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Article

Stability of totally-positive switched linear systems with mode-dependent average dwell time switching

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Abstract: Totally-positive switched linear systems (TSLSs), as one of the special switched system classes, have both the complex dynamic behavior of switched systems and their special dynamic properties of totally positive dynamical systems. Recently, TSLSs have attracted scientists' extensive attention, due to their wide applications, such as economics, biology, communication, and electronic information engineering. The research focuses on the stability issue of TSLSs. Several new exponential stability criteria of TSLSs in both continuous-time and discrete-time cases are obtained by combining the strategy of mode-dependent average dwell time (MDADT) and the multiple linear co-positive Lyapunov function approach. Those stability criteria obtained are presented in the form of linear constraints, making them easy to verify and apply through tools such as linear programming (LP). Since the MDADT framework only limits the average dwell time (ADT) of each subsystem and does not impose restrictions on the switching order or subsystem activation frequency, the conclusion of this paper is robust for switching sequences. The corresponding ADT stability criteria have also been inferred. Furthermore, it is pointed out that the stability issue under arbitrary switching can be solved by the common linear co-positive Lyapunov function (CLCLF) method. Finally, the efficiency of the results is verified by two numerical examples. One of them is from the epidemiological models, which provides a practically motivated TSLSs to make the validation more convincing.

Keywords: totally-positive switched linear systems; mode-dependent average dwell time; multiple Lyapunov functions; stability

1. Introduction

In the past few years, switched systems have received a lot of attention and have had many applications, such as Boolean control networks [1–4], positive (multi-agent) systems [5–8] and fractional-order systems [9]. For a switched system, its stability is a significant issue [10–13], which is related to the switching signal [14–16]. Generally speaking, switching signals contain two types: the constrained switching [17, 18] and the arbitrary switching [9,19–21]. Compared to arbitrary switching, the system stability under constrained switching has received widespread attention due to the complexity of its research. Dwell time (DT) switching is a natural constraint switching technique for stable subsystems, which requires that the activation time of all subsystems be no less than a given constant. Although the DT form is simple, the strict constraint on subsystem activation time limits its application. Then the paper [22] put forward the average dwell time (ADT) switching, one of the widely constrained switching, which means DT is not less than a constant in some average sense. Noticing that ADT ignores differences among subsystems, Zhao et al. [23] came up with the mode-dependent average dwell time (MDADT), which requires every subsystem to have a separate ADT. Afterwards, ADT and MDADT became two main switching techniques for studying the stability of switched systems with stable subsystems.

Recently, totally positive dynamical systems (TDSs) have attracted scientists' extensive attention, due to their wide applications, such as economics, biology, communication, and electronic information engineering [24]. The TDSs was first proposed and investigated by Schwarz in 1970. Consider the continuous-time system $Z(s) = A(s)Z(s)$ with $A(s)$ and $Z(s)$ being the $n \times n$ system matrix function and the state matrix function of s, respectively. Then it is called TDS if $Z(s)$ is a totally positive (TP) matrix for every $s \geq s_0$ with some initial time s_0 and $Z(s_0) = I$. Totally-positive switched linear systems (TSLSs), as one of the special switched system classes, have both the complex dynamic behavior of switched systems and their special dynamic properties of TDSs [25]. Thus, it is necessary to study the stability of TSLSs.

The multiple linear co-positive Lyapunov function (MLCLF) is the main approach for studying the stability of TSLSs. Wang et al. [25] investigated the stability of TSLSs by using MLCLF with ADT switching. It is worth noting that MDADT can be seen as an expansion of ADT, since MDADT switching makes each subsystem have its own ADT. Considering this, some special stability results with the MDADT switching may be obtained for TSLSs. To our knowledge, there are few stability results based on MDADT switching for TSLSs, which inspires us to pursue this subject.

The main contributions are as follows. This paper first introduces the MDADT strategy to the stability study of TSLSs in both the continuous-time and discrete-time cases, and combines the Lyapunov function method to obtain some stability criteria with lower conservatism. The main conclusions obtained are presented in the form of linear constraints, making them easy to verify and apply through tools such as linear programming (LP). The corresponding ADT stability criteria have also been inferred. Finally, it is pointed out that the stability issue under arbitrary switching can be solved by the common linear co-positive Lyapunov function (CLCLF) method.

The following is the paper's outline. Section 2 introduces the necessary system description and preliminaries, and Section 3 gives the stability results of TSLSs in both continuous-time and discrete-time cases by the MDADT strategy and the ML-CLF approach, then Section 4 illustrates the validity of the new approach by two numerical examples. Lastly, Section 5 gives the conclusion.

The notions of the paper are used as shown in **Table 1** below.

Table 1. Notions and symbols.

R^{+}	all nonnegative numbers
R^{n}_{+}	all n dimensional positive vectors
$\mathrm{R}^{n\times n}$	the set of $n \times n$ real matrices
$\mathrm{P} \geq 0$ (> 0)	P to be a positive semi-definite (definite) matrix
I	identity matrix with appropriate dimensions
T	transposition for matrix or vector
$\|A\|$	the row norm of the matrix A
$\in$ $(\forall)$	in (for all)

Table 1. *Cont.*

$\lambda_{\min}(Q)$ $(\lambda_{\max}(Q))$	minimum (maximum) eigenvalue for the matrix Q
$A \succcurlyeq B$ $(A \succ B)$	$a_{ij} \geq b_{ij}$ $(a_{ij} > b_{ij})$, where a_{ij} (b_{ij}) means entry (i, j) of matrix A (B)
$\triangleq$	is the shorthand of
$diag$	diagonal matrix

2. System descriptions and preliminaries

Consider the continuous-time TSLS:

$$\dot{Z}(s) = \sum\nolimits_{p=1}^{W} \delta_p(\sigma(s))A_p Z(s), \quad Z(s_0) \succcurlyeq \mathbf{0} \tag{1}$$

where $A_p \in \mathrm{R}^{n\times n}$, $s \in [s_0, +\infty)$, s_0 is the initial time, $Z(s)$ is the $n \times n$ state matrix function of s, $Z(s_0)$ is the initial value, and $\mathbf{0}$ is a zero matrix. Let $Z(s, s_0)$ denote the trajectory of system (1) with the initial value $Z(s_0)$, in the absence of ambiguity, it is abbreviated as $Z(s)$. System (1) is called totally-positive when the trajectory $Z(s, s_0)$ is a totally-positive matrix. The right continuous piecewise constant function $\sigma(s)$ denotes a switching signal that takes its values in $\tilde{W} = \{1, 2, \ldots, W\}$, where $W \in \mathbb{R}^+$ represents the subsystem number. Suppose the switching time sequence is denoted as $s_1, s_2, \ldots, s_i, \ldots$ in $(s_0, +\infty)$, then the $\sigma(s_i)^{th}$ subsystem is activated for $s \in [s_i, s_{i+1})$, $i = 0, 1, 2, \ldots$. If $\sigma(s) = p$, take $\delta_p(\sigma(s)) = 1$, otherwise $\delta_p(\sigma(s)) = 0$.

Consider the discrete-time TSLS:

$$Z(k+1) = \sum\nolimits_{p=1}^{W} \delta_p(\sigma(k))A_p Z(k), \quad Z(k_0) \succcurlyeq \mathbf{0} \tag{2}$$

where A_p, W, $\mathbf{0}$, $\sigma(\cdot)$, and $\delta_p(\cdot)$ are defined as above; k_0 and $Z(k_0)$ denote the initial time and initial value, respectively; $Z(k)$ is the $n \times n$ state matrix function of k, $k = k_0 + 1$, $k_0 + 2$, $\ldots$; Let $Z(k, k_0)$ denote the trajectory of system (2) with the initial value $Z(k_0)$, in the absence of ambiguity, it is abbreviated as $Z(k)$.

Without loss of generality, the initial time s_0 of the continuous-time case (k_0 of the discrete-time case) is often set to 0 in practice.

Definition 1. *Given a matrix $\Lambda \in \mathrm{R}^{n\times n}$, if the determinant of every square sub-matrix of the matrix Λ is positive, then it is called totally positive (TP) [24].*

Definition 2. *System (1) (2) is said to be a totally-positive switched linear system (TSLS) if the trajectory $Z(s,\, s_0)$ ($Z(k,\, k_0)$) is TP for all $s > s_0$ ($k > k_0$) [24].*

Definition 3. *System (1) (2) is exponentially stable if there exist constants $\varepsilon > 0$ and $\omega > 0$ ($0 < \varrho < 1$), such that the system's trajectory $\|Z(s)\| \leq \varepsilon e^{-\omega(s-s_0)}\|Z(s_0)\|$ ($\|Z(k)\| \leq \varepsilon \varrho^{k-k_0}\|Z(k_0)\|$) for all $s \geq s_0$ ($k \geq k_0$), where $Z(s_0)$ ($Z(k_0)$) is the initial condition [25].*

The purpose of this article is to find a set of MDADT or ADT switching signals to stabilize the switched system. Let us recall the definitions of MDADT and ADT.

Definition 4. *Given $\sigma(s)$ and $\forall d > c \geq 0$, let $N_\sigma(d, c)$ be the switching number of $\sigma(s)$ over the interval (c, d) [22]. $\sigma(s)$ is said to have an ADT τ_a, if there exist $N_0 > 0$*

and $\tau_a > 0$ *such that* $N_\sigma(d,c) \le N_0 + \frac{d-c}{\tau_a}$.

Definition 5. *Given* $\sigma(s)$ *and* $\forall d > c \ge 0$, *let* $N_{\sigma p}(d,c)$ *and* $S_p(d,c)$, *respectively, be the* p^{th} *subsystem's switching number and total running time over* (c,d), $p \in W\widetilde{W}$. $\sigma(s)$ *is said to have a MDADT* τ_a *if there exist* $N_{0p} > 0$ *and* $\tau_{ap} > 0$ *[23] such that* $N_{\sigma p}(d,c) \le N_{0p} + \frac{S_p(c,d)}{\tau_{ap}}$.

3. Stability results based on MDADT strategy

3.1. Continuous-time model

Theorem 1. *Suppose there exist* $\alpha_p > 0$, $\lambda_p \ge 1$, *and* $\nu_p = [\nu_{p1}, \nu_{p2}, \ldots, \nu_{pn}]^T \in \mathrm{R}_+^n$ *satisfying*

$$\sum\nolimits_{i=1}^{n} a_{ij}^p \nu_{pi} \le -\alpha_p \nu_{pj} \tag{3}$$

$$\nu_{pj} \le \lambda_p \nu_{qj} \tag{4}$$

where $p,q \in W\widetilde{W}$, *and* a_{ij}^p *denotes the* (i,j) *element of* A_p *with* $i,j \in J = \{1,2,\ldots,n\}$. *Then system (1) is exponentially stable under the following MDADT switching.*

$$\tau_{ap} \ge \tau_{ap}^* = \frac{2\ln\lambda_p}{\alpha_p} \tag{5}$$

Proof of Theorem 1. Not to lose generality, it is assumed here that the initial time $s_0 = 0$. Mark the switching time sequence $s_1, s_2, \ldots, s_i, \ldots, s_{N_\sigma(s',0)}$ in the interval $(0, s')$, where $s' > 0$ represents the endpoint time of the considered time interval and s_i is the i^{th} switching time for $i = 1,2,\ldots,N_\sigma(s',0)$. Take the following candidate MLCLF for the system (1)

$$V_{\sigma(s)}(s, Z(s)) = \sum\nolimits_{p=1}^{W} \nu_p^T Z^T(s) \delta_p(\sigma(s)) \nu_p \tag{6}$$

where $Z(s)$ and $\delta_p(\sigma(s))$ are given in the system (1), and $\nu_p \in \mathrm{R}_+^n$ with its components satisfying inequalities (3) and (4). On the basis of (4), we get

$$\nu_p \preccurlyeq \lambda_p \nu_q, \quad \nu_p^T \preccurlyeq \lambda_p \nu_q^T \tag{7}$$

where $p,q \in W\widetilde{W}$. Then

$$\nu_p^T Z^T(s_i)\nu_p \le \nu_p^T Z^T(s_i^-)\lambda_p \nu_q \le \lambda_p \nu_q^T Z^T(s_i^-)\lambda_p \nu_q \le \lambda_p^2 \nu_q^T Z^T(s_i^-)\nu_q \tag{8}$$

where $(\sigma(s_i), \sigma(s_i^-)) = (p,q) \in W\widetilde{W} \times W\widetilde{W}$, and $Z^T(s_i^-)$ and $\sigma(s_i^-)$ represent the left limit of $Z^T(s)$ and $\sigma(s)$ at the point s_i, respectively. Owing to the values of $\sigma(s)$, we get

$$\sum\nolimits_{p=1}^{W} \delta_p(\sigma(s_i)) = \sum\nolimits_{q=1}^{W} \delta_q(\sigma(s_i^-)) = 1 \tag{9}$$

By (8), we obtain

$$\sum\nolimits_{p=1}^{W} \delta_p(\sigma(s_i))\nu_p^T Z^T(s_i)\nu_p \le \lambda_p^2 \sum\nolimits_{q=1}^{W} \delta_q(\sigma(s_i^-))\nu_q^T Z^T(s_i^-)\nu_q \tag{10}$$

Therefore,

$$V_{\sigma(s_i)}(s_i, Z(s_i)) \leq \lambda^2_{\sigma(s_i)} V_{\sigma(s_i^-)}(s_i^-, Z(s_i^-)) \tag{11}$$

where $V_{\sigma(s_i^-)}(s_i^-, Z(s_i^-))$ represents the left limit of $V_{\sigma(s)}(s_i, Z(s))$ at the point s_i.

On the basis of (3), we get

$$\begin{bmatrix} \sum_{i=1}^n a_{i1}^p \nu_{pi} \\ \sum_{i=1}^n a_{i2}^p \nu_{pi} \\ \dots \\ \sum_{i=1}^n a_{in}^p \nu_{pi} \end{bmatrix} \preccurlyeq -\alpha_p \begin{bmatrix} \nu_{p1} \\ \nu_{p2} \\ \dots \\ \nu_{pn} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} a_{11}^p & a_{21}^p & \dots & a_{n1}^p \\ a_{12}^p & a_{22}^p & \dots & a_{n2}^p \\ \dots & \dots & \dots & \dots \\ a_{1n}^p & a_{2n}^p & \dots & a_{nn}^p \end{bmatrix} \preccurlyeq -\alpha_p \begin{bmatrix} \nu_{p1} \\ \nu_{p2} \\ \dots \\ \nu_{pn} \end{bmatrix}$$

Then

$$A_p^T \nu_p \preccurlyeq -\alpha_p \nu_p \tag{12}$$

and

$$\nu_p^T Z^T(s) A_p^T \nu_p \leq -\alpha_p \nu_p^T Z^T(s) \nu_p \tag{13}$$

which means that

$$\dot{V}_{\sigma(s)}(s, Z(s)) \leq -\alpha_{\sigma(s)} V_{\sigma(s)}(s, Z(s)) \tag{14}$$

By integrating this for any $s \in [s_i, s_{i+1})$, we have

$$V_{\sigma(s)}(s, Z(s)) \leq e^{-\alpha_{\sigma(s_i)}(s-s_i)} V_{\sigma(s_i)}(s_i, Z(s_i)) \tag{15}$$

Combined with (5), (11), (15) and Definition 5, we get

$$\begin{aligned}
&V_{\sigma(s')}(s', Z(s')) \\
&\leq e^{-\alpha_{\sigma(s_{N_\sigma})}(s'-s_{N_\sigma})} V_{\sigma(s_{N_\sigma})}(s_{N_\sigma}, Z(s_{N_\sigma})) \\
&\leq \lambda^2_{\sigma(s_{N_\sigma})} e^{-\alpha_{\sigma(s_{N_\sigma})}(s'-s_{N_\sigma})} V_{\sigma(s^-_{N_\sigma})}(s^-_{N_\sigma}, Z(s^-_{N_\sigma})) \\
&\leq \lambda^2_{\sigma(s_{N_\sigma})} e^{-\alpha_{\sigma(s_{N_\sigma})}(s'-s_{N_\sigma})} e^{-\alpha_{\sigma(s_{N_\sigma-1})}(s_{N_\sigma}-s_{N_\sigma-1})} V_{\sigma(s_{N_\sigma-1})}(s_{N_\sigma-1}, Z(s_{N_\sigma-1})) \\
&\leq \dots \\
&\leq \left(\prod_{j=1}^{N_\sigma} \lambda^2_{\sigma(s_j)}\right) e^{-\alpha_{\sigma(s_{N_\sigma})}(s'-s_{N_\sigma}) - \alpha_{\sigma(s_{N_\sigma-1})}(s_{N_\sigma}-s_{N_\sigma-1}) - \dots - \alpha_{\sigma(s_0)}(s_1-s_0)} V(0) \\
&\leq \left(\prod_{p=1}^{W} \lambda_p^{2N_{\sigma p}}\right) e^{-\sum_{p=1}^W \alpha_p S_p} V(0) \\
&= e^{\sum_{p=1}^W 2N_{\sigma p} \ln \lambda_p - \sum_{p=1}^W \alpha_p S_p} V(0) \\
&\leq e^{\sum_{p=1}^W 2N_{0p} \ln \lambda_p} e^{\sum_{p=1}^W \left(\frac{2\ln \lambda_p}{\tau_{ap}} - \alpha_p\right) S_p} V(0) \\
&\leq c e^{\max\left\{\frac{2\ln \lambda_p}{\tau_{ap}} - \alpha_p\right\} s'} V(0)
\end{aligned} \tag{16}$$

where $c = e^{\sum_{p=1}^W 2N_{0p} \ln \lambda_p}$, $\sum_{p=1}^W S_p = s'$, $V(0) \triangleq V_{\sigma(0)}(0, Z(0))$, $N_\sigma \triangleq N_\sigma(0, s')$, $N_{\sigma p} \triangleq N_{\sigma p}(0, s')$ and $S_p \triangleq S_p(0, s')$. Let $\gamma_1 = \min_{(j,p) \in J \times W\widetilde{W}} \{\nu_{pj}\}$ and $\gamma_2 =$

$\max_{(j,p)\in J\times W\widetilde{W}}\{\nu_{pj}\}$, then

$$\|Z(s')\|\leq \frac{\gamma_2^2}{\gamma_1^2}ce^{\max\left\{\frac{2\ln\lambda_p}{\tau_{ap}}-\alpha_p\right\}s'}\|Z(0)\| \tag{17}$$

Thus, system (1) is exponentially stable under the MDADT (5). □

Corollary 1. *Suppose there exist* $\alpha > 0$*,* $\lambda \geq 1$*, and* $\nu_p = [\nu_{p1}, \nu_{p2}, \ldots, \nu_{pn}]^{\mathrm{T}} \in \mathrm{R}_+^n$ *satisfying*

$$\sum\nolimits_{i=1}^{n} a_{ij}^p \nu_{pi} \leq -\alpha\nu_{pj} \tag{18}$$

$$\nu_{pi} \leq \lambda\nu_{qj} \tag{19}$$

where $p, q \in W\widetilde{W}$*,* $i, j \in J = \{1, 2, \ldots, n\}$*, and* a_{ij}^p *are the same as above. Then system (1) is exponentially stable under the following ADT switching*

$$\tau_a \geq \tau_a^* = \frac{2\ln\lambda}{\alpha} \tag{20}$$

Proof of Corollary 1. Since the proof is similar to Theorem 1 by taking $\alpha_p = \alpha$ and $\lambda_p = \lambda$, we omit it here. □

Remark 1. *If* $\nu_p = \nu = [\nu_1, \nu_2, \ldots, \nu_n]^T$ *for all* $p \in W\widetilde{W}$*, we can choose CLCLF as* $V(s, Z(s)) = \nu^T Z^T(s)\nu$ *and replace (18) and (19) with* $\sum_{i=1}^{n} a_{ij}^p \nu_i < 0$*. Then the system stability condition for arbitrary switching can be obtained.*

3.2. Discrete-time model

Theorem 2. *Suppose there exist* $0 < \alpha_p < 1$*,* $\lambda_p \geq 1$*, and* $\nu_p = [\nu_{p1}, \nu_{p2}, \ldots, \nu_{pn}]^T \in \mathrm{R}_+^n$ *satisfying*

$$\sum_{i=1}^{n} a_{ij}^p \nu_{pi} \leq (1-\alpha_p)\nu_{pj} \tag{21}$$

$$\nu_{pi} \leq \lambda_p\nu_{qj} \tag{22}$$

where $p, q \in W\widetilde{W}$*,* $i, j \in J = \{1, 2, \ldots, n\}$*, and* a_{ij}^p *are the same as above. Then system (2) is exponentially stable under the following MDADT switching*

$$\tau_{ap} \geq \tau_{ap}^* = -\frac{2\ln\lambda_p}{\ln(1-\alpha_p)} \tag{23}$$

Proof of Theorem 2. Not loss of generality, suppose the initial time $k_0 = 0$ and denote the switching time in sequence to be $k_1, k_2, \ldots, k_i, \ldots, k_{N_\sigma(k',0)}$ in the interval $(0, k')$, where $k' > 0$ represents the endpoint time of the considered time interval and k_i is the i^{th} switching time for $i = 1, 2, \ldots, N_\sigma(k', 0)$. Consider the following candidate MLCLF for the system (2)

$$V_{\sigma(k)}(k, Z(k)) = \sum\nolimits_{p=1}^{W} \nu_p^T Z^T(k)\delta_p(\sigma(k))\nu_p \tag{24}$$

where $Z(k)$ and $\delta_p(\sigma(k))$ are given in the system (2), and $\nu_p \in \mathrm{R}_+^n$ with its components satisfying inequalities (21) and (22).

According to (22), we have

$$\nu_p \preccurlyeq \lambda_p \nu_q, \quad \nu_p^T \preccurlyeq \lambda_p \nu_q^T \tag{25}$$

Then

$$\begin{aligned} \nu_p^T Z^T(k_i)\nu_p \le \nu_p^T Z^T(k_i-1)\lambda_p\nu_q \le \lambda_p\nu_q^T Z^T(k_i-1)\lambda_p\nu_q \\ \le \lambda_p^2\nu_q^T Z^T(k_i-1)\nu_q \end{aligned} \tag{26}$$

where $(\sigma(k_i), \sigma(k_i-1)) = (p,q) \in W\widetilde{W} \times W\widetilde{W}$. Because of the values of $\sigma(k)$, we get

$$\sum\nolimits_{p=1}^{W} \delta_p(\sigma(k_i)) = \sum\nolimits_{q=1}^{W} \delta_q(\sigma(k_i-1)) = 1 \tag{27}$$

In view of (26), we obtain

$$\sum\nolimits_{p=1}^{W} \delta_p(\sigma(k_i))\nu_p^T Z^T(k_i)\nu_p \le \lambda_p^2 \sum\nolimits_{q=1}^{W} \delta_p(\sigma(k_i-1))\nu_q^T Z^T(k_i-1)\nu_q \tag{28}$$

Consequently,

$$V_{\sigma(k_i)}(k_i, Z(k_i)) \le \lambda_{\sigma(k_i)}^2 V_{\sigma(k_i-1)}(k_i-1, Z(k_i-1)) \tag{29}$$

According to (21), we obtain

$$\begin{bmatrix} \sum_{i=1}^n a_{i1}^p \nu_{pi} \\ \sum_{i=1}^n a_{i2}^p \nu_{pi} \\ \cdots \\ \sum_{i=1}^n a_{in}^p \nu_{pi} \end{bmatrix} \preccurlyeq (1-\alpha_p) \begin{bmatrix} \nu_{p1} \\ \nu_{p2} \\ \cdots \\ \nu_{pn} \end{bmatrix}$$

Therefore

$$\begin{bmatrix} a_{11}^p & a_{21}^p & \cdots & a_{n1}^p \\ a_{12}^p & a_{22}^p & \cdots & a_{n2}^p \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n}^p & a_{2n}^p & \cdots & a_{nn}^p \end{bmatrix} \begin{bmatrix} \nu_{p1} \\ \nu_{p2} \\ \cdots \\ \nu_{pn} \end{bmatrix} \preccurlyeq (1-\alpha_p) \begin{bmatrix} \nu_{p1} \\ \nu_{p2} \\ \cdots \\ \nu_{pn} \end{bmatrix}$$

So we get

$$A_p^T \nu_p \preccurlyeq (1-\alpha_p)\nu_p \tag{30}$$

and

$$\nu_p^T Z^T(k) A_p^T \nu_p \le (1-\alpha_p)\nu_p^T Z^T(k)\nu_p \tag{31}$$

which implies that for $\sigma(k) = \sigma(k+1)$, it follows

$$\begin{aligned} \sum\nolimits_{p=1}^{W} \delta_p(\sigma(k+1))\nu_p^T Z^T(k+1)\nu_p = \sum\nolimits_{p=1}^{W} \delta_p(\sigma(k))\nu_p^T Z^T(k) A_p^T \nu_p \\ \le \sum\nolimits_{p=1}^{W} (1-\alpha_p)\delta_p(\sigma(k))\nu_p^T Z^T(k)\nu_p \end{aligned} \tag{32}$$

Then $V_{\sigma(k+1)}(k+1, Z(k+1)) \le (1-\alpha_{\sigma(k)})V_{\sigma(k)}(k, Z(k))$.

For any $k \in [k_i, k_{i+1}]$, we can get

$$V_{\sigma(k)}(k, Z(k)) \le (1-\alpha_{\sigma(k)})^{k-k_i} V_{\sigma(k_i)}(k_i, Z(k_i)) \tag{33}$$

Combined with (23), (29) and (33), we get

$$\begin{aligned}
V_{\sigma(k')}(k', Z(k')) &\le (1-\alpha_{\sigma(k_{N_\sigma})})^{k'-k_{N_\sigma}} V_{\sigma(k_{N_\sigma})}(k_{N_\sigma}, Z(k_{N_\sigma})) \\
&\le \lambda^2_{\sigma(k_{N_\sigma})}(1-\alpha_{\sigma(k_{N_\sigma})})^{k'-k_{N_\sigma}} V_{\sigma(k_{N_\sigma-1})}(k_{N_\sigma-1}, Z(k_{N_\sigma-1})) \\
&\le \dots \\
&\le \left(\prod_{j=0}^{N_\sigma} \lambda_{\sigma(k_j)}\right)(1-\alpha_{\sigma(k_{N_\sigma})})^{k'-k_{N_\sigma}}(1-\alpha_{\sigma(k_{N_\sigma-1})})^{k_{N_\sigma}-k_{N_\sigma-1}} \\
&\quad \times \dots (1-\alpha_{\sigma(k_0)})^{k_1-k_0} V(0) \\
&\le \prod_{p=1}^{W}\left[\lambda_p^{2N_{\sigma p}}(1-\alpha_p)^{K_p}\right] V(0) \\
&\le e^{\sum_{p=1}^{W}(\ln \lambda_p^{2N_{\sigma p}} + \ln(1-\alpha_p)K_p)} V(0) \\
&\le e^{\sum_{p=1}^{W} 2N_{0p}\ln\lambda_p} e^{\sum_{p=1}^{W}\left(\frac{2\ln\lambda_p}{\tau_{ap}} + \ln(1-\alpha_p)\right)K_p} V(0) \\
&\le d e^{\max\left\{\frac{2\ln\lambda_p}{\tau_{ap}} + \ln(1-\alpha_p)\right\}k'} V(0)
\end{aligned} \tag{34}$$

Where $d = e^{\sum_{p=1}^{W} 2N_{0p}\ln\lambda_p}$, $\sum_{p=1}^{W} K_p = k'$, $V(0) \triangleq V_{\sigma(0)}(0, Z(0))$, $N_\sigma \triangleq N_\sigma(0, k')$, $N_{\sigma p} \triangleq N_{\sigma p}(0, k')$, and $K_p \triangleq S_p(0, k')$. Let $\gamma_1 = \min_{(j,p)\in J\times W}\{\nu_{pj}\}$ and $\gamma_2 = \max_{(j,p)\in J\times W}\{\nu_{pj}\}$, then

$$\| Z(k') \| \le \frac{\gamma_2^2}{\gamma_1^2} d e^{\max\left\{\frac{2\ln\lambda_p}{\tau_{ap}} + \ln(1-\alpha_p)\right\}k'} \| Z(0) \| \tag{35}$$

Thus, system (2) is exponentially stable under the MDADT (23). □

Remark 2. *Similar to Corollary 1, Theorem 2 can be obtained, which is based on the ADT switching [25]. The following corollary offers some details.*

Corollary 2. *Suppose there exist* $0 < \alpha < 1$, $\lambda \ge 1$, *and* $v_p = [\nu_{p1}, \nu_{p2}, \dots, \nu_{pn}]^T \in \mathrm{R}_+^n$ *satisfying*

$$\sum_{i=1}^{n} a_{ij}^p v_{pi} \le (1-\alpha) v_{pj} \tag{36}$$

$$v_{pj} \le \lambda v_{qi} \tag{37}$$

where $p, q \in \tilde{W}$, $i, j \in J = \{1, 2, \dots, n\}$, *and* a_{ij}^p *are the same as above. Then system (2) is exponentially stable under the following ADT switching.*

$$\tau_a \ge \tau_a^* = -\frac{2\ln\lambda}{\ln(1-\alpha)} \tag{38}$$

Proof of Corollary 2. Let $\alpha_p = \alpha$ and $\lambda_p = \lambda$, it can be easily proved by the same process of Theorem 2. □

Remark 3. *Similarly, one can get stability exponentially under arbitrary switching for the system (2) by taking* $v_p = v = [v_1, v_2, \dots, v_n]^T$ *for all* $p \in \tilde{W}$ *and choosing CLCLF as* $V(k, Z(k)) = v^T Z^T(k) v$ *and using* $\sum_{i=1}^{n} a_{ij}^p v_i \le v_j$ *instead of (36) and (37).*

Remark 4. *In the practical application of Theorems 1 and 2, due to the coupling of these parameters α_p, λ_p and υ_p, it is necessary to preset the values of parameters α_p and λ_p, and then solve the corresponding υ_p through linear inequalities (3–4)/(21–22). If these inequalities are not feasible, parameters α_p and λ_p need to be adjusted before solving. Therefore, these parameters are not unique.*

Remark 5. *In fact, the condition $\alpha_p > 0$ in Theorem 1 ($0 < \alpha_p < 1$ in Theorem 2) is to ensure the stability of the subsystem. As both ADT and MDADT strategies are mainly applied to switched systems with stable subsystems, this condition is important in this paper. However, it is worth mentioning that some improved MDADT strategies, such as fast MDADT and weighted ADT [26], can be used for the situation of some unstable subsystems, thereby relaxing the requirements for this condition. This is one of our future research topics.*

Remark 6. *The MDADT framework only limits the ADT of each subsystem and does not impose restrictions on the switching order or subsystem activation frequency. Therefore, the conclusion of this paper is robust for switching sequences that meet the MDADT framework. In order to better discuss the sensitivity of switching sequences, the concept of the distance between switching sequences needs to be introduced. It will be our next research direction.*

Remark 7. *According to Theorems 1 and 2 and Corollaries 1 and 2, MDADT τ_{ap} (ADT τ_a) depends on α_p (α) and λ_p (λ). Generally, in the case where the inequality conditions of theorems and corollaries are feasible, smaller λ_p (λ) and larger α_p (α) can achieve better MDADT τ_{ap} (ADT τ_a).*

4. Two numerical examples

Example 1. *The following example is from the epidemiological models [27]. This population epidemiological model is divided into n groups with each group having $S_j(s)$ susceptibles and $I_j(s)$ infectives two classes for $j = 1, 2, \ldots, n$. Assumption that $S_j(s) + I_j(s) = N_j$ is constant for all $s \geq 0$, and denoting $x_j(s) = I_j(s)/N_j$, the model can write $\dot{x}_j(s) = (1 - x_j(s)) \sum_{i=1}^{n} \frac{a_{ji} N_i}{N_j} x_i(s) - (c_j + d_j) x_j(s)$, where $a_{ji} > 0$ denotes the rate of the susceptibles in the j^{th} group being infected by the infectives in the i^{th} group, c_j and d_j represent the cure rate and death rate of the j^{th} group, respectively. The linearized system can be given by $\dot{x}(s) = \sum_{p=1}^{W} \delta_p(\sigma(s)) A_p x(s)$, where A_p is the tridiagonal matrix with positive entries on the sub- and super-diagonal. It follows from Theorem 6 by Margaliot and Sontag [24] that the system (1) is TSLS.*

Suppose the system (1) with two subsystems and their matrices,

$$A_1 = \begin{bmatrix} -5.4 & 2.3 & 0 & 0 \\ 2.3 & -6.5 & 3.1 & 0 \\ 0 & 0.9 & -8.9 & 3.2 \\ 0 & 0 & 3.2 & -15.7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5.7 & 1.2 & 0 & 0 \\ 1.1 & -6.9 & 2.2 & 0 \\ 0 & 2.1 & -8.8 & 3.2 \\ 0 & 0 & 3.2 & -15.2 \end{bmatrix}$$

where $Z = (Z_1, Z_2, Z_3, Z_4)^T$ and $\sigma(s) = p \in \{1, 2\}$. We assign $Z(0) =$

diag$(2,4,1,3)$.

Some comparisons are given in **Table 2**. It can be seen that since $\alpha = \min\{\alpha_1,\alpha_2\} = \min\{1.02, 1.27\} = 1.02$ and $\lambda = \max\{\lambda_1,\lambda_2\} = \max\{2.50, 1.70\}$ = 2.50, each subsystem ADT ($\tau_{a1}^* \approx 1.80$ and $\tau_{a2}^* \approx 0.84$) is no more than the total ADT ($\tau_a^* \approx 1.80$). This is to show that MDADT is superior to ADT to some extent. By taking $\tau_{a1} = 2$, $\tau_{a2} = 1$ as an example, it is clear that the signal satisfies the MDADT condition but does not satisfy the ADT condition. Therefore, the stability of the system with the signal can be determined based on the MDADT strategy, but it cannot be determined under the ADT one. Here, a periodic switching signal $\sigma^1(s)$ is adopted with $\tau_{a1} = 2$, $\tau_{a2} = 1$, the corresponding switching time sequence ($s_1 = 2$, $s_2 = 3$, ..., $s_{2i-1} = 3i - 1$, $s_{2i} = 3i$, ...with $\sigma^1(s_{2i-1}) = 1$ and $\sigma^1(s_{2i}) = 2$) is easily determined for $i = 1, 2, \ldots$. The state response of the system converges to 0 under the switching signal $\sigma^1(s)$ shown in the middle of **Figure 1**.

Table 2. Comparisons between ADT and MDADT strategies.

Strategy	**ADT**	**MDADT**
α	$\alpha = 1.02$	$\alpha_1 = 1.02, \alpha_2 = 1.27$
λ	$\lambda = 2.50$	$\lambda_1 = 2.50, \lambda_2 = 1.70$
Signal design	$\tau_a \geq \tau_a^* = \frac{2\ln 2.50}{1.02} \approx 1.80$	$\tau_{a1} \geq \tau_{a1}^* = \frac{2\ln 2.50}{1.02} \approx 1.80$, $\tau_{a2} \geq \tau_{a2}^* = \frac{2\ln 1.70}{1.27} \approx 0.84$

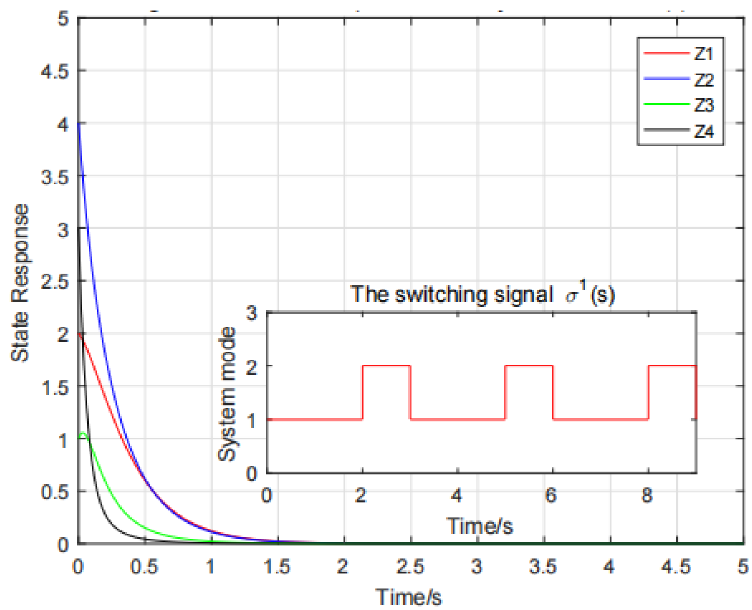

Figure 1. The state response of the system under $\sigma^1(s)$.

Example 2. *Consider the system (2) with subsystem matrices*

$$A_1 = \begin{bmatrix} 0.6 & -0.1 & 0 & 0 \\ -0.2 & 0.5 & 0 & 0 \\ 0 & 0.1 & 0.5 & 0.25 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.3 & 0 & 0 \\ -0.42 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0.6 & -0.1 \\ 0 & 0 & -0.2 & 0.5 \end{bmatrix}$$

where $Z = (Z_1, Z_2, Z_3, Z_4)^T$ *and* $\sigma(s) = p \in \{1, 2\}$. *We assign* $Z(0) = diag(2, 3, 4, 1)$, $\alpha_1 = 0.45$, $\alpha_2 = 0.65$, $\lambda_1 = 2.07$, $\lambda_2 = 2.05$. *Thus, the exponentially stable for any switching with MDADT* $\tau_{a1} \geq \tau^*_{a1} = -\frac{2\ln 2.07}{\ln 0.55} \approx 2.43$, $\tau_{a2} \geq \tau^*_{a2} = -\frac{2\ln 2.05}{\ln 0.35} \approx 1.37$. *Here, a periodic switching signal* $\sigma^2(k)$ *is adopted with* $\tau_{a1} = 3 > 2.43$, $\tau_{a2} = 2 > 1.37$, *the corresponding switching time sequence (*$k_1 = 3$, $k_2 = 5$, ..., $k_{2i-1} = 5i - 2$, $k_{2i} = 5i$, *...with* $\sigma^2(k_{2i-1}) = 1$ *and* $\sigma^2(k_{2i}) = 2$*) is easily determined for* $i = 1, 2, \ldots$. *The state response of the system converges to 0 under the switching signal* $\sigma^2(k)$ *shown in the middle of* ***Figure 2****. Other corresponding symbols and explanations are similar to Example 1.*

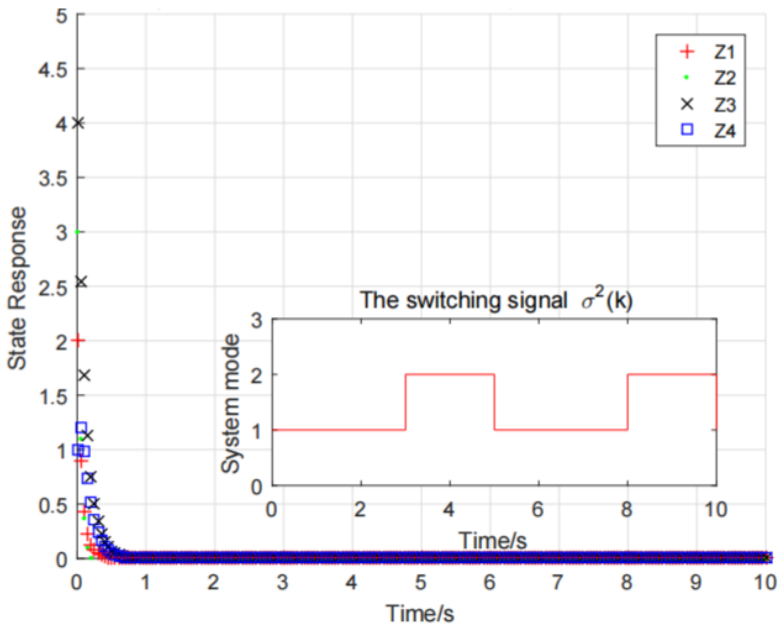

Figure 2. The state response of the system under $\sigma^2(k)$.

5. Conclusion

This paper studies the stability of TSLSs in both continuous-time and discrete-time cases. By the MLCLF approach and MDADT switching, some new stability criteria of TSLSs are obtained. The new criteria provide a larger allowable range for switching signals compared to existing literature, thus exhibiting lower conservatism. Then, we extend them to continuous-time and discrete-time cases with ADT switching and arbitrary switching as two corollaries and two remarks, respectively. Finally, two numerical examples are given to verify the availability of our results.

It is worth noting that both ADT and MDADT were initially proposed for situations where all subsystems are stable. When there exist unstable subsystems, there are mainly two ways to solve this situation. One is to design a controller for an unstable subsystem to make the closed-loop subsystem stable, and then design ADT or MDADT switching. The second is to improve MDADT by proposing a fast MDADT for handling unstable subsystems. This is one of the future research directions. On the other hand, very recently, several new switching strategies: weighted ADT [26], edge-dependent ADT [28], persistent dwell-time (PDT) [29], average PDT [30], and edge-dependent PDT [31] were proposed. All of them can obtain more general stability results than the classical ADT and MDADT. In the future, we will investigate the stability of TSLSs based on these new strategies.

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