16} + \frac{2}{C_p^2} - \frac{1-m}{C_U} - k_0 D_{11}$.

After moving terms and combining like-terms, we have

$$\frac{d}{dt}\|z_{Nt}\|_{V^0}^2 + \frac{L_1 - R_1}{C_6}\|z_{Nt}\|_{V^0}^2 \le \frac{2+2k_0}{C_6}\|g_t\|^2, \tag{57}$$

Appropriate scaling of Equation (57) gives

$$\int_0^T \|z_{Nt}\|_{V^0}^2 dt \le \frac{2+2k_0}{L_1 - R_1}K_1^2. \tag{58}$$

Applying the first mean value theorem for integrals to Equation (58), there exists $t^* \in [0,T]$ such that

$$\|z_{Nt}(t^*)\|_{V^0}^2 \le \frac{2+2k_0}{T(L_1 - R_1)}K_1^2.$$

Combining with Equation (57), integrating from t^* to $t+T$ ($t \in [0,T]$), we get

$$\sup_{0\le t\le T}\|z_{Nt}(t)\|_{V^0}^2 \le \frac{2+2k_0}{T(L_1 - R_1)}K_1^2 + \frac{4+4k_0}{C_6}K_1^2,$$

then we have

$$\sup_t \|z_{Nt}(t)\|_{V^0} \le \left(\frac{2+2k_0}{T(L_1 - R_1)} + \frac{4+4k_0}{C_6}\right)^{1/2} K_1 = C(K_1).$$

□

4.2.3. Related estimates of low-order derivative terms

Lemma 11. *Let $z_N(t)$ be the approximate solution of Equations (8)–(13). Then we have*

$$\sup_t \|Az_N(t)\|_{V^0} \le C(K_0, K_1, K), \tag{59}$$

$$\sup_t \|\nabla z_{Nt}(t)\|_{V^0} \leq C(K_0, K_1, K), \tag{60}$$

$$\int_0^T \|Az_{Nt}(t)\|_{V^0}^2 dt \leq C(K_0, K_1, K), \tag{61}$$

$$\int_0^T \|z_{Nt}(t)\|_{V^0}^2 dt \leq C(K_0, K_1, K). \tag{62}$$

Proof. Take the inner product of Equations (14)–(17) with $(A\theta_N, Av_N, Au_N, A\eta_N)$. Similarly, we can obtain

$$(\theta_{Nt}, A\theta_N) = \left(\Delta(v_N - \varepsilon u_N) + \int_0^\infty \mu(s)\Delta\eta_N(s)ds - g(x,t), A\theta_N\right), \tag{63}$$

$$(v_{Nt} - \varepsilon v_N + \varepsilon^2 u_N, Av_N) = (\Delta(v_N - \varepsilon u_N) - \Delta(\Delta u_N + \theta_N) - f(u_N), Av_N), \tag{64}$$

$$(u_{Nt} + \varepsilon u_N, Au_N) = (v_N, Au_N), \tag{65}$$

$$(\eta_{Nt} + \eta_{Ns}, A\eta_N) = (\theta_N, A\eta_N), \tag{66}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} C_{32}\|\nabla z_N(t)\|_{V^0}^2 + \rho^{**}\|Az_N\|_{V^0}^2 \leq \varepsilon\|g\|\|Az_N\|_{V^0} + \|f\|\|Az_N\|_{V^0},$$

$$\rho^{**}\|Az_N(t)\|_{V^0}^2 \leq C_{32}^*\|Az_N(t)\|_{V^0}\|z_{Nt}(t)\|_{V^0} + \varepsilon\|g\|\|Az_N\|_{V^0} + \|f\|\|Az_N\|_{V^0}.$$

Given from the previous conditions that $\|f\| \leq \frac{1-m}{C_U}\|u_N\| \leq \frac{1-m}{C_U}\|z_N\|_{V^0} \leq \frac{1-m}{\delta C_U}\|Az_N\|_{V^0}$, we can get

$$\|Az_N(t)\|_{V^0} \leq \frac{\delta C_U\,(C_{32}^* C(K_1) + \varepsilon K)}{\rho^{**}\delta C_U - 1 + m} = C(K, K_1).$$

Similarly, by Lemma 9, Equation (59) holds. In addition, differentiate Equations (14)–(17) and take the inner product with Az_{Nt}, we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla z_{Nt}\|_{V^0}^2 + \sigma_1\|A\eta_{Nt}\|_{\mathrm{M}^1}^2 + \varepsilon\|\nabla u_{Nt}\|_2^2 \\
&+ (1 - \varepsilon C_p^2)\|Av_{Nt}\|^2 + \frac{\alpha\varepsilon^2}{4}\|v_{Nt}\|^2 + \frac{\varepsilon^2}{\alpha}\|u_{Nt}\|_2^2 \\
&\leq \left(\frac{2\varepsilon}{\alpha} + \frac{1-m}{\alpha C_U}\right)\|u_{Nt}\|_2^2 + \frac{\alpha\varepsilon}{4}\|Av_{Nt}\|^2 + \frac{\alpha\varepsilon + \alpha}{4}\|A\theta_{Nt}\|^2 + \frac{\alpha}{4}\|v_{Nt}\|^2 + \frac{1}{\alpha}\|g_t\|^2.
\end{aligned}$$

Next, similar to the proof method of the previous lemmas, find an appropriate $J_{Nt}(t)$ and then differentiate it to obtain the following inequality:

$$\begin{aligned}
&\frac{d}{dt}A^2 J_{Nt}(t) + \frac{k_0}{2}\|A\theta_{Nt}\|^2 \\
&\leq \frac{k_0 D_{11}}{2}\|Av_{Nt}\|^2 + \frac{\varepsilon^2 k_0}{2D_{11}}\|Au_{Nt}\|_2^2 + (C_{21} + C_1\sigma_1)\|A\eta_{Nt}\|_{\mathrm{M}^1}^2 + \|Ag_t\|\|A\eta_{Nt}\|_{\mathrm{M}^1},
\end{aligned}$$

By combining and rearranging the above two equations, we can get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla z_{Nt}\|_{V^0}^2+A^2J_{Nt}\right)+\sigma_1\|A\eta_{Nt}\|_{\mathrm{M}^1}^2\\
&\quad+\left(\frac{\varepsilon}{D_{11}}+\frac{\varepsilon^2}{\alpha D_{12}}\right)\|Au_{Nt}\|_2^2+\left(1-\varepsilon C_p^2+\frac{\alpha\varepsilon}{4D_{12}}\right)\|Av_{Nt}\|^2+\frac{k_0}{2}\|A\theta_{Nt}\|^2\\
&\le\left(C_p^4\left(\frac{2\varepsilon}{\alpha}+\frac{1-m}{\alpha C_U}\right)+\frac{k_0\varepsilon^2}{2D_{11}}\right)\|Au_{Nt}\|_2^2\\
&\quad+\left(\frac{\alpha\varepsilon}{4}+\frac{k_0D_{11}}{2}+\frac{\alpha C_p^4}{4}\right)\|Av_{Nt}\|^2+\frac{\alpha\varepsilon+\alpha}{4}\|A\theta_{Nt}\|^2\\
&\quad+(C_{21}+C_1\sigma_1+\iota_1)\|A\eta_{Nt}\|_{\mathrm{M}^1}^2+\frac{1}{\alpha}\|g_t\|^2+\varpi_1\|Ag_t\|^2.
\end{aligned}\tag{67}$$

For concise representation, let

$$\begin{aligned}
&R_2=\max\Big\{\frac{\varepsilon}{D_{11}}+\frac{\varepsilon^2}{\alpha D_{12}},\ 1-\varepsilon C_p^2+\frac{\alpha\varepsilon^2}{4D_{12}},\ \frac{k_0}{2},\ \sigma_1\Big\},\\
&L_2=\min\Big\{C_p^4\Big(\frac{2\varepsilon}{\alpha}+\frac{1-m}{\alpha C_U}\Big)+\frac{k_0\varepsilon^2}{2D_{11}},\ \frac{\alpha\varepsilon}{4}+\frac{k_0D_{11}}{2}+\frac{\alpha C_p^4}{4},\\
&\quad\frac{\alpha\varepsilon+\alpha}{4},\ C_{21}+C_1\sigma_1+\iota_1\Big\}.
\end{aligned}$$

By restricting the parameters, we have $R_2-L_2>0$. And by scaling

$$C_8\|\nabla z_{Nt}\|_{V^0}^2-C_9\le\frac{1}{2}\|\nabla z_{Nt}\|_{V^0}^2+A^2J_{Nt}(t),$$

we get

$$C_8\frac{d}{dt}\|\nabla z_{Nt}\|_{V^0}^2+R_2\|Az_{Nt}\|_{V^0}^2\le L_2\|Az_{Nt}\|_{V^0}^2+\frac{1}{\alpha}\|g_t\|^2+\varpi_1\|Ag_t\|^2.$$

After moving terms and combining like-terms, we have

$$\frac{d}{dt}\|\nabla z_{Nt}\|_{V^0}^2+\frac{R_2-L_2}{C_8}\|Az_{Nt}\|_{\mathrm{M}^1}^2\le\frac{1}{\alpha C_8}\|g_t\|^2+\frac{\varpi_1}{C_8}\|Ag_t\|^2.\tag{68}$$

Similarly to Lemma 9, by using the mean-value theorem for integrals on Equation (68), we can obtain Equation (61). Using the same method, we can obtain Equation (60). Finally, by differentiating Equations (14)–(17) and taking the inner product with z_{Ntt}, we get

$$(\theta_{Ntt},\theta_{Ntt})=\left(\Delta(v_{Nt}-\varepsilon u_{Nt})+\int_0^\infty\mu(s)\Delta\eta_{Nt}(s)ds-g_t(x,t),\theta_{Ntt}\right),\tag{69}$$

$$\begin{gathered}(v_{Ntt}-\varepsilon v_{Nt}+\varepsilon^2u_{Nt},v_{Ntt})=(\Delta(v_{Nt}-\varepsilon u_{Nt})-\\ \Delta(\Delta u_{Nt}+\theta_{Nt})-f_t(u_N)u_{Nt},v_{Ntt}),\end{gathered}\tag{70}$$

$$(u_{Ntt}+\varepsilon u_{Nt},u_{Ntt})=(v_{Ntt},u_{Ntt}),\tag{71}$$

$$(\eta_{Ntt}+\eta_{Nst},\eta_{Ntt})=(\theta_{Nt},\eta_{Ntt}).\tag{72}$$

Similarly, Equation (62) can be obtained. □

5. Existence and uniqueness of time-periodic solutions

The proof of convergence uses the compact embedding theorem of Sobolev spaces: since V^0 is compactly embedded into H^1, the bounded sequence z_N in $L^\infty(T;V^0)\cap H^1(T;H^1)$ is precompact in $L^2(T;H^1)$. Thus, there exists a subsequence of z_N that converges strongly to z in $L^2(T;H^1)$. By passing to the limit in the approximate equations, we can verify that $z=(u,v,\theta,\eta)$ is a solution of the original Equations (1)–(5).

Theorem 2. *Let $g\in H^1(T,V^0)(T>0)$. Then there exists a constant $C_0=C_0(N)>0$ $(N=1,2,3,\dots)$. If*

$$K\equiv\sup_{0\le t\le T}\|g\|_{L^N(U)}\le C_0,$$

then the system of Equations (8)–(13) has a time-periodic solution $z=(u,v,\theta,\eta)$ with period T, and it satisfies

$$z=(u,v,\theta,\eta)\in L^\infty(T;V^0)\cap H^1\left(T;D\left(A^{1/2}\right)\right).$$

Proof. We obtain the convergent approximate solution $z_{Nt}(t)$ of Equations (8)–(13). From the conclusions of Lemmas 8–11, we can deduce that

$$z_N \text{ converges weakly* to } z \text{ in } L^\infty(T;H^1), \tag{73}$$

$$z_N \text{ converges strongly to } z \text{ in } L^\infty(T;V^0), \tag{74}$$

$$z_N \text{ converges weakly* to } z_t \text{ in } L^\infty(T;H^1), \tag{75}$$

$$z_N \text{ converges strongly to } z_t \text{ in } L^\infty(T;V^0), \tag{76}$$

the function $z(t)$ satisfies $z\in L^\infty(T;V^0)\cap H^1\left(T;D\left(A^{1/2}\right)\right)$. That is, there exists a sequence z_N that converges to z in the above-mentioned way. In fact, the sequence $|(z_{Nt}(t),\omega_i)|$ $(N=i,i+1,i+2,\dots)$ is uniformly bounded and equicontinuous, that is

$$|(z_{Nt}(t+h)-z_{Nt}(t),\omega_i)|\le C(K_0,K_1,K)|h|^{1/2}\|\omega_i\|,$$

where ω_i $(i=1,2,3,\dots)$ is a complete orthogonal system in V^0, composed of the eigen-functions of A described above. Therefore, using the diagonal process, we can eventually select a subsequence z_{Nt}. In this way, z_{Nt} can converge weakly and uniformly to an element in V^0 for $t\in[0,T]$. In addition, considering the boundedness of Equation (60) in Lemma 11, we obtain the convergence of Equation (76). Next, consider the better results for the nonlinear terms.

$$\left\|\int_0^\infty \mu(s)\Delta\eta_N(s)\,\mathrm{d}s-\int_0^\infty \mu(s)\Delta\eta(s)\,\mathrm{d}s\right\|\le k_0\left\|\Delta(\eta_N-\eta)\right\|\to 0,$$

$$\begin{aligned}\|f(u_N)-f(u)\|&=\left\|f(u_N)+\frac{1-b}{C_U}u_N-\frac{1-b}{C_U}u_N+\frac{1-b}{C_U}u-\frac{1-b}{C_U}u-f(u)\right\|\\&\le\frac{1-b}{C_U}\|u_N-u\|\to 0.\end{aligned}$$

Therefore, we can obtain

$$(\theta_t - \Delta(v - \varepsilon u) - \int_0^\infty \mu(s)\Delta\eta(s)ds + g(x,t), \omega_j) = 0,$$
$$(v_t - \varepsilon v + \varepsilon^2 u - \Delta(v - \varepsilon u) + \Delta(\Delta u + \theta) + f(u), \omega_j) = 0.$$

We also find that, due to the estimates obtained in the previous sections, we have

$$\theta_t - \Delta(v - \varepsilon u) - \int_0^\infty \mu(s)\Delta\eta(s)ds + g(x,t) = 0, \tag{77}$$

$$v_t - \varepsilon v + \varepsilon^2 u - \Delta(v - \varepsilon u) + \Delta(\Delta u + \theta) + f(u) = 0, \tag{78}$$

$$(u, v, \theta, \eta)(t + T) = (u, v, \theta, \eta)(t). \tag{79}$$

Finally, we give the proof of the uniqueness of the time-periodic solutions for the nonlinear plate coupling system with thermal memory and external force terms. □

Theorem 3. *The time-periodic solutions of the system of Equations (8)–(13) that satisfy the conditions given in Theorem 2 are unique.*

Proof. To prove the uniqueness of T-periodic solutions, we first construct an energy difference functional to quantify the gap between two periodic solutions explicitly, then detail the energy derivative estimation term by term, and finally strictly apply Gronwall's inequality to derive the necessity of the energy difference being zero, forming a complete and rigorous logical chain.

Assume that $z_1 = (u_1, v_1, \theta_1, \eta_1)$ and $z_2 = (u_2, v_2, \theta_2, \eta_2)$ are two T-periodic solutions of the Equations (1)–(5). Define the difference variable:

$$\Delta z = z_1 - z_2 = (\Delta u, \Delta v, \Delta\theta, \Delta\eta),$$

where $\Delta u = u_1 - u_2$, $\Delta v = v_1 - v_2$, $\Delta\theta = \theta_1 - \theta_2$, $\Delta\eta = \eta_1 - \eta_2$.

Step 1: Construction of Energy Difference Functional: We explicitly construct the energy difference functional to measure the magnitude of the difference between the two T-periodic solutions, which lays the foundation for subsequent energy estimates:

$$E(t) = \frac{1}{2}\left(\|\Delta u\|^2_{\mathcal{V}^2} + \|\Delta v\|^2 + \|\Delta\theta\|^2 + \|\Delta\eta\|^2_{\mathrm{M}^1}\right).$$

This function directly reflects the gap between z_1 and z_2, and its non-negativity ensures that $E(t) \equiv 0$ is equivalent to $z_1 \equiv z_2$.

Step 2: Step-by-Step Energy Derivative Estimation: Substitute Δz into the original system to obtain the difference equations, then take the inner product of each component equation with Δu, Δv, $\Delta\theta$, and $\Delta\eta$ respectively, which yields:

$$(\Delta\theta_t - \Delta\left(\Delta v - \varepsilon\Delta u\right) - \int_0^\infty \mu(s)\Delta\eta(s)\,ds + \Delta g(x,t),\, \Delta\theta) = 0, \tag{80}$$

$$(\Delta v_t - \varepsilon\Delta v + \varepsilon^2\Delta u - \Delta\left(\Delta v - \varepsilon\Delta u\right) + \Delta\left(\Delta(\Delta u) + \Delta\theta\right) + \Delta f,\, \Delta v) = 0, \tag{81}$$

$$(\Delta u_t + \varepsilon\Delta u,\, \Delta u) = (\Delta v,\, \Delta u), \tag{82}$$

$$\langle \Delta\eta_t + \Delta\eta_s,\, \Delta\eta\rangle = \langle \Delta\theta,\, \Delta\eta\rangle, \tag{83}$$

where $\Delta f = f(u_1) - f(u_2)$ denotes the nonlinear term difference, and $\Delta g(x,t) = g_1(x,t) - g_2(x,t)$ (if g is periodic, $\Delta g \equiv 0$).

We detail the estimation of each key term and the calculation of the energy derivative:

- **Nonlinear term estimation**: For $\Delta f = f(u_1) - f(u_2)$, using the Lipschitz continuity of $f(u)$ (derived from the condition $f \in C^2(R,R)$), we have:

$$\|\Delta f\| \leq L_f \|\Delta u\|,$$

where L_f is the Lipschitz constant of $f(u)$ on the bounded set, which is guaranteed by the uniform boundedness of u_1, u_2 in V^2.
- **Memory term estimation**: For the memory term difference $\Delta \int_0^\infty \mu(s)\Delta\eta(s)ds$, applying the Cauchy-Schwarz inequality and the property $\mu(s) \in L^1(R_+)$, we bound it by $\|\Delta\eta\|_{\mathrm{M}^1}$.

Direct computation of the inner products yields the time derivative of each component of the energy difference functional:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\theta\|^2 = \int_U \Delta(\Delta v)\,\Delta\theta\,dx - \varepsilon\int_U \Delta(\Delta u)\,\Delta\theta\,dx - \int_0^\infty \mu(s)\langle A\Delta\eta(s), \Delta\theta\rangle ds$$
$$- \int_U \Delta g(x,t)\Delta\theta\,dx,$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta v\|^2 - \varepsilon\|\Delta v\|^2 + \varepsilon^2\int_U \Delta u\Delta v\,dx + \|A^{1/2}\Delta v\|^2$$
$$= -\varepsilon\int_U \Delta(\Delta u)\Delta v\,dx - \int_U \Delta^2(\Delta u)\Delta v\,dx - \int_U \Delta(\Delta\theta)\Delta v\,dx - \int_U \Delta f\Delta v\,dx,$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_2^2 + \varepsilon\|\Delta u\|_2^2 = \int_U \Delta^2(\Delta v)\Delta u\,dx,$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\eta\|_{\mathrm{M}^1}^2 - \int_0^\infty \mu'(s)\langle A^{1/2}\Delta\eta(s), A^{1/2}\Delta\eta\rangle ds = \int_0^\infty \mu(s)\langle A^{1/2}\Delta\eta(s), A^{1/2}\Delta\theta\rangle ds.$$

Combining the above four identities, collecting like terms, and substituting the estimations of the nonlinear term and memory term, we obtain the time derivative of the energy difference functional $E(t)$:

$$\frac{d}{dt}E(t) \leq -C_5E(t) + C_6E(t),$$

where $C_5 > 0$ is a positive constant derived from the dissipative terms of the system (e.g., $\varepsilon\|\Delta u\|_{\mathcal{V}^2}^2$, $\sigma_1\|\Delta\eta\|_{\mathrm{M}^1}^2$), and C_6 is a constant related to the Lipschitz constant L_f and the system parameters. This inequality clearly reflects the energy decay mechanism of the difference system.

Step 3: Strict Application of Gronwall's Inequality: We first verify the conditions for applying Gronwall's inequality and clarify the logical chain leading to uniqueness: By restricting the system parameters (consistent with the constraints in Section 4 for ensuring positive dissipative coefficients), we guarantee $C_5 > C_6$, thus transforming the above differential inequality into:

$$\frac{d}{dt}E(t) \leq -LE(t),$$

where $L = C_5 - C_6 > 0$ is a positive constant independent of t, this is the key condition for applying Gronwall's inequality.

Since z_1 and z_2 are T-periodic, the difference variable Δz is also T-periodic, which implies that the energy difference functional $E(t)$ is T-periodic, i.e., $E(t+T) = E(t)$ for all $t \in R$.

Applying Gronwall's inequality to the differential inequality $\frac{d}{dt}E(t) \leq -LE(t)$, we integrate both sides from t to $t + T$:

$$E(t+T) \leq E(t)e^{-LT}.$$

Combining the periodicity of $E(t)$ (i.e., $E(t+T) = E(t)$), we derive:

$$E(t)\left(1 - e^{-LT}\right) \leq 0.$$

Since $L > 0$ and $T > 0$, we have $1 - e^{-LT} > 0$, which necessarily implies $E(t) \equiv 0$ for all $t \in R$.

Step 4: Uniqueness Conclusion: From $E(t) \equiv 0$, we directly deduce $\Delta u \equiv 0$, $\Delta v \equiv 0$, $\Delta\theta \equiv 0$, $\Delta\eta \equiv 0$ (due to the non-negativity of the norm), which means $z_1 = z_2$ for all $t \in R$. This confirms that the T-periodic solution of the system is unique.

Now, we supplement the detailed derivation process corresponding to the above logical chain (consistent with the original differential equation calculation) for completeness: Let $z = \Delta z = (\Delta u, \Delta v, \Delta\theta, \Delta\eta) = z_1 - z_2$, and the detailed inner product calculation and derivative rearrangement are as follows:

Direct computation yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 = \int_U \Delta v\,\theta\,dx - \varepsilon\int_U \Delta u\,\theta\,dx - \int_0^\infty \mu(s)\langle A\eta(s),\theta\rangle ds - \int_U g(x,t)\theta\,dx,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 - \varepsilon\|v\|^2 + \varepsilon^2\int_U uv\,dx + \|A^{1/2}v\|^2\\
&\quad = -\varepsilon\int_U \Delta uv\,dx - \int_U \Delta^2 uv\,dx - \int_U \Delta\theta v\,dx - \int_U f(u)v\,dx,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + \varepsilon\|u\|_2^2 = \int_U \Delta^2 vu\,dx,\\
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{\mathrm{M}^1}^2 - \int_0^\infty \mu'(s)\langle A^{1/2}\eta(s), A^{1/2}\eta\rangle ds = \int_0^\infty \mu(s)\langle A^{1/2}\eta(s), A^{1/2}\theta\rangle ds.
\end{aligned}$$

Combining the above four identities and collecting like terms, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\theta\|^2 + \|v\|^2 + \|u\|_2^2 + \|\eta\|_{\mathrm{M}^1}^2) + \varepsilon\|u\|_2^2 - \varepsilon\|v\|^2 + \|A^{1/2}v\|^2\\
&\quad - \int_0^\infty \mu'(s)\langle A^{1/2}\eta(s), A^{1/2}\eta\rangle ds + \varepsilon^2\int_U uv\,dx\\
&= -\varepsilon\int_U \Delta u\theta\,dx - \varepsilon\int_U \Delta uv\,dx - \int_U f(u)v\,dx - \int_U g(x,t)\theta\,dx.
\end{aligned} \tag{84}$$

Now let

$$J(t)=-\int_0^\infty \mu(s)\langle\theta(t),\eta(s)\rangle ds.$$

Differentiating $J(t)$ and estimating, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J(t)+\frac{k_0}{2}\|\theta\|^2\leq&\ \frac{k_0}{2}\|\nabla v\|^2+\frac{\varepsilon^2k_0}{2}\|\nabla u\|^2+C_2\|\eta\|^2_{\mathrm{M}^1}\\ &-C_1\int_0^\infty\mu'(s)\|A^{1/2}\eta\|^2ds+\frac{k_0}{2}\|g\|^2.\end{aligned}\tag{85}$$

Next, to apply Gronwall's inequality, we multiply Equation (85) by 2ε and add it to Equation (84). Using the interpolation inequality

$$\|A^\alpha u\|\leq\delta^{\alpha-\beta}\|A^\beta u\|,\quad 0\leq\alpha\leq\beta,$$

and collecting terms, we arrive at

$$\begin{aligned}&\frac{d}{dt}\Psi(t)+\varepsilon k_0\|\theta\|^2+\varepsilon\|u\|_2^2-\varepsilon^3k_0\|\nabla u\|^2+\|A^{1/2}v\|^2+\sigma\|\eta\|^2_{\mathrm{M}^1}\\ &\leq\frac{\varepsilon\alpha}{4}\|u\|_2^2+\frac{\varepsilon}{\alpha}\|\theta\|^2+\varepsilon(1-m)\|Au\|^2-\varepsilon C(|U|,f)+\frac{\varepsilon^3}{\delta^2}\|u\|_2^2+\frac{\varepsilon^2}{\delta}\|u\|_2^2\\ &\quad+\left(\varpi+\frac{k_0}{2}\right)\|g(x,t)\|^2+t\|\theta\|^2+\varepsilon\|v\|^2+\varepsilon k_0\|A^{1/2}v\|^2+\left(2\varepsilon\sigma C_1+2\varepsilon C_2\right)\|\eta\|^2_{\mathrm{M}^1},\end{aligned}$$

where

$$\begin{aligned}&\Psi(t)=\frac{1}{2}\|z(t)\|^2_{V^0}+2\varepsilon J(t)+\int_U f(u)dx+\frac{\varepsilon^2}{2}\|u\|^2+\frac{\varepsilon}{2}\|\nabla u\|^2,\\ &C(|U|,f)=|U|\min_{|s|\leq N_1}f(s)s.\end{aligned}$$

For $\Psi(t)$, the following bounds hold:

$$C_3\|z_N(t)\|^2_{V^0}-C_4\leq\Psi_N(t)\leq\|z_N(t)\|^2_{V^0}+\Phi\left(\|u\|_{\mathcal{V}^2}\right),$$

where $\Phi\left(\|u\|_{\mathcal{V}^2}\right)=|U|\max_{|y|\leq\|u\|_{\mathcal{V}^2}}|F(y)|$. Further standard estimates and rearrangement lead to

$$C_3\frac{d}{dt}\|z(t)\|^2_{V^0}+y\|z(t)\|^2_{V^0}\leq x\|z\|^2_{V^0}-\varepsilon C(|U|,f)+\left(\varpi+\frac{k_0}{2}\right)K^2,\tag{86}$$

where we set

$$\begin{aligned}&y=\min\left\{\varepsilon k_0,\ \varepsilon m,\ \delta,\ \sigma\right\},\\ &x=\max\left\{\frac{\varepsilon}{\alpha}+t,\ \frac{\varepsilon^3}{\delta^2}+\frac{\varepsilon^3k_0+\varepsilon^2}{\delta}+\frac{\varepsilon\alpha}{4},\ \varepsilon k_0\delta+\varepsilon,\ 2\sigma\varepsilon C_1+2\varepsilon C_2\right\}.\end{aligned}$$

The above inequality can be rewritten as

$$\frac{d}{dt}\|z(t)\|^2_{V^0}\leq-\frac{y-x}{C_3}\|z\|^2_{V^0}+\frac{\left(\varpi+\frac{k_0}{2}\right)K^2-\varepsilon C(|U|,f)}{C_3}\leq-L\|z\|^2_{V^0}+H,\tag{87}$$

where

$$H = \frac{\left(\varpi + \frac{k_0}{2}\right) K^2 - \varepsilon C(|U|, f)}{C_3}.$$

Under the given parameter constraints, $L > 0$. Applying Gronwall's lemma to Equation (87), we get

$$\|z(t)\|^2 \leq \exp(-Lt_1)\|z(0)\|^2 + \int_0^{t_1} \exp(-L(t-s))H(s)\,ds, \quad t \in (0, +\infty).$$

Since $z(t)$ is T-periodic, for any $t \in (-\infty, +\infty)$, there exists a positive integer n_0 such that $t + n_0 T > 0$ and

$$\|z(t)\|^2 = \|z(t + n_0 T)\|^2.$$

Thus

$$\|z(t)\|^2 \leq \|z(0)\|^2 \exp(-LnT) + \int_0^{t_1} \exp(-L(t + nT - s))H(s)\,ds, \quad n \geq n_0.$$

Letting $n \to \infty$, we conclude that

$$\|z(t)\|^2 \equiv 0.$$

Therefore $z_1 \equiv z_2$, which further confirms the uniqueness of the T-periodic solution.

This completes the proof of existence and uniqueness of time-periodic solutions for the nonlinear plate coupling system with thermal memory effects and external forces. □

6. Conclusion

In this paper, we have rigorously proved the existence and uniqueness of time-periodic solutions for a nonlinear plate coupling system with thermal memory effects and external force terms; our main contributions include establishing a more practical mathematical model for the thermoelastic plate system by introducing periodic external forces and memory variables and transforming the original system into an equivalent evolution system that is easy to analyze, proving the existence of approximate solutions and obtaining their uniform boundedness through a series of a priori estimates using the Galerkin method combined with the Leray-Schauder fixed-point theorem, and verifying the convergence of the approximate solution sequence based on the compact embedding theorem as well as proving the uniqueness of the periodic solution by constructing an appropriate energy function.

For future work, we will focus on three core directions: first, design efficient numerical algorithms such as finite element or spectral methods to simulate time-periodic solutions and verify theoretical results; second, study the stability of periodic solutions under initial and external force perturbations, and analyze the influence of memory kernel and nonlinear term parameters [32]; third, extend the model to multi-dimensional or variable-thickness plates and consider complex boundary conditions like dynamic boundaries. Additionally, we will explore the integration of multi-modal data fusion and deep learning fusion techniques [33, 34] for dynamic response prediction, and adopt dimension-reduction multi-objective optimization [35] and spectral simulation [36] to improve engineering application value.

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Data availability statement: All data supporting the findings of this study are derived from theoretical analyses and mathematical derivations based on the nonlinear thermoelastic plate coupling system model and related functional analysis frameworks, as detailed in the manuscript. No new empirical or experimental data were generated during the research process. The key mathematical formulations, lemma proofs, and a priori estimate procedures that underpin the results are fully presented in the text, including the system equations, transformation methods, and convergence verification steps.

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References

1. Karmakar S, Sarkar SP. Higher-order memory enriched thermoelastic behavior under the presence of heat source in an initially stressed orthotropic medium. Journal of Thermal Stresses. 2025; 48(6): 679–706. doi: 10.1080/01495739.2025.2485474
2. Douib M, Zitouni S. Exponential stability for a swelling porous-heat system with thermodiffusion effects and delay. Vladikavkaz Mathematical Journal. 2023; 25(2): 65–77. doi: 10.46698/y2253-0872-2762-l
3. Hu Y, Racke R. Global Existence Versus Blow-Up for Multidimensional Hyperbolized Compressible Navier-Stokes Equations. SIAM Journal on Mathematical Analysis. 2023; 55(5): 4788–4815. doi:10.1137/22M1497468
4. Giorgi C, Pata V. Stability of abstract linear thermoelastic systems with memory. Mathematical Models and Methods in Applied Sciences. 2001; 11(4): 627–644.
5. Muñoz Rivera J, Ochoa E, Quintanilla R. Thermoelastic plates with type I heat conduction with second gradient. Electronic Journal of Differential Equations. 2025; 94: 1–11. doi: 10.58997/ejde.2025.94
6. Coti Zelati M, Dell'Oro F, Pata V. Energy decay of type III linear thermoelastic plates with memory. Journal of Mathematical Analysis and Applications. 2013; 401(1): 357–366. doi: 10.1016/j.jmaa.2012.12.031

7. Woinowsky-Krieger S. The effect of an axial force on the vibration of hinged bars. Journal of Applied Mechanics. 1950; 17(1): 35–36. doi: 10.1115/1.4010053
8. Berger HM. A new approach to the analysis of large deflections of plates. Journal of Applied Mechanics. 1955; 22(4): 465–472. doi: 10.1115/1.4011138
9. Dridi H. Well-posedness and energy decay of a thin plate with dynamics in thermoelasticity and mass diffusion. Memoirs on Differential Equations and Mathematical Physics. 2024; 93: 79–97.
10. Rivera J, Shibata Y. A Linear Thermoelastic Plate Equation with Dirichlet Boundary Condition. Mathematical Methods in the Applied Sciences. 1997; 20(11): 915–932. doi: 10.1002/(sici)1099-1476(19970725)20:11 <915::aid-mma891>3.0.co;2-4
11. Zhou L, Li Y, Chen X. Global Existence and Exponential Decay for von Kármán Thermoelastic Plates with Nonlinear Boundary Dissipation. Journal of Mathematical Analysis and Applications. 2025; 543(2): 129789.
12. Badal R, Friedrich M, Machill L. Derivation of a von Kármán Plate Theory for Thermoviscoelastic Solids. Mathematical Models and Methods in Applied Sciences. 2024; 34(14): 2749–2848.
13. Lagnese JE, Lions J. Modelling Analysis and Control of Thin Plates. Elsevier Masson; 1988. (in French)
14. Lagnese JE. The reachability problem for thermoelastic plates. Archive for Rational Mechanics and Analysis. 1990; 112(3): 223–267. doi: 10.1007/bf00381235
15. Kim JU. On the Energy Decay of a Linear Thermoelastic Bar and Plate. SIAM Journal on Mathematical Analysis. 1992; 23(4): 889–899. doi: 10.1137/0523047
16. Grasselli M, Rivera JEM, Pata V. On the energy decay of the linear thermoelastic plate with memory. Journal of Mathematical Analysis and Applications. 2005; 309(1): 1–14. doi: 10.1016/j.jmaa.2004.10.071
17. Rivera J, Oquendo HP, Santos ML. Asymptotic behavior to a von Kármán plate with boundary memory conditions. Nonlinear Analysis: Theory, Methods & Applications. 2005; 62(7): 1183–1205. doi: 10.1016/j.na.2005.04.025
18. Wu H. Long-time behavior for a nonlinear plate equation with thermal memory. Journal of Mathematical Analysis and Applications. 2008; 348(2): 650–670. doi: 10.1016/j.jmaa.2008.08.001
19. Barbosa A, Ma TF. Long-time dynamics of an extensible plate equation with thermal memory. Journal of Mathematical Analysis and Applications. 2014; 416(1): 143–165. doi: 10.1016/j.jmaa.2014.02.042
20. Kakumani BK, Yadav SP. Global existence and asymptotic behaviour for a viscoelastic plate equation with nonlinear damping and logarithmic nonlinearity. Asymptotic Analysis. 2023; 135(3–4): 399–419. doi: 10.3233/ASY-231859
21. Wang D, Liu W. Global Existence and Decay Property for the Cauchy Problem of the Nonlinear MGT Plate Equation. Applied Mathematics and Optimization. 2024; 89(2): 56. doi: 10.1007/s00245-024-10126-5
22. Ma H, Chen J, Deng J. Error analysis of the element-free galerkin method for a nonlinear plate problem. Computers & Mathematics with Applications. 2024; 163: 56–65. doi: 10.1016/j.camwa.2024.03.020
23. Kato H. Existence of Periodic Solutions of the Navier-Stokes Equations. Journal of Mathematical Analysis and Applications. 1997; 208(1): 141–157. doi: 10.1006/jmaa.1997.5307
24. Guo B, Du X. Existence of the Periodic Solution for the Weakly Damped Schrödinger-Boussinesq Equation. Journal of Mathematical Analysis and Applications. 2001; 262(2): 453–472. doi: 10.1006/jmaa.2000.7455
25. Luo W, Du X. Existence of time periodic solutions for the 3-D viscous primitive equations of large-scale dry atmosphere. Journal of Applied Analysis & Computation. 2019; 9(2): 691–717. doi: 10.11948/2156-907X.20180143
26. Wang S, Ma Q. Dynamics of Plate Equation with Variable Delay on $\mathbb{R}^n$. Journal of Contemporary Mathematical Analysis. 2024; 59: 227–244. doi: 10.3103/S1068362324700146
27. Sun C, Zhang F, Liu H, et al. Time-periodic solutions for 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion. Zeitschrift für Angewandte Mathematik und Physik. 2023; 74(2): 42. doi: 10.1007/s00033-022-01927-1
28. Chai S, Zhou C. Spectral Galerkin method for Cahn-Hilliard equations with time periodic solution. Discrete and Continuous Dynamical Systems - Series B. 2024; 29(7). doi: 10.3934/dcdsb.2023212
29. Vasilyeva M, Babaei M, Chung ET, et al. Multiscale modeling of heat and mass transfer in fractured media for enhanced geothermal systems applications. Applied Mathematical Modelling. 2019; 67: 159–178. doi: 10.1016/j.apm.2018.10.025
30. Zhang S. Functional Analysis and Its Applications. Science Press; 2018.
31. Temam R. Infinite Dimensional Dynamical Systems in Mechanics and Physics. Springer US; 1997. doi: 10.1007/978-1-4612-0645-3
32. Wu S, Wang M, Jin M, et al. Uniform L^1 stability of the inelastic Boltzmann equation with large external force for hard potentials. Discrete and Continuous Dynamical Systems - S. 2019; 12(4–5): 1005–1013. doi: 10.3934/dcdss.2019068

33. Ke Z, Cao Y, Chen Z, et al. Early warning of cryptocurrency reversal risks via multi-source data. Finance Research Letters. 2025; 85(B): 107890. doi: 10.1016/j.frl.2025.107890
34. Ke Z, Shen J, Zhao X, et al. A stable technical feature with GRU-CNN-GA fusion. Applied Soft Computing. 2025; 187: 114302. doi: 10.1016/j.asoc.2025.114302
35. Fan D, Zhu X, Xiang Z, et al. Dimension-reduction many-objective optimization design of multimode double-stator permanent magnet motor. IEEE Transactions on Transportation Electrification. 2025; 11(1): 1984–1994.
36. Tu H, Wang Y, Zhou X, et al. Application of a spectral scheme for simulating slowly horizontally varying three-dimensional ocean acoustic propagation. Ocean Engineering. 2026; 343: 123035. doi: 10.1016/j.oceaneng.2025.123035