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Article

Unified framework of four Caputo fractional differences for initial and final value problems in discrete fractional calculus with variable bounds

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Abstract: Fractional calculus has emerged as a powerful tool for characterizing non-classical dynamic phenomena, yet its discretization remains fragmented, with existing studies primarily focusing on single combinations of variable bounds and difference directions. To address this gap, this paper proposes a unified theoretical framework for discrete fractional calculus by systematically introducing four novel Caputo fractional difference definitions, which integrate variable upper/lower-limit sums with forward/backward difference operations. First, we rigorously derive the fundamental properties of these four definitions, including the commutativity of fractional sums and differences, and their consistency with integer-order difference operations. Second, we construct fractional difference equations for each definition, establish their equivalence to Volterra sum equations, and provide explicit solutions and strict proofs for their corresponding initial and final value problems. To validate the theoretical results, we design four targeted computational cases and numerical simulations, confirm the consistency between theoretical solutions and numerical results, and intuitively demonstrate the long-memory effect of fractional-order discrete systems. Furthermore, we present a concise comparison of the four definitions, clarifying their suitability for discrete systems with distinct boundary conditions and dynamic characteristics. This work not only completes the theoretical system of discrete fractional calculus with variable bounds but also provides standardized and targeted mathematical tools for modeling complex discrete dynamic processes, laying a solid foundation for the practical application of discrete fractional calculus in fields such as engineering control, infectious disease modeling, and economic dynamics.

Keywords: Caputo fractional difference equation; forward and backward differences; variable upper and variable lower-limit sums; initial and final value problems

1. Introduction

In 1695, French mathematician L'Hôpital first mentioned fractional calculus in a letter to German mathematician Leibniz, with Leibniz's response, stating that "it will one day be a very useful result," becoming a classic [1]. Although fractional calculus was mentioned, it remained merely a subject of mathematical discourse and conjecture for over a century. It was not until 1832 that Liouville introduced the concept of fractional derivatives and successfully applied them [2]. The first conference on fractional calculus and its applications was held at Yale University in 1974, the same year Oldham and Spanier published a monograph on fractional calculus, discussing the general properties of difference and integral operators and their applications in mathematics and other fields [3]. In recent years, it has become increasingly recognized that fractional calculus can characterize certain non-classical phenomena in the natural sciences and engineering applications that integer-order calculus cannot, such as complex network phenomena [4], memristive circuit effects [5], the spread of COVID-19, and biological phenomena [6,7]. Moreover, behind these phenomena, fractional calculus has found extensive applications in numerous fields, including mechanics [8, 9], biomedical engineering [10], control theory [11,12], and economics [13].

In fact, most variables in life exist in discrete forms, and many studies of natural models require the discretization of continuous variables. With the rapid advancement of digital technology, the discretization of fractional calculus has become an increasingly significant area of research. As both theoretical models involving discrete variables and computational models requiring the discretization of continuous variables evolve, difference equations naturally emerge as an indispensable tool. Difference equations primarily originate from the discretization of differential equations and the study of discrete models in practical problems, such as infectious disease models [14, 15], signal processing [16], chaotic memristors [17, 18], and economic models [19,20]. Early theoretical research on difference equations focused mainly on stability analysis and the stability of solutions. In 1892, the renowned Russian mathematician Lyapunov introduced concepts and properties related to the solutions of differential and difference equations, laying the foundation for the stability of differential and difference equations [21]. In 1959, Kalman and Bertram were the first to study the stability of discrete dynamical systems in detail [22]. Subsequently, since the 1990s, many scholars have conducted systematic qualitative studies on the theory of difference equations, resulting in numerous monographs on the subject. For example, in 1993, Kocic and others compiled a comprehensive work on the global behavior of nonlinear difference equations, from which much of the fundamental theory of difference equations originated [23]. In 2000, Agarwal systematically described the relevant symbols, basic concepts, and fundamental properties of difference equations [24].

Compared to the well-established research on continuous fractional differentiation and integration, the study of discrete fractional differences and sums has been relatively delayed. Just as integer-order differentiation and differences share similarities, the definitions of fractional differences, akin to those of fractional derivatives, include the Grünwald-Letnikov (G-L) definition [25], the Riemann-Liouville (R-L) definition [26], and the Caputo definition [27]. Based on the theory of fractional difference equations, scholars have undertaken extensive work. Goodrich, by combining discrete calculus with forward fractional difference theory, extensively explored the theoretical foundations and practical applications of discrete fractional calculus [28]. Aledj employed fixed-point theory to investigate the existence and Ulam stability of implicit fractional difference equations in Banach spaces and Banach algebras [29]. Hattaf proposed a predictor-corrector compact difference scheme for nonlinear fractional differential equations [30].

Regarding the definitions of discrete fractional calculus, the Caputo definition is widely adopted in real-world problems because it allows for the provision of initial conditions in the form of integer-order derivatives, thereby facilitating more convenient and intuitive modeling and integral transformation processes. Allouch, for instance, investigated the existence of solutions to boundary value problems for fractional q-difference equations involving Caputo-type q-derivatives and nonlinear integral boundary conditions in Banach spaces [31]. Alili, based on the Caputo fractional difference definition, studied the numerical solution method for the time-fractional Cattaneo equation, proposed an efficient finite difference scheme, and verified its accuracy and efficiency through numerical examples [32]. Using the Caputo definition and difference operators, Nong explored the existence of solutions to hybrid and non-hybrid discrete fractional three-point boundary value inclusion problems for the elastic beam equation. By employing fixed-point theory, existence results were established, and examples were provided to support the conclusions [33]. In reality, the Caputo fractional difference definition is a combination of fractional sums and integer-order differences: in the theory of calculus, there exist variable upper limit integrals and variable lower limit integrals [34]. Similarly, for fractional sums, there should be variable upper and lower limit sums, while integer-order differences can be divided into forward differences and backward differences [35]. Theoretically, there should exist four distinct definitions of Caputo fractional differences, each leading to initial and final value problems for forward and backward fractional difference equations within the broader context of fractional difference equations. However, most of the existing studies primarily focus on the combination of variable upper-limits and forward differences, specifically addressing forward difference equations with known initial conditions. While Cheng [36] introduced a fractional difference definition that combines backward differences with variable upper limits and provided examples of solving fractional difference equations under this definition, a standardized form for the equation's solution was not presented. Furthermore, research on other possible forms of fractional differences remains exceedingly limited, and systematic development in this area has been virtually nonexistent.

Therefore, this paper aims to systematically propose and analyze four definitions of fractional differences based on the Caputo definition. Furthermore, we examine the initial and terminal value problems of fractional difference equations under both forward and backward difference definitions. Through the fractional difference method, this study aims to extend and optimize the sole combination of "variable upper-limit sums and forward differences" originally proposed specifically for initial value problems [28], while systematically addressing the other three logically indispensable combinations of variable bounds (upper/lower) and difference directions (forward/backward) [36]—thus yielding more precise models for characterizing complex dynamic processes in discrete systems. Particularly when dealing with discrete systems that exhibit long memory effects, non-local interactions, or complex behaviors, fractional difference methods demonstrate their unique advantages and vast application prospects. The organization of this paper is as follows: In Section

2, we provide some notations and basic preliminaries. In Section 3, we introduce the definitions of forward differences, backward differences, variable limit integrals, as well as fractional differences and fractional integrals. In Section 4, we derive several important properties of fractional differences and integrals. In Section 5, we establish Caputo fractional difference equations based on the four different definitions and present related results. Finally, Section 6 concludes the paper.

2. Preliminaries

To establish a robust foundation for our analysis, we present a series of basic definitions and notations, together with their associated fundamental theorems, which form the foundation for the subsequent analysis.

Definition 1. *Let $x \in \mathbb{R}_+$, the Gamma function is defined as [28]:*

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\, dt.$$

with the convention that $0! = 1$. It possesses the following properties:

1. $\Gamma(x+1) = x\Gamma(x)$,

2. $\Gamma(n) = (n-1)!$, *where $n \in \mathbb{N}_+$.*

Definition 2. *Let $n, m \in \mathbb{N}_+$, define the rising factorial as [28]:*

$$\begin{aligned} n^{\overline{m}} &= n(n+1)(n+2)\cdots(n+m-1) \\ &= \tfrac{(n-1)!n(n+1)(n+2)\cdots(n+m-1)}{(n-1)!} \\ &= \tfrac{(n+m-1)!}{(n-1)!} = \tfrac{\Gamma(n+m)}{\Gamma(n)}. \end{aligned}$$

And the down factorial as:

$$\begin{aligned} n^{\underline{m}} &= n(n-1)(n-2)\cdots(n-m+1) \\ &= \tfrac{n(n-1)(n-2)\cdots(n-m+1)(n-m)!}{(n-m)!} \\ &= \tfrac{n!}{(n-m)!} = \tfrac{\Gamma(n+1)}{\Gamma(n-m+1)}. \end{aligned}$$

Extending n to the range of real numbers, the rising and down factorials of x are defined as:

$$x^{\overline{m}} = x(x+1)(x+2)\cdots(x+m-1) = \frac{\Gamma(x+m)}{\Gamma(x)},$$

$$x^{\underline{m}} = x(x-1)(x-2)\cdots(x-m+1) = \frac{\Gamma(x+1)}{\Gamma(x-m+1)}.$$

and when $m = 0,\ x^{\overline{0}} = x^{\underline{0}} = 1,\ when\ m < 0,\ x^{\overline{m}} = x^{\underline{m}} = 0.$

Definition 3. *Let $n \in \mathbb{N}_+$, $\alpha \in \mathbb{R}_+$, fractional factorial is defined as [28]:*

$$n^{\overline{\alpha}} = \frac{\Gamma(n+\alpha)}{\Gamma(n)}, n^{\underline{\alpha}} = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}.$$

Note: In the following parts of this paper, generally, a and b are constants, while n and t are variables.

3. Difference and sum

According to the fundamental theory of calculus:

1. $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ and $\frac{d}{dx}\int_x^b f(t)\,dt = -f(x)$,
2. $\int_a^x f^{'}(t)\,dt = f(x) - f(a)$ and $\int_x^b f^{'}(t)\,dt = f(b) - f(x)$,
3. $\int_a^x dt = x - a$ and $\frac{d}{dx}(x-a) = 1$,
4. $\int_x^b dt = b - x$ and $\frac{d}{dx}(b-x) = -1$.

Theoretically, analogous to the relationship between differentiation and integration, the interplay between difference and sum should similarly exhibit an analogous correspondence. Unlike differentiation, which does not possess directional distinctions, the concept of difference bifurcates into backward and forward differences. Accordingly, this section introduces definitions for both backward and forward differences, as well as for backward and forward variable upper and lower-limit sums, all grounded in the four foundational principles of calculus. The correlation between difference and sum is thereby substantiated, and the definitions of fractional sums in various forms are elucidated.

3.1. Backward difference and sum

This section commences with the introduction of the backward difference definition, accompanied by the corresponding definitions and properties of variable upper and lower-limits related to backward differences. Subsequently, the order of these variable upper and lower-limits is extended to derive fractional-order variable upper and lower-limits pertinent to backward differences.

Definition 4. *(Backward difference) We define the first-order backward difference of $x(n)$ as [28]:*

$$\nabla x(n) = x(n) - x(n-1).$$

and the k-order backward difference of $x(n)$ as

$$\nabla^k x(n) = \nabla\left(\nabla^{k-1} x(n)\right).$$

where $n \in \mathbb{Z}, k \in \mathbb{N}_+$.

Definition 5. *(Backward variable upper-limit sum) Let $a, n \in \mathbb{Z}$, $k \in \mathbb{N}_+$ and $a < n$, we call [28]:*

$${}_a\nabla^{-1} x(n) = \sum_{i=a+1}^{n} x(i) = x(a+1) + \cdots + x(n).$$

the first order backward variable upper-limit sum of $x(n)$, and we call

$${}_a\nabla^{-k} x(n) = {}_a\nabla^{-1}\left[{}_a\nabla^{-(k-1)}\right] x(n).$$

the k-order backward variable upper-limit sum of $x(n)$.

To validate the correctness of the backward variable upper-limit sum and backward difference operations as defined in Definition 5, we first establish their direct correspondence to foundational relationships in continuous calculus. Specifically, these discrete operations align with the continuous integral identity $\int_a^x dt = x - a$ and its associated derivative result $\frac{d(x-a)}{dx} = 1$. Furthermore, the dual relationships—i.e., the backward difference of a backward variable upper-limit sum, and the backward variable upper-limit sum of a backward difference—mirror two key calculus correspondences for variable upper-limit integrals: the derivative of a variable upper-limit integral $\frac{d}{dx}\int_a^x f(t)dt = f(t)$, and the integral of a derivative over a variable upper-limit domain $\int_a^x f'(t)dt = f(x) - f(a)$. To formalize and rigorously confirm these discrete-continuous analogies, we present the following theorem.

Theorem 1. *Assume* $a, n \in Z$, *and* $a < n$, *the following four equations hold:*

1. $\ {}_a\nabla^{-1}1 = \sum_{i=a+1}^{n} 1 = 1 + 1 + \cdots + 1 = n - a$,
2. $\nabla(n-a) = (n-a) - (n-1-a) = 1$,
3. $\nabla\left[\ {}_a\nabla^{-1}x(n)\right] = \nabla\left[\sum_{i=a+1}^{n} x(i)\right] = \sum_{i=a+1}^{n} x(i) - \sum_{i=a+1}^{n-1} x(i) = x(n)$,
4. $\ {}_a\nabla^{-1}\left[\nabla x(n)\right] = \sum_{i=a+1}^{n}\left(x(n) - x(n-1)\right) = x(n) - x(a)$.

By Definition 5, we have

$$
\begin{aligned}
{}_a\nabla^{-2}x(n) = \quad & {}_a\nabla^{-1}\left[\ {}_a\nabla^{-1}x(n)\right] = {}_a\ \nabla^{-1}\left[\sum_{i=a+1}^{n} x(i)\right] \\
&= \sum_{r=a+1}^{n}\sum_{i=a+1}^{r} x(i) = {}_a\ \nabla^{-1}\left[x(a+1) + x(a+2) + \cdots + x(n)\right] \\
&= (n-a)x(a+1) + (n-a-1)x(a+2)\cdots + x(n) \\
&= \sum_{i=a+1}^{n} (n-i+1)x(i).
\end{aligned}
$$

then

$$
\begin{aligned}
{}_a\nabla^{-3}x(n) \quad &= \sum_{r=a+1}^{n}\sum_{j=a+1}^{r}\sum_{i=a+1}^{j} x(i) \\
&= \tfrac{(n-a)(n-a+1)}{2}x(a+1) + \tfrac{(n-a-1)(n-a)}{2}x(a+2) \\
&+ \cdots + \tfrac{1\times 2}{2}x(n) \\
&= \tfrac{1}{2!}\sum_{i=a+1}^{n} (n-i+1)(n-i-i+2)x(i).
\end{aligned}
$$

by analogy, and based on the statement of Definition 2, we have

$$
\begin{aligned}
{}_a\nabla^{-m}x(n) \quad &= \tfrac{1}{(m-1)!}\sum_{i=a+1}^{n} (n-i+1)(n-i+2)\cdots(n-i+m-1)x(i) \\
&= \tfrac{1}{(m-1)!}\sum_{i=a+1}^{n} (n-i+1)^{\overline{m-1}}x(i) \\
&= \tfrac{1}{\Gamma(m)}\sum_{i=a+1}^{n} \tfrac{\Gamma(n-i+m)}{\Gamma(n-i+1)}x(i),\ m \in \mathbb{N}_+.
\end{aligned}
\tag{1}
$$

By extending the order of integer-order sum in Equation (1) to fractional-order, and in conjunction with the statement of Definition 3, we can derive the following theorem.

Definition 6. *(Fractional backward variable upper-limit sum) Let $a, n \in \mathbb{Z}$, $a < n$ and $\alpha > 0$, we call [28]:*

$$ {}_a\nabla^{-\alpha}x(n) = \frac{1}{\Gamma(\alpha)}\sum_{i=a+1}^{n}(n-i+1)^{\overline{\alpha-1}}x(i) = \frac{1}{\Gamma(\alpha)}\sum_{i=a+1}^{n}\frac{\Gamma(n-i+\alpha)}{\Gamma(n-i+1)}x(i). $$

the α-order backward variable upper-limit sum of $x(n)$.

Definition 7. *(Backward variable lower-limit sum) Let $b, n \in \mathbb{Z}$, $k \in \mathbb{N}_+$ and $b > n$, we call [28]:*

$$ \nabla_b^{-1}x(n) = \sum_{i=n+1}^{b}x(i) = x(n+1) + \cdots x(b-1) + x(b). $$

the first order backward variable lower-limit sum of $x(n)$, and we call

$$ \nabla_b^{-k}x(n) = \nabla_b^{-1}\nabla_b^{-(k-1)}x(n). $$

the k-order backward variable lower-limit sum of $x(n)$.

To validate the correctness of the backward variable lower-limit summation defined in Definition 7, we first establish its correspondence to relevant formulas in continuous calculus. Specifically, the identity $\int_x^b dt = b - x$ and its derivative counterpart $\frac{d(b-x)}{dx} = -1$. We further verify that the relationship between the backward difference of the variable lower-limit summation and the variable lower-limit summation under backward difference operations aligns with the analogous calculus relationships for variable lower-limit integrals, namely $\frac{d}{dx}\int_x^b f(t)dt = -f(x)$ and $\int_x^b f'(t)dt = f(b) - f(x)$. To formalize these correspondences, the following theorem is stated.

Theorem 2. *Assume $b, n \in Z$, and $b > n$. The following four relations are satisfied:*

1. ${}_b\nabla^{-1}1 = \sum_{i=n+1}^{b} 1 = 1 + 1 + \cdots + 1 = b - n$,
2. $\nabla(b-n) = (b-n) - [b-(n-1)] = -1$,
3. $\nabla\left[{}_b\nabla^{-1}x(n)\right] = \nabla\left[\sum_{i=n+1}^{b} x(i)\right] = \sum_{i=n+1}^{b} x(i) - \sum_{i=n}^{b} x(i) = -x(n)$,
4. ${}_b\nabla^{-1}\left[\nabla x(n)\right] = \sum_{i=n+1}^{b}(x(n) - x(n-1)) = x(b) - x(n)$.

By Definition 7, we have

$$ \begin{aligned} \nabla_b^{-2}x(n) &= \sum_{i=n+1}^{b}\sum_{j=i+1}^{b}x(j) = \sum_{i=n+1}^{b}[x(n+1) + \cdots x(b-1) + x(b)] \\ &= x(n+2) + 2x(n+3)\cdots(b-n-2)x(b-1) + (b-n-1)x(b) \\ &= \sum_{i=n+2}^{b}(i-n-1)x(i). \end{aligned} $$

then

$$\nabla_b^{-3}x(n)=\sum_{i=n+1}^{b}\sum_{j=i+1}^{b}\sum_{k=j+1}^{b}x(k)=\frac{1}{2!}\sum_{i=n+3}^{b}(i-n-1)(i-n-2)x(i).$$

by analogy, and based on the statement of Definition 2, we can get

$$\begin{aligned}\nabla_b^{-m}x(n)&=\tfrac{1}{(m-1)!}\sum_{i=n+m}^{b}(i-n-1)(i-n-2)\cdots[i-n-(m-1)]\,x(i)\\&=\tfrac{1}{(m-1)!}\sum_{i=n+m}^{b}(i-n-1)^{\underline{m-1}}x(i)\\&=\tfrac{1}{\Gamma(m)}\sum_{i=n+m}^{b}\tfrac{\Gamma(i-n)}{\Gamma(i-n-m+1)}x(i),\ m\in\mathbb{N}_+.\end{aligned}\tag{2}$$

By extending the order of integer-order sum in Equation (2) to fractional-order, and in conjunction with the statement of Definition 3, we can derive the following theorem.

Definition 8. *(Fractional backward variable lower-limit sum) Let* $b, n\in\mathbb{Z}$*,* $b<n$ *and* $\alpha>0$*, we call [28]:*

$$\nabla_b^{-\alpha}x(n)=\frac{1}{\Gamma(\alpha)}\sum_{i=n+\alpha}^{b}(i-n-1)^{\underline{\alpha-1}}x(i)=\frac{1}{\Gamma(\alpha)}\sum_{i=n+\alpha}^{b}\frac{\Gamma(i-n)}{\Gamma(i-n-\alpha+1)}x(i).$$

the α*-order backward variable lower-limit sum of* $x(n)$*.*

3.2. Forward difference and sum

This section presents the definition of the forward difference, along with the definitions and properties of the variable upper and lower-limits integral corresponding to the forward difference. We then generalize the order of these variable limits to fractional orders, thereby achieving the fractional-order variable upper and lower-limits integral associated with the forward difference.

Definition 9. *(Forward difference) Let* $n\in\mathbb{Z}$*,* $k\in\mathbb{N}_+$*, we call [28]:*

$$\Delta x(n)=x(n+1)-x(n).$$

the first order forward difference of $x(n)$*, and we call*

$$\Delta^k x(n)=\Delta\Delta^{k-1}x(n).$$

the k-order forward difference of $x(n)$*.*

Definition 10. *(Forward variable upper-limit sum) Let* $a, n\in\mathbb{Z}$*,* $k\in\mathbb{N}_+$ *and* $a<n$*, we call*

$$_a\Delta^{-1}x(n)=\sum_{i=a}^{n-1}x(i)=x(a)+x(a+1)+\cdots+x(n-1).$$

the first order forward variable upper-limit sum of $x(n)$, and we call

$$ {}_a\Delta^{-k}x(n) =_a \Delta^{-1}\left[{}_a\Delta^{-(k-1)}\right] x(n). $$

the k-order forward variable upper-limit sum of $x(n)$.

To validate the correctness of the forward variable upper-limit sum defined in Definition 10, we first link it to its counterpart in continuous calculus: specifically, the integral identity $\int_a^x dt = x - a$ and its derivative $\frac{d(x-a)}{dx} = 1$. We further establish that two key relationships in the discrete domain—namely, the forward difference of a forward variable upper-limit sum, and the forward variable upper-limit sum of a forward difference—align with the analogous principles for variable upper-limit integrals in continuous calculus: the derivative of a variable upper-limit integral $\frac{d}{dx}\int_a^x f(t)dt = f(t)$, and the integral of a derivative over a variable upper-limit domain $\int_a^x f'(t)dt = f(x) - f(a)$. To formalize these discrete-continuous correspondences, we introduce the following theorem.

Theorem 3. *Assume $a, n \in Z$, and $a < n$, The following four formulas are true:*

1. ${}_a\Delta^{-1}1 = \sum_{i=a}^{n-1} 1 = 1 + 1 + \cdots + 1 = n - a,$
2. $\Delta(n-a) = [(n+1) - a] - (n-a) = 1,$
3. $\Delta\left[{}_a\Delta^{-1}x(n)\right] = \Delta\left[\sum_{i=a}^{n-1} x(i)\right] = \sum_{i=a}^{n} x(i) - \sum_{i=a}^{n-1} x(i) = x(n),$
4. ${}_a\Delta^{-1}\left[\Delta x(n)\right] = \sum_{i=a}^{n-1}\left(x(n+1) - x(n)\right) = x(n) - x(a).$

By Definition 10, we have

$$\begin{aligned} {}_a\Delta^{-2}x(n) &=_a \Delta^{-1}\left[\sum_{i=a}^{n-1} x(i)\right] = \sum_{r=a}^{n-1}\sum_{i=a}^{r-1} x(i) \\ &= \Delta^{-1}\left[x(a) + x(a+1) + \cdots + x(n-1)\right] \\ &= (n-a-1)x(a) + (n-a-2)x(a+1) + \cdots + x(n-2) \\ &= \sum_{i=a}^{n-2}(n-i-1)x(i). \end{aligned}$$

then

$$\begin{aligned} {}_a\Delta^{-3}x(n) &= {}_a\Delta^{-1}\left[{}_a\Delta^{-2}x(n)\right] =_a \Delta^{-1}\left[\sum_{i=a}^{n-2}(n-i-1)x(i)\right] \\ &= \sum_{i=a}^{n-1}\left[(n-a-1)x(a) + (n-a-2)x(a+1) + \cdots + x(n-2)\right] \\ &= \tfrac{(n-a-1)(n-a-2)}{2}x(a) + \tfrac{(n-a-2)(n-a-3)}{2}x(a+1) + \tfrac{2\times 1}{2}x(n-3) \\ &= \tfrac{1}{2!}\sum_{i=a}^{n-3}(n-i-1)(n-i-2)x(i). \end{aligned}$$

by analogy, and based on the statement of Definition 2, we can get

$$
\begin{aligned}
{}_a\Delta^{-m}x(n) &= \tfrac{1}{(m-1)!}\sum_{r=a}^{n-m}(n-i-1)(n-i-2)\cdots[n-i-(m-1)]\,x(i)\\
&= \tfrac{1}{(m-1)!}\sum_{i=a}^{n-m}(n-i-1)^{\underline{m-1}}\,x(i)\\
&= \tfrac{1}{\Gamma(m)}\sum_{i=a}^{n-m}\tfrac{\Gamma(n-i)}{\Gamma(n-i-m+1)}x(i),\; m\in\mathbb{N}_+.
\end{aligned}
\tag{3}
$$

By extending the order of integer-order sum in Equation (3) to fractional-order, and in conjunction with the statement of Definition 3, we can derive the following theorem.

Definition 11. *(Fractional forward variable upper-limit sum) Let $a, n \in \mathbb{Z}$, $a < n$ and $\alpha > 0$, we call*

$$
{}_a\Delta^{-\alpha}x(n) = \frac{1}{\Gamma(\alpha)}\sum_{i=a}^{n-\alpha}(n-i-1)^{\underline{\alpha-1}}x(i) = \frac{1}{\Gamma(\alpha)}\sum_{i=a}^{n-\alpha}\frac{\Gamma(n-i)}{\Gamma(n-i-\alpha+1)}x(i).
$$

the α-order forward variable upper-limit sum of $x(n)$.

Definition 12. *(Forward variable lower-limit sum) Let $b, n \in \mathbb{Z}$, $k \in \mathbb{N}_+$ and $b > n$, we call*

$$
\Delta_b^{-1}x(n) = \sum_{i=n}^{b-1}x(i) = x(n) + x(n+1)\cdots x(b-1).
$$

the first order forward variable lower-limit sum of $x(n)$, and we call

$$
\Delta_b^{-k}x(n) = \Delta_b^{-1}\Delta_b^{-(k-1)}x(n).
$$

the k-order forward variable lower-limit sum of $x(n)$.

To validate the correctness of the backward variable lower-limit sum defined in Definition 7, we establish its direct correspondence to foundational relationships in continuous calculus: specifically, the integral identity $\int_x^b dt = b - x$ and its derivative counterpart $\frac{d(b-x)}{dx} = -1$. We further verify that two critical relationships in the discrete domain—the backward difference of a backward variable lower-limit sum, and the backward variable lower-limit sum of a backward difference—align with the analogous principles for variable lower-limit integrals in continuous calculus: the derivative of a variable lower-limit integral $\frac{d}{dx}\int_x^b f(t)dt = -f(x)$, and the integral of a derivative over a variable lower-limit domain $\int_x^b f'(t)dt = f(b) - f(x)$. To formalize these discrete-continuous correspondences, the subsequent theorem is stated as follows.

Theorem 4. *Assume $b, n \in Z$, and $b > n$, The following four equalities are established:*

1. ${}_a\Delta^{-1}1 = \sum_{i=n}^{b-1}1 = 1 + 1 + \cdots + 1 = b - n$,
2. $\Delta(b-n) = [b-(n+1)] - (b-n) = -1$,
3. $\Delta\left[{}_a\Delta^{-1}x(n)\right] = \Delta\left[\sum_{i=n}^{b-1}x(i)\right] = \sum_{i=n+1}^{b-1}x(i) - \sum_{i=n}^{b-1}x(i) = -x(n)$,
4. ${}_a\Delta^{-1}\left[\Delta x(n)\right] = \sum_{i=n}^{b-1}(x(n+1) - x(n)) = x(b) - x(n)$.

By Definition 12, we have

$$\begin{aligned}\Delta_b^{-2}x(n) &= \sum_{i=n}^{b-1}\sum_{j=i}^{b-1}x(j) = \Delta_b^{-1}\left[x(n)+x(n+1)\cdots x(b-1)\right] = \\ &= x(n)+2x(n+1)+\cdots+(b-n)x(b-1) = \sum_{i=n}^{b-1}(i-n+1)x(i).\end{aligned}$$

then

$$\Delta_b^{-3}x(n) = \sum_{i=n}^{b-1}\sum_{j=i}^{b-1}\sum_{k=j}^{b-1}x(k) = \frac{1}{2!}\sum_{i=n}^{b-1}(i-n+1)(i-n+2)x(i).$$

by analogy, and based on the statement of Definition 2, we can get

$$\begin{aligned}\Delta_b^{-m}x(n) &= \tfrac{1}{(m-1)!}\sum_{i=n}^{b-1}(i-n+1)(i-n+2)\cdots\left[i-n+(m-1)\right]x(i) \\ &= \tfrac{1}{(m-1)!}\sum_{i=n}^{b-1}(i-n+1)^{\overline{m-1}}\,x(i) \\ &= \tfrac{1}{\Gamma(m)}\sum_{i=n}^{b-1}\tfrac{\Gamma(i-n+m)}{\Gamma(i-n+1)}x(i),\ m\in\mathbb{N}_+.\end{aligned}\tag{4}$$

By extending the order of integer-order sum in Equation (4) to fractional-order, and in conjunction with the statement of Definition 3, we can derive the following theorem.

Definition 13. *(Fractional forward variable lower-limit sum) Let* $b, n \in \mathbb{Z}$*,* $b > n$ *and* $\alpha > 0$*, we call*

$$\nabla_b^{-\alpha}x(n) = \frac{1}{\Gamma(\alpha)}\sum_{i=n}^{b-1}(i-n+1)^{\overline{\alpha-1}}\,x(i) = \frac{1}{\Gamma(\alpha)}\sum_{i=n}^{b-1}\frac{\Gamma(i-n+\alpha)}{\Gamma(i-n+1)}x(i).$$

the α*-order forward variable lower-limit sum of* $x(n)$*.*

Theorems 1–4 not only validate the correctness of the definitions of variable upper-limit and variable lower-limit summations under the given forward and backward difference definitions (Definitions 5, 7, 10 and 12), including the properties of constant and linear functions in the discrete context. They also verify the consistency of the difference and variable upper-limit summation operations in the discrete case, ensuring their alignment with the corresponding operations of integrals and derivatives in the continuous case. Specifically, they confirm the correctness of the difference of the variable upper-limit summation and the variable upper-limit summation under the difference operation, which mirrors the derivative of a variable upper-limit integral and the integral of a derivative in the continuous case.

Note: Here, we set the backward and forward sum operators with initial value a as ${}_a\nabla^1$ and ${}_a\Delta^1$ respectively, and the backward and forward sum operators with final value b as ${}_b\nabla^1$ and ${}_b\Delta^1$.

The relationship between the relevant definitions and theorems in Section 3 and the key Theorems 22–25 is shown in **Figure 1**.

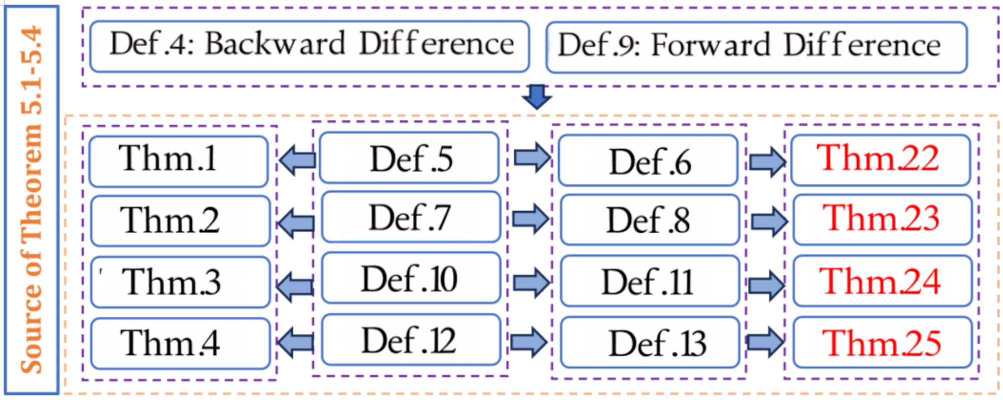

Figure 1. Relationship between the Section 3 and the Theorems 22–25.

4. Some basic properties

In order to prove the initial and final value problems of fractional difference equations, this section gives the properties of difference and sum and fractional difference and sum.

To maintain the consistency of the right side of the equation in Theorems 1–4, negative difference operators ($-\nabla$ and $-\Delta$) are introduced for the variable lower bound difference. Theorems 2 and 4 become the following form.

Theorem 5. *Let b, $n \in Z$ and $b > n$, the following two equalities hold:*

1. $(-\nabla)\nabla_b^{-1}x(n) = -\nabla\left[\sum_{i=n+1}^{b} x(i)\right] = -\left[\sum_{i=n+1}^{b} x(i) - \sum_{i=n}^{b} x(i)\right] = x(n)$,
2. $\nabla_b^{-1}\left[(-\nabla)x(n)\right] = \sum_{i=n+1}^{b}\left(-x(n) + x(n-1)\right) = x(n) - x(b)$.

Theorem 6. *Let b, $n \in Z$ and $b > n$, the following two equalities hold:*

1. $(-\Delta)\Delta_b^{-1}x(n) = -\Delta\left[\sum_{i=n}^{b-1} x(i)\right] = -\left[\sum_{i=n+1}^{b-1} x(i) - \sum_{i=n}^{b-1} x(i)\right] = x(n)$,
2. $\Delta_b^{-1}\left[(-\Delta)x(n)\right] = \sum_{i=n}^{b-1}\left(-x(n+1) + x(n)\right) = x(n) - x(b)$.

Therefore, the difference operators corresponding to sums ${}_a\nabla^{-1}$, ∇_b^{-1}, ${}_a\Delta^{-1}\Delta_b^{-1}$ *are* $\nabla, -\nabla, \Delta, -\Delta$.

Note:

(1) When the common property of forward difference and backward difference are indicated, the difference operator is indicated by D.

(2) [·] represents an integer close to zero.

Definition 14. *(Fractional difference) Let $n \in Z$,$\alpha > 0$ and assume that $m = [\alpha] + 1$. Define [27]:*

$$^{RL}D^{\alpha}x(n) = D^{m}D^{-(m-\alpha)}x(n).$$

as α order Riemann-Loiuville(R-L) type fractional difference. Meantime, define

$$^{C}D^{\alpha}x(n) = D^{-(m-\alpha)}D^{m}x(n).$$

as α order Caputo type fractional difference.

Theorem 7. *Let $n \in Z$ and for any $\alpha > 0$,$\beta > 0$, then [27]:*

$$D^{-\alpha} D^{-\beta} x(n) = D^{-(\alpha+\beta)} x(n) = D^{-\beta} D^{-\alpha} x(n).$$

Theorem 8. *For any $\alpha > 0$,$\beta \in R$, it follows that [27]:*

$$D^{\alpha} D^{\beta} = D^{\alpha+\beta}.$$

Combined with the fundamental theorem of calculus, similar to Theorems 1, 3, 5, and 6, and the definition of the Caputo-type fractional difference (Definition 14) and theorems (Theorems 7 and 8), we can derive the following results for fractional differences and sums.

Theorem 9. *Let a, b, $n \in Z$, a < n, b > n and $\alpha > 0$, $m = [\alpha] + 1$, the following four equalities hold:*

1. $\nabla^{\alpha} \left[{}_a\nabla^{-\alpha} \right] x(n) = \nabla^{m} \left[{}_a\nabla^{-(m-\alpha)} \right] \left[{}_a\nabla^{-\alpha} \right] x(n) = \nabla^{m} \nabla^{-m} x(n) = x(n)$,
2. $-\nabla^{\alpha} \nabla_b^{-\alpha} x(n) = (-\nabla)^{m} \nabla_b^{-(m-\alpha)} \nabla_b^{-\alpha} x(n) = (-\nabla)^{m} \nabla_b^{-m} x(n) = x(n)$,
3. $\Delta^{\alpha} \left[{}_a\Delta^{-\alpha} \right] x(n) = \Delta^{m} \left[{}_a\Delta^{-(m-\alpha)} \right] \left[{}_a\Delta^{-\alpha} \right] x(n) = \Delta^{m} \Delta^{-m} x(n) = x(n)$,
4. $-\Delta^{\alpha} \Delta_b^{-\alpha} x(n) = (-\Delta)^{m} \Delta_b^{-(m-\alpha)} \Delta_b^{-\alpha} x(n) = (-\Delta)^{m} \Delta_b^{-m} x(n) = x(n)$.

Theorem 10. *Let $m, k \in \mathbb{Z}, m \leq k$ and assume that $x(n)$ is the backward difference of $y(n), \nabla y(n) = x(n)$. So $y(n)$ is the sum corresponding to $x(n)$, denoted by $y(n) = \sum_i x(i)$, then [37]:*

$$\sum_{i=m}^{k} x(i) = y(k) - y(m-1).$$

Theorem 11. *Let $m, k \in \mathbb{Z}$, $m \leq k$ and assume that $x(n)$ is the forward difference of $z(n)$, $\nabla z(n) = x(n)$. So $z(n)$ is the sum corresponding to $x(n)$, denoted by $z(n) = \sum_i x(i)$, then [37]:*

$$\sum_{i=m}^{k} x(i) = z(k+1) - z(m).$$

Theorem 12. *Let a, b, $n \in Z$, $m \in \mathbb{N}_{+}$, and a < n, b > n, we define that:*

1. ${}_a\nabla^{-1} \frac{(n-a)^{\overline{m-1}}}{(m-1)!} = \frac{(n-a)^{\overline{m}}}{m!}$,
2. $\nabla_b^{-1} \frac{(b-n)^{\underline{m-1}}}{(m-1)!} = \frac{(b-n)^{\underline{m}}}{m!}$,
3. ${}_a\Delta^{-1} \frac{(n-a)^{\underline{m-1}}}{(m-1)!} = \frac{(n-a)^{\underline{m}}}{m!}$,
4. $\Delta_b^{-1} \frac{(b-n)^{\overline{m-1}}}{(m-1)!} = \frac{(b-n)^{\overline{m}}}{m!}$.

Proof.

1. Since

$$\begin{aligned}\nabla \frac{(n-a)^{\overline{m}}}{m!} &= \nabla \frac{(n-a)(n-a+1)(n-a+2)\cdots(n-a+m-1)}{m!} \\ &= \frac{(n-a)(n-a+1)(n-a+2)\cdots(n-a+m-1)}{m!} \\ &= \frac{(n-a-1)(n-a)(n-a+2)\cdots(n-a+m-2)}{m!}\left[(n-a+m-1)-(n-a-1)\right] \\ &= \frac{(n-a)(n-a+1)(n-a+2)\cdots(n-a+m-2)}{(m-1)!}\left[(n-a+m-2)\right] \\ &= \frac{(n-a)(n-a+1)(n-a+2)\cdots(n-a+m-2)}{(m-1)!} \\ &= \frac{(n-a)^{\overline{m-1}}}{(m-1)!}.\end{aligned}$$

Then we can obtain ${}_a\nabla^1 \frac{(na)^{\overline{m1}}}{(m1)!} = \frac{(n-a)^{\overline{m}}}{m!}$.

2. Since

$$\begin{aligned}(-\nabla)\frac{(b-n)^{\underline{m}}}{m!} &= (-\nabla)\frac{(b-n)(b-n-1)(b-n-2)\cdots[b-n-(m-2)]}{m!} \\ &= -\frac{(b-n+1)(b-n)(b-n-1)(b-n-2)\cdots[b-n-(m-2)]}{m!} \\ &+ \frac{(b-n)(b-n-1)(b-n-2)\cdots[b-n-2)]}{m!} \\ &= \frac{(b-n)(b-n-1)(b-n-2)\cdots[b-n-(m-2)]}{m!}\left[-(b-n-m+1)+(b-n+1)\right] \\ &= \frac{(b-n)(b-n-2)\cdots[b-n-(m-2)]}{(m-1)!} = \frac{(b-n)^{\underline{m}}}{(m-1)!}.\end{aligned}$$

Then we can obtain $\nabla_b^{-1}\frac{(b-n)^{\underline{m-1}}}{(m-1)!} = \frac{(b-n)^{\underline{m}}}{m!}$. □

The proofs of item 3 and 4 follow directly from the proofs of item 1 and 2. Specifically, by applying the established properties in item 1 and 2, we can extend the results to demonstrate the validity of item 3 and 4.

Theorem 13. *Let a, b, $n \in Z$, k, $m \in \mathbb{N}$ and a < n, b > n, we find that:*

1. $\nabla^k\left({}_a\nabla^{-m}\right)\frac{(n-a)^{\overline{m}}}{m!} = \frac{(n-a)^{\overline{m-k}}}{(m-k)!}$,
2. $(\nabla)^k\left({}_b\nabla^m\right)\frac{(bn)^{\underline{m}}}{m!} = \frac{(bn)^{\underline{mk}}}{(mk)!}$,
3. $\Delta^k\left({}_a\Delta^m\right)\frac{(na)^{\underline{m}}}{m!} = \frac{(na)^{\underline{mk}}}{(m-k)!}$,
4. $(\Delta)^k\left({}_b\Delta^m\right)\frac{(bn)^{\overline{m}}}{m!} = \frac{(bn)^{\overline{mk}}}{(mk)!}$,
5. $\nabla^m\left({}_a\nabla^m\right)\frac{(na)^{\overline{m}}}{m!} = \frac{(na)^{\overline{0}}}{0!} = 1$,
6. $(\nabla)^m\left({}_b\nabla^m\right)\frac{(bn)^{\underline{m}}}{m!} = \frac{(bn)^{\underline{0}}}{0!} = 1$,
7. $\Delta^m\left({}_a\Delta^m\right)\frac{(na)^{\underline{m}}}{m!} = \frac{(na)^{\underline{0}}}{0!} = 1$,
8. $(\Delta)^m\left({}_b\Delta^m\right)\frac{(bn)^{\overline{m}}}{m!} = \frac{(bn)^{\overline{0}}}{0!} = 1$,
9. $\nabla^k\frac{(na)^{\overline{m}}}{m!} = (\nabla)^k\frac{(bn)^{\underline{m}}}{m!} = \Delta^k\frac{(na)^{\underline{m}}}{m!} = (\Delta)^k\frac{(bn)^{\overline{m}}}{m!} = 0$.

When $k < m$, the items 1, 2, 3, and 4 hold, when $k=m$, the item 5, 6, 7, and 8 hold, when $k>m$, the item 9 holds.

Theorem 14. *Let a, $n \in Z$, $m \in \mathbb{N}$ and a < n, we have:*

$$ {}_a\nabla^{-m}\nabla^m x(n) = x(n) - \sum_{k=0}^{m-1} \frac{(n-a)^{\overline{k}}}{k!}\nabla^k x(a). $$

Proof. According to the fourth equation of Theorem 1, we can obtain

$$ {}_a\nabla^{-1}\nabla x(n) ={}_a \nabla^{-1}\left[x(n) - x(n-1)\right] = \sum_{i=a+1}^{n}\left[x(i) - x(i-1)\right] = x(n) - x(a). $$

then

$$ {}_a\nabla^{-2}\nabla^2 x(n) = {}_a\nabla^{-2}\left\{x(n) - x(n-1) - \left[x(n-1) - x(n-2)\right]\right\} = \\ {}_a\nabla^{-1}\left[x(n) - x(n-1) - \nabla x(a)\right] = x(n) - x(0) - (n-a)\nabla x(a), $$

$$ \begin{aligned} {}_a\nabla^{-3}\nabla^3 x(n) &= {}_a\nabla^{-3}\left\{x(n) - x(n-1) - 2\left[x(n-1) - x(n-2)\right] + x(n-2) - x(n-3)\right\} \\ &= {}_a\nabla^{-2}\left[x(n) - 2x(n-1) + x(n-2) - \nabla^2 x(a)\right] \\ &= {}_a\nabla^{-1}\left[x(n) - x(n-1) - \nabla x(a) - (n-a)\nabla^2 x(a)\right] \\ &= x(n) - x(a) - (n-a)\nabla x(a) - \tfrac{(n-a)(n-a+1)}{2}\nabla^2 x(a). \end{aligned} $$

by analogy, based on Theorems 10–13, we can get

$$ \begin{aligned} {}_a\nabla^{-m}\nabla^m x(n) &= x(n) - (n-a)\nabla x(a) - \tfrac{(n-a)(n-a+1)}{2}\nabla^2 x(a) \\ &- \tfrac{(n-a)(n-a+1)(n-a+2)}{3!}\nabla^3 x(a) \\ &- \cdots - \tfrac{(n-a)(n-a+1)(n-a+2)\cdots(n-a+m-1-1)}{(m-1)!}\nabla^{m-1} x(a) \\ &= x(n) - \textstyle\sum_{k=0}^{m-1}\tfrac{(n-a)^{\overline{k}}}{k!}\nabla^k x(a). \end{aligned} $$

□

Theorem 15. *Let a, $n \in Z$, a < n and $\alpha > 0$, $m = [\alpha] + 1$, we have:*

$$ {}_a\nabla^{-\alpha}\nabla^\alpha x(n) = x(n) - \sum_{k=0}^{m-1} \frac{(n-a)^{\overline{k}}}{k!}\nabla^k x(a). $$

Proof. In view of Definition 14 (Caputo fractional difference) and Theorem 14, we can obtain

$$ a\nabla^{-\alpha}\nabla^\alpha x(n) ={}_a \nabla^{-\alpha}\left[a\nabla^{-(m-\alpha)}\right]\nabla^m x(n) ={}_a \nabla^{-\alpha-(m-\alpha)}\nabla^m x(n) = \\ {}_a\nabla^{-m}\nabla^m x(n) = x(n) - \textstyle\sum_{k=0}^{m-1}\tfrac{(n-a)^{\overline{k}}}{k!}\nabla^k x(a)\,. $$

□

Theorem 16. *Let b, $n \in Z$, $m \in \mathbb{N}$, and b > n, we have:*

$$ \nabla_b^{-m}(-\nabla)^m x(n) == x(n) - \sum_{k=0}^{m-1} \frac{(b-n)^{\underline{k}}}{k!}(-\nabla)^k x(b). $$

Proof. According to the fourth equation of Theorem 2, we can obtain

$$\begin{aligned}\nabla_b^{-2}(-\nabla)^2x(n) &= \nabla_b^{-2}\left[x(i-2)-2x(i-1)+x(i)\right] = x(n)-x(b)-\sum_{i=n+1}^{b}\left[-\nabla x(b)\right]\\ &= x(n)-x(b)-(b-n)\left[-\nabla x(b)\right],\\ &\nabla_b^{-3}\left[(-\nabla)^3x(n)\right] = x(n)-x(b)-(b-n)\nabla x(b)-\frac{(b-n)(b-n-1)}{2}(-\nabla)^2x(b).\end{aligned}$$

by analogy, based on Theorems 10–13, we can get

$$\begin{aligned}\nabla_b^{-m}(-\nabla)^m x(n) &= x(n)-(b-n)(-\nabla)x(b)-\tfrac{(b-n)(b-n-1)}{2}(-\nabla)^2x(b)\\ &-\tfrac{(b-n)(b-n-2)}{3!}(-\nabla)^3x(b)\\ &-\cdots-\tfrac{(b-n)(b-n-1)\cdots(b-n-m+1)}{(m-1)!}(-\nabla)^{m-1}x(b)\\ &= x(n)-\textstyle\sum_{k=0}^{m-1}\tfrac{(b-n)^{\underline{k}}}{k!}(-\nabla)^k x(b).\end{aligned}$$

□

Theorem 17. *Let b, $n \in Z$, b > n and $\alpha > 0$, $m = [\alpha] + 1$, we have:*

$$\nabla_b^{-\alpha}(-\nabla)^\alpha x(n) = x(n)-\sum_{k=0}^{m-1}\frac{(b-n)^{\underline{k}}}{k!}(-\nabla)^k x(b).$$

Proof. In view of Definition 14 (Caputo fractional difference) and Theorem 16, we can obtain

$$\begin{aligned}\nabla_b^{-\alpha}(-\nabla)^\alpha x(n) &= \nabla_b^{-\alpha}\nabla_b^{-(m-\alpha)}(-\nabla)^m x(n)\\ &= \nabla_b^{-m}(-\nabla)^m x(n)\\ &= x(n)-\sum_{k=0}^{m-1}\tfrac{(b-n)^{\underline{k}}}{k!}(-\nabla)^k x(b).\end{aligned}$$

□

Theorem 18. *Let a, $n \in Z$, $m \in \mathbb{N}$ and a < n, we have:*

$${}_a\Delta^{-m}\Delta^m x(n) = x(n)-\sum_{k=0}^{m-1}\frac{(n-a)^{\underline{k}}}{k!}\Delta^k x(a).$$

Proof. According to the fourth equation of Theorem 3, we can obtain

$${}_a\Delta^{-1}\Delta x(n) = {}_a\Delta^{-1}\left[x(n+1)-x(n)\right] = \sum_{i=a}^{n-1}\left[x(i+1)-x(i)\right] = x(n)-x(a).$$

then

$$\begin{aligned}{}_a\Delta^{-2}\Delta^2 x(n) = {}_a\Delta^{-2}\left\{x(n+2)-x(n+1)-\left[x(n+1)-x(n)\right]\right\} =\\ {}_a\Delta^{-1}\left[x(n+1)-x(n)-\Delta x(a)\right] = x(n)-(n-a)\Delta x(a).\end{aligned}$$

$$\begin{aligned}
&{}_a\Delta^{-3}\Delta^3 x(n) =_a \Delta^{-3}\left\{x(n+3)-3x(n+2)+3x(n+1)-x(n)\right\}\\
&= {}_a\Delta^{-2}\left[x(n+2)-2x(n+1)+x(n)-\Delta^2 x(a)\right]\\
&= {}_a\Delta^{-1}\left[x(n+1)-x(n)-\nabla x(a)-(n-a)\nabla^2 x(a)\right]\\
&= x(n)-x(a)-(n-a)\Delta x(a)-\tfrac{(n-a)(n-a-1)}{2}\Delta^2 x(a).
\end{aligned}$$

by analogy, based on Theorems 10–13, we can get

$$\begin{aligned}
{}_a\Delta^{-m}\Delta^m x(n) &= x(n)-x(a)-(n-a)\Delta x(a)-\tfrac{(n-a)(n-a-1)}{2}\Delta^2 x(a)\\
&-\tfrac{(n-a)(n-a-1)(n-a-2)}{3!}\Delta^3 x(a)\\
&-\cdots-\tfrac{(n-a)(n-a-1)(n-a-2)\cdots[n-a-(m-1)]}{(m-1)!}\Delta^{m-1}x(a)\\
&= x(n)-\sum_{k=0}^{m-1}\tfrac{(n-a)^{\underline{k}}}{k!}\Delta^k x(a).
\end{aligned}$$

□

Theorem 19. *Let a, $n \in Z$, a < n, and $\alpha > 0$, $m =$ [α] $+ 1$, we have:*

$$ {}_a\Delta^{-\alpha}\Delta^{\alpha}x(n) = x(n)-\sum_{k=0}^{m-1}\frac{(n-a)^{\underline{k}}}{k!}\Delta^k x(a). $$

***Note**: The proof of Theorem 19 is similar to that of Theorem 15.*

Theorem 20. *Let b, $n \in Z$, $m \in \mathbb{N}$, and b > n, we have:*

$$ \Delta_b^{-m}(-\Delta)^m x(n) = x(n)-\sum_{k=0}^{m-1}\frac{(b-n)^{\overline{k}}}{k!}(-\Delta)^k x(b). $$

Proof. According to the fourth equation of Theorem 4, we can obtain

$$ \Delta_b^{-1}(-\Delta)x(n) = \sum_{i=n}^{b-1}[x(i)-x(i+1)] = x(n)-x(b). $$

then

$$\begin{aligned}
\Delta_b^{-2}(-\Delta)^2 x(n) &= \Delta_b^{-2}[x(n)-2x(n+1)+x(n+2)] = x(n)-x(b)-\sum_{i=n}^{b-1}[-\Delta x(b)]\\
&= x(n)-x(b)-(b-n)[-\Delta x(b)],
\end{aligned}$$

$$ \Delta_b^{-3}\left[(-\Delta)^3 x(n)\right] = x(n)-x(b)-(b-n)\Delta x(b)-\frac{(b-n)(b-n+1)}{2}(-\Delta)^2 x(b). $$

by analogy, based on Theorems 10–13, we can get

$$\begin{aligned}
\Delta_b^{-m}(-\Delta)^m x(n) &= x(n)-x(b)-(b-n)(-\Delta)x(b)-\tfrac{(b-n)(b-n+1)}{2}(-\Delta)^2 x(b)\\
&-\tfrac{(b-n)(b-n+1)(b-n+2)}{3!}(-\Delta)^3 x(b)-\ldots\\
&-\tfrac{(b-n)(b-n+1)(b-n+2)\cdots(b-n+m-1)}{(m-1)!}(-\Delta)^{m-1}x(b)\\
&= x(n)-\textstyle\sum_{k=0}^{m-1}\tfrac{(b-n)^k}{k!}(-\Delta)^k x(b).
\end{aligned}$$

□

Theorem 21. *Let b, n ∈ Z , b > n, and* $\alpha > 0$, $m = [\alpha] + 1$, *we have:*

$$_a\Delta^{-\alpha}\Delta^{\alpha}x(n) = x(n) - \sum_{k=0}^{m-1}\frac{(n-a)^{\underline{k}}}{k!}\Delta^k x(a).$$

Note: The proof of Theorem 21 is similar to that of Theorem 17.

And the derivation relationship between the theorems in Section 4 and the key Theorems 22–25 is shown in **Figure 2**.

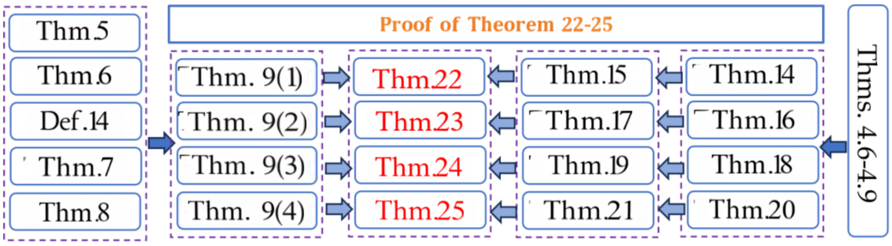

Figure 2. Derivation relationship between the theorems in Section 4 and the Theorems 22–25.

5. The initial and final value problems of the solution of Caputo fractional difference equations

Building on the four Caputo fractional difference definitions and their fundamental properties established in Sections 3 and 4, this section focuses on two core objectives: first, to rigorously derive the equivalent Volterra sum equations for each type of fractional difference equation, and systematically prove the solvability of their corresponding initial or final value problems; second, to verify the accuracy of the theoretical solutions and demonstrate the practical applicability of the proposed framework through targeted computational examples and numerical simulations. Unless otherwise specified, the fractional difference operators employed in this chapter refer to Caputo fractional difference operators.

5.1. Theorems and proofs

Theorem 22. *Let a, t ∈ Z , a < t, and* $\alpha > 0$, $m = [\alpha] + 1$. *u(t) and f[t, u(t)] are given functions. Backward fractional difference equation is defined as:*

$$\begin{cases} \nabla^{\alpha}u(t) = f\left[t, u(t)\right], \\ \nabla^k u(a) = u_k, \ (k = 0,\ 1,\ 2,\ \cdots,\ m-1). \end{cases} \tag{5}$$

The Cauchy initial value problem (Equation (5)) is equivalent to volterra sum equation

$$u(t) = \sum_{k=0}^{m-1}\frac{(t-a)^{\bar{k}}}{k!}u_k + \frac{1}{\Gamma(\alpha)}\sum_{i=a+1}^{t}(t-i+1)^{\overline{\alpha-1}}f\left[i, u(i)\right]. \tag{6}$$

Proof. To establish the necessity, we apply the fractional summation operator $_a\nabla^{-\alpha}$ to both sides of Equation (5) relying on Definition 6 and Theorem 15, leading to the subsequent result

$$\begin{aligned} {}_a\nabla^{-\alpha}\nabla^{\alpha}u(t) = \quad & {}_a\nabla^{-\alpha}\left[a\nabla^{-(m-\alpha)}\right]\nabla^{m}u(t) =_a \nabla^{-m}\nabla^{m}u(t) \\ & = u(t) - \textstyle\sum_{k=0}^{m-1}\frac{(t-a)^{\bar{k}}}{k!}\nabla^{k}u(a) \\ & = \nabla^{-\alpha}f\left[t,u(t)\right] = \frac{1}{\Gamma(\alpha)}\textstyle\sum_{i=a+1}^{t}(t-i+1)^{\overline{\alpha-1}}f\left[i,u(i)\right]. \end{aligned}$$

Then

$$u(t) = \sum_{k=0}^{m-1}\frac{(t-a)^{\bar{k}}}{k!}u_k + \frac{1}{\Gamma(\alpha)}\sum_{i=a+1}^{t}(t-i+1)^{\overline{\alpha-1}}f\left[i,u(i)\right].$$

Having established the necessity, we now proceed to prove the sufficiency.

Taking the fractional difference operator ∇^{α} into both sides of the Equation (6), we can obtain

$$\nabla^{\alpha}u(t) = \nabla^{\alpha}\sum_{k=0}^{m-1}\frac{(t-a)^{\bar{k}}}{k!}u_k + \nabla^{\alpha}\frac{1}{\Gamma(\alpha)}\sum_{i=a+1}^{t}(n-i+1)^{\overline{\alpha-1}}f\left[i,u(i)\right].$$

According to Definition 14 (Caputo type fractional difference) and Theorem 13, we have

$$\nabla^{\alpha}\frac{(t-a)^{\bar{k}}}{k!} = \nabla^{-(m-\alpha)}\left(\nabla^{m}\frac{(t-a)^{\bar{k}}}{k!}\right) = 0, (k = 0, 1, 2, \cdots, m-1).$$

And from the first equation of Theorem 9, we have

$$\nabla^{\alpha}\left\{{}_a\nabla^{-\alpha}f\left[t,u(t)\right]\right\} = f\left[t,u(t)\right].$$

then

$$\nabla^{\alpha}u(t) = f\left[t,u(t)\right].$$

In addition, taking the n (n = 1, 2, $\cdots$, $[\alpha]$) order difference into both sides of Equation (6), we get

$$\nabla^{n}u(t) = \nabla^{n}\sum_{k=0}^{m-1}\frac{(t-a)^{\bar{k}}}{k!}u_k + \nabla^{n}\frac{1}{\Gamma(\alpha)}\sum_{i=a+1}^{t}(t-i+1)^{\overline{\alpha-1}}f\left[i,u(i)\right]. \quad (7)$$

Let initial value $t = a$, The first term on the right side of the Equation (7), when $n \neq k$

$$\nabla^{n}\frac{(t-a)^{\bar{k}}}{k!} = \frac{(t-a)^{\overline{k-n}}}{(k-n)!} = 0.$$

when $n = k$

$$\nabla^{n}\frac{(t-a)^{\bar{k}}}{k!} = \frac{(t-a)^{\overline{k-k}}}{(k-k)!} = \frac{(t-a)^{\bar{0}}}{0!} = 1.$$

And when $t = a$, set $\frac{1}{(\alpha)}\sum_{i=a+1}^{a}(t-i+1)^{\overline{\alpha-1}}f\left[i,u(i)\right] = 0$, the second term on the right side of the Equation (7), we can obtain

$$\nabla^{k}u(a) = u_k$$

where k = 0, 1, 2, $\cdots$, $m-1$. And adequacy is proven. □

Specifically, when $\alpha = 1$, we have

$$u(t) = u(a) + \sum_{i=a+1}^{t} f\left[i, u(i)\right].$$

Similarly, in the following theorem, we can also easily obtain the expression of $u(t)$ when $a = 1$.

Theorem 23. *Let* $b,\ t \in Z,\ b > t$ *and* $\alpha > 0,\ m = [\alpha] + 1.$ $u(t)$ *and* $f[t-\alpha+1,\ u(t-\alpha+1)]$ *are given functions. Negative backward fractional difference equation is defined as*

$$\begin{cases} (-\nabla)^{\alpha}u(t) = f\left[t-\alpha+1, u(t-\alpha+1)\right], \\ (-\nabla)^{k}u(b) = u_k,\ (k = 0, 1, 2, \cdots, m-1). \end{cases} \tag{8}$$

The Cauchy final value problem (Equation (8)) is equivalent to the volterra sum equation

$$u(t) = \sum_{k=0}^{m-1} \frac{(b-t)^{\underline{k}}}{k!} u_k + \frac{1}{\Gamma(\alpha)} \sum_{i=t+1}^{b} (i-t)^{\overline{\alpha-1}} f\left[i, u(i)\right]. \tag{9}$$

Proof. To establish the necessity, we apply the fractional summation operator $\nabla_b^{-\alpha}$ to both sides of Equation (8), relying on Definition 8 and Theorem 17, leading to the subsequent result.

$$\begin{aligned} \nabla_b^{-\alpha}(-\nabla)^{\alpha}u(t) &= \nabla_b^{-m}(-\nabla)^{m}u(t) = u(t) - \sum_{k=0}^{m-1} \tfrac{(b-t)^{\underline{k}}}{k!}(-\nabla)^{k}u(b) \\ &= \nabla^{-\alpha} f\left[t-\alpha+1, u(t-\alpha+1)\right] \\ &= \tfrac{1}{\Gamma(\alpha)} \sum_{i=t+1}^{b} \tfrac{\Gamma[i-(t-\alpha+1)]}{\Gamma[i-(t-\alpha+1)-\alpha+1]} f\left[i, u(i)\right] \\ &= \tfrac{1}{\Gamma(\alpha)} \sum_{i=t+1}^{b} \tfrac{\Gamma(i-t+\alpha-1)}{\Gamma(i-t)} f\left[i, u(i)\right] \\ &= \tfrac{1}{\Gamma(\alpha)} \sum_{i=t+1}^{b} (i-t))^{\overline{\alpha-1}}\ f\left[i, u(i)\right]. \end{aligned}$$

Then

$$u(t) = \sum_{k=0}^{m-1} \frac{(b-t)^{\underline{k}}}{k!} u_k + \frac{1}{\Gamma(\alpha)} \sum_{i=t+1}^{b} (i-t)^{\overline{\alpha-1}} f\left[i, u(i)\right].$$

Having established the necessity, we now proceed to prove the sufficiency.

Taking the fractional difference operator $(-\nabla)^{\alpha}$ into both sides of the first equation of Equation (9), we can obtain

$$(-\nabla)^{\alpha}u(t) = (-\nabla)^{\alpha}\sum_{k=0}^{m-1}\frac{(b-t)^{\underline{k}}}{k!}u_k + (-\nabla)^{\alpha}\frac{1}{\Gamma(\alpha)}\sum_{i=t+1}^{b}(i-t)^{\overline{\alpha-1}}f\left[i,u(i)\right].$$

According to Definition 14 (Caputo type fractional difference) and Theorem 13, we have

$$(-\nabla)^{\alpha}\frac{(b-t)^{\underline{k}}}{k!} = \nabla^{-(m-\alpha)}\left((-\nabla)^{m}\frac{(b-t)^{\underline{k}}}{k!}\right) = 0\ (k=0,1,2,\cdots,m-1).$$

Set $j = i + \alpha - 1$, and by invoking the second equation of Theorem 9, the subsequent result is derived.

$$\begin{aligned}
&(-\nabla)^{\alpha}\tfrac{1}{\Gamma(\alpha)}\sum_{i=t+1}^{b}(\quad i-t)^{\overline{\alpha-1}}f\left[i,u(i)\right]\\
&=(-\nabla)^{\alpha}\tfrac{1}{\Gamma(\alpha)}\sum_{i=t+1}^{b}\tfrac{\Gamma(i-t+\alpha-1)}{\Gamma(i-t)}f\left[i,u(i)\right]\\
&=(-\nabla)^{\alpha}\tfrac{1}{\Gamma(\alpha)}\sum_{j=t+1}^{b}\tfrac{\Gamma(j+1-\alpha-1)}{\Gamma(j+1-\alpha-t)}f\left[j+1-\alpha,u(j+1-\alpha)\right]\\
&=(-\nabla)^{\alpha}\tfrac{1}{\Gamma(\alpha)}\sum_{j=t+\alpha}^{b}\tfrac{\Gamma(j-t+\alpha-1)}{\Gamma(j+1-\alpha-t)}f\left[j+1-\alpha,u(j+1-\alpha)\right]\\
&=(-\nabla)^{\alpha}\nabla^{-\alpha}f\left(t+1-\alpha,u(t+1-\alpha)\right]\\
&=f\left[t+1-\alpha,u(t+1-\alpha)\right].
\end{aligned}$$

Then

$$(-\nabla)^{\alpha}u(t) = f\left[t-\alpha+1,u(t-\alpha+1)\right].$$

In addition, taking the $n(n = 1,2,\cdots,[\alpha])$ order difference into both sides of Equation (9), we get

$$(-\nabla)^{n}u(t) = (-\nabla)^{n}\sum_{k=0}^{m-1}\frac{(b-t)^{\underline{k}}}{k!}u_k + (-\nabla)^{n}\frac{1}{\Gamma(\alpha)}\sum_{i=t+1}^{b}(i-t)^{\overline{\alpha-1}}f[i,u(i)]. \tag{10}$$

Let final value $t = b$, the first term on the right side of the Equation (10), when $n \neq k$

$$(-\nabla)^{n}\frac{(b-t)^{\underline{k}}}{k!} = \frac{(b-t)^{\underline{k-n}}}{(k-n)!} = 0,$$

when $n = k$

$$(-\nabla)^{n}\frac{(b-t)^{\underline{k}}}{k!} = \frac{(b-t)^{\underline{k-k}}}{(k-k)!} = \frac{(b-t)^{\underline{0}}}{0!} = 1.$$

And when $t = a$, set $\frac{1}{(\alpha)}\sum_{i=b+1}^{b}(i-t)^{\overline{\alpha-1}}f\left[i,u(i)\right] = 0$, the second term on the right side of the Equation (10), we can obtain

$$(-\nabla)^{k}u(b) = u_k$$

where k = 0, 1, 2, $\cdots$, m−1. And adequacy is proven. □

Theorem 24. *Let a, t* $\in Z$ *, a < t, and* $\alpha > 0$*,* $m = [\alpha] + 1$*. u(t) and* $f[t + \alpha - 1,\ u(t + \alpha - 1)]$ *are given functions. Forward fractional difference equation is defined as:*

$$\begin{cases} \Delta^{\alpha} u(t) = f\left[t + \alpha - 1, u(t + \alpha - 1)\right], \\ \Delta^{k} u(a) = u_k, (k = 1, 2, 3, \cdots, m - 1). \end{cases} \tag{11}$$

The Cauchy initial value problem (Equation (11)) is equivalent to volterra sum equation

$$u(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{\underline{k}}}{k!} u_k + \frac{1}{\Gamma(\alpha)} \sum_{i=a}^{t-1} (t-i)^{\overline{\alpha-1}} f\left[i, u(i)\right]. \tag{12}$$

Proof. To establish the necessity, we apply the fractional summation operator ${}_a\nabla^{\alpha}$ to both sides of Equation (11), relying on Definition 11 and Theorem 19, leading to the subsequent result.

$$\begin{aligned} {}_a\Delta^{-\alpha}\Delta^{\alpha} u(t) = \quad & {}_a\Delta^{-m}\Delta^{m} u(t) \\ & = u(t) - \sum_{k=0}^{m-1} \tfrac{(t-a)^{\underline{k}}}{k!} \nabla^{k} u(a) \\ & = \Delta^{-\alpha} f\left[t + \alpha - 1, u(t + \alpha - 1)\right] \\ & = \tfrac{1}{\Gamma(\alpha)} \sum_{i=a}^{t+\alpha-1-\alpha} \tfrac{\Gamma(t+\alpha-1-i)}{\Gamma(t+\alpha-1-i-\alpha+1)} f\left[i, u(i)\right] \\ & = \tfrac{1}{\Gamma(\alpha)} \sum_{i=a}^{t-1} \tfrac{\Gamma(t-i+\alpha-1)}{\Gamma(t-i)} f\left[i, u(i)\right] \\ & = \tfrac{1}{\Gamma(\alpha)} \sum_{i=a}^{t-1} (t-i)^{\overline{\alpha-1}} f\left[i, u(i)\right]. \end{aligned}$$

Then

$$u(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{\underline{k}}}{k!} u_k + \frac{1}{\Gamma(\alpha)} \sum_{i=a}^{t-1} (t-i)^{\overline{\alpha-1}} f\left[i, u(i)\right].$$

Having established the necessity, we now proceed to prove the sufficiency.

Taking the fractional difference operator ∇^{α} into both sides of the first equation of Equation (12), we can obtain

$$\Delta^{\alpha} u(t) = \Delta^{\alpha} \sum_{k=0}^{m-1} \frac{(t-a)^{\underline{k}}}{k!} u_k + \Delta^{\alpha} \frac{1}{\Gamma(\alpha)} \sum_{i=a}^{t-1} (t-i)^{\overline{\alpha-1}} f\left[i, u(i)\right].$$

According to Definition 14 (Caputo type fractional difference) and Theorem 13, we have

$$\Delta^{\alpha} \frac{(t-a)^{\underline{k}}}{k!} = \Delta^{-(m-\alpha)} \left(\Delta^{m} \frac{(t-a)^{\underline{k}}}{k!} \right) = 0, (k = 0, 1, 2, \cdots, m - 1)$$

and

$$\Delta^{\alpha}\tfrac{1}{\Gamma(\alpha)}\sum_{i=a}^{t-1}(t-i)^{\underline{\alpha-1}}f\left[i,u(i)\right]$$
$$=\Delta^{\alpha}\tfrac{1}{\Gamma(\alpha)}\sum_{i=a}^{t-1}\tfrac{\Gamma(t-i+\alpha-1)}{\Gamma(t-i)}f\left[i,u(i)\right]$$
$$=\Delta^{\alpha}\tfrac{1}{\Gamma(\alpha)}\sum_{i=a}^{t+\alpha-1-\alpha}\tfrac{\Gamma(t-i+\alpha-1)}{\Gamma(t-i)}f\left[i,u(i)\right]$$
$$=\Delta^{\alpha}\tfrac{1}{\Gamma(\alpha)}\sum_{i=a}^{t+\alpha-1-\alpha}\tfrac{\Gamma(t+\alpha-1-i)}{\Gamma(t+\alpha-1-\alpha+1-i)}f\left[i,u(i)\right].$$

Set $t+\alpha-1=p$, The above formula can be transformed into

$$\Delta^{\alpha}\frac{1}{\Gamma(\alpha)}\sum_{i=a}^{p-\alpha}\frac{\Gamma(p-i)}{\Gamma(p-\alpha+1-i)}f\left[i,u(i)\right]=\Delta^{\alpha}\Delta^{-\alpha}f\left[p,u(p)\right].$$

Set $p=t+\alpha-1$, and from the third equation of Theorem 9, $\Delta^{\alpha}\left\{\Delta^{-\alpha}f\left[t,u(t)\right]\right\}=f\left[t,u(t)\right]$, we can obtain

$$\Delta^{\alpha}\frac{1}{\Gamma(\alpha)}\sum_{i=a}^{t-1}\frac{\Gamma(t-i+\alpha-1)}{\Gamma(t-i)}f\left[i,u(i)\right]=\Delta^{\alpha}\Delta^{-\alpha}f\left[p,u(p)\right]=f\left[t+\alpha-1,u(t+\alpha-1)\right].$$

Then

$$\Delta^{\alpha}u(t)=f\left[t+\alpha-1,u(t+\alpha-1)\right].$$

In addition, taking the $n(n=1,2,\cdots,[\alpha])$ order difference into both sides of Equation (12), we get

$$\Delta^{n}u(t)=\Delta^{n}\sum_{k=0}^{m-1}\frac{(t-a)^{\underline{k}}}{k!}u_k+\Delta^{n}\frac{1}{\Gamma(\alpha)}\sum_{i=a}^{t-1}(t-i)^{\overline{\alpha-1}}f\left[i,u(i)\right]. \tag{13}$$

let initial value $t=a$, The first term on the right side of the Equation (13), when $n\neq k$

$$\Delta^{n}\frac{(t-a)^{\underline{k}}}{k!}=\left.\frac{(t-a)^{\underline{k-n}}}{(k-n)!}\right|_a^t=0,$$

when $n=k$

$$\Delta^{k}\frac{(t-a)^{\underline{k}}}{k!}=\frac{(t-a)^{\underline{k-k}}}{(k-k)!}=\frac{(t-a)^{\underline{0}}}{0!}=1.$$

Set

$$\frac{1}{\Gamma(\alpha)}\sum_{i=a}^{a-1}(t-i)^{\overline{\alpha-1}}f\left[i,u(i)\right]=0,$$

we can obtain

$$\Delta^{k}u(a)=u_k,$$

where, $k=0,1,2,\cdots,m-1$. And adequacy is proven. □

Theorem 25. *Let* b, $t\in Z$, $b>t$ *and* $\alpha>0$, $m=[\alpha]+1$. $u(t)$ *and* $f[t,u(t)]$ *are* *given functions. Negative forward fractional difference equation is defined as*

$$\begin{cases} (-\Delta)^{\alpha}u(t) = f\left[t, u(t)\right], \\ (-\Delta)^{k}u(b) = u_k,\ (k = 0, 1, 2, \cdots, m-1). \end{cases} \tag{14}$$

The Cauchy initial value problem (Equation (14)) is equivalent to the volterra sum equation

$$u(t) = \sum_{k=0}^{m-1} \frac{(b-t)^{\overline{k}}}{k!} u_k + \frac{1}{\Gamma(\alpha)} \sum_{i=t}^{b-1} (i-t+1)^{\overline{\alpha-1}} f\left[i, u(i)\right]. \tag{15}$$

Proof. To establish the necessity, we apply the fractional summation operator $\nabla_b^{-\alpha}$ to both sides of Equation (14), relying on Definition 13 and Theorem 21, leading to the subsequent result.

$$\begin{aligned} \Delta_b^{-\alpha}(-\Delta)^{\alpha}u(t) &= \Delta_b^{-m}(-\Delta)^{m}u(t) \\ &= u(t) - \sum_{k=0}^{m-1} \tfrac{(b-t)^{\overline{k}}}{k!}(-\Delta)^{k}u(b) \\ &= \Delta_b^{-\alpha} f\left[t, u(t)\right] = \tfrac{1}{\Gamma(\alpha)} \sum_{i=n}^{b-1} (i-t+1)^{\overline{\alpha-1}} f\left[i, u(i)\right]. \end{aligned}$$

then

$$u(t) = \sum_{k=0}^{m-1} \frac{(b-t)^{\overline{k}}}{k!} u_k + \frac{1}{\Gamma(\alpha)} \sum_{i=t}^{b-1} (i-t+1)^{\overline{\alpha-1}} f\left[i, u(i)\right].$$

Having established the necessity, we now proceed to prove the sufficiency.

Taking the fractional difference operator $(-\Delta)^{\alpha}$ into both sides of the first equation of Equation (15), we can obtain

$$(-\Delta)^{\alpha}u(t) = (-\Delta)^{\alpha} \sum_{k=0}^{m-1} \frac{(b-t)^{\overline{k}}}{k!} u_k + (-\Delta)^{\alpha} \frac{1}{\Gamma(\alpha)} \sum_{i=n}^{b-1} (i-t+1)^{\overline{\alpha-1}} f\left[i, u(i)\right].$$

According to Definition 14 (Caputo type fractional difference) and Theorem 13, we have

$$(-\Delta)^{\alpha} \frac{(b-t)^{\overline{k}}}{k!} = \Delta^{-(m-\alpha)} \left((-\Delta)^{m} \frac{(b-t)^{\overline{k}}}{k!} \right) = 0 \quad (k = 0, 1, 2, \cdots, m-1),$$

and from the fourth equation of Theorem 9, we have

$$(-\Delta)^{\alpha} \frac{1}{\Gamma(\alpha)} \sum_{i=t}^{b-1} (i-t+1)^{\overline{\alpha-1}} f\left[i, u(i)\right] = (-\Delta)^{\alpha}\Delta^{-\alpha} f\left[t, u(t)\right] = f\left[t, u(t)\right].$$

Then

$$(-\Delta)^{\alpha}u(t) = f\left[t, u(t)\right].$$

In addition, taking the $n(n = 1, 2, \cdots, [\alpha]\,)$ order difference into both sides of Equation (15), we get

$$(-\Delta)^n u(t) = (-\Delta)^n \sum_{k=0}^{m-1} \frac{(b-t)^{\overline{k}}}{k!} u_k + (-\Delta)^n \frac{1}{\Gamma(\alpha)} \sum_{i=n}^{b-1} (i-t+1)^{\overline{\alpha-1}} f\left[i, u(i)\right]. \tag{16}$$

Let final value $t = b$, The first term on the right side of the Equation (16), when $n \neq k$

$$(-\Delta)^n \frac{(b-t)^{\overline{k}}}{k!} = \frac{(b-t)^{\overline{k-n}}}{(k-n)!} = 0.$$

when $n = k$

$$(-\Delta)^k \frac{(b-t)^{\overline{k}}}{k!} = \frac{(b-t)^{\overline{k-k}}}{(k-k)!} = \frac{(b-t)^{\overline{0}}}{0!} = 1.$$

Set

$$\frac{1}{\Gamma(\alpha)} \sum_{i=b}^{b-1} (i-t+1)^{\overline{\alpha-1}} f\left[i, u(i)\right] = 0,$$

we can obtain

$$(-\Delta)^k u(b) = u_k,$$

where $k = 0, 1, 2, \cdots, m-1$. And adequacy is proven. □

5.2. Computational cases and numerical simulations

5.2.1. Example 1 of Theorem 22: Solution of backward Caputo fractional difference equation for initial value problem

Consider the following parameter settings for the initial value problem of the backward Caputo fractional difference equation. Nonhomogeneous term: $f\left(t, u(t)\right) = sin(t)$. Initial point: $a = 0$. Initial condition: $u(a) = u_0 = 0$. Fractional orders: $\alpha = 1$, 0.9, 0.8, 0.6.

For a fractional order α, let $[\alpha]$ denote the integer part of α; the order of integer-order difference required for the initial condition is $m = [\alpha] + 1$. For $\alpha = 1$ (integer order), $m = [1] + 1 = 2$; for non-integer α = (0.9, 0.8, 0.6), $m = [\alpha] + 1$, which is consistent with the single initial condition $u_0 = 0$ adopted in this example.

The backward Caputo fractional difference equation (Equation (5)) for the above initial value problem is formulated as

$$\begin{cases} \nabla^{\alpha} u(t) = \sin(t), \\ u(0) = u_0 = 0. \end{cases}$$

The Cauchy initial value problem in Equation (5) is equivalent to the following Volterra sum equation (derived from Theorem 22):

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{n} \frac{\Gamma(n-i+\alpha)}{\Gamma(n-i+1)} \sin(t_i),$$

where t is discretized into a sequence $t(n) = n \cdot dt$ with a constant step size dt, $n = 1,\ 2,\ 3, \cdots$ denotes the discrete time index, and $t_i = i \cdot dt(i.e.\ (t(i) - t(i-1) = dt))$.

For $\alpha = 0.8$ (non-integer order, $m = 1$) and discretization step size $dt = 0.05$, the initial condition $u_0 = 0$ satisfies the requirement of the backward fractional difference equation. The explicit solution of $u(t(n))$ is

$$u(t(n)) = \frac{1}{\Gamma(0.8)} \sum_{i=1}^{n} \frac{\Gamma(n-i+0.8)}{\Gamma(n-i+1)} \sin(0.05i).$$

We calculate the values of $u(t(n))$ for the first three discrete time points (n = 1, 2, 3) as follows

Calculation for $n = 1$

$$\begin{aligned} u(t(1)) &= u(0.05) \\ &= \frac{1}{\Gamma(0.8)} \sum_{i=1}^{1} \frac{\Gamma(1-i+0.8)}{\Gamma(1-i+1)} \sin(0.05i) \\ &= \frac{1}{\Gamma(0.8)} \cdot \frac{\Gamma(0.8)}{\Gamma(1)} \cdot \sin(0.05 \times 1) \\ &= \frac{\Gamma(0.8) \cdot \sin(0.05)}{\Gamma(0.8) \cdot 1} \\ &= \sin(0.05) \approx 0.04998. \end{aligned}$$

Calculation for $n = 2$

$$\begin{aligned} u(t(2)) &= u(0.10) \\ &= \frac{1}{\Gamma(0.8)} \left[\frac{\Gamma(2-1+0.8)}{\Gamma(2-1+1)} \sin(0.05) + \frac{\Gamma(2-2+0.8)}{\Gamma(2-2+1)} \sin(0.10) \right] \\ &= \frac{1}{\Gamma(0.8)} \left[\frac{\Gamma(1.8)}{\Gamma(2)} \sin(0.05) + \frac{\Gamma(0.8)}{\Gamma(1)} \sin(0.10) \right] \\ &\approx \frac{1}{0.93138} \left[\frac{0.89358}{1} \times 0.04998 + \frac{0.93138}{1} \times 0.09983 \right] \approx 0.1495. \end{aligned}$$

Calculation for $n = 3$

$$\begin{aligned} u(t(3)) &= u(0.15) \\ &= \frac{1}{\Gamma(0.8)} \left[\frac{\Gamma(2.8)}{\Gamma(3)} \sin(0.05) + \frac{\Gamma(1.8)}{\Gamma(2)} \sin(0.10) + \frac{\Gamma(0.8)}{\Gamma(1)} \sin(0.15) \right] \\ &\approx \frac{1}{0.93138} \left[\frac{2.5082}{2} \times 0.04998 + \frac{0.89358}{1} \times 0.09983 + \frac{0.93138}{1} \times 0.14944 \right] \\ &\approx 0.2980. \end{aligned}$$

Numerical simulations for the full range of fractional orders (α = 1, 0.9, 0.8, 0.6) were implemented in MATLAB, as shown in **Figure 3**.

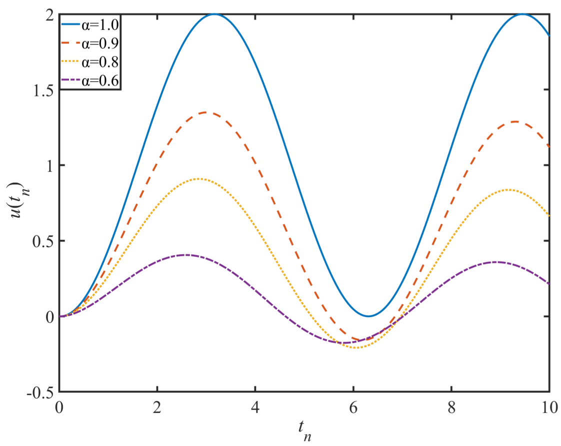

Figure 3. Numerical simulation results of Example 1.

The simulation results illustrate the evolution of $u(t)$ with discrete time t under different fractional orders, as α decreases from 1 to 0.6, the response of $u(t)$ exhibits a smaller trend, which reflects the "memory effect" characteristic of fractional-order dynamic systems.

5.2.2. Example 2 of Theorem 23: Solution of negative backward Caputo fractional difference equation for final value problem

Based on Theorem 23, we consider the final value problem of the negative backward fractional difference equation with the following parameter settings. Nonhomogeneous term: $f\,(t, u(t)) = cos(t)$. Final point: Discrete index $b = 200$, corresponding to continuous time $t(b) = 10$ (consistent with discretization step size $dt = 0.05$, i.e., $(t(b) = b{\cdot}dt = 200 \times 0.05 = 10)$. Final condition: $u(t(b)) = u_b = 0$. Fractional order: $\alpha = 0.8$ (non-integer order, $m = [\alpha]{+}1 = [0.8]{+}1 = 1$), which matches the 0-th order final condition $u(t(b)) = 0)$.

For the negative backward Caputo fractional difference equation, the Cauchy final value problem (Equation (8)) corresponding to Theorem 23 is equivalent to the following Volterra sum equation

$$u(t) = u_b + \frac{1}{\Gamma(\alpha)} \sum_{i=t+1}^{b} (i-t)^{\overline{\alpha-1}} \cos(i \cdot dt),$$

where $(i-t)^{\overline{\alpha-1}} = \frac{\Gamma(i-t+\alpha-1)}{\Gamma(i-t)}$ denotes the rising factorial (consistent with Definition 2), and i represents the discrete time index.

With fractional order $\alpha = 0.8$ (yielding $\alpha - 1 = -0.2$) and discretization step size $dt = 0.05$), the final condition $u(t(200)) = u(10) = 1$ satisfies the requirement of $m = 1$ for the final value problem. The explicit solution of $u(t(n))$ $(where\ (t(n) = n \cdot dt)$, $n = 1, 2, \ldots, 200)$ is

$$u(t(n)) = \frac{1}{\Gamma(0.8)} \sum_{i=n+1}^{200} (i-n)^{\overline{-0.2}} \cos(0.05i)\}.$$

To highlight the terminal constraint effect (core characteristic of final value problems), we calculate $u\,(t(n))$ for the discrete points near the final value (n = 198, 199).

Calculation for $n = 199$ $(t = 9.95)$

$$\begin{aligned} u(t(199)) &= u(9.95) \\ &= \tfrac{1}{\Gamma(0.8)} \sum_{i=200}^{200} (i-199)^{\overline{-0.2}} \cos(0.05i) \\ &= \tfrac{1}{0.93138} \cdot (200-199)^{\overline{-0.2}} \cdot \cos(0.05 \times 200) \\ &= \tfrac{1}{0.93138} \cdot \tfrac{\Gamma(1-0.2)}{\Gamma(1)} \cdot \cos(10) \\ &= \tfrac{1}{0.93138} \cdot 0.93138 \cdot (-0.8391) \approx 0.1609. \end{aligned}$$

Calculation for $n = 198$ $(t = 9.90)$

$$\begin{aligned} u(t(198)) &= u(9.90) \\ &= \frac{1}{\Gamma(0.8)} \sum_{i=199}^{200} (i-198)^{\overline{-0.2}} \cos(0.05i) \\ &= \frac{1}{0.93138}\left[(199-198)^{\overline{-0.2}} \cos(9.95) + (200-198)^{\overline{-0.2}} \cos(10.0)\right] \\ &= \frac{1}{0.93138}\left[\frac{\Gamma(1-0.2)}{\Gamma(1)} \cdot (-0.8293) + \frac{\Gamma(2-0.2)}{\Gamma(2)} \cdot (-0.8391)\right] \\ &= \frac{1}{0.93138}\left[0.93138 \times (-0.8293) + \frac{1.78716}{1} \times (-0.8391)\right] \\ &\approx \frac{1}{0.93138}(-0.772 + (-1.500)) \approx -2.440. \end{aligned}$$

Numerical simulations for the negative backward fractional difference equation (with $\alpha = 0.6,\ 0.8,\ 0.9,\ 1.0$ for comparison) were implemented in MATLAB as shown in **Figure 4**.

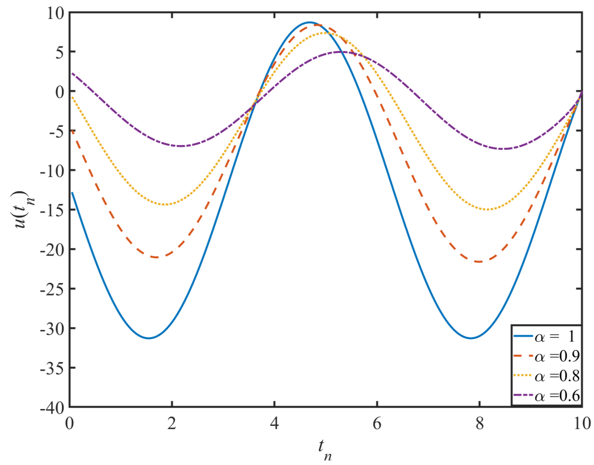

Figure 4. Numerical simulation results of Example 2.

The simulation results illustrate the dynamic behavior of $u\,(t))$ under the final value constraint, as the discrete index$n$approaches the final point $(n\rightarrow 200)$,$u\,(t(n))$ converges to the terminal condition 0, which verifies the correctness of the equivalent Volterra sum equation (Equation (9)) derived from Theorem 23. Compared with the integer order $(\alpha = 1.0)$, the non-integer order leads to a smoother convergence to the initial value.

5.2.3. Example 3 of Theorem 24: Solution of forward Caputo fractional difference equation for final value problem

Based on Theorem 24, we investigate the initial value problem of the forward fractional difference equation with standardized and corrected parameter settings. Nonhomogeneous term: $f\,(t, u(t)) = e^t$. Initial point: Discrete index $a = 0$, corresponding to continuous time $t(a) = 0$ (consistent with the discretization step size $dt = 0.05$, i.e., $(t(a) = a \cdot dt = 0)$. Initial condition: $u\,(t(a)) = u_0 = 0$. Fractional order: $\alpha = 0.8$ (non-integer order, $m = [\alpha] + 1 = [0.8] + 1 = 1$), matching the 0-th order initial condition $u\,(t(a)) = 0)$.

For the forward Caputo fractional difference equation, the Cauchy initial value problem (Equation (11)) corresponding to Theorem 24 is equivalent to the following

Volterra sum equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=a}^{t-1} (t-i)^{\overline{\alpha-1}} e^{i \cdot dt},$$

where $((t-i)^{\overline{\alpha-1}} = \frac{\Gamma(t-i+\alpha-1)}{\Gamma(t-i)})$ denotes the rising factorial (consistent with Definition 2), i is the discrete time index.

With fractional order $\alpha = 0.8$ (yielding $\alpha - 1 = -0.2$) and discretization step size $dt = 0.05$, the initial condition $u(t(0)) = u(0) = 0$ satisfies the requirement of $m = 1$ for the initial value problem. The explicit solution of $u(t(n))$ (where $(t(n) = n \cdot dt), \ (n = 1, 2, \ldots)$ is

$$u\left(t(n)\right) = \frac{1}{\Gamma(0.8)} \sum_{i=0}^{n-1} (n-i)^{\overline{-0.2}} e^{(0.05i)},$$

where $(\Gamma(0.8) \approx 0.93138)$.We calculate the values of $u(t(n))$ for the first three discrete time points (n = 1, 2, 3) with detailed numerical derivations.

Calculation for $n = 1$ $(t = 0.05)$

$$\begin{aligned} u(t(1)) &= u(0.05) \\ &= \tfrac{1}{\Gamma(0.8)} \sum_{i=0}^{0} (1-i)^{\overline{-0.2}} e^{(0.05i)} \\ &= \tfrac{1}{0.93138} \cdot (1-0)^{\overline{-0.2}} \cdot e^0 \\ &= \tfrac{1}{0.93138} \cdot \tfrac{\Gamma(1-0.2)}{\Gamma(1)} \cdot 1 \\ &= \tfrac{1}{0.93138} \cdot 0.93138 \cdot 1 = 1.0000. \end{aligned}$$

Calculation for $n = 2$ $(t = 0.10)$

$$\begin{aligned} u(t(2)) &= u(0.10) \\ &= \tfrac{1}{\Gamma(0.8)} \sum_{i=0}^{0} (2-i)^{\overline{-0.2}} \ e^{0.05i} \\ &= \tfrac{1}{0.93138} \left[(2-0)^{\overline{-0.2}} e^0 + (2-1)^{\overline{-0.2}} e^{0.05} \right] \\ &= \tfrac{1}{0.93138} \left[\tfrac{\Gamma(2-0.2)}{\Gamma(2)} \cdot 1 + \tfrac{\Gamma(1-0.2)}{\Gamma(1)} \cdot 1.0513 \right] \\ &= \tfrac{1}{0.93138} \left[\tfrac{1.78716}{1} \times 1 + 0.93138 \times 1.0513 \right] \\ &= \tfrac{1}{0.93138} (1.78716 + 0.9791) \approx 2.970. \end{aligned}$$

Calculation for $n = 3$ $(t = 0.15)$

$$\begin{aligned} u(t(3)) &= u(0.15) \\ &= \tfrac{1}{\Gamma(0.8)} \sum_{i=0}^{2} (3-i)^{\overline{-0.2}} e^{0.05i} \\ &= \tfrac{1}{0.93138} \left[(3-0)^{\overline{-0.2}} e^0 + (3-1)^{\overline{-0.2}} e^{0.05} + (3-2)^{\overline{-0.2}} e^{0.10} \right] \\ &= \tfrac{1}{0.93138} \left[\tfrac{\Gamma(3-0.2)}{\Gamma(3)} \cdot 1 + \tfrac{\Gamma(2-0.2)}{\Gamma(2)} \cdot 1.0513 + \tfrac{\Gamma(1-0.2)}{\Gamma(1)} \cdot 1.1052 \right] \\ &= \tfrac{1}{0.93138} \left[\tfrac{2.6807}{2} \times 1 + \tfrac{1.78716}{1} \times 1.0513 + 0.93138 \times 1.1052 \right] \\ &= \tfrac{1}{0.93138} (1.3403 + 1.8789 + 1.0294) \approx 4.564. \end{aligned}$$

Numerical simulations for the forward fractional difference equation (with $\alpha = 0.6$, 0.8, 0.9, 1.0 for comparative analysis) were implemented in MATLAB as shown in **Figure 5**.

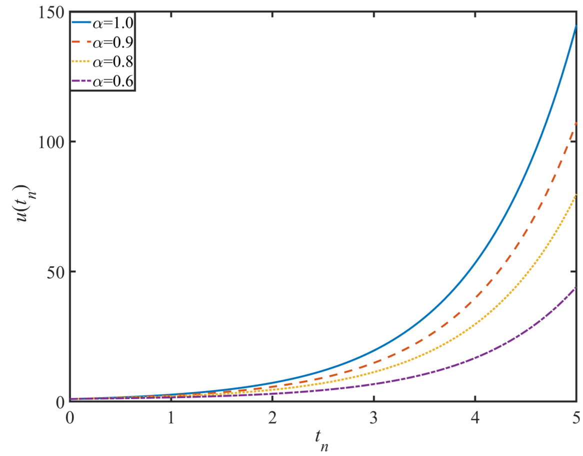

Figure 5. Numerical simulation results of Example 3.

The simulation results demonstrate the forward evolution characteristic of the solution under the initial value constraint, as the discrete index n increases (forward in time), $u\left(t(n)\right)$ exhibits an exponential growth trend consistent with the nonhomogeneous term e^t. Compared with the integer order ($\alpha = 1.0$), the non-integer order results in a slower growth rate, verifies the correctness of the equivalent Volterra sum equation (Equation (12)) derived from Theorem 24 and the adaptability of the forward Caputo fractional difference definition to initial value problems with forward-time evolution characteristics.

5.2.4. Example 4 of Theorem 25: Solution of negative forward Caputo fractional difference equation for final value problem

Based on Theorem 25, we analyze the final value problem of the negative forward fractional difference equation with rigorously corrected parameter settings. Nonhomogeneous term: $f\left(t, u(t)\right) = \frac{1}{t+\epsilon}$ (avoid division by zero at $t = 0$; $\epsilon = 10^6$ is introduced as a small positive constant for numerical stability, and $t = idt$ for discrete systems). Final point: Discrete index $b = 200$, corresponding to continuous time $t(b) = 10$ (consistent with discretization step size $dt = 0.05$, i.e., $t(b) = b \cdot dt = 200 \times 0.05 = 10$). Final condition: $u\left(t(b)\right) = u_b = 0$). Fractional order: $\alpha = 0.8$ (non-integer order, $m = [\alpha] + 1 = [0.8] + 1 = 1$), matching the 0-th order final condition $u\left(t(b)\right) = 0$).

For the negative forward Caputo fractional difference equation, the Cauchy final value problem Equation (14) corresponding to Theorem 25 is equivalent to the following Volterra sum equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=t}^{b-1} (i - t + 1)^{\overline{\alpha-1}} \frac{1}{i \cdot dt + \epsilon}.$$

where $((i - t + 1)^{\overline{\alpha-1}} = \frac{\Gamma(i-t+1+\alpha-1)}{\Gamma(i-t+1)} = \frac{\Gamma(i-t+\alpha)}{\Gamma(i-t+1)})$ (consistent with Definition 2 for rising factorial).

With fractional order $\alpha = 0.8$ (yielding $\alpha - 1 = -0.2$) and discretization step size $(dt = 0.05$ $(\epsilon = 10^{-6})$, the final condition $u(t(200)) = u(10) = 0$ satisfies the requirement of $m = 1$ for the final value problem. The explicit solution of $u(t(n))$ (where $t(n) = n \cdot dt,\ n = 1,\ 2, \ldots,\ 200)$ is simplified (by setting $u_0 = 0$) as

$$u\left(t(n)\right) = \frac{1}{\Gamma(0.8)} \sum_{i=n}^{199} \left(i - n + 1\right)^{\overline{-0.2}} \frac{1}{0.05i + 10^{-6}}.$$

To reflect the terminal constraint characteristic of final value problems (core logic of negative forward fractional differences), we calculate $u(t(n))$ for discrete points near the final value.

Calculation for $n = 199$ $(t = 9.95)$

$$\begin{aligned} u(t(199)) &= u(9.95) \\ &= \tfrac{1}{\Gamma(0.8)} \sum_{i=199}^{199} \left(i - 199 + 1\right)^{\overline{-0.2}} \tfrac{1}{0.05i+10^{-6}} \\ &= \tfrac{1}{0.93138} \cdot \left(199 - 199 + 1\right)^{\overline{-0.2}} \cdot \tfrac{1}{0.05\times 199+10^{-6}} \\ &= \tfrac{1}{0.93138} \cdot \tfrac{\Gamma(1-0.2)}{\Gamma(1)} \cdot \tfrac{1}{9.950001} \\ &= \tfrac{1}{0.93138} \cdot 0.93138 \cdot 0.10050 \approx 0.1005. \end{aligned}$$

Calculation for $n = 198$ $(t = 9.90)$

$$\begin{aligned} u(t(198)) &= u(9.90) \\ &= \tfrac{1}{\Gamma(0.8)} \sum_{i=198}^{199} \left(i - 198 + 1\right)^{\overline{-0.2}} \tfrac{1}{0.05i+10^{-6}} \\ &= \tfrac{1}{0.93138} \left[\left(198 - 198 + 1\right)^{\overline{-0.2}} . \tfrac{1}{0.05\times 198+10^{-6}} \right] \\ &+ \left(199 - 198 + 1\right)^{\overline{-0.2}} \cdot \tfrac{1}{0.05\times 199+10^{-6}} \\ &= \tfrac{1}{0.93138} \left[\tfrac{\Gamma(1-0.2)}{\Gamma(1)} \cdot \tfrac{1}{9.900001} + \tfrac{\Gamma(2-0.2)}{\Gamma(2)} \cdot \tfrac{1}{9.950001} \right] \\ &= \tfrac{1}{0.93138} [0.93138 \times 0.10101 + 1.78716 \times 0.10050] \\ &= \tfrac{1}{0.93138} (0.09406 + 0.17961) \approx 0.2938. \end{aligned}$$

Numerical simulations for the negative forward fractional difference equation (with α = 0.6, 0.8, 0.9, 1.0 for comparative analysis) were implemented in MATLAB, as shown in **Figure 6**.

The simulation results illustrate the key characteristics of the solution under final value constraints: as the discrete index n approaches the final point $(n \to 200)$, $u\left(t(n)\right)$ converges to the terminal condition 0, which validates the correctness of the equivalent Volterra sum equation Equation (15) derived from Theorem 25. This confirms the adaptability of the negative forward Caputo fractional difference definition to final value problems with reverse-time evolution characteristics.

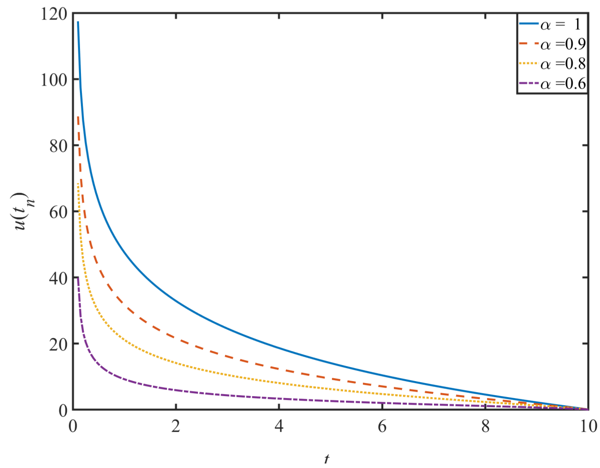

Figure 6. Numerical simulation results of Example 4.

The theoretical analysis in this section provides standardized solution forms for discrete fractional dynamic systems with diverse boundary conditions (initial or final value constraints) and dynamic characteristics. The subsequent numerical validation not only confirms the consistency between theoretical derivations and computational results but also offers intuitive guidance for the application of the four fractional difference definitions in real-world scenarios.

6. Conclusion

Through the innovative integration of variable upper-limit summation and variable lower-limit summation, combined with forward and backward differences, this paper systematically introduces four distinct definitions of Caputo fractional differences. Furthermore, it thoroughly investigates the key properties inherent in these definitions. Building upon this foundation, the paper successfully constructs four fractional difference equations based on these definitions and conducts rigorous derivations and validations of the initial and terminal value problems associated with the solutions to these equations. The results demonstrate that, when the order $\alpha = 1$, the fractional difference equations presented in this paper converge to the results derived from conventional integer-order difference equations. This alignment confirms that integer-order difference equations are, in fact, a special case of fractional difference equations under certain conditions.

To facilitate readers in selecting the most appropriate definition for their specific research scenarios, a concise comparison of the four Caputo fractional difference definitions is provided as follows:

1. The backward variable upper-limit fractional difference (∇^{α}) is tailored for initial value problems $(a < t)$ and is suitable for discrete systems dependent on historical states (e.g., infectious disease spread, economic growth with long memory);
2. The negative backward variable lower-limit fractional difference $(\nabla)^{\alpha})$ is designed for final value problems $(b > t)$ and excels in systems influenced by terminal states (e.g., optimal control, dynamic programming with terminal constraints);
3. The forward variable upper-limit fractional difference (Δ^{α}) adapts to initial

value problems $(a < t)$ and is preferred for predictive models requiring forward evolution (e.g., signal processing, chaotic system synchronization);

4. The negative forward variable lower-limit fractional difference $((\Delta)^{\alpha})$ is applicable to final value problems $(b > t)$ and is well-suited for terminal goal-oriented dynamic systems with forward evolution demands (e.g., engineering optimization, terminal-constrained dynamic systems).

This comparison highlights the complementary nature of the four definitions, covering both initial/final value conditions and historical/future-dependent dynamics. Looking ahead, future work will build upon the four definitions presented in this paper to solve for $u(t)$ based on its position. In particular, while the backward difference terminal value (with variable lower-limit summation) and the forward difference initial value (with variable upper-limit summation) definitions only involve $u(t)$ on the left-hand side of the equation, both backward initial value and forward terminal value problems involve $u(t)$ on both sides of the equation, necessitating the solution of more complex equations. These challenges open up avenues for further exploration and refinement in solving fractional difference equations and their applications in real-world problems.

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