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Article

Time-optimal control with bang-bang property for strongly coupled nonlinear microwave heating systems

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Abstract: This paper makes a rigorous mathematical analysis of the time-optimal control problem for a nonlinear microwave heating system described by coupled partial differential equations. The work extends the established theory for linear models to a physically realistic nonlinear regime, where magnetic field permeability exhibits a nonlinear dependence on the material's evolving temperature field. The first result establishes the exact controllability of this nonlinear distributed parameter system. This is achieved by applying the Kakutani Fixed-Point Theorem to an appropriately defined solution operator, proving that the system state can be driven from any admissible initial temperature distribution to a specified target profile within a finite time horizon using suitable control inputs. Leveraging this controllability foundation and employing crucial a priori energy estimates, we subsequently prove the existence of at least one time-optimal control via minimizing sequences and weak compactness arguments. The central contribution is the rigorous analytic proof of the bang-bang property for these time-optimal controls. This structural property is demonstrated by contradiction, using a pivotal quantitative relation—derived from the controllability analysis—that links the minimum achievable control time to the L^2-norm of the control force. The proof conclusively shows that any time-optimal control must saturate the prescribed control constraints almost everywhere in the time-space domain, taking values only at the extremes of the admissible set. These results lay a firm theoretical foundation for optimal control protocol design in nonlinear microwave heating, confirming that efficient strategies are inherently of switching type and offering a benchmark for future numerical and experimental work as well.

Keywords: nonlinear microwave heating system; strongly coupled PDE system; time-optimal control; controllability; bang-bang property

1. Introduction

Microwave heating is widely employed in various applications, such as food processing and sterilization, due to its effectiveness, convenience, rapidity, and cleanliness [1, 2]. However, two significant challenges persist. The first is the nonuniform distribution of heat generated during the microwave heating process, which can lead to localized overheating. This may engender temperature spikes that exceed desired thresholds, potentially damaging the heated material. The second challenge stems from the inherent nonlinearity of microwave heating systems, where the electromagnetic field distribution varies with the temperature of the material. This nonlinear behavior complicates the assurance that a prescribed control input can drive the system to a desirable state.

Typically modeled as a coupled system of Maxwell's equations and the heat equation, microwave heating dynamics have attracted considerable research interest. While significant work has addressed the solvability and optimal control of such systems [3–6], the time-optimal control (TOC) problem remains relatively underexplored [7–9], primarily due to the difficulties introduced by system nonlinearity in both control design and controllability analysis. In practical applications, a key objective is to achieve a target heating profile as quickly as possible without causing thermal damage, which naturally aligns with the TOC framework. Given the advantages of microwave heating in speed and efficiency, TOC problems in this context trigger much interest in further research among engineers and researchers.

In this paper, similar to the study by Luo et al. [8], we look at the TOC problem for a class of nonlinear microwave heating systems and establish the bang-bang property of the corresponding optimal controls. Prior studies [8, 10–12] considered a weakly coupled linear model under the assumption that the magnetic field distribution is temperature-independent. In practice, however, the material properties of the heating device vary with temperature, resulting in a strongly coupled nonlinear system. Owing to this nonlinearity, controllability cannot be deduced via standard methods such as the Carleman inequality, commonly used for linear systems [13–16]. Instead, we employ the Kakutani Fixed-Point Theorem to set up controllability. The proof of the existence of a time-optimal control also requires techniques distinct from those for linear cases, despite some parallels with prior works [17–20]. Unlike the methods used for linear systems [21–24], we prove this property by contradiction, utilizing a quantitative relation between the control duration and the norm of the control force. This key inequality is derived directly from the controllability analysis.

The remainder of this paper is organized as follows. Section 2 is devoted to the rigorous mathematical formulation of the microwave heating process, culminating in the precise statement of the time-optimal control problem, denoted as $(\mathbf{P})$. In Section 3, we establish a crucial observability estimate for the system, which serves as the foundation for deriving the controllability of problem $(\mathbf{P})$. Building upon the controllability result, Section 4 addresses the existence of a time-optimal control via the method of minimizing sequences. Finally, Section 5 presents the main contribution of this work: a proof by contradiction of the bang-bang property for the time-optimal control.

2. Mathematical model for the time optimal control problem of microwave heating

This section is devoted to the rigorous mathematical formulation of microwave heating. In accordance with the standard approach widely adopted in the literature [5–8], the microwave heating process is mathematically described as a coupled system of Maxwell's equations and the heat conduction equation. To facilitate the analysis, we adopt the assumption [3, 8], that the object heated is placed in a microwave heating processor cavity denoted by $\Omega \subset R^3$ with C^1- boundary $\partial\Omega$, and let $\mathbf{E}(x,t)$ and

$\mathbf{H}(x,t)$ denote the electric and magnetic fields at $x \in \Omega$ and time t correspondingly. Through Faraday Law, Ampere Theorem and Ohm Principle for electromagnetics, the distribution of the electric and magnetic field strengths can be exemplified by Maxwell Equation as follows [1–3]:

$$\varepsilon \mathbf{E}_t(x,t) + \sigma \mathbf{E}(x,t) = \nabla \times \mathbf{H}(x,t), \quad x \in \Omega, t > 0,$$

$$\mu \mathbf{H}_t(x,t) + \nabla \times \mathbf{E}(x,t) = 0, \quad x \in \Omega, t > 0,$$

$$\nabla \cdot \mathbf{H}(x,t) = 0, \quad x \in \Omega, t > 0,$$

where ε, μ and σ are the electric permittivity, magnetic permeability, and electric conductivity, respectively. It is noted that $\nabla \cdot \mathbf{H} = 0$ holds automatically as long as an initial field $\mathbf{H}(x,0)$ satisfies the same condition.

Due to the high frequency of the microwaves, the scale of the time variable in electromagnetic fields is different from that of heat conduction. It is usually assumed that the electric and magnetic fields are time-harmonic, in practice, with fixed frequency ω, then

$$\begin{cases} \mathbf{E}_t(x,t) = \hat{\mathbf{E}}(x) e^{j\omega t}, \\ \mathbf{H}_t(x,t) = \hat{\mathbf{H}}(x) e^{j\omega t}, \end{cases}$$

where j denotes the unit complex number, on the basis of assumptions that Maxwell's equations can be reduced to a Helmholtz type of system, and simplified as equations containing a single electronic field $\mathbf{E}$ (or only a magnetic field $\hat{\mathbf{H}}$). Here we only consider the electronic field system denoted by $\mathbf{E}$, taking the place of $\mathbf{E}$, then the simplified time-harmonic Maxwell's equation, namely, the Helmholz system is obtained, which is a single system germane to the electromagnetic field $\mathbf{E}$ [3–7].

$$\nabla \times (\frac{1}{\mu} \nabla \times \mathbf{E}) + \omega(-\omega\varepsilon + j\sigma)\mathbf{E} = 0.$$

Take account of the dissipative effect and assume the dielectric constant ε and permeability constant μ both are complex numbers [3]:

$$\varepsilon = \varepsilon_0(\varepsilon_1 - j\varepsilon_2), \qquad \mu = \mu_0(\mu_1 - j\mu_2),$$

where ε_0 is the permittivity in free space, ε_1 is the relative electric permittivity, ε_2 is the effective loss factor of electric energy, μ_0 is the permeability in free space, μ_1 represents the relative magnetic permeability, μ_2 is the relative magnetic loss factor. Set $\gamma = \frac{1}{\mu}$, $\xi = -\varepsilon_0\varepsilon_1\omega + j(\varepsilon_0\varepsilon_2\omega + \sigma) \triangleq -a_1 + ja_2$, then the above equation can be expressed as follows:

$$\nabla \times [\gamma \nabla \times \mathbf{E}] + \xi \mathbf{E} = 0.$$

During the heating process, both induced current and eddy current will act on the heated object, according to Ohm's law: $\mathbf{J}_\varepsilon = \varepsilon(j\omega)\mathbf{E}$ and $\mathbf{J}_\sigma = \sigma\mathbf{E}$, then it holds that

$$\mathbf{J} = \mathbf{J}_\varepsilon + \mathbf{J}_\sigma = \varepsilon(j\omega)\mathbf{E} + \sigma\mathbf{E} = (a_2 - ja_1)\mathbf{E}.$$

Therefore, the heat energy generated by a microwave per unit time and unit volume is

$$W(x,t) = \frac{1}{2}Re[\mathbf{E}\cdot\mathbf{J}^*] = \frac{1}{2}a_2|\mathbf{E}|^2,$$

where $\mathbf{J}^*$ is the complex conjugate function of $\mathbf{J}$. According to the Fourier Law, the distribution of the temperature $y(x,t)$ satisfies the following

$$\rho_0 c_0 y_t(x,t) - \nabla\cdot(k(x)\nabla y(x,t)) = \frac{1}{2}a_2|\mathbf{E}|^2, \quad x\in\Omega, t>0,$$

where $\rho_0 > 0$ is the density of matter, $c_0 > 0$, specific heat coefficient, here it is assumed to be constant; $k(x)$ is the thermal conductivity, which is generally related to its position, to non-thermal materials, the value is independent of temperature and time. Set $\rho_0 c_0 = 1$, then the mathematical model of microwave heating can be similarly formulated as a coupled system with corresponding initial and boundary valuable condition [8].

$$\begin{cases} \nabla\times(\gamma\nabla\times\mathbf{E}) + (-a_1 + ia_2)\mathbf{E} = 0, \quad (x,t)\in\Omega\times[0,T], \\ \mathbf{n}\times\mathbf{E} = \mathbf{n}\times\mathbf{G}, \quad (x,t)\in\partial\Omega\times[0,T], \\ y_t - \nabla\cdot(k\nabla y) = \frac{1}{2}a_2|\mathbf{E}|^2, \quad (x,t)\in\Omega\times[0,T], \\ y(x,t) = 0, \quad (x,t)\in\partial\Omega\times[0,T], \\ y(x,0) = y_0(x), \quad x\in\Omega, \end{cases} \tag{1}$$

where $\mathbf{n}$ is the external unit normal vector on $\partial\Omega$, and $\mathbf{G}$ is the electric field generated by the external photoelectric device. If $\gamma = \gamma(x)$, $a_1 = a_1(x)$, $a_2 = a_2(x)$, $k = k(x)$, the system above is a weekly coupled system. In practical microwave heating, virtually all experimental results demonstrate that the relative electric permittivity ε_1, the effective loss factor of electric energy ε_2, the relative magnetic permeability μ_1 and the relative magnetic loss factor μ_2 are usually relative to the temperature y, consequently, the corresponding variables should be formulated in the form of $\gamma(x,y) = \gamma_1(x,y) + j\gamma_2(x,y), \xi(x,y) = a_1(x,y) + a_2(x,y)$, which implies their inherent dependence on temperature y, then the system above becomes a strongly coupled one. In this paper, we consider a class of strongly coupled microwave heating systems.

In the context of microwave heating, a primary engineering objective is to achieve a desired heating profile within the shortest possible time, which naturally leads to the formulation of a time-optimal control (TOC) problem. From a practical standpoint, a common operational constraint is the binary nature of the control input, where the heating process is regulated simply by switching the microwave power on or off. Mathematically, this constraint implies that the control force is spatially uniform, i.e., independent of the spatial variable. Consequently, the admissible control set is formally defined as follows:

$$\boldsymbol{K} = \{g : [0,T]\to R \text{ is measurable and } 0\le g(t)\le 1\},$$

where $g\in\boldsymbol{K}$, $g(t) = 0$ means the switch is off and $g(t) = 1$ means the switch is on.

Following the standard approach adopted in previous studies [5–7,25], to facilitate a mathematically tractable formulation, we introduce a time-dependent control function

$g(t)$, which multiplies the source term on the right-hand side of the heat equation. This function serves to modulate the intensity of the microwave heating input. The resulting controlled mathematical model for the microwave heating process is thus formally expressed as follows:

$$\begin{cases} \nabla \times (\gamma(x,y)\nabla \times \mathbf{E}) + (-a_1(x,y) + ia_2(x,y))\mathbf{E} = 0, \quad (x,t) \in \Omega \times [0,T], \\ \mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}, \quad (x,t) \in \partial\Omega \times [0,T], \\ y_t - \nabla \cdot (k(x)\nabla y) = \frac{1}{2}a_2(x,y)|\mathbf{E}|^2 g(t), \quad (x,t) \in \Omega \times [0,T], \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \\ y(x,0) = y_0(x), \quad x \in \Omega. \end{cases} \tag{2}$$

To establish a well-posed mathematical framework for the coupled microwave heating system, we adopt the following assumptions, consistent with the methodology in studies [8,26–28]. These assumptions are designed to ensure solution existence while deliberately abstracting from the complexities introduced by boundary conditions, even external force, thereby focusing the analysis on the core nonlinear coupling mechanism.

H1. *Functions $y_0(\cdot), y_T(\cdot) \in L^2(\Omega)$, function $k : \Omega \to R$ is $C^1(\Omega)$ and there exist positive constants μ_0 and μ_1 such that $0 < \mu_0 \le k(x) \le \mu_1$, for any $x \in \Omega$.*

H2. *(i) Function $a_1(\cdot)$ and $a_2(\cdot)$ are real measurable, and there exist positive numbers b_0 and a_0 such that $0 < a_0 \le a_1(x)$, $a_2(x) \le b_0$, $\forall x \in \Omega$;*

(ii) Function $\gamma(\cdot) = \frac{1}{\mu(\cdot)} := \gamma_1(\cdot) + i\gamma_2(\cdot)$ is complex value, and there exists a constant $\gamma_0 > 0$ such that $\gamma_1(x), \gamma_2(x) \ge \gamma_0 > 0$, $\forall x \in \Omega$;

(iii) Function $\boldsymbol{G} : \partial\Omega \to R^3$ can be extended to Ω, and the extension $\overline{\boldsymbol{G}}$ satisfies

$$\|\overline{\boldsymbol{G}}\|_{H(curl,\Omega)} \le c_0 \|\boldsymbol{G}\|_{L^2(\partial\Omega)},$$

where c_0 is a constant only depending on Ω.

H2a. *The functions $\gamma_1(x,y), \gamma_2(x,y), a_1(x,y)$ and $a_2(x,y)$ are real positive measurable, where $\gamma(x,y) := \gamma_1(x,y) + j\gamma_2(x,y)$ and $\xi(x,y) := a_1(x,y) + ja_2(x,y)$, are uniformly Lipschitz continuous to the variable y. For any $x \in \Omega, y \in R$, there exist positive constants a_0 and A_0, such that*

$$\gamma_1(x,y), a_2(x,y) \ge a_0 > 0, \quad |\gamma(x,y)| + |\xi(x,y)| \le A_0.$$

To the system, if the switch is off, it is trivial that $\mathbf{E} \equiv 0$; hence, the system solution of the Equation (2) is obvious. According to the conditions for the uniqueness of solution for the coupled system in papers [3,27], Equation (2) naturally has a unique electric field intensity.

To be convenient to discuss the TOC problem later, several inner product spaces are defined [3,8] as the following:

$$H(curl,\Omega) = \{\mathbf{E} \in (L^2(\Omega))^3 : \nabla \times \mathbf{E} \in (L^2(\Omega))^3\},$$

$$H_0(curl,\Omega) = \{\mathbf{E} \in (L^2(\Omega))^3 : \nabla \times \mathbf{E} \in [L^2(\Omega)]^3, \mathbf{n} \times \mathbf{E} = 0 \text{ on } \partial\Omega\},$$

$$H(div,\Omega) = \{\mathbf{E} \in (L^2(\Omega))^3 : \nabla \cdot \mathbf{E} \in L^2(\Omega)\},$$

then the spaces $H_0(curl,\Omega)$ and $H(curl,\Omega)$ are Hilbert spaces with inner product

$$\langle \mathbf{E},\mathbf{G}\rangle = \int_\Omega [(\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{G}^*)+\mathbf{E}\cdot\mathbf{G}^*]dx,$$

where $\mathbf{G}^*$ is the complex dual to $\mathbf{G}$. Since the norm is derived by inner product, then the function spaces $H(curl,\Omega)$, $H_0(curl,\Omega)$ and $H(div,\Omega)$ are Banach spaces.

Then the TOC problem $(\mathbf{P})$ of the strongly coupled microwave heating system is to find a control $g\in\mathbf{K}$, to lead the temperature $y(t,x;y_0,g)$ to some expected temperature $y_T(\cdot)\in L^2(\Omega)$ as soon as possible, where $y(t,x;y_0,g)$ is the solution of Equation (2) corresponding to the control $g\in\mathbf{K}$, and the optimal time $T^*>0$ is defined as follows:

$$T^* := inf\{T:\ y(T,x;y_0,g)=y_T(x), a.e., x\in\Omega,\ \forall g\in\mathbf{K}\}.$$

Prior to addressing the time-optimal control (TOC) problem, we introduce several preliminary results that are instrumental to the subsequent analysis. First, we recall the statement of the Kakutani Fixed-Point Theorem [26], a cornerstone result in nonlinear functional analysis that will play a pivotal role in establishing the controllability of the system.

Lemma 1 (Kakutani Fixed-Point Theorem)**.** *Suppose that*
(i) *the multivalued map* $T: K\to 2^K$ *is upper semicontinuous;*
(ii) *K is a nonempty, compact and convex set in a locally convex space X;*
(iii) *the set* $T(X)$ *is nonempty, closed and convex for all* $x\in K$.
Then T has a fixed point.

The following section presents key results pertaining to linear parabolic systems, which serve as foundational tools for establishing both the controllability and the existence of a time-optimal control for the microwave heating system. For linear parabolic equations, the Carleman inequality [8, 13] provides a powerful method for proving controllability. In what follows, we consider the initial-boundary value problem for a homogeneous linear parabolic system:

$$\begin{cases} y_t-\nabla\cdot(k(x)\nabla y)=0,\ (x,t)\in\Omega\times[T_0,T],\\ y(x,t)=0,\ (x,t)\in\partial\Omega\times[T_0,T],\\ y(x,T_0)=y_{T_0}(x),\ x\in\Omega,\end{cases}\tag{3}$$

then there is the proposition.

Lemma 2. *Set* ω *to be a nonempty open subset* Ω*, if H1 holds, then there exists a positive constant* $C=C(\Omega,\omega)$ *only depending on* Ω *and* ω*, for any given time* $T>0$ *and a initial value* $y_0\in L^2(\Omega)$*, such that the solution* $y\in C([0,T];L^2(\Omega))$ *of Equation* (3) *satisfies*

$$\|y(\cdot,T)\|_{L^2(\Omega)}\le \exp[C(1+\frac{1}{T})]\int_0^T\int_\omega |y(x,t)|^2dxd\tau.\tag{4}$$

According to the inequality above, we have the controllability for the following similar system [8,13].

$$\begin{cases} y_t - \nabla\cdot(k(x)\nabla y) = v(x,t), & x\in\Omega, t>0,\\ y(x,t)=0, & (x,t)\in\partial\Omega, t>0,\\ y(x,0)=y_0(x), & x\in\Omega. \end{cases} \tag{5}$$

Lemma 3. *Assume ω to be any nonempty subset of Ω, and $T > T_0 \geq 0$ to be a constant decided, if $y(T_0)\in L^2(\Omega)$ and H1 holds, then there exists a control function $v\in L^\infty(T_0,T;L^2(\Omega))$ such that the solution $y\in W[T_0,T]\cap C([T_0,T];L^2(\Omega))$ of the following system:*

$$\begin{cases} y_t - \nabla\cdot(k(x)\nabla y) = \chi_I(t)\chi_\omega(x)v(x,t),\ (x,t)\in\Omega\times[T_0,T],\\ y=0,\ (x,t)\in\partial\Omega\times[T_0,T],\\ y(x,T_0)=y_{T_0}(x),\ x\in\Omega, \end{cases} \tag{6}$$

satisfies

$$\|y(\cdot,T)\|_{L^2(\Omega)}=0,$$

and

$$\|v\|_{L^\infty(T_0,T;L^2(\Omega))} \leq 2\exp[C(1+\frac{1}{T-T_0})]\|y(T_0)\|_{L^2(\Omega)},$$

where $C > 0$ is a constant only depending on Ω,ω,I; the function $\chi_I(\cdot)$ and the function $\chi_\omega(\cdot)$ are character functions of time t on I and space x on space ω, correspondingly.

Proof. Consider the Equation (3) is a dual equation,

$$\begin{cases} p_t + \nabla\cdot(k(x)\nabla p) = 0,\ (x,t)\in\Omega\times[T_0,T],\\ p(x,t)=0,\ (x,t)\in\partial\Omega\times[T_0,T],\\ p(x,T)\in L^2(\Omega), \end{cases} \tag{7}$$

set $s=T-t$ and $\widehat{p}(x,s)=p(x,T-s)$, then the Equation (7) is transformed to

$$\begin{cases} \widehat{p}_t - \nabla\cdot(k(x)\nabla \widehat{p}) = 0,\ (x,t)\in\Omega\times[T_0,T],\\ \widehat{p}(x,t)=0,\ (x,t)\in\partial\Omega\times[T_0,T],\\ \widehat{p}(x,T_0)\in L^2(\Omega), \end{cases} \tag{8}$$

and for any $\widehat{p}(x,T_0)\in L^2(\Omega)$, there exists a unique solution $\widehat{p}\in C([T_0,T];L^2(\Omega))$ of Equation (8) by Lemma 2. Consequently,

$$\|\widehat{p}(\cdot,T)\|_{L^2(\Omega)} \leq \exp[C(1+\frac{1}{T-T_0})]\int_{T_0}^{T}\int_\omega |\widehat{p}(x,s)|^2dxds,$$

and

$$\|p(\cdot,T_0)\|_{L^2(\Omega)} \leq \exp[C(1+\frac{1}{T-T_0})]\int_{T_0}^{T}\int_\omega |p(x,t)|^2dxdt. \tag{9}$$

To the equation

$$y_t - \nabla \cdot (k(x)\nabla y) = \chi_I(t)\chi_\omega(x)v,$$

multiply both sides by inner production for p, and integrate both sides on $\Omega \times [T_0, T]$, then, applying integration by parts yields the following identity:

$$\begin{aligned} &\langle y(\cdot, T), p(\cdot, T)\rangle_{L^2(\Omega)} - \langle y(\cdot, T_0), p(\cdot, T_0)\rangle_{L^2(\Omega)} \\ &\qquad = 2\int_{T_0}^{T}\int_\Omega \chi_I(t)\chi_\omega(x)v(x,t)p(x,t)dxdt. \end{aligned} \tag{10}$$

To show $y(\cdot, T) \equiv 0$ in the $L^2(\Omega)$, it only needs a $v \in L^\infty(T_0, T; L^2(\Omega))$ such that

$$-\langle y(\cdot, T_0), p(\cdot, T_0)\rangle_{L^2(\Omega)} = 2\int_{T_0}^{T}\int_\Omega \chi_I(t)\chi_\omega(x)v(x,t)p(x,t)dxdt.$$

In the space

$$Q = \{\chi_I(t)\chi_\omega(x)p(x,t) : p(\cdot, T) \in L^2(\Omega)\},$$

where $p(x,t)$ is the solution of the Equation (7), a linear functional

$$F : Q \subset L^1(T_0, T; L^2(\Omega)) \to \mathbf{R},$$

is defined by

$$F(\chi_I\chi_\omega p) = -\langle y(T_0)(\cdot), p(\cdot, T_0)\rangle_{L^2(\Omega)},$$

where $p \in L^1(T_0, T; L^2(\Omega))$ is the solution of Equation (7), then

$$\begin{aligned} |F(\chi_I\chi_\omega p)|^2 &\le \|y(T_0)\|^2_{L^2(\Omega)}\|p(\cdot, T_0)\|^2_{L^2(\Omega)} \\ &\le 2\exp[C(1 + \frac{1}{T - T_0})]\|y(T_0)\|^2_{L^2(\Omega)}\|\chi_I\chi_\omega p\|^2_{L^1(T_0,T;L^2(\Omega))} \end{aligned} \tag{11}$$

namely

$$\|F\|^2 \le 2\exp[C(1 + \frac{1}{T - T_0})]\|y(T_0)\|^2_{L^2(\Omega)},$$

where $\|F\|$ is the norm of the linear operator F, then F is a bounded linear functional. According to Hahn-Banach theorem [26], there exists a boundary linear functional

$$\widehat{F} : L^1(T_0, T; L^2(\Omega)) \to \mathbf{R},$$

such that

$$\widehat{F}|_Q = F,$$

and

$$\|\widehat{F}\|^2 = \|F\|^2 \le 2\exp[C(1 + \frac{1}{T - T_0})]\|y(T_0)\|^2_{L^2(\Omega)}.$$

According to Riesz Theorem [26], there exists a funciton $v \in L^\infty(T_0, T; L^2(\Omega))$ such that

$$\widehat{F}(\chi_I \chi_\omega p) = \int_{T_0}^{T} \int_\Omega \chi_I \chi_\omega p v dx dt,$$

and

$$\|v\|^2_{L^\infty(T_0,T;L^2(\Omega))} = \|\widehat{F}\|^2 \le 2\exp[C(1+\frac{1}{T-T_0})]\|y(T_0)\|^2_{L^2(\Omega)},$$

therefore,

$$-\langle y(\cdot,T_0), p(\cdot,T_0)\rangle_{L^2(\Omega)} = 2\int_{T_0}^{T}\int_\Omega \chi_I(t)\chi_\omega(x)v(x,t)p(x,t)dxdt.$$

□

3. Controllability of the microwave heating system

This section is devoted to the analysis of controllability for the coupled microwave heating system. As a preliminary step toward establishing the controllability of the controlled system Equation (23), we first present and prove two essential corollaries. The subsequent analysis will be developed within the framework of the system described below:

$$\begin{cases} \nabla \times [\gamma(x,t)\nabla \times \mathbf{E}] + \xi(x,t)\mathbf{E} = 0, \quad (x,t) \in \Omega \times [0,T], \\ \mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}, \quad (x,t) \in \partial\Omega \times [0,T]. \end{cases} \tag{12}$$

Consequently, we arrive at the following conclusion regarding this matter.

Lemma 4. *Assume that H2 and H2a hold, it is obvious that*

$$\gamma_n(x,t) \to \gamma(x),$$

$$\xi_n(x,t) \to \xi(x,t),$$

in $L^2(0,T;L^2(\Omega))$*, as* $n \to +\infty$*, and*

$$\boldsymbol{E}_n(x,t) \to \boldsymbol{E}(x,t) \quad in \quad L^2(0,T;L^2(\Omega)),$$

where $\boldsymbol{E}_n$ *and* $\boldsymbol{E}$ *are the solutions of the Equation* (23) *corresponding to* γ_n*,* ξ_n*,*γ *and* ξ*.*

Proof. Set $\mathbf{W}_n(x,t) = \mathbf{E}_n(x,t) - \mathbf{E}(x,t)$, then $\mathbf{W}_n(x,t)$ satisfies the following:

$$\begin{cases} \nabla \times [\gamma_n(x,t)\nabla \times \mathbf{W}_n] + \xi_n(x,t)\mathbf{W}_n \\ = -\nabla \times [(\gamma_n(x,t) - \gamma(x,t))\nabla \times \mathbf{E}] - (\xi_n(x,t) - \xi(x,t))\mathbf{E}, (x,t) \in \Omega \times [0,T], \\ \mathbf{n} \times \mathbf{W}_n = 0, \quad (x,t) \in \partial\Omega \times [0,T], \end{cases} \tag{13}$$

multiply both sides of the above equation by $\mathbf{W}_n$ and integrate it on Ω, then

$$\begin{aligned} &-\int_\Omega \gamma_n(x,t)|\nabla \times \mathbf{W}_n|^2 dx + \int_\Omega \xi_n(x,t)|\mathbf{W}_n|^2 dx \\ &= \int_\Omega (\gamma_n(x,t) - \gamma(x,t))(\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{W}_n) dx - \int_\Omega (\xi_n(x,t) - \xi(x,t))\mathbf{E}\mathbf{W}_n dx \\ &\equiv I_1 + I_2. \end{aligned} \tag{14}$$

Take into account both the real and imaginary parts, then

$$Re(I_1+I_2)=\int_\Omega \gamma_{1n}(x,t)|\nabla\times\mathbf{W}_n|^2dx-\int_\Omega a_{1n}(x,t)|\mathbf{W}_n|^2dx,$$

and

$$Im(I_1+I_2)=\int_\Omega \gamma_{2n}(x,t)|\nabla\times\mathbf{W}_n|^2dx-\int_\Omega a_{2n}(x,t)|\mathbf{W}_n|^2dx.$$

From the Cauchy-Schwarz inequality, it is known as follows:

$$\begin{aligned}|I_1|&=|\int_\Omega[\gamma_n(x,t)-\gamma(x,t)](\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{W}_n)|\\&\le\delta_1\int_\Omega|\nabla\times\mathbf{W}_n|^2dx+C_1(\delta_1)\int_\Omega|\gamma_n(x,t)-\gamma(x,t)|^2|\nabla\times\mathbf{E}|^2dx,\end{aligned}\tag{15}$$

and

$$\begin{aligned}|I_2|&=|\int_\Omega[\xi_n(x,t)-\xi(x,t)]\mathbf{E}\cdot\ \mathbf{W}_ndx|\\&\le\delta_2\int_\Omega|\mathbf{W}_n|^2dx+C_2(\delta_2)\int_\Omega|\xi_n(x,t)-\xi(x,t)|^2|\ \mathbf{E}|^2dx,\end{aligned}\tag{16}$$

where δ_1 and δ_2 are small enough positive constants. According to H2a, if the constants δ_1 and δ_2 are small enough, then

$$\begin{aligned}\textstyle\int_\Omega|\nabla\times\mathbf{W}_n|^2dx+\int_\Omega|\mathbf{W}_n|^2dx\le C_1\int_\Omega|\gamma_n-\gamma|^2|\nabla\times\mathbf{E}|^2dx+\\ C_2\textstyle\int_\Omega|\xi_n-\xi|^2|\mathbf{E}|^2dx,\end{aligned}\tag{17}$$

integrate both sides of the formula above on the interval $[0,T]$, then

$$\begin{aligned}&\int_0^T\int_\Omega|\nabla\times\mathbf{W}_n|^2dxdt+\int_0^T\int_\Omega|\mathbf{W}_n|^2dxdt\\&\quad\le C_1\int_0^T\int_\Omega|\gamma_n-\gamma|^2|\nabla\times\mathbf{E}|^2dxdt+C_2\int_0^T\int_\Omega|\xi_n-\xi|^2|\mathbf{E}|^2dxdt.\end{aligned}\tag{18}$$

An application of the Lebesgue dominated convergence theorem [13,27–29] yields

$$\int_0^T\int_\Omega|\gamma_n-\gamma|^2|\nabla\times\mathbf{E}|^2dxdt\to0,\tag{19}$$

and

$$\int_0^T\int_\Omega|\xi_n-\xi|^2\to0.\tag{20}$$

Therefore,

$$\nabla\times\mathbf{W}_n\to0\text{ in }L^2(0,T;L^2(\Omega)),\mathbf{W}_n\to0\ \text{ in }L^2(0,T;L^2(\Omega)).$$

Furthermore,

$$\mathbf{E}_n\to\mathbf{E}\ \text{ in }L^2(0,T;L^2(\Omega)),\ \ \nabla\times\mathbf{E}_n\to\nabla\times\mathbf{E}\ \text{ in }L^2(0,T;L^2(\Omega)).$$

□

Without loss of generality, assume $y_T(\cdot) = 0$. The controllability of Equation (2) is established by the following theorem.

Theorem 1. *If the hypotheses H1 and H2a hold, for any $T > T_0 \geq 0$ and $y(T_0) \in L^2(\Omega)$, there exists a $g \in L^\infty[T_0, T]$ such that*

$$\| \frac{1}{2}a_2(x,y)|\boldsymbol{E}|^2 g(t) \|_{L^\infty(T_0,T;L^2(\Omega))} \leq exp(C(1+\frac{1}{T-T_0}))\|y(T_0)\|_{L^2(\Omega)} \tag{21}$$

and

$$(\boldsymbol{E}, y) \in L^\infty(T_0, T; H(curl, \Omega)) \times (W[T_0, T] \cap C([T_0, T]; L^2(\Omega))),$$

satisfy the following equation:

$$\begin{cases} \nabla \times (\gamma(x,y)\nabla \times \boldsymbol{E}) + \xi(x,y)\boldsymbol{E} = 0, \quad (x,t) \in \Omega \times [T_0, T], \\ y_t - \nabla \cdot (k(x)\nabla y) = \frac{1}{2}\chi_I(t)\chi_\omega(x)a_2(x,y)|\boldsymbol{E}|^2 g(t), \quad (x,t) \in \Omega \times [T_0, T], \\ \boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{n} \times \boldsymbol{G}, \quad (x,t) \in \partial\Omega \times [T_0, T], \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [T_0, T], \\ y(x, T_0) = y_{T_0}(x), \quad x \in \Omega, \\ y(x,T) = 0, \quad x \in \Omega, \end{cases} \tag{22}$$

where C is a constant, I is any measurable subset of the interval $[T_0, T]$ with positive measure $|I| > 0$, any set $\omega \subset \Omega$, and $\chi_A(\cdot)$ is the character function on set A with corresponding variable.

Proof. To show the above result with Kakutanis Fixed-Point Theorem [25], the set is defined by

$$\Pi = \{u(x,t) \in W[T_0, T] | \|u\|_{L^2(T_0,T;L^2(\Omega))} \leq k_0\},$$

where the constant k_0 will be decided later for some certain goal. Obviously, that is $\Pi \subset L^2(T_0, T; L^2(\Omega))$. To any $u \in \Pi$, considering the following weakly coupled system

$$\begin{cases} \nabla \times (\gamma(x,u)\nabla \times \mathbf{E}) + \xi(x,u)\mathbf{E} = 0, \quad (x,t) \in \Omega \times [T_0, T], \\ y_t - \nabla \cdot (k(x)\nabla y) = \frac{1}{2}\chi_I(t)\chi_\omega(x)a_2(x,u)|\mathbf{E}|^2 g(t), \quad (x,t) \in \Omega \times [T_0, T], \\ \mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}, \quad (x,t) \in \partial\Omega \times [T_0, T], \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [T_0, T], \\ y(x, T_0) = y_{T_0}(x), \quad x \in \Omega, \\ y(x,T) = 0, \quad x \in \Omega, \end{cases} \tag{23}$$

according to Lemma 3, there exists a solution

$$\mathbf{E} \times y \in L^\infty(T_0, T; H(curl, \Omega)) \times (W[T_0, T] \cap C([T_0, T]; L^2(\Omega)))$$

with

$$g \in L^\infty(T_0, T),$$

which satisfies

$$\| \frac{1}{2}a_2(x,u)|\mathbf{E}|^2 g(t) \|_{L^\infty(T_0,T;L^2(\Omega))} \leq exp(C(1+\frac{1}{T-T_0}))\|y(T_0)\|_{L^2(\Omega)}. \quad (24)$$

Define the mapping

$$\Phi : \Pi \to L^2(T_0,T;L^2(\Omega)),$$

where $\Phi(u) = \{y|$ for $u \in \Pi\}$ and y satisfies the Equation (24), and where y is the solution to the Equation (23), then

$$\forall u \in \Pi, \Phi(u) \neq \phi.$$

(i) To show $\Pi \subset L^2(T_0,T;L^2(\Omega))$ to be convex. Since the embedding $W[T_0,T] \hookrightarrow L^2(T_0,T;L^2(\Omega))$ is compact, then Π is convex and compact in $L^2(T_0,T;L^2(\Omega))$, therefore, $\Phi(u)$ is convex and compact in $L^2(T_0,T;L^2(\Omega))$.

(ii) For any $u \in \Pi$, there exists a $g \in L^\infty[T_0,T]$, such that the solution y satisfies the Equation (23), and according to Lemma 4, $\forall t \in [T_0,T]$,

$$\int_\Omega |\nabla \times \mathbf{E}|^2 dx + \int_\Omega |\mathbf{E}|^2 dx \leq C_1,$$

where $C_1 > 0$ depends on Ω and $\overline{\mathbf{G}}$.

For $N = 3$, according to the embedding theorem in Sobolev space [13, 21, 28], there exists a $K_2 > 0$, such that

$$\int_\Omega |\mathbf{E}|^6 dx \leq K_2, \forall t \in [T_0,T]. \quad (25)$$

From the heat Equation (23), for any $t \in [T_0,T]$, then

$$\begin{aligned} \frac{d}{dt}\int_\Omega y^2 dx + \int_\Omega k(x)|\nabla y|^2 dx &= \int_\Omega \frac{1}{2}\chi_I \chi_\omega a_2(x,u)|\mathbf{E}|^2 g(t) dx \\ &\leq K_2(\int_\Omega y^2 dx + \int_\Omega |\mathbf{E}|^4 dx)|g(t)|_I, \end{aligned} \quad (26)$$

and

$$\frac{d}{dt}\int_\Omega y^2 dx \leq K_2(\int_\Omega y^2 dx + \int_\Omega |\mathbf{E}|^4 dx)|g(t)|_I,$$

for $k(x) \geq \mu_0 > 0$.

By Gronwall's inequality in the differential form, then

$$\int_\Omega y^2 dx \leq exp(K_2 \int_0^t |g(s)|ds)[\int_\Omega |y(T_0)|^2 dx + \int_{T_0}^t |\mathbf{E}|^4 dx)], a.et \in [T_0,T].$$

According to Equations (24) and (25), it shows that there exists a constant $K_3 > 0$ only depending on $y(T_0)$ and Ω, such that

$$\int_\Omega y^2 dx \leq K_3,$$

where $K_3 = \frac{k_0}{(T-T_0)}$. This means $\Phi(\Pi) \subset \Pi$.

(iii) If Φ is upper semicontinuous in the space $L^2(T_0,T;L^2(\Omega))$, it only needs

$$u_m \in \Pi \to u \quad in \quad L^2(T_0,T;L^2(\Omega))$$

and

$$y_m \in \Phi(u_m) \to y \quad in \quad L^2(T_0,T;L^2(\Omega)),$$

then $y \in \Phi(u)$.

By Lemma 4, for

$$\mathbf{E}_m(x,t) \to \mathbf{E}(x,t) \quad in \quad L^2(T_0,T;L^2(\Omega)),$$

then there exists a subsequence of $\{\mathbf{E}_m(x,t)\}$ still denoted by $\{\mathbf{E}_m(x,t)\}$, such that

$$\{\mathbf{E}_m(x,t)\} \to \mathbf{E}(x,t), a.e \quad [x,t] \in \Omega \times [T_0,T].$$

For subhypothesis H2a, it is known that

$$a_2(x,u_m) \to a_2(x,u), a.e \quad [x,t] \in \Omega \times [T_0,T].$$

Considering

$$\| \ \frac{1}{2}a_2(x,u)|\mathbf{E}|^2 g_m(t) \ \|_{L^\infty(T_0,T;L^2(\Omega))} \leq exp(C(1+\frac{1}{T-T_0}))\|y(T_0)\|_{L^2(\Omega)},$$

a_2 and $|\mathbf{E}|^2$ are bounded, then $g_m \in L^\infty[T_0,T] \subset L^p[T_0,T]$ for any $p \geq 1$, and g is bounded, therefore, there exists subsequence $\{g_m\}$, which is still named by $\{g_m\}$, and

$$g_m \rightharpoonup g \quad in \quad L^1(T_0,T;L^2(\Omega)),$$

where $\rightharpoonup$ means weak convergence. Then for any $\phi(x,t) \in C^\infty(\Omega \times [T_0,T])$, it will satisfy

$$\frac{1}{2}\int_{T_0}^{T}\int_\Omega \chi_I(\tau)\chi_\omega(x)a_2(x,u_m)|\mathbf{E}_m|^2 g_m(\tau)\phi(x,t)dxd\tau \to$$
$$\frac{1}{2}\int_{T_0}^{T}\int_\Omega \chi_I(\tau)\chi_\omega(x)a_2(x,u)|\mathbf{E}|^2 g(\tau)\phi(x,t)dxd\tau.$$

For $y_m = \Phi(u_m)$, then

$$\begin{aligned} &\tfrac{d}{dt}y_m - \nabla\cdot(k(x)\nabla y_m) = \\ &\tfrac{1}{2}\chi_I(t)\chi_\omega(x)a_2(x,u_m)|\mathbf{E}_m|^2 g_m(t), (x,t) \in \Omega \times [T_0,T]. \end{aligned} \tag{27}$$

Since $t \in [T_0,T]$, then

$$\begin{aligned} &\int_\Omega |y_m(x,t)|^2 dx + 2\int_{T_0}^{t}\int_\Omega k(x)|\nabla y_m|^2 dxd\tau \\ &\quad \leq \int_\Omega |y_{T_0}|^2 dx + \int_{T_0}^{T}\int_\Omega \chi_I(\tau)\chi_\omega(x)a_2(x,u_m)|\mathbf{E}_m|^2 g_m dxd\tau. \end{aligned} \tag{28}$$

Therefore, there exists a constant $K_5 > 0$ such that

$$\int_{\Omega} |y_m(x,t)|^2 dx + \int_{T_0}^{t} \int_{\Omega} k(x) |\nabla y_m|^2 dx d\tau \leq K_5,$$

so $\{y_m\}$ is a bounded sequence in the space $W[T_0, T]$. Therefore, there exists a subsequence of $\{y_m\}$, still denoted by $\{y_m\}$ for simplicity, such that

$$y_m \to y \quad in \quad L^2(T_0, T; L^2(\Omega)),$$

$$\nabla y_m \to \nabla y \quad in \quad L^2(T_0, T; L^2(\Omega)).$$

According to both sides of the Equation (27), there is a weak solution such that

$$y_t - \nabla(k(x)\nabla y) = \frac{1}{2}\chi_I(t)\chi_\omega(x) a_2(x,u)|\mathbf{E}|^2 g(t), (x,t) \in \Omega \times [T_0, T], \tag{29}$$

as $m \to \infty$, namely, $y \in \Phi(u)$.

In view of the preceding analysis from (i)–(iii), all the conditions required for the application of the Kakutani Fixed-Point Theorem are satisfied. Consequently, there exists a point $y \in \Pi$ such that $y = \Phi(y)$, thereby completing the proof of the theorem.

□

4. Existence of time-optimal control for the microwave heating system

In this section, we establish the existence of a time-optimal control for problem (**P**), governed by the strongly coupled microwave heating Equation (2). By Theorem 2, the admissible control set is nonempty, and controllability is verified via the method of minimizing sequences. Owing to the strong coupling inherent in the system, the analysis necessitates techniques tailored to nonlinear problems. Following the approach employed in studies [8, 28, 30, 31], we first derive key estimates for solutions of the strongly coupled system.

Lemma 5. *Assume that H1, H2 and H2a hold, then for any* $g \in \boldsymbol{K}$*, the controlled system Equation* (2) *has a solution* $(\boldsymbol{E}, y) \in L^\infty(0, T; H(curl, \Omega)) \times W[0, T] \cap C([0, T]; L^2(\Omega))$ *and it satisfies*

$$\int_{\Omega} |\boldsymbol{E}|^2 dx \leq C_0, \tag{30}$$

$$\int_{\Omega} |\nabla \times \boldsymbol{E}|^2 dx + \int_{\Omega} |\boldsymbol{E}|^6 dx \leq C_1, \tag{31}$$

$$\int_{\Omega} [|\nabla \cdot (a_1 \boldsymbol{E})|^2 + |\nabla \cdot (a_2 \boldsymbol{E})|^2] dx \leq C_2, \tag{32}$$

$$\sup_{0 \leq t \leq T} \int_{\Omega} |y|^2 dx + \int_{0}^{T} \int_{\Omega} |\nabla y|^2 dx dt \leq C_3, \tag{33}$$

where positive constants C_0*,* C_1*,* C_2 *and* C_3 *only depend on some known data.*

The controllable set is defined by

$$\mathbf{K}_{ad} = \{g | \exists T > 0, \text{ s.t. } y(x,T;g) = 0\},$$

where $y(x,T;g)$ is the solution of the Equation (2) corresponding to $g \in \mathbf{K}$.

Based on Theorem 1 and Lemma 5, we establish the following result, following an approach analogous to that of Theorem 2 [8].

Theorem 2. *Assume that the conditions H1, H2 and H2a hold, for any given* $y_0 \in L^2(\Omega) \setminus \{0\}$*, there exist a positive number* $T > 0$ *and a control* $g \in \boldsymbol{K}$ *such that the strongly coupled system Equation* (2) *corresponding to the control* g *has a unique solution*

$$(\boldsymbol{E}, y) \in L^\infty(0,T; H(curl,\Omega)) \times W[0,T] \cap C([0,T]; L^2(\Omega)),$$

and y *satisfies*

$$\|y(\cdot,T;y_0,g)\|_{L^2(\Omega)} = 0.$$

Based on Theorem 1, which establishes that the admissible control set $\mathbf{K}_{ad}$ is nonempty (i.e., the strongly coupled system is controllable), we now proceed to demonstrate the existence of a time-optimal control for problem ($\mathbf{P}$).

Theorem 3. *Assuming that H1, H2 and H2a hold, for any initial statement* $y_0 \in L^2(\Omega) \setminus \{0\}$*, the time optimal control problem* ($\boldsymbol{P}$) *of Equation* (2) *has a solution, namely, there exists the optimal time* $T^* > 0$*, which is defined by*

$$T^* := inf\{T | y(\cdot,T;y_0,g) = 0, g \in \boldsymbol{K}\},$$

this control $g \in \boldsymbol{K}$ *is defined as the time-optimal control, denoted by* g^**. For the time-optimal control, it is established that the corresponding state* y *fulfills the following relation:*

$$\|y(\cdot,T^*;y_0,g^*)\|_{L^2(\Omega)} = 0.$$

Proof. According to Theorem 1, we know $\mathbf{K}_{ad} \neq \emptyset$, then

$$\{T > 0 | y(\cdot,T;y_0,g) = 0, g \in \mathbf{K}\} \neq \emptyset$$

is obvious. By the definition of the optimal time,

$$T^* := inf\{T > 0 | y(\cdot,T;y_0,g) = 0, g \in \mathbf{K}\},$$

there exists a time sequence $\{T_m\}_{m>1}$, such that

$$T^* = \lim_{m\to\infty} T_m,$$

the corresponding $g_m \in \mathbf{K}$, $E_m \in L^\infty(0,T; H(curl,\Omega)$, $y_m \in W[0,T] \cap$

$C([0,T];L^2(\Omega))$, and $T_m > 0$ is large enough constants and satisfies

$$\begin{cases} \nabla \times (\gamma(x,y_m) \times \mathbf{E}_m) + \xi(x,y_m)\mathbf{E}_m = 0, \quad (x,t) \in \Omega \times [0,T_m], \\ (y_m)_t - \nabla \cdot (k(x)\nabla y_m) = \frac{1}{2}a_2(x,y_m)|\mathbf{E}_m|^2 g_m(t), \quad (x,t) \in \Omega \times [0,T_m], \\ \mathbf{n} \times \mathbf{E}_m = \mathbf{n} \times \mathbf{G}, \quad (x,t) \in \partial\Omega \times [0,T_m], \\ y_m(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T_m], \\ y_m(x,T_0) = y(T_0), \quad x \in \Omega, \\ y_m(x,T_m) = 0, \quad x \in \Omega. \end{cases} \tag{34}$$

According to Lemma 5, then $\mathbf{E}_m \in L^\infty(0,T;H(curl,\Omega) \cap L^6(\Omega \times [0,T])$ and $y_m \in W[0,T]$ are bounded, such that

$$\max_{0 \le t \le T} \|y_m(\cdot t)\|_{L^2(\Omega)} \le C$$

holds for a constant $C > 0$. Since the embeddings $H(curl,\Omega) \hookrightarrow L^2(\Omega)$ and $W[0,T] \hookrightarrow L^2(0,T;L^2(\Omega))$ are compact, it indicates that there exists a subsequence of $(\mathbf{E}_m, y_m)$, still denoted by $(\mathbf{E}_m, y_m)$ and $\mathbf{E}^* \in L^\infty(0,T;H(curl,\Omega)) \cap L^\infty(0,T;L^6(\Omega)$, $y^* \in W[0,T]$, such that they satisfy

$$\mathbf{E}_m \rightharpoonup \mathbf{E}^* \quad \text{in} \quad L^p(0,T;H(curl,\Omega)),$$

$$\nabla y_m \rightharpoonup \nabla y^* \quad \text{in} \quad L^2(0,T;H^{-1}(\Omega)),$$

$$\mathbf{E}_m(\cdot,t) \longrightarrow \mathbf{E}^*(\cdot,t) \quad \text{in} \quad L^2(\Omega), \quad a.e. \ t \in [0,T],$$

and

$$y_m \longrightarrow y^* \quad \text{in} \quad L^2(\Omega), \quad \forall t \in [0,T], \tag{35}$$

as $m \to +\infty$.

In space L^2, strong convergence implies that there exists a subsequence of the pairs $(\mathbf{E}_m, y_m)$, which, for notational simplicity, is still denoted by $(\mathbf{E}_m, y_m)$, such that the following relation holds:

$$\mathbf{E}_m(x,t) \longrightarrow \mathbf{E}^*(x,t) \quad \text{a.e.} \quad (x,t) \in \Omega \times [0,T],$$

$$y_m(x,t) \longrightarrow y^*(x,t) \quad \text{for any} \quad t \in [0,T], \quad \text{a.e.} \quad x \in \Omega, \quad \text{as} \quad m \to +\infty.$$

From condition H2a and Lemma 4, then it yields

$$\begin{aligned} &\int_0^T \int_\Omega |[a_2(x,y_m)|\mathbf{E}_m|^2 - a_2(x,y)|\mathbf{E}^*|^2]|^2 dxdt \\ &\le \int_0^T \int_\Omega |[a_2(x,y_m)(|\mathbf{E}_m|^2 - |\mathbf{E}^*|^2)]|^2 dxdt \\ &+ \int_0^T \int_\Omega |(a_2(x,y_m) - a_2(x,y))|\mathbf{E}^*|^2]|^2 dxdt \\ &\le A_0 \int_0^T \int_\Omega (|\mathbf{E}_m|^2 - |\mathbf{E}|^2)]^2 dxdt + \int_0^T \int_\Omega |(a_2(x,y_m) - a_2(x,y))|^2 |\mathbf{E}^*|^4 dxdt \\ &\longrightarrow 0, (\quad m \longrightarrow +\infty). \end{aligned} \tag{36}$$

To the equation

$$(y_m)_t - \nabla \cdot (k(x)\nabla y_m) = \frac{1}{2}a_2(x, y_m)|\mathbf{E}_m|^2 g_m(t), \tag{37}$$

multiply its both sides by $y_m(x,t) - y^*(x,t)$, and integrate it over $[0,T] \times \Omega$, by Green integral formula, then it satisfies

$$\begin{aligned} & \int_0^T \int_\Omega (y_m)_t (y_m - y^*) dxdt + \int_0^T \int_\Omega k(x)\nabla y_m \cdot \nabla(y_m - y^*) dxdt \\ = & \int_0^T \int_\Omega \frac{1}{2} a_2(x, y_m)|\mathbf{E}_m|^2 g_m(t)(y_m - y^*) dxdt. \end{aligned}$$

Since $\{g_m\} \subset \mathbf{K}$ is a bounded subsequence in $L^p(0,T), 1 \leq p \leq +\infty$, then there exists a weakly converge subsequence, which is also denoted by $\{g_m\}$, namely, there exists $g^* \in \mathbf{K}$ such that

$$g_m \rightharpoonup g^* \quad \text{in} \quad L^p(0,T).$$

Therefore, y^* satisfies the following system

$$(y^*)_t - \nabla \cdot (k(x)\nabla y^*) = \frac{1}{2}a_2(x, y^*)|\mathbf{E}^*|^2 g^*(t), \tag{38}$$

in the sense of weak convergence.

For the embedding $W[0,T] \hookrightarrow C([0,T]; L^2(\Omega))$ is continuous, then it satisfies

$$\|y_m(\cdot,t)\|_{L^2(\Omega)} \longrightarrow \|y^*(\cdot,t)\|_{L^2(\Omega)}, m \to +\infty,$$

for any arbitrary $t \in [0,T]$.

Therefore, it holds

$$y^*(x,0) = y_0(x) \quad \text{in} \quad L^2(\Omega),$$

and for the $0 < t_m < T$, there exists a $y^*(x,t_m)$ such that it yields

$$y_m(x,t_m) \to y^*(x,t_m) \quad \text{in} \quad L^2(\Omega),$$

as $m \to \infty$. For $y^* \in C((0,T], L^2(\Omega)), T_m \to T^*$, we can obtain $y^*(\cdot, T^*) = 0$ in the space $L^2(\Omega)$.

On the other hand, to the following equation

$$\nabla \times (\gamma(x, y_m)\nabla \times \mathbf{E}_m) + \xi(x, y_m)\mathbf{E}_m = 0, \tag{39}$$

multiply its both sides by $\Psi \in H_0(curl, \Omega)$ and integrate the variable x over the space Ω, then it follows that

$$-\int_\Omega \gamma(x, y_m)(\nabla \times \mathbf{E}_m) \cdot (\nabla \times \Psi) dx + \int_\Omega \xi(x, y_m)\mathbf{E}_m \cdot \Psi dx = 0. \tag{40}$$

Assume that γ and ξ are uniformly Lipschitz continuous to the statement variable y as they are assumed in H2a, then y_m in the Equation (35) converges, and $\mathbf{E}^*$ satisfies

$$-\int_\Omega \gamma(x,y^*)(\nabla\times\mathbf{E}^*)\cdot(\nabla\times\Psi)dx+\int_\Omega \xi(x,y^*)\mathbf{E}^*\cdot\Psi dx=0. \tag{41}$$

From the above, it is known that $(\mathbf{E}^*,y^*)\in L^\infty(0,T;H(curl,\Omega))\times W[0,T]$ is the solution to the controlled system corresponding to the control $g^*\in\mathbf{K}$ and y^* satisfying

$$\|y^*(\cdot,T^*;y_0,g^*)\|_{L^2(\Omega)}=0.$$

□

5. Bang-bang property of the time optimal control of problem (P)

Based on the preceding analysis, the existence of a time-optimal control for the strongly coupled microwave heating system described in problem $(\mathbf{P})$ has been established. Inspired by previous studies [8,14,29,30], this section is devoted to proving that this time-optimal control possesses the bang-bang property. The bang-bang property is of significant practical importance, as it not only determines the structural form of the optimal control law but also elucidates the mechanism by which the control force acts upon the system to achieve a specified target in minimal time.

Theorem 4 (Bang-bang property)**.** *To the strongly coupled microwave heating system Equation* (2)*, assume that the conditions H1, H2, and H2a hold, then there exists a time optimal control* g^* *such that it satisfies* $g^*=0$ *or* $g^*=1$ *a.e on* $[0,T^*]$*, and*

$$\begin{cases}\nabla\times(\gamma(x,y^*)\nabla\times\boldsymbol{E})+(-a_1(x,y)+ia_2(x,y^*))\boldsymbol{E}=0, & (x,t)\in\Omega\times[0,T^*],\\ \boldsymbol{n}\times\boldsymbol{E}=\boldsymbol{n}\times\boldsymbol{G}, & (x,t)\in\partial\Omega\times[0,T^*],\\ y_t^*-\nabla\cdot(k(x)\nabla y^*)=\frac{1}{2}a_2(x,y^*)|\boldsymbol{E}|^2g^*(t), & (x,t)\in\Omega\times[0,T^*],\\ y^*(x,t)=0, & (x,t)\in\partial\Omega\times[0,T^*],\\ y^*(x,0)=y_0(x), & x\in\Omega,\\ y^*(x,T^*)=0, & x\in\Omega.\end{cases} \tag{42}$$

Proof. By contradiction, set the optimal time as T^*, optimal control as $g^*\in\mathbf{K}$ and the corresponding optimal solution as $(\mathbf{E}^*,y^*)\in L^\infty(0,T^*;H(curl,\Omega))\times W[0,T^*]\cap C([0,T^*];L^2(\Omega))$ to the coupled Equation (2), namely,

$$\begin{cases}\nabla\times(\gamma(x,y^*)\nabla\times\mathbf{E}^*)+\xi(x,y^*)\mathbf{E}^*=0, & (x,t)\in\Omega\times[0,T^*],\\ \mathbf{n}\times\mathbf{E}^*=\mathbf{n}\times\mathbf{G}, & (x,t)\in\partial\Omega\times[0,T^*],\\ y_t^*-\nabla\cdot(k(x)\nabla y^*)=\frac{1}{2}a_2(x,y^*)|\mathbf{E}^*|^2g^*, & (x,t)\in\Omega\times[0,T^*],\\ y^*(x,t)=0, & (x,t)\in\partial\Omega\times[0,T^*],\\ y^*(x,0)=y_0(x), & x\in\Omega,\\ y^*(x,T^*)=0. \end{cases} \tag{43}$$

If it contradicts the above, then there exists a constant $0<\mu_1<1$ and a measurable set $I^*\subset[0,T^*]$ with $|I^*|>0$, satisfying

$$0\le g^*\le 1-\mu_1,\quad \forall t\in I^*.$$

In the following, it will show that there exists a $\delta > 0$ and an optimal control $g_\delta \in \mathbf{K}$, such that the solution $(\mathbf{E}_\delta, y_\delta)$ of the Equation (2) corresponding to the time optimal control g_δ satisfy

$$\begin{cases} \nabla \times (\gamma(x, y_\delta)\nabla \times \mathbf{E}_\delta) + \xi(x, y_\delta)\mathbf{E}_\delta = 0, \quad (x,t) \in \Omega \times [0, T^* - \delta], \\ \mathbf{n} \times \mathbf{E}_\delta = \mathbf{n} \times \mathbf{G}, \quad (x,t) \in \partial\Omega \times [0, T^* - \delta], \\ (y_\delta)_t - \nabla \cdot (k(x)\nabla y_\delta) = \frac{1}{2}a_2(x, y_\delta)|\mathbf{E}_\delta|^2 g_\delta, \quad (x,t) \in \Omega \times [0, T^* - \delta], \\ y_\delta(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, T^* - \delta], \\ y_\delta(x, 0) = y_0(x), \quad x \in \Omega, \\ y_\delta(x, T^* - \delta) = 0. \end{cases} \tag{44}$$

At first, taking account of the heat equation in the controlled coupled system, we assume $g(t) = 0, t \in [0, T_0]$, then it is

$$\begin{cases} y_t - \nabla \cdot (k(x)\nabla y) = 0, (x,t) \in \Omega \times [0, T_0], \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, T_0], \\ y(x,0) = y_0, \quad x \in \Omega. \end{cases} \tag{45}$$

According to Lemma 4 in the studies [6,8], then we have

$$\|y(\cdot, T_0)\|_{L^2(\Omega)} \leq \exp(-\lambda T_0/2)\|y_0(\cdot)\|_{L^2(\Omega)}. \tag{46}$$

Furthermore, for any $g \in L^\infty(0,T)$, it can be induced that the solution

$$(\mathbf{E}, y) \in L^\infty(0, T; H(curl, \Omega)) \times (W[0,T] \cap C([0,T]; L^2(\Omega)))$$

of the strongly coupled system Equation (2) such that there exists a constant $r > 0$, for any $t \in [0,T]$ a.e, satisfying

$$\int_\Omega a_2(x,y)|\mathbf{E}|^2 dx \geq r. \tag{47}$$

For the subhypothesis H2a, then there exists an $a_2(x,y)$ with $a_2(x,y) \geq a_0$, and to show the Equation (47) is equal to the following inequality

$$\int_\Omega |\mathbf{E}|^2 dx \geq r. \tag{48}$$

By contradiction, for any positive integer n, there exists a corresponding solution

$$(\mathbf{E}_n, y_n) \in L^\infty(0, T; H(curl, \Omega)) \times W[0,T]$$

to the Equation (2), such that

$$\int_\Omega |\mathbf{E}_n|^2 dx \leq \frac{1}{n}, n = 1, \cdots. \tag{49}$$

According to the lemma as in the study by Wei et al. [3], y_n are bounded and compactly embed in the space $W[0,T]$, namely,

$$W[0,T] \hookrightarrow L^2(0,T;L^2(\Omega)),$$

then there exists a subsequence of (y_n) still denoted by (y_n) and a function $\bar{y} \in W[0,T]$, such that

$$y_n \longrightarrow \bar{y}, \text{ in } L^2(0,T;L^2(\Omega)), \quad n \to +\infty.$$

According to the subhypothesis H2a, γ and ξ both are Lipschize continuous to the variable y, therefore, it yields

$$\gamma(x,y_n) \to \gamma(x,\bar{y}), \qquad \xi(x,y_n) \to \xi(x,\bar{y}).$$

From Lemma 4 and the convergence of the sequence y_n , we have the result following:

$$\int_\Omega |\overline{\mathbf{E}}|^2 dx = 0, \qquad \int_\Omega |\nabla \times \overline{\mathbf{E}}|^2 dx = 0.$$

Since $(\mathbf{E}_n, y_n)$ satisfies the Equation (2), then the limitations of them $(\overline{\mathbf{E}}, \bar{y})$ are the solution to the coupled system Equation (2), correspondingly. Consequently, we can conclude that $\mathbf{G} \neq 0$ in the space $L^2(\partial\Omega)$ for a.e $t \in [0,T]$, and $\|\overline{\mathbf{E}}\|_{L^2(\Omega)} \neq 0$, this contradicts the assumption, and it means the result holds.

Furthermore, to show the theorem, let δ_0 to be a certain positive number to satisfy $\delta_0 \leq \frac{|I^*|}{2}$, then it leads to the result

$$| \, I^* \cap (\delta_0, T^*] \, | \geq | \, I^* \, | - \delta_0 \geq \frac{|I^*|}{2}.$$

Let $I = I^* \cap (\delta_0, T^*)$, then $|I| > 0$ is obvious, where $|I|$ is the measure of set I. If it follows $0 < T_0 < T_1 < T^*$, $\omega = \Omega$, and $I \subset (T_1, T^*)$, according to Theorem 1, then there exists $\tilde{g} \in L^\infty(0,T^*)$, the corresponding solution

$$(\widetilde{\mathbf{E}}, \tilde{y}) \in L^\infty(T_0,T^*;H(curl,\Omega)) \times W[T_0,T] \cap C([T_0,T];L^2(\Omega))$$

will satisfy

$$\begin{cases} \nabla \times (\gamma(x,\tilde{y})\nabla \times \widetilde{\mathbf{E}}) + \varepsilon\xi(x,\tilde{y})\widetilde{\mathbf{E}} = 0, \quad (x,t) \in \Omega \times [T_0,T^*], \\ \mathbf{n} \times \widetilde{\mathbf{E}} = \mathbf{n} \times \mathbf{G}, \quad (x,t) \in \partial\Omega \times [T_0,T^*], \\ \tilde{y}_t - \nabla \cdot (k(x)\nabla\tilde{y}) = \frac{1}{2}\chi_I(t)a_2(x,\tilde{y})|\widetilde{\mathbf{E}}|^2\tilde{g}(t), \quad (x,t) \in \Omega \times [T_0,T^*], \\ \tilde{y}(x,t) = 0, \quad (x,t) \in \partial\Omega \times [T_0,T^*], \\ \tilde{y}(x,T_0) = y(x,T_0), \quad x \in \Omega, \\ \varepsilon\tilde{y}(x,T^*) = 0, \end{cases} \tag{50}$$

and

$$\frac{1}{2}\|a_2(x,\tilde{y})|\widetilde{\mathbf{E}}|^2\tilde{g}(t)\|_{L^\infty(T_0,T;L^2(\Omega))} \leq exp(C(1 + \frac{1}{T^* - T_0}))\|y(\cdot,T_0)\|_{L^2(\Omega)}. \tag{51}$$

From the property of the solution above and the hypothesis $a_2(x,y) \geq a_0 > 0$, the control satisfies

$$|\tilde{g}(t)| \leq \frac{2}{r \cdot a_0} exp(C(1 + \frac{1}{T^* - T_0}))\|y(\cdot, T_0)\|_{L^2(\Omega)}. \tag{52}$$

Combining the Equation (46), it can be established that

$$\lim_{T_0 \to +\infty} \|y(\cdot, T_0)\|_{L^2(\Omega)} = 0,$$

and let T_0 to be large enough such that

$$\frac{2}{r \cdot a_0} exp(C(1 + \frac{1}{T^* - T_0}))\|y(\cdot, T_0)\|_{L^2(\Omega)} \leq 1,$$

and this means $\tilde{g} \in \mathbf{K}$.

Assume that $T_0 = \delta, T_1 = \delta_0$ and

$$\hat{g}(t) = \begin{cases} g^*(t), & t \in [T_0, T^*]\backslash I^*, \\ \tilde{g}(t), & t \in I^*, \end{cases} \tag{53}$$

then we have $\hat{g} \in \mathbf{K}$, and

$$\begin{cases} \nabla \times (\gamma(x, \hat{y}) \times \widehat{\mathbf{E}}) + \xi(x, \hat{y})\widehat{\mathbf{E}} = 0, \quad (x, t) \in \Omega \times [\delta, T^*], \\ \hat{y}_t - \nabla \cdot (k(x)\nabla \hat{y}) = \frac{1}{2}\hat{a}_2(x, \hat{y})|\widehat{\mathbf{E}}|^2 \hat{g}(t), (x, t) \in \Omega \times [\delta, T^*], \\ \mathbf{n} \times \widehat{\mathbf{E}} = \mathbf{n} \times \mathbf{G}, \quad (x, t) \in \partial\Omega \times [\delta, T^*], \\ \hat{y}(x, t) = 0, \quad (x, t) \in \partial\Omega \times [\delta, T^*], \\ \hat{y}(x, \delta) = \hat{y}(\delta), \quad x \in \Omega, \\ \hat{y}(x, T^*) = 0, \quad x \in \Omega. \end{cases} \tag{54}$$

Let $y(\cdot, t) = \hat{y}(\cdot, \delta + t), \mathbf{E}(\cdot, t) = \widehat{\mathbf{E}}(\cdot, \delta + t)$, then it satisfies

$$\begin{cases} \nabla \times (\gamma(x, y) \times \mathbf{E}) + \xi(x, y)\mathbf{E} = 0, \quad (x, t) \in \Omega \times [0, T^* - \delta], \\ y_t - \nabla \cdot (k(x)\nabla y) = \frac{1}{2}\hat{a}_2(x, y)|\mathbf{E}|^2 \hat{g}(t), (x, t) \in \Omega \times [0, T^* - \delta], \\ \mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{G}, \quad (x, t) \in \partial\Omega \times [0, T^* - \delta], \\ y(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T^* - \delta], \\ y(x, 0) = y(0) = y_0, \quad x \in \Omega, \\ y(x, T^* - \delta) = 0, \quad x \in \Omega. \end{cases} \tag{55}$$

According to Theorem 3, this means that $T^* - \delta$ is a reachable time for the time-optimal control problem ($\mathbf{P}$), it contract that T^* is the optimal time for the time-optimal control problem ($\mathbf{P}$) governed by the Equation (2), then $g(t) = 0$ or $g(t) = 1$, $t \in [0, T^*]$ holds. □

6. Conclusion

Based on the preceding analysis and the results [8, 29–31], it is theoretically confirmed that microwave heating can achieve a rapid temperature rise in a target material. Specifically, for both linear and nonlinear microwave heating systems—that is, regardless of whether the permeability and magnetic loss factor depend on temperature—the desired final temperature can be attained in a time-optimal manner. Moreover, the bang-bang property of the time-optimal controls implies that, in essence, only one switching action is required to reach the target in the shortest possible time. Thus, this study not only provides a theoretical foundation for the application of microwave heating but also offers practical guidance for its optimal control.

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