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Article

Global martingale solutions to a stochastic superlinear cross-diffusion population system

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Abstract: In this work we try to show a martingale solution exists to a stochastic cross-diffusion population system. The transition rate is superlinear. The diffusion matrix does not satisfy the local Lipschitz property. Once the diffusion matrix does not satisfy the local Lipschitz property, we can not apply the existence and uniqueness theorem to derive approximated solutions of this stochastic population system. We have to regularize the diffusion matrix in order to apply the existence and uniqueness theorem, and this is the key idea of this work. By applying the existence and uniqueness theorem, we derive a sequence of approximated solutions. We rely on the $It\widehat{o}$ formula to estimate approximated solutions. Then we derive the tightness of the approximated sequence in a topological space, with its limit a martingale solution of a stochastic cross-diffusion system. The diffusion matrix of this stochastic cross-diffusion system is a regularization of the original diffusion matrix. The limit of this sequence of regularized diffusion matrices is the diffusion matrix of the original stochastic population system. We show that the limit of this sequence of martingale solutions is also the martingale solution of the original stochastic population system. Nonnegative property for the martingale solution is proved via a standard Stampacchia-type argument.

Keywords: martingale solutions; tightness criterion; stochastic superlinear Shigesada-Kawasaki-Teramoto type population system

1. Introduction

The cross-diffusion system is a powerful tool in describing the motions of interacting population species. A typical example is the deterministic Shigesada-Kawasaki-Teramoto population system [1]. Generally transition rates are nonlinear.

The existence of global weak solutions to deterministic cross-diffusion systems has been shown in several papers: Shigesada-Kawasaki-Teramoto population system [2–7], the Maxwell-Stefan type cross-diffusion system [8, 9], the degenerate cross-diffusion system [10, 11]. Those papers mainly adopt a so-called entropy method with an application of the Aubin-Lions Lemma. In this paper we take the random factor into consideration.

Let us consider a system of stochastic differential equations, that for $u=(u_1, \ldots, u_n)$,

$$du_i - div\left(\sum_{j=1}^{n} A_{ij}(\mathbf{u})\nabla u_j\right)dt = \sum_{j=1}^{n}\sigma_{ij}(\mathbf{u})dW_j(t) \quad in \quad \mathcal{O},\ t>0,\ 1\le i\le n, \tag{1}$$

and

$$\sum_{j=1}^{n} A_{ij}(\mathbf{u})\nabla u_j \cdot \nu = 0 \quad on \quad \partial\mathcal{O},\ t > 0,\ u_i(0) = u_i^0 \quad in \quad \mathcal{O},\ 1 \le i \le n, \tag{2}$$

where $O \subset R^d$, $d \le 3$ is a bounded domain with Lipschitz boundary. ν is an exterior unit normal vector to ∂O and u_i^0 is the initial datum. The concentrations $u_i(\omega, x, t)$ are defined on $\Omega \times O \times [0, T]$, where $\omega \in \Omega$ represents the stochastic factor, $x \in O$ the spatial variable, and $t \in [0, T]$ the time index.

External perturbations or a lack of necessary information motivate the introduction of noise terms $\sigma(\mathbf{u}) = (\sigma_{ij}(\mathbf{u}))$. The diffusion matrix $A(\mathbf{u}) = (A_{ij}(\mathbf{u}))$ is

$$A_{ii}(\mathbf{u}) = a_{i0} + (s+1)a_{ii}u_i^s + \sum_{k=1, k\neq i}^{n} a_{ik}u_k^s, \text{ and } A_{ij}(\mathbf{u}) = s a_{ij} u_i u_j^{s-1}, \text{ if } i \neq j, \tag{3}$$

with $a_{i0} > 0$, $a_{ik} > 0$ and $s > 0$.

If $s = 2$, the existence of a global martingale solution to (1)–(3) has been shown [12]. We extend the result to the case if $1 < s < 2$. We declare in the following discussion, constants $C > 0$ are independent of variables, and their values are subject to change.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a complete right continuous filtration $\mathcal{F} = (\mathcal{F}_t)_{t\ge 0}$. The space $L^2(\mathcal{O})$ comprises of square integrable functions $u : \mathcal{O} \to R$, and for $u, v \in L^2(\mathcal{O})$, $\langle u, v\rangle = \int_{\mathcal{O}} uvdx$. The space $H^1(\mathcal{O})$ comprises of $u \in L^2(\mathcal{O})$, and the distributional derivatives $\partial u/\partial x_1, \ldots, \partial u/\partial x_n$ belong to $L^2(\mathcal{O})$. Let H be a Hilbert space, $L^2(\Omega; H)$ refers to the space of all H-valued random variables u with $E\|u\|_H^2 = \int_\Omega \|u(\omega)\|_H^2 P(d\omega) < \infty$.

The $L^2(\mathcal{O})$ norm of the vector $\mathbf{u} = (u_1, \ldots, u_n)$ is defined as $\|\mathbf{u}\|^2_{L^2(\mathcal{O})} = \sum_{i=1}^n \|u_i\|^2_{L^2(\mathcal{O})}$. We fix a Hilbert basis $(e_k)_{k\in N}$ to the space $L^2(\mathcal{O})$. For any separable Hilbert space Y with orthonormal basis $(\eta_k)_{k\in N}$, we denote

$$L(Y; L^2(\mathcal{O})) = \left\{ L : Y \to L^2(\mathcal{O}) \text{ linear continuous} : \sum_{k=1}^{\infty} \|L\eta_k\|^2_{L^2(\mathcal{O})} < \infty \right\}$$

as the space of Hilbert-Schmidt operators from Y to $L^2(\mathcal{O})$. The norm of the space is defined as $\|L\|^2_{L(Y;L^2(\mathcal{O}))} = \sum_{k=1}^{\infty} \|L\eta_k\|^2_{L^2(\mathcal{O})}$.

Let $(\beta_{jk})_{j=1,\ldots,n,k\in N}$ be a sequence of independent one-dimensional Brownian motions and for $j = 1, \ldots, n$,

$$\begin{aligned} W_j(x,t,\omega) &= \sum_{k\in\mathbb{N}} \eta_k(x)\beta_{jk}(t,\omega), \\ \sigma_{ij}(\mathbf{u})dW_j(t) &= \sum_{k,l\in\mathbb{N}} \sigma_{ij}^{kl}(\mathbf{u}) e_l d\beta_{jk}(t), \\ \text{with } \sigma_{ij}^{kl}(\mathbf{u}) &= (\sigma_{ij}(\mathbf{u})\eta_k, e_l)_{L^2(\mathcal{O})}, \end{aligned} \tag{4}$$

and $\mathbf{W} = (W_1, \ldots, W_n)$ takes values in another separable Hilbert space Y_0 that $Y \subset Y_0$.

Let us give assumptions on multiplicative noise terms. Noise terms $\sigma = \sigma_{ij}(\mathbf{u})$

are $B(L^2(O) \otimes [0,T] \otimes F; B(L(Y; L^2(O))))$-measurable and F-adapted. For every $\mathbf{u}, \mathbf{v} \in L^2(O)$, $1 \leq i, j \leq n$, we have

$$\begin{aligned} &\|\sigma_{ij}(\mathbf{u})\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))} \leq C(1 + \|\mathbf{u}\|^2_{L^2(\mathcal{O})}), \\ &\|\sigma_{ij}(\mathbf{u}) - \sigma_{ij}(\mathbf{v})\|_{\mathcal{L}(Y;L^2(\mathcal{O}))} \leq C\|\mathbf{u} - \mathbf{v}\|_{L^2(\mathcal{O})}, \\ &\sum_{j=1}^{n} \|\sigma_{ij}(\mathbf{u})\|_{\mathcal{L}(Y;L^2(\mathcal{O}))} \leq C\|u_i\|_{L^2(\mathcal{O})}. \end{aligned} \tag{5}$$

Let us give the definition of the solution to (1)–(3).

Definition 1. Let $T > 0$, the system $(\widetilde{U}, \widetilde{\mathbf{W}}, \widetilde{\mathbf{u}})$ is a global martingale solution to (1)–(3) if $\widetilde{U} = (\widetilde{\Omega}, \widetilde{F}, \widetilde{P}, \widetilde{F})$ is a stochastic basis with filtration $\widetilde{F} = (\widetilde{F}_t)_{t\in(0,T)}$, $\widetilde{\mathbf{W}}$ is a cylindrical Wiener process, $\widetilde{\mathbf{u}}(t) = (\widetilde{u}_1(t), \ldots, \widetilde{u}_n(t))$ is an $\widetilde{F}_t$-adapted stochastic process for every $t \in (0,T)$, and for $1 \leq i \leq n$,

$$\widetilde{u}_i \in L^2(\widetilde{\Omega}; C^0([0,T]; L^2_\omega(O))) \cap L^2(\widetilde{\Omega}; L^2(0,T; H^1(O))).$$

The law of $\widetilde{u}_i(0)$ is identical to u_i^0, $\widetilde{\mathbf{u}}$ satisfies that for every $\phi_i \in H^1(O)$,

$$\langle \widetilde{u}_i(t), \phi_i\rangle = \langle \widetilde{u}_i(0), \phi_i\rangle + \sum_{j=1}^{n} \int_0^t \langle \mathrm{div}(A_{ij}(\widetilde{\mathbf{u}}(r))\nabla \widetilde{u}_j(r)), \phi_i\rangle dr + \langle \sum_{j=1}^{n} \int_0^t \sigma_{ij}(\widetilde{\mathbf{u}}(r)) d\widetilde{W}_j(r), \phi_i\rangle.$$

The topological space $C^0([0,T]; L^2_\omega(\mathcal{O}))$ represents all weakly continuous functions $u : [0,T] \to L^2(O)$ such that $\sup_{0<t<T} \|u(t)\|_{L^2(O)} < \infty$. In Section 5, several topological spaces will be introduced.

The major assumption is given in (6). There exists a sequence of positive real numbers $\pi = (\pi_1, \ldots, \pi_n)$ such that

$$\pi_i a_{ij} = \pi_j a_{ji}, \quad (s+1)a_{ii} > \frac{s^2}{4} \sum_{j=1, j\neq i}^{n} a_{ij}, \text{ for every } 1 \leq i, j \leq n. \tag{6}$$

The relation $\pi_i a_{ij} = \pi_j a_{ji}$ is called the “detailed balance” condition. Assumption (6) also requires that self-diffusion “dominates” the cross-diffusion factor, which is called the “weak cross-diffusion” condition. How to show a martingale solution exists for (a_{ij}) weaker than (6) is a big challenge and remains the central topic for future investigations.

For each vector $\mathbf{u} = (u_1, \ldots, u_n)$, we adopt the notation $\mathbf{v} = \mathbf{u}^s$, $\mathbf{v} = (v_1, \ldots, v_n)$, such that $v_i = u_i^s$. The final assumption is given as

$$u_i^0 \geq 0 \; a.e. \; in \; \mathcal{O}, \; \mathbb{P} - a.s. \quad \mathbb{E}\|(\mathbf{u}^0)^{\frac{s}{2}}\|^p_{L^2(\mathcal{O})} < \infty, \quad p = \frac{24}{4-d}, \quad d \leq 3. \tag{7}$$

We notice that in uniform estimates Lemma 5, higher order moment estimates are required, and assumption (7) plays a major role during estimation. After these preparations, let us state the main theorem.

Theorem 1. *Let $T > 0$, and let $\sigma = (\sigma_{ij})_{i,j=1}^n$ with $\sigma_{ij} : L^2(O) \times [0,T] \times \Omega \to L(Y; L^2(O))$. If (5), (6) and (8) hold, $1 < s < 2$, then a global martingale solution exists to (1)–(3), which is non-negative $\widetilde{P}$-a.s.*

2. Ideas of the proof

We adopt the Galerkin approximation method, by fixing an orthonormal basis $(e_k)_{k\geq 1}$ of $L^2(O)$. For every $N \in$ N, the space $H_N = span\{e_1, \ldots, e_N\}$ satisfies $H_N \subset H^1(O) \cap L^\infty(O)$. We introduce the projection operator $\Pi_N : L^2(O) \to H_N$, with

$$\Pi_N(\mathbf{v}) = \sum_{i=1}^{N} \langle \mathbf{v}, e_i \rangle e_i, \quad \mathbf{v} \in L^2(O).$$

The existence and uniqueness theorem for a stochastic differential equation has been applied to derive approximated solutions [12]. This existence and uniqueness theorem is Theorem 3.1.1 [13], with its original edition [14]. It has also been cited as Theorem 22 [12]. A key assumption to apply this theorem is the so-called local weak monotonicity condition.

Once $1 < s < 2$, the diffusion matrix does not satisfy the local Lipschitz property, and we are not able to verify the local weak monotonicity condition, in order to apply this existence and uniqueness theorem. This is the main difficulty in showing a martingale solution exists for (1)–(3), if $1 < s < 2$.

In this situation, we consider a regularization of the diffusion matrix in (3). Let us denote $\chi_j(K, \mathbf{u}) = 1$ if $|u_j| \geq \frac{1}{K}$, and $\chi_j(K, \mathbf{u}) = K|u_j|$, if $|u_j| < \frac{1}{K}$, $K \in N$. We also denote $P = \text{diag}(\pi_1, \ldots, \pi_n)$ a diagonal matrix, $a = (a_1, \ldots, a_n)$, $b = (b_{ij})_{1\leq i,j\leq n}$, and consider

$$Pd\mathbf{u} = a(K, \mathbf{u})dt + b(\mathbf{u})d\mathbf{W}(t), \quad t > 0, \quad u_i(0) = |\Pi_N(u_i^0)|, \quad 1 \leq i \leq n, \tag{8}$$

where

$$a_i(K, \mathbf{u}) = \Pi_N div\left(\sum_{j=1}^{n} \pi_i M_{ij}(K, \mathbf{u})\nabla u_j\right), \quad b_{ij}(\mathbf{u}) = \pi_i \Pi_N \sigma_{ij}(\mathbf{u}). \tag{9}$$

The diffusion matrix $M(K, \mathbf{u}) = (M_{ij}(K, \mathbf{u}))$ is

$$\begin{aligned} M_{ij}(K, \mathbf{u}) &= a_{i0} + (s+1)a_{ii}|u_i|^s + \sum_{k=1,k\neq i}^{n} a_{ik}|u_k|^s, && \text{if } i = j, \\ M_{ij}(K, \mathbf{u}) &= s a_{ij} u_i \cdot |u_j|^{s-1}\chi_j(K, \mathbf{u}), && \text{if } i \neq j. \end{aligned} \tag{10}$$

We remark that $\Pi_N(\mathbf{u}^0)$ may have no sign, so we instead consider the initial value $u_i(0) = |\Pi_N(u_i^0)|$, and the Hilbert-Schmidt operator $\sigma_{ij}(\mathbf{u})$, $1 \leq i, j \leq n$ has been projected. In Lemma 2, we show that $M(K, \mathbf{u}) = (M_{ij}(K, \mathbf{u}))$ satisfies the local Lipschitz property, which allows us to show a unique strong (in the probabilistic sense) solution $\mathbf{u}^{(N)}(K, t)$ exists to (8)–(10), i.e.

$$\pi_i u_i^{(N)}(K, t) = \pi_i u_i^{(N)}(0) + \int_0^t a_i(K, \mathbf{u}^{(N)}(K, r))dr + \int_0^t \sum_{j=1}^{n} b_{ij}(\mathbf{u}^{(N)}(K, r))dW_j(r).$$

If $\mathbf{u}^{(N)}(K, t)$ is non-negative P-a.s. then for

$$Pd\mathbf{u} = a'(K,\mathbf{u})dt + b(\mathbf{u})d\mathbf{W}(t), \quad t>0, \quad u_i(0) = |\Pi_N(u_i^0)|, \quad 1 \le i \le n, \tag{11}$$

where $a' = (a_1', \dots, a_n')$,

$$a_i'(K,\mathbf{u}) = \Pi_N div \left(\sum_{j=1}^n \pi_i A_{ij}(K,\mathbf{u}) \nabla u_j \right), \quad b_{ij}(\mathbf{u}) = \pi_i \Pi_N \sigma_{ij}(\mathbf{u}), \tag{12}$$

and the diffusion matrix $A(K,\mathbf{u}) = (A_{ij}(K,\mathbf{u}))$ is

$$\begin{aligned} A_{ij}(K,\mathbf{u}) &= a_{i0} + (s+1)a_{ii}u_i^s + \sum_{k=1,k\neq i}^n a_{ik}u_k^s, \quad &&\text{if } i=j, \\ A_{ij}(K,\mathbf{u}) &= sa_{ij}u_iu_j^{s-1}\chi_j(K,\mathbf{u}), &&\text{if } i \neq j, \end{aligned} \tag{13}$$

$\mathbf{u}^{(N)}(K,t)$ is also a strong solution (in the probabilistic sense) to (11)–(13).

In this work, we let $N \to \infty$ first to derive a sequence of approximated solutions indexed by $K \in N$. Then we let $K \to \infty$, and show that the sequence indexed by $K \in N$ converges to a martingale solution of (1)–(3).

Proving strategies [12,15,16]: Firstly, we prove the existence of a unique strong (in the probabilistic sense) solution to (8)–(10), by applying the existence and uniqueness result for a stochastic differential equation. Then by a standard Stampacchia-type argument [17], we derive that this strong (in the probabilistic sense) solution is non-negative, P-a.s.

The second step is uniform estimates. Then by some fundamental tools in stochastic analysis [12, 18–20], we find another sequence possessing same laws to the existing one, and show that the approximated sequence indeed converges to a martingale solution of (1)–(3).

In this work, we have shown that once the power of transition rate s satisfies that $1 < s < 2$, then a martingale solution exists to (1)–(3). For the general $s > 0$ case, the situation might be different.

We rely on Lemma 1 to estimate approximated solutions, and we notice that Lemma1 does not hold if $0 < s < 1$.

Though Lemma 1 holds if $s > 2$, we do not extend the main result to the case when $s > 2$ The stochastic Galerkin method can not be applied if the transformation between variables is nonlinear.

In Lemma 5, we have only considered linear transformations between variables. Once $s > 2$, we may not be able to derive strong enough estimations by merely relying on linear transformations. So we mainly consider the case when $1 < s < 2$ in this paper.

3. Stochastic Galerkin approximation

We first of all consider an important matrix analysis result.

Lemma 1. *For every* $\mathbf{z} = (z_1, z_2, \dots, z_n) \in R^n$ *and* $\mathbf{u} = (u_1, u_2, \dots, u_n) \in R^n$, *there exist constants* $\alpha_1 > 0$, $\alpha_2 > 0$ *such that*

$$\sum_{i,j=1}^{n} \pi_i M_{ij}(K,\mathbf{u}) z_i z_j \geq \alpha_1 \sum_{i=1}^{n} z_i^2 + \alpha_2 \sum_{i=1}^{n} |u_i|^s z_i^2.$$

Proof. We define a matrix $\bar{M}(K,\mathbf{u}) = (\bar{M}_{ij}(K,\mathbf{u}))$, with $\bar{M}_{ii}(K,\mathbf{u}) = \frac{s^2}{4}\sum_{k=1,k\neq i}^{n} a_{ik}|u_i|^s + \sum_{k=1,k\neq i}^{n} a_{ik}|u_k|^s = \sum_{k=1,k\neq i}^{n}\left(\frac{s^2}{4}a_{ik}|u_i|^s + a_{ik}|u_k|^s\right)$, and $\bar{M}_{ij}(K,\mathbf{u}) = M_{ij}(K,\mathbf{u})$ if $i \neq j$.

By assumption (6), $(s+1)a_{ii} > \frac{s^2}{4}\sum_{k=1,k\neq i}^{n} a_{ik}$, there exist positive constants $\{\beta_i\}_{i=1,\ldots,n}$, such that for every $\pi_i > 0$, $\pi_i M_{ii}(K,\mathbf{u}) - \pi_i \bar{M}_{ii}(K,\mathbf{u}) \geq \pi_i a_{i0} + \beta_i |u_i|^s$, and

$$\sum_{i,j=1}^{n} \pi_i M_{ij}(K,\mathbf{u}) z_i z_j \geq \sum_{i,j=1}^{n} \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j + \sum_{i=1}^{n} \pi_i a_{i0} z_i^2 + \sum_{i=1}^{n} \beta_i |u_i|^s z_i^2. \tag{14}$$

Provided that $\sum_{i,j=1}^{n} \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j \geq 0$, then by (14), we have

$$\sum_{i,j=1}^{n} \pi_i M_{ij}(K,\mathbf{u}) z_i z_j \geq \sum_{i=1}^{n} \pi_i a_{i0} z_i^2 + \sum_{i=1}^{n} \beta_i |u_i|^s z_i^2,$$

choose $\alpha_1 = \min\{\pi_i a_{i0} : 1 \leq i \leq n\}$, $\alpha_2 = \min\{\beta_i : 1 \leq i \leq n\}$, we can show this lemma.

We are left to show that $\sum_{i,j=1}^{n} \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j \geq 0$. Once we denote \$$\bar{M}_{ii}^k(K,\mathbf{u}) = \frac{s^2}{4} a_{ik}|u_i|^s + a_{ik}|u_k|^s, k \neq i$, then $\bar{M}_{ii}(K,\mathbf{u}) = \sum_{k=1,k\neq i}^{n} \bar{M}_{ii}^k(K,\mathbf{u})$, and therefore,

$$\begin{aligned}
&\sum_{i,j=1}^{n} \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j = \sum_{i=1}^{n} \pi_i \bar{M}_{ii}(K,\mathbf{u}) z_i^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j \\
&= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} (\pi_i \bar{M}_{ii}^j(K,\mathbf{u}) z_i^2 + \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j) \\
&= \sum_{i=1}^{n}\sum_{j=1,j<i}^{n} (\pi_i \bar{M}_{ii}^j(K,\mathbf{u}) z_i^2 + \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j) + \sum_{i=1}^{n}\sum_{j=1,j>i}^{n} (\pi_i \bar{M}_{ii}^j(K,\mathbf{u}) z_i^2 + \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j) \\
&= \sum_{i=1}^{n}\sum_{j=1,j<i}^{n} (\pi_i \bar{M}_{ii}^j(K,\mathbf{u}) z_i^2 + \pi_i \bar{M}_{ij}(K,\mathbf{u}) z_i z_j) + \sum_{i=1}^{n}\sum_{j=1,j<i}^{n} (\pi_j \bar{M}_{jj}^i(K,\mathbf{u}) z_j^2 + \pi_j \bar{M}_{ji}(K,\mathbf{u}) z_j z_i) \\
&= \sum_{i=1}^{n}\sum_{j=1,j<i}^{n} \left[\pi_i \bar{M}_{ii}^j(K,\mathbf{u}) z_i^2 + (\pi_i \bar{M}_{ij}(K,\mathbf{u}) + \pi_j \bar{M}_{ji}(K,\mathbf{u})) z_i z_j + \pi_j \bar{M}_{jj}^i(K,\mathbf{u}) z_j^2\right].
\end{aligned}$$

Let us show that if $i \neq j$, either $u_i \neq 0$ or $u_j \neq 0$, then $\pi_i \bar{M}_{ii}^j(K,\mathbf{u}) z_i^2 + (\pi_i \bar{M}_{ij}(K,\mathbf{u}) + \pi_j \bar{M}_{ji}(K,\mathbf{u})) z_i z_j + \pi_j \bar{M}_{jj}^i(K,\mathbf{u}) z_j^2 \geq 0$, which is equivalent to show

$$(\pi_i \bar{M}_{ij}(K,\mathbf{u}) + \pi_j \bar{M}_{ji}(K,\mathbf{u}))^2 \leq 4\pi_i \bar{M}_{ii}^j(K,\mathbf{u}) \pi_j \bar{M}_{jj}^i(K,\mathbf{u}). \tag{15}$$

We notice that

$$\pi_i \bar{M}_{ij}(K,\mathbf{u}) + \pi_j \bar{M}_{ji}(K,\mathbf{u}) = s\pi_i a_{ij}(u_i|u_j|^{s-1}\chi_j(K,\mathbf{u}) + u_j|u_i|^{s-1}\chi_i(K,\mathbf{u})),$$

since $u_i u_j |u_i|^{s-1}|u_j|^{s-1}\chi_i(K,\mathbf{u})\chi_j(K,\mathbf{u}) \leq |u_i|^s |u_j|^s$, so

$$(\pi_i\bar{M}_{ij}(K,\mathbf{u})+\pi_j\bar{M}_{ji}(K,\mathbf{u}))^2\le\pi_i^2a_{ij}^2(s^2|u_i|^{2s-2}u_j^2+2s^2|u_i|^s|u_j|^s+s^2u_i^2|u_j|^{2s-2}).$$

Also,

$$4\pi_i\bar{M}_{ii}^j(K,\mathbf{u})\pi_j\bar{M}_{jj}^i(K,\mathbf{u})=\pi_i^2a_{ij}^2\left(s^2|u_i|^{2s}+\left(\frac{s^4}{4}+4\right)|u_i|^s|u_j|^s+s^2|u_j|^{2s}\right).$$

So long as

$$|u_i|^{2s}+|u_j|^{2s}-|u_i|^{2s-2}u_j^2-u_i^2|u_j|^{2s-2}=(|u_i|^{2s-2}-|u_j|^{2s-2})(u_i^2-u_j^2),$$

$s>1$, thus

$$|u_i|^{2s}+|u_j|^{2s}-|u_i|^{2s-2}u_j^2-u_i^2|u_j|^{2s-2}\ge0,$$

so

$$s^2|u_i|^{2s}+s^2|u_j|^{2s}\ge s^2|u_i|^{2s-2}u_j^2+s^2u_i^2|u_j|^{2s-2}.$$

The fact that \$ $\frac{s^4}{4}+4\ge2s^2$ indicates $\left(\frac{s^4}{4}+4\right)|u_iu_j|^s\ge2s^2|u_iu_j|^s$, and we conclude that (15) holds. If $i\ne j, u_i=u_j=0$, then

$$\pi_i\bar{M}_{ii}^j(K,\mathbf{u})z_i^2+(\pi_i\bar{M}_{ij}(K,\mathbf{u})+\pi_j\bar{M}_{ji}(K,\mathbf{u}))z_iz_j+\pi_j\bar{M}_{jj}^i(K,\mathbf{u})z_j^2=0,$$

and we have shown the lemma. □

Lemma 2. *The diffusion matrix $M(K,\mathbf{u})$ in (10) satisfies that for every $K\in N$, $\mathbf{y},\mathbf{z}\in R^n$, $\mathbf{y},\mathbf{z}\in H_N$ and $\|\mathbf{y}\|_{H_N},\|\mathbf{z}\|_{H_N}<R$,*

$$\|M(K,\mathbf{y})-M(K,\mathbf{z})\|_{L^2(\mathcal{O})}\le C\|\mathbf{y}-\mathbf{z}\|_{L^2(\mathcal{O})} \tag{16}$$

with $C>0$ independent of $\mathbf{y},\mathbf{z}$.

Proof. We notice that if $\|\mathbf{y}\|_{H_N},\|\mathbf{z}\|_{H_N}<R$, then for a.e. $x\in O, |y_i|,|z_i|<C$. We have

$$M_{ij}(K,\mathbf{y})-M_{ij}(K,\mathbf{z})=(s+1)a_{ii}(|y_i|^s-|z_i|^s)+\sum_{k=1,k\ne i}^{n}a_{ik}(|y_k|^s-|z_k|^s),\quad i=j,$$

$$M_{ij}(K,\mathbf{y})-M_{ij}(K,\mathbf{z})=sa_{ij}(y_i|y_j|^{s-1}\chi_j(K,\mathbf{y})-z_i|z_j|^{s-1}\chi_j(K,\mathbf{z}))$$
$$=sa_{ij}(y_i-z_i)|y_j|^{s-1}\chi_j(K,\mathbf{y})+sa_{ij}(|y_j|^{s-1}\chi_j(K,\mathbf{y})-|z_j|^{s-1}\chi_j(K,\mathbf{z}))z_i,\quad i\ne j.$$

The key factor in the proof is to show that

$$\big||y_j|^{s-1}\chi_j(K,\mathbf{y})-|z_j|^{s-1}\chi_j(K,\mathbf{z})\big|\le C|y_j-z_j|,$$

with $C>0$ independent of $\mathbf{y},\mathbf{z}$. We observe that $|y_j|^{s-1}\chi_j(K,\mathbf{y})-|z_j|^{s-1}\chi_j(K,\mathbf{z})$ can be divided into four cases: $\chi_j(K,\mathbf{y})=0$ or $\chi_j(K,\mathbf{y})=1$; $\chi_j(K,\mathbf{z})=0$ or $\chi_j(K,\mathbf{z})=1$. Let us present the key computation for each case.

(i) If $\frac{1}{K} \leq |y_j|, |z_j| < C$, there exists ρ_j, with

$$\frac{1}{K} \leq \min\{|y_j|, |z_j|\} \leq \rho_j \leq \max\{|y_j|, |z_j|\} < C,$$

such that

$$\left||y_j|^{s-1}\chi_j(K,\mathbf{y}) - |z_j|^{s-1}\chi_j(K,\mathbf{z})\right| = \left||y_j|^{s-1} - |z_j|^{s-1}\right| = (s-1)\left|\rho_j^{s-2}(|y_j| - |z_j|)\right| \leq C|y_j - z_j|.$$

(ii) If $|y_j|, |z_j| < \frac{1}{K}$, there exists ζ_j, with

$$0 \leq \zeta_j \leq \max\{|y_j|, |z_j|\} \leq \frac{1}{K},$$

such that

$$\left||y_j|^{s-1}\chi_j(K,\mathbf{y}) - |z_j|^{s-1}\chi_j(K,\mathbf{z})\right| = K\left||y_j|^s - |z_j|^s\right| = Ks\left|\zeta_j^{s-1}(|y_j| - |z_j|)\right| \leq C|y_j - z_j|.$$

(iii) If $|y_j| < \frac{1}{K}, \frac{1}{K} \leq |z_j| < C$, then we have

$$2K|y_j|^s \leq |z_j|^{s-1}(K|z_j| + 1),$$

Then

$$2K|y_j|^s|z_j|^{s-1}(K|z_j| - 1) \leq |z_j|^{2s-2}(K^2|z_j|^2 - 1),$$

Thus

$$K^2|y_j|^{2s} - 2K|y_j|^s|z_j|^{s-1} + |z_j|^{2s-2} \leq K^2|y_j|^{2s} - 2K^2|y_j|^s|z_j|^s + K^2|z_j|^{2s}.$$

There exists κ_j, with $0 \leq \kappa_j < C$, such that
$||y_j|^s - |z_j|^s| = s|_j^{s-1}(|y_j| - |z_j|)| \leq C|y_j - z_j|$,
And

$$\left||y_j|^{s-1}\chi_j(K,\mathbf{y}) - |z_j|^{s-1}\chi_j(K,\mathbf{z})\right| = \left|K|y_j|^s - |z_j|^{s-1}\right| \leq K\left||y_j|^s - |z_j|^s\right| \leq C|y_j - z_j|.$$

Combining these computations, we derive that

$$\left||y_j|^{s-1}\chi_j(K,\mathbf{y}) - |z_j|^{s-1}\chi_j(K,\mathbf{z})\right| \leq C|y_j - z_j|.$$

We notice that actually there exists another case: $|z_j| < \frac{1}{K}, \frac{1}{K} \leq |y_j| < C$. Its discussion is completely identical to (iii), and we omit it.

For every $|y_i|, |z_i| < C$, there exists λ_i, with $0 \leq \lambda_i < C$, that

$$\left||y_i|^s - |z_i|^s\right| = s\left|\lambda_i^{s-1}(|y_i| - |z_i|)\right| \leq C\left||y_i| - |z_i|\right| \leq C|y_i - z_i|,$$

which follows if i=j,

$$
\begin{aligned}
&\|M_{ij}(K,\mathbf{y})-M_{ij}(K,\mathbf{z})\|^2_{L^2(\mathcal{O})}\\
&\quad=\left\|(s+1)a_{ii}(|y_i|^s-|z_i|^s)+\sum_{k=1,k\neq i}^{n}a_{ik}(|y_k|^s-|z_k|^s)\right\|^2_{L^2(\mathcal{O})}\\
&\quad\le C\left(\||y_i|^s-|z_i|^s\|^2_{L^2(\mathcal{O})}+\sum_{k=1,k\neq i}^{n}\||y_k|^s-|z_k|^s\|^2_{L^2(\mathcal{O})}\right)\\
&\quad\le C\sum_{i=1}^{n}\|y_i-z_i\|^2_{L^2(\mathcal{O})}=C\|\mathbf{y}-\mathbf{z}\|^2_{L^2(\mathcal{O})},
\end{aligned}
\tag{17}
$$

and if $i\neq j$, since $|\chi_j(K,\mathbf{y})|\le 1$, then $\||y_j|^{s-1}\chi_j(K,\mathbf{y})\|_{L^\infty(O)}<C$, and

$$
\begin{aligned}
&\|M_{ij}(K,\mathbf{y})-M_{ij}(K,\mathbf{z})\|^2_{L^2(\mathcal{O})}\\
&\le C\left\|(y_i-z_i)|y_j|^{s-1}\chi_j(K,\mathbf{y})+(|y_j|^{s-1}\chi_j(K,\mathbf{y})-|z_j|^{s-1}\chi_j(K,\mathbf{z}))z_i\right\|^2_{L^2(\mathcal{O})}\\
&\le C\left(\left\|(y_i-z_i)|y_j|^{s-1}\chi_j(K,\mathbf{y})\right\|^2_{L^2(\mathcal{O})}+\left\|(|y_j|^{s-1}\chi_j(K,\mathbf{y})-|z_j|^{s-1}\chi_j(K,\mathbf{z}))z_i\right\|^2_{L^2(\mathcal{O})}\right)\\
&\le C\left\||y_j|^{s-1}\chi_j(K,\mathbf{y})-|z_j|^{s-1}\chi_j(K,\mathbf{z})\right\|^2_{L^2(\mathcal{O})}\|z_i\|^2_{L^\infty(\mathcal{O})}\\
&+C\left\||y_j|^{s-1}\chi_j(K,\mathbf{y})\right\|^2_{L^\infty(\mathcal{O})}\|y_i-z_i\|^2_{L^2(\mathcal{O})}\le C\sum_{i=1}^{n}\|y_i-z_i\|^2_{L^2(\mathcal{O})}=C\|\mathbf{y}-\mathbf{z}\|^2_{L^2(\mathcal{O})}.
\end{aligned}
\tag{18}
$$

By (17)–(18), we can show that (16) holds and we finish the proof of this lemma. □

After we have shown the preparation Lemma 2, we apply the existence and uniqueness result in Lemma 3.

Lemma 3. *For every $T>0$, there exists a unique strong (in the probabilistic sense) solution $\mathbf{u}^{(N)}(K,t)\in H_N$, $0<t<T$ to (8)–(10), P-a.s.*

Proof. Let $R>0$, $T>0$, $\omega\in\Omega$ and let $\mathbf{y}=(y_1,\dots,y_n)\in R^n$, $\mathbf{z}=(z_1,\dots,z_n)\in R^n$, $\mathbf{y},\mathbf{z}\in H_N$ with $\|\mathbf{y}\|_{H_N},\|\mathbf{z}\|_{H_N}\le R$. By Lemma 1, $PM(K,\mathbf{y})$, $\mathbf{y}\in R^n$ is positive semi-definite.

Norms are equivalent in finite dimensional spaces, so $\|\nabla(y_i-z_i)\|_{L^2(O)}\le\|y_i-z_i\|_{H^1(O)}\le C\|y_i-z_i\|_{H_N}\le C\|\mathbf{y}-\mathbf{z}\|_{H_N}$. By Lemma 2, we derive that

$$
\begin{aligned}
&\langle a(K,\mathbf{y})-a(K,\mathbf{z}),\mathbf{y}-\mathbf{z}\rangle=-\sum_{i,j=1}^{n}\int_{\mathcal{O}}\pi_iM_{ij}(K,\mathbf{y})\nabla(y_i-z_i)\cdot\nabla(y_j-z_j)\,dx\\
&\quad+\sum_{i,j=1}^{n}\int_{\mathcal{O}}\pi_i(M_{ij}(K,\mathbf{z})-M_{ij}(K,\mathbf{y}))\nabla(y_i-z_i)\cdot\nabla z_j\,dx\\
&\le C\sum_{i,j=1}^{n}\|M_{ij}(K,\mathbf{y})-M_{ij}(K,\mathbf{z})\|_{L^2(\mathcal{O})}\|\nabla(y_i-z_i)\|_{L^2(\mathcal{O})}\|\nabla z_j\|_{L^\infty(\mathcal{O})}\\
&\le C\sum_{i=1}^{n}\|y_i-z_i\|^2_{H_N}\le C\|\mathbf{y}-\mathbf{z}\|^2_{H_N},
\end{aligned}
$$

and $\|\mathbf{b}(\mathbf{y})-\mathbf{b}(\mathbf{z})\|^2_{L(Y;H_N)}\le C\|\sigma(\mathbf{y})-\sigma(\mathbf{z})\|^2_{L(Y;H_N)}\le C\|\mathbf{y}-\mathbf{z}\|^2_{H_N}$. For the weak coercivity condition, we take $\mathbf{y} \in H_N$ with $\|\mathbf{y}\|_{H_N} \le R$, and

$$\langle a(K,\mathbf{y}),\mathbf{y}\rangle + \|b(\mathbf{y})\|^2_{\mathcal{L}(Y;H_N)} = -\sum_{i,j=1}^{n}\int_{\mathcal{O}} \pi_i M_{ij}(K,\mathbf{y})\nabla y_i \cdot \nabla y_j\,dx + \|P\sigma(\mathbf{y})\|^2_{\mathcal{L}(Y;H_N)} \le C(1+\|\mathbf{y}\|^2_{H_N}).$$

The existence and uniqueness result in Theorem 3.1.1 [13] indicates that for every $N \in$ N, a unique strong (in the probabilistic sense) solution $\mathbf{u}^{(N)}(K,t)$ to (8)–(10) exists. □

The proof of Lemma 4 [12, 17]: We replace the diffusion matrix $M(K,\mathbf{u}) = (M_{ij}(K,\mathbf{u}))$ in (10) by

$$\begin{aligned} M^+_{ij}(K,\mathbf{u}) &= a_{i0} + (s+1)a_{ii}|u_i|^s + \sum_{k=1,k\neq i}^{n} a_{ik}|u_k|^s, \quad \text{if } i=j,\\ M^+_{ij}(K,\mathbf{u}) &= s a_{ij} u_i^+ \cdot |u_j|^{s-1}\chi_j(K,\mathbf{u}), \qquad \text{if } i\neq j, \end{aligned} \tag{19}$$

where $z^+ = \max\{0,z\}$ is the positive part of $z \in R$. Rest procedures for this Stampacchia truncation method [17] and Section 2.6. [12].

Lemma 4. *For every $1 \le i \le n$, $u_i^{(N)}(K,t) \ge 0$ in O, for a.e. $t \in [0,T]$, P-a.s. $\mathbf{u}^{(N)}(K,t)$ is also a strong (in the probabilistic sense) solution to (11)–(13).*

Proof. The idea of proof is to approximate the function $f(z) = z^- = \max\{0,-z\}, z \in R$, then we use the $It\hat{o}$ formula. We define as in Section 2.4. [17] the following functions:

for $\varepsilon > 0$,

$$f_\varepsilon(z) = -z, \quad \text{if } z \le -\varepsilon,$$

and

$$f_\varepsilon(z) = -3\left(\frac{z}{\varepsilon}\right)^4 z - 8\left(\frac{z}{\varepsilon}\right)^3 z - 6\left(\frac{z}{\varepsilon}\right)^2 z, \quad \text{if } -\varepsilon \le z \le 0,$$

and

$$f_\varepsilon(z) = 0, \quad \text{if } z \ge 0.$$

Then f_ε has at most linear growth, i.e. $|f_\varepsilon(z)| \le C|z|$ for every $z \in R$. f'_ε and $\psi_\varepsilon = f_\varepsilon f''_\varepsilon + (f'_\varepsilon)^2$ are bounded in R.

We set $F_\varepsilon(v) = \int_O f_\varepsilon(v(x))^2 dx$, for square-integrable functions $v : O \to R$. We replace diffusion coefficients $M_{ij}(K,\mathbf{u}^{(N)})$ by modified coefficients in (19). We observe that generally, $M^+_{ij}(K,\mathbf{u}) \neq M_{ij}(K,\mathbf{u})$, but if $u_i \ge 0$ for every $1 \le i \le n$, then we obtain that $M^+_{ij}(K,\mathbf{u}) = M_{ij}(K,\mathbf{u})$.

By the $It\hat{o}$ formula, we have

$$\begin{aligned}
&F_\varepsilon(u_i^{(N)}(K,t)) = F_\varepsilon(u_i^{(N)}(0)) \\
&\quad + 2\int_0^t \int_{\mathcal{O}} f_\varepsilon(u_i^{(N)}) f'_\varepsilon(u_i^{(N)}) \Pi_N \left(\sum_{j=1}^n \sigma_{ij}(\mathbf{u}^{(N)}) \right) dx\, dW_j(r) \\
&\quad - 2\int_0^t \int_{\mathcal{O}} \psi_\varepsilon(u_i^{(N)}) \sum_{j=1}^n M_{ij}^+(K,\mathbf{u}^{(N)}) \nabla u_i^{(N)} \cdot \nabla u_j^{(N)} dx\, dr \\
&\quad + \int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty \psi_\varepsilon(u_i^{(N)}) e_k e_l \sigma_{ij}^{mk}(\mathbf{u}^{(N)}) \sigma_{ij}^{ml}(\mathbf{u}^{(N)}) dx\, dr \\
&= I_{\varepsilon,0}^{(N)} + I_{\varepsilon,1}^{(N)} + I_{\varepsilon,2}^{(N)} + I_{\varepsilon,3}^{(N)},
\end{aligned} \tag{20}$$

with notations in $I_{\varepsilon,3}^{(N)}$ refer to (4).

Let us show that the integral $I_{\varepsilon,2}^{(N)}$ is non-positive. Indeed, we write

$$\begin{aligned}
I_{\varepsilon,2}^{(N)} &= -2\int_0^t \int_{\mathcal{O}} \psi_\varepsilon(u_i^{(N)}) M_{ii}^+(K,\mathbf{u}^{(N)}) |\nabla u_i^{(N)}|^2 dx\, dr \\
&\quad - 2\int_0^t \int_{\mathcal{O}} \psi_\varepsilon(u_i^{(N)}) \sum_{j\neq i} M_{ij}^+(K,\mathbf{u}^{(N)}) \nabla u_i^{(N)} \cdot \nabla u_j^{(N)} dx\, dr.
\end{aligned} \tag{21}$$

We notice that the second term on the right-hand-side of (21) vanishes since $\psi_\varepsilon(u_i^{(N)}) = 0$ if $u_i^{(N)} \geq 0$, and $M_{ij}^+(K,\mathbf{u}^{(N)}) = 0$ if $u_i^{(N)} \leq 0$. The first term on the right-hand-side of (21) is non-positive, which follows that $I_{\varepsilon,2}^{(N)} \leq 0$. Let us take expected values in (20), the stochastic integral term vanishes, and

$$\mathbb{E}F_\varepsilon(u_i^{(N)}(K,t)) \leq \mathbb{E}F_\varepsilon(u_i^{(N)}(0)) + \mathbb{E}\int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty \psi_\varepsilon(u_i^{(N)}) e_k e_l \sigma_{ij}^{mk}(\mathbf{u}^{(N)}) \sigma_{ij}^{ml}(\mathbf{u}^{(N)}) dx\, dr. \tag{22}$$

It is shown in Section 3.4. [17] that as $\varepsilon \to 0$, P-a.s.

$$\begin{aligned}
&\mathbb{E}F_\varepsilon(u_i^{(N)}(K,t)) \to \mathbb{E}\|(u_i^{(N)}(K,t))^-\|_{L^2(\mathcal{O})}^2, \mathbb{E}F_\varepsilon(u_i^{(N)}(0)) \to \mathbb{E}\|(u_i^{(N)}(0))^-\|_{L^2(\mathcal{O})}^2, \\
&\mathbb{E}\int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty \psi_\varepsilon(u_i^{(N)}) e_k e_l \sigma_{ij}^{mk}(\mathbf{u}^{(N)}) \sigma_{ij}^{ml}(\mathbf{u}^{(N)}) dx\, dr \\
&\quad \to \mathbb{E}\int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty e_k e_l \sigma_{ij}^{mk}(-(\mathbf{u}^{(N)})^-) \sigma_{ij}^{ml}(-(\mathbf{u}^{(N)})^-) dx\, dr.
\end{aligned}$$

As $\varepsilon \to 0$ in (22), we have

$$\begin{aligned}
&\mathbb{E}\|(u_i^{(N)}(K,t))^-\|_{L^2(\mathcal{O})}^2 \\
&\quad \leq \mathbb{E}\|(u_i^{(N)}(0))^-\|_{L^2(\mathcal{O})}^2 + \mathbb{E}\int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty e_k e_l \sigma_{ij}^{mk}(-(\mathbf{u}^{(N)})^-) \sigma_{ij}^{ml}(-(\mathbf{u}^{(N)})^-) dx\, dr \\
&\quad \leq \mathbb{E}\|(u_i^{(N)}(0))^-\|_{L^2(\mathcal{O})}^2 + \mathbb{E}\int_0^t \sum_{j=1}^n \|\sigma_{ij}(-(\mathbf{u}^{(N)})^-)\|_{\mathcal{L}(Y;L^2(\mathcal{O}))}^2 dr.
\end{aligned} \tag{23}$$

The first term on the right-hand-side of (23) vanishes since $u_i^{(N)}(0) = |\Pi_N(u_i^0)| \geq 0$. For the second term, by assumption (5), we derive that

$$\mathbb{E}\|(u_i^{(N)}(K,t))^-\|^2_{L^2(\mathcal{O})} \leq C\mathbb{E}\int_0^t \|(u_i^{(N)}(K,r))^-\|^2_{L^2(\mathcal{O})}dr.$$

The Gronwall lemma implies that $E\|(u_i^{(N)}(K,t))^-\|^2_{L^2(O)} = 0$ for $t \in (0,T)$. Then for every $1 \leq i \leq n$, $u_i^{(N)}(K,t) \geq 0$ in O, P-a.s. for a.e. $t \in [0,T]$, and we have shown this lemma. □

4. Energy estimates of $\mathbf{u}^{(N)}(K,t)$

Lemma 5. *For every $T > 0$, there exists a constant $C > 0$, which does not depend on N, such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left(\sup_{t\in(0,T)} \|\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right) \leq C, \tag{24}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left(\int_0^T \|\nabla\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\right) \leq C, \tag{25}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left(\int_0^T \|\nabla(\mathbf{u}^{(N)}(K,r))^s\|^2_{L^2(\mathcal{O})}dr\right) \leq C, \tag{26}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left(\int_0^T \|(\mathbf{u}^{(N)}(K,r))^s\|^3_{L^2(\mathcal{O})}dr\right) \leq C. \tag{27}$$

Proof. Let us apply the Itô formula [21,22] to the process $\mathbf{X}(t) = \mathbf{u}^{(N)}(K,t)$, $P^{\frac{1}{2}} = \text{diag}(\pi_1^{\frac{1}{2}}, \ldots, \pi_n^{\frac{1}{2}})$, $t \in [0,T]$, and

$$\frac{1}{2}\|P^{\frac{1}{2}}\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})} - \frac{1}{2}\|\Pi_N(P^{\frac{1}{2}}\mathbf{u}^0)\|^2_{L^2(\mathcal{O})} = \frac{1}{2}\int_0^t \|\Pi_N(P^{\frac{1}{2}}\sigma(\mathbf{u}^{(N)}(K,r)))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr$$
$$+ \sum_{i,j=1}^n \int_0^t \langle \Pi_N div(\pi_i A_{ij}(K,\mathbf{u}^{(N)}(K,r))\nabla u_j^{(N)}(K,r)), u_i^{(N)}(K,r)\rangle dr$$
$$+ \sum_{i,j=1}^n \int_0^t \langle \Pi_N(\pi_i\sigma_{ij}(\mathbf{u}^{(N)}(K,r)))dW_j(r), u_i^{(N)}(K,r)\rangle,$$

which implies that

$$\begin{aligned}
&\frac{1}{2}\|P^{\frac{1}{2}}\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})} - \frac{1}{2}\|\Pi_N(P^{\frac{1}{2}}\mathbf{u}^0)\|^2_{L^2(\mathcal{O})} \\
&= \frac{1}{2}\int_0^t \|\Pi_N(P^{\frac{1}{2}}\sigma(\mathbf{u}^{(N)}(K,r)))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr \\
&- \sum_{i,j=1}^n \int_0^t \langle \pi_i A_{ij}(K,\mathbf{u}^{(N)}(K,r))\nabla u_j^{(N)}(K,r), \nabla u_i^{(N)}(K,r)\rangle dr \\
&+ \sum_{i,j=1}^n \int_0^t \langle \pi_i\sigma_{ij}(\mathbf{u}^{(N)}(K,r))dW_j(r), u_i^{(N)}(K,r)\rangle.
\end{aligned} \tag{28}$$

By Lemma 1, if for every $1 \leq i \leq n$, u_i is non-negative, then for $z_i, z_j \in R$, there exist constants $\alpha_1 > 0, \alpha_2 > 0$, such that

$$\sum_{i,j=1}^{n} \pi_i A_{ij}(K,\mathbf{u}) z_i z_j = \sum_{i,j=1}^{n} \pi_i M_{ij}(K,\mathbf{u}) z_i z_j \geq \alpha_1 \sum_{i=1}^{n} z_i^2 + \alpha_2 \sum_{i=1}^{n} u_i^s z_i^2.$$

We have shown that for every $1 \leq i \leq n, N \in N, u_i^{(N)}(K,t)$ is non-negative P-a.s. thus

$$\begin{aligned}
&\sum_{i,j=1}^{n} \langle \pi_i A_{ij}(K,\mathbf{u}^{(N)}(K,r)) \nabla u_j^{(N)}(K,r), \nabla u_i^{(N)}(K,r) \rangle \\
&\quad \geq \alpha_1 \sum_{i=1}^{n} \int_{\mathcal{O}} |\nabla u_i^{(N)}(K,r)|^2 dx + \alpha_2 \sum_{i=1}^{n} \int_{\mathcal{O}} |u_i^{(N)}(K,r)|^s |\nabla u_i^{(N)}(K,r)|^2 dx \\
&\quad \geq \alpha_1 \|\nabla \mathbf{u}^{(N)}(K,r)\|_{L^2(\mathcal{O})}^2 + \alpha_2 \|\nabla (\mathbf{u}^{(N)}(K,r))^{\frac{s+2}{2}}\|_{L^2(\mathcal{O})}^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{2}\|P^{\frac{1}{2}}\mathbf{u}^{(N)}(K,t)\|_{L^2(\mathcal{O})}^2 + \alpha_1 \int_0^t \|\nabla \mathbf{u}^{(N)}(K,r)\|_{L^2(\mathcal{O})}^2 dr + \alpha_2 \int_0^t \|\nabla (\mathbf{u}^{(N)}(K,r))^{\frac{s+2}{2}}\|_{L^2(\mathcal{O})}^2 dr \\
&\quad \leq \frac{1}{2}\|P^{\frac{1}{2}}\mathbf{u}^0\|_{L^2(\mathcal{O})}^2 + \frac{1}{2}\int_0^t \|P^{\frac{1}{2}}\sigma(\mathbf{u}^{(N)}(K,r))\|_{\mathcal{L}(Y;L^2(\mathcal{O}))}^2 dr \\
&\quad + \sum_{i,j=1}^{n} \int_0^t \langle \pi_i \sigma_{ij}(\mathbf{u}^{(N)}(K,r)) dW_j(r), u_i^{(N)}(K,r) \rangle,
\end{aligned} \tag{29}$$

and for the second term on the right-hand-side of (29), by assumption (5),

$$\frac{1}{2}\int_0^t \|P^{\frac{1}{2}}\sigma(\mathbf{u}^{(N)}(K,r))\|_{\mathcal{L}(Y;L^2(\mathcal{O}))}^2 dr \leq C \int_0^T (1 + \|\mathbf{u}^{(N)}(K,r)\|_{L^2(\mathcal{O})}^2) dr.$$

For the third term on the right-hand-side of (29), the process

$$\mu^{(N)}(t) = \sum_{i,j=1}^{n} \int_0^t \langle \pi_i \sigma_{ij}(\mathbf{u}^{(N)}(K,r)) dW_j(r), u_i^{(N)}(K,r) \rangle, \quad t \in [0,T],$$

is an F_t-martingale. By the Burkholder-Davis-Gundy inequality, if we denote $\langle \mu^{(N)}(T) \rangle$ as the quadratic variation of $\mu^{(N)}(T)$, then

$$E\left(\sup_{0 \leq t \leq T} |\mu^{(N)}(t)|\right) \leq CE\left(\langle \mu^{(N)}(T) \rangle^{\frac{1}{2}}\right),$$

which follows that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]} \left| \sum_{i,j=1}^{n} \int_0^t \langle \pi_i \sigma_{ij}(\mathbf{u}^{(N)}(K,r)) dW_j(r), u_i^{(N)}(K,r) \rangle \right| \right) \\
&\quad \leq C\mathbb{E}\left(\sum_{i,j=1}^{n} \int_0^T \left(\int_{\mathcal{O}} \pi_i \sigma_{ij}(u^{(N)}(K,r)) u_i^{(N)}(K,r) dx\right)^2 dr\right)^{\frac{1}{2}} \\
&\quad \leq C\mathbb{E}\left(\sum_{i,j=1}^{n} \int_0^T \left(\int_{\mathcal{O}} \sigma_{ij}(u^{(N)}(K,r)) u_i^{(N)}(K,r) dx\right)^2 dr\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C\mathbb{E}\left(\int_0^T\left(\int_{\mathcal{O}}\sum_{i,j=1}^n\sigma_{ij}^2(u^{(N)}(K,r))dx\right)\cdot\left(\int_{\mathcal{O}}\sum_{i=1}^n(u_i^{(N)}(K,r))^2dx\right)dr\right)^{\frac{1}{2}}$$
$$=C\mathbb{E}\left(\int_0^T\|u^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}\|\sigma(u^{(N)}(K,r))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr\right)^{\frac{1}{2}}.$$

For every $\varepsilon_0>0$, by assumption (5), we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in(0,T)}\left|\sum_{i,j=1}^n\int_0^t\langle\pi_i\sigma_{ij}(u^{(N)}(K,r))dW_j(r),u_i^{(N)}(K,r)\rangle\right|\right)\\
&\leq C\mathbb{E}\left(\int_0^T\|u^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}\|\sigma(u^{(N)}(K,r))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}\right)^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left(\left(\sup_{t\in(0,T)}\|u^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)^{\frac{1}{2}}\left(\int_0^T(1+\|u^{(N)}(K,r)\|^2_{L^2(\mathcal{O})})dr\right)^{\frac{1}{2}}\right)\\
&\leq C\varepsilon_0\mathbb{E}\left(\sup_{t\in(0,T)}\|u^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)+\frac{C}{4\varepsilon_0}\left(T+\mathbb{E}\int_0^T\|u^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\right).
\end{aligned}\tag{30}$$

By (29)–(30), for every $\varepsilon_0>0$, and $t\in[0,T]$,

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left(\sup_{t\in[0,T]}\|P^{\frac{1}{2}}\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)+\alpha_1\mathbb{E}\int_0^t\|\nabla\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\\
&+\alpha_2\mathbb{E}\int_0^t\|\nabla(\mathbf{u}^{(N)}(K,r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr\\
&\leq\frac{1}{2}\mathbb{E}\|P^{\frac{1}{2}}\mathbf{u}^0\|^2_{L^2(\mathcal{O})}+C\mathbb{E}\int_0^T(1+\|\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})})dr\\
&+C\varepsilon_0\mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)+\frac{C}{4\varepsilon_0}\left(T+\mathbb{E}\int_0^T\|\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\right)\\
&\leq C+C\left(1+\frac{1}{4\varepsilon_0}\right)\mathbb{E}\int_0^T\|\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr+C\varepsilon_0\mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right),
\end{aligned}$$

thus we conclude that there exist $\alpha_1>0$, $\alpha_2>0$, $\alpha_3>0$, that

$$\begin{aligned}
&\alpha_3\mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)+\alpha_1\mathbb{E}\int_0^t\|\nabla\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\\
&+\alpha_2\mathbb{E}\int_0^t\|\nabla(\mathbf{u}^{(N)}(K,r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr\\
&\leq C+C\left(1+\frac{1}{4\varepsilon_0}\right)\mathbb{E}\int_0^T\left(\sup_{\tau\in[0,r]}\|\mathbf{u}^{(N)}(K,\tau)\|^2_{L^2(\mathcal{O})}\right)dr\\
&+C\varepsilon_0\mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right).
\end{aligned}$$

Let us choose this ε_0 such that $C\varepsilon_0<\alpha_3$, then for some positive constants $C_0,C_1,\alpha_1,\alpha_2,\alpha_3$, we have

$$\begin{aligned}
&\alpha_3\mathbb{E}\left(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)+\alpha_1\mathbb{E}\int_0^t\|\nabla\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\\
&+\alpha_2\mathbb{E}\int_0^t\|\nabla(\mathbf{u}^{(N)}(K,r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr\leq C_0+C_1\mathbb{E}\int_0^T\left(\sup_{\tau\in[0,r]}\|\mathbf{u}^{(N)}(K,\tau)\|^2_{L^2(\mathcal{O})}\right)dr,
\end{aligned}\tag{31}$$

and by the Gronwall lemma,

$$\mathbb{E}\left(\sup_{t\in(0,T)}\|\mathbf{u}^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)\leq\frac{C_0}{\alpha_3}\left(1+\frac{C_1T}{\alpha_3}e^{\frac{C_1T}{\alpha_3}}\right), \tag{32}$$

with all constants $C_0, C_1, \alpha_1, \alpha_2, \alpha_3$ independent of N. Then we have shown (24). By (24) and (31), we can show that (25) holds, and

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left(\int_0^T\|\nabla(\mathbf{u}^{(N)}(K,r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr\right)<C. \tag{33}$$

In order to show (26), we notice that since $1<s<2$, then

$$\int_0^T\|\nabla(\mathbf{u}^{(N)}(K,r))^s\|^2_{L^2(\mathcal{O})}dr\leq\int_0^T\|\nabla\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr+\int_0^T\|\nabla(\mathbf{u}^{(N)}(K,r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr,$$

which follows (combining 33)

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left(\int_0^T\|\nabla(\mathbf{u}^{(N)}(K,r))^s\|^2_{L^2(\mathcal{O})}dr\right)\leq\sup_{N\in\mathbb{N}}\mathbb{E}\left(\int_0^T\|\nabla\mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\right)$$
$$+\sup_{N\in\mathbb{N}}\mathbb{E}\left(\int_0^T\|\nabla(\mathbf{u}^{(N)}(K,r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr\right)<C,$$

and we have shown that (26) holds.

In order to show (27), still relying on $1<s<2$, then by (24), we have

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left(\sup_{t\in(0,T)}\|(\mathbf{u}^{(N)}(K,t))^{\frac{s}{2}}\|^2_{L^2(\mathcal{O})}\right)\leq C. \tag{34}$$

We also need a higher order moment estimate, which is (35). The proof of (35) can be referred to Lemma 6 [12]. Let $p=\frac{24}{4-d}$ and by assumption (7), $\mathbb{E}\|(\mathbf{u}^0)^{\frac{s}{2}}\|^p_{L^2(\mathcal{O})}<\infty$, then

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left(\sup_{t\in(0,T)}\|(\mathbf{u}^{(N)}(K,t))^{\frac{s}{2}}\|^p_{L^2(\mathcal{O})}\right)\leq C. \tag{35}$$

Let us choose $\theta=\frac{d}{d+2}$, by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
&\mathbb{E}\int_0^T\|(\mathbf{u}^{(N)}(K,r))^s\|^3_{L^3(\mathcal{O})}dr\\
&\leq C\mathbb{E}\int_0^T\|\nabla(u^{(N)}(K,r))^s\|^{2\theta}_{L^2(\mathcal{O})}\|(u^{(N)}(K,r))^s\|^{2(1-\theta)}_{L^1(\mathcal{O})}dr\\
&=C\mathbb{E}\int_0^T\|\nabla(u^{(N)}(K,r))^s\|^{\frac{2d}{d+2}}_{L^2(\mathcal{O})}\|(u^{(N)}(K,r))^s\|^{\frac{4}{d+2}}_{L^1(\mathcal{O})}dr\\
&\leq C\mathbb{E}\left\{\left(\int_0^T\|\nabla(u^{(N)}(K,r))^s\|^2_{L^2(\mathcal{O})}dr\right)^{\frac{d}{d+2}}\left(\int_0^T\|(u^{(N)}(K,r))^{\frac{s}{2}}\|^4_{L^2(\mathcal{O})}dr\right)^{\frac{2}{d+2}}\right\},
\end{aligned}$$

and $\int_0^T\|(\mathbf{u}^{(N)}(K,r))^{\frac{s}{2}}\|^4_{L^2(\mathcal{O})}dr\leq T\sup_{t\in(0,T)}\|(\mathbf{u}^{(N)}(K,t))^{\frac{s}{2}}\|^4_{L^2(\mathcal{O})}$, thus

$$\begin{aligned}
&\mathbb{E}\int_0^T \|(\mathbf{u}^{(N)}(K,r))^s\|_{L^3(\mathcal{O})}^3 dr \\
&\le CT^{\frac{2}{d+2}}\mathbb{E}\left\{\left(\sup_{t\in(0,T)}\|(u^{(N)}(K,t))^{\frac{s}{2}}\|_{L^2(\mathcal{O})}^4\right)\left(\int_0^T\|\nabla(u^{(N)}(K,r))^s\|_{L^2(\mathcal{O})}^2dr\right)^{\frac{d}{d+2}}\right\} \\
&\le CT^{\frac{2}{d+2}}\left(\mathbb{E}\sup_{t\in(0,T)}\|(u^{(N)}(K,t))^{\frac{s}{2}}\|_{L^2(\mathcal{O})}^4\right)^{\frac{2}{d+2}}\left(\mathbb{E}\int_0^T\|\nabla(u^{(N)}(K,r))^s\|_{L^2(\mathcal{O})}^2dr\right)^{\frac{d}{d+2}} \\
&\le C.
\end{aligned} \tag{36}$$

By (26) and (36), we derive that

$$\sup_{N\in\mathbb{N}}\mathbb{E}\int_0^T\|(\mathbf{u}^{(N)}(K,t))^s\|_{H^1(\mathcal{O})}^2dr\le C. \tag{37}$$

Since $d\le 3$, we deduce that

$$\begin{aligned}
&\mathbb{E}\int_0^T\|(u^{(N)}(K,r))^s\|_{L^2(\mathcal{O})}^3dr\le C\mathbb{E}\int_0^T\|(u^{(N)}(K,r))^s\|_{H^1(\mathcal{O})}^{\frac{3d}{2+d}}\|(u^{(N)}(K,r))^s\|_{L^1(\mathcal{O})}^{\frac{6}{2+d}}dr \\
&\le C\mathbb{E}\left(\sup_{t\in(0,T)}\|(u^{(N)}(K,t))^{\frac{s}{2}}\|_{L^2(\mathcal{O})}^{\frac{12}{2+d}}\int_0^T\|(u^{(N)}(K,r))^s\|_{H^1(\mathcal{O})}^{\frac{3d}{2+d}}dr\right) \\
&\le C\left(\mathbb{E}\left(\sup_{t\in(0,T)}\|(u^{(N)}(K,t))^{\frac{s}{2}}\|_{L^2(\mathcal{O})}^{\frac{24}{4-d}}\right)\right)^{\frac{4-d}{2(2+d)}} \\
&\cdot\left(\mathbb{E}\int_0^T\|(u^{(N)}(K,r))^s\|_{H^1(\mathcal{O})}^2dr\right)^{\frac{3d}{2(2+d)}},
\end{aligned}$$

by (35) and (37), we conclude that

$$\mathbb{E}\int_0^T\|(\mathbf{u}^{(N)}(K,r))^s\|_{L^2(\mathcal{O})}^3dr\le C,$$

and we have shown the lemma. □

5. Existence of a martingale solution proof

We introduce those topological spaces [12]:

$$Z_T=C^0([0,T];H^3(O)')\cap L^2_\omega(0,T;H^1(O))\cap L^2(0,T;L^2(O))\cap C^0([0,T];L^2_\omega(O)),$$

endowed with the topology T, with T the maximum one of above topological spaces.

In Lemma 6, we show that the approximated sequence is tight in Z_T. Details for the compactness criterion [15,16,23,24]. Theorem 10 [12] provides criterions for the tightness of approximated sequence $(\mathbf{u}^{(N)}(K,t))_{N\in\mathbb{N}}$ in Z_T, and we have verified two of those criterions in Lemma 5, which are

$$\sup_{N\in\mathbb{N}}\mathbb{E}\left(\sup_{t\in(0,T)}\|\mathbf{u}^{(N)}(K,t)\|_{L^2(\mathcal{O})}^2\right)\le C,\quad \sup_{N\in\mathbb{N}}\mathbb{E}\left(\int_0^T\|\mathbf{u}^{(N)}(K,t)\|_{H^1(\mathcal{O})}^2dr\right)\le C.$$

By the fact that embeddings $H^3(\mathcal{O})\hookrightarrow H^1(\mathcal{O})\hookrightarrow L^2(\mathcal{O})$ are dense and continuous and the embedding $H^1(\mathcal{O})\hookrightarrow L^2(\mathcal{O})$ is compact when $d\le 3$, it remains to show that $(\mathbf{u}^{(N)}(K,t))_{N\in\mathbb{N}}$ satisfies the Aldous condition in $H^3(\mathcal{O})'$.

For a detailed explanation of the Aldous condition, please refer to Definition 3 [12]. Let $(X_N)_{N\in N}$ be a sequence of stochastic processes on a complete separable metric space S defined on the probability space (Ω, F, P), with filtration $F = (F_t)_{t\in[0,T]}$. We say that $(X_N)_{N\in N}$ satisfies the Aldous condition if and only if for every $\varepsilon > 0$, there exists $\kappa > 0$, such that for every $\delta > 0$ and each sequence $(\tau_N)_{N\in N}$ of F-stopping times with $\tau_N \le T$, it holds that

$$\sup_{N\in\mathbb{N}} \sup_{0<\theta<\delta} \mathbb{P}\left\{d(X_N(\tau_N+\theta), X_N(\tau_N)) \ge \kappa\right\} \le \varepsilon.$$

In Lemma 6, we choose $d(X_N(\tau_N+\theta), X_N(\tau_N)) = \|X_N(\tau_N+\theta) - X_N(\tau_N)\|_{H^3(\mathcal{O})'}$ with $X_N(t) = u^{(N)}(K,t)$, and try to show that

$$\sup_{N\in\mathbb{N}} \sup_{0<\theta<\delta} \mathbb{P}\left(\|\mathbf{u}^{(N)}(K,\tau_N+\theta) - \mathbf{u}^{(N)}(K,\tau_N)\|_{H^3(\mathcal{O})'} \ge \kappa\right) \le \varepsilon.$$

Lemma 6. *Let us denote the law of* $\mathbf{u}^{(N)}(K,t)$ *as* $L(\mathbf{u}^{(N)}(K,t))$*, then the set of measures* $\{L(\mathbf{u}^{(N)}(K,t)) : N \in N\}$ *is tight on* $(Z_T, \mathcal{T})$.

Proof. Let $t \in (0,T)$ and $\phi_i \in H^3(O)$, then

$$\begin{aligned}\langle u_i^{(N)}(K,t), \phi_i\rangle &= \langle \Pi_N(u_i^0), \phi_i\rangle - \sum_{j=1}^n \int_0^t \langle A_{ij}(K, \mathbf{u}^{(N)}(K,r))\nabla u_j^{(N)}(K,r), \nabla \Pi_N \phi_i\rangle dr \\ &+ \sum_{j=1}^n \int_0^t \langle \Pi_N(\sigma_{ij}(\mathbf{u}^{(N)}(K,r)))dW_j(r), \phi_i\rangle = J_0^{(N)} + J_1^{(N)}(t) + J_2^{(N)}(t).\end{aligned}$$

If we denote $I_1 = \left\{\omega\in\Omega : 0 \le \int_0^T \|(\mathbf{u}^{(N)}(K,r))^s\|^3_{L^2(O)} dr \le 1\right\}$, with the complement of I_1 given by $I_1^c = \left\{\omega\in\Omega : \int_0^T \|(\mathbf{u}^{(N)}(K,r))^s\|^3_{L^2(O)} dr > 1\right\}$. Then

$$\begin{aligned}\mathbb{E}\left(\|(\mathbf{u}^{(N)}(K,t))^s\|^2_{L^3(0,T;L^2(\mathcal{O}))}\right) &= \mathbb{E}\left(\int_0^T \|(\mathbf{u}^{(N)}(K,r))^s\|^3_{L^2(\mathcal{O})} dr\right)^{\frac{2}{3}} \\ &= \int_{I_1\cup I_1^c} \left(\int_0^T \|(\mathbf{u}^{(N)}(K,r))^s\|^3_{L^2(\mathcal{O})} dr\right)^{\frac{2}{3}} \mathbb{P}(d\omega) \\ &\le 1 + \mathbb{E}\int_0^T \|(\mathbf{u}^{(N)}(K,r))^s\|^3_{L^2(\mathcal{O})} dr \le C.\end{aligned}$$

By the H\" older inequality, we have

$$\begin{aligned}&\left(\mathbb{E}\left(\|(\mathbf{u}^{(N)}(K,t))^s\|_{L^3(0,T;L^2(\mathcal{O}))} \|\nabla \mathbf{u}^{(N)}(K,t)\|_{L^2(0,T;L^2(\mathcal{O}))}\right)\right)^2 \\ &\le \mathbb{E}\left(\|(\mathbf{u}^{(N)}(K,t))^s\|^2_{L^3(0,T;L^2(\mathcal{O}))}\right) \mathbb{E}\left(\|\nabla \mathbf{u}^{(N)}(K,t)\|^2_{L^2(0,T;L^2(\mathcal{O}))}\right) \\ &\le C\mathbb{E}\int_0^T \|\nabla \mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})} dr.\end{aligned}$$

Using the continuous embedding of $H^3(\mathcal{O}) \hookrightarrow W^{1,\infty}(\mathcal{O})$, when $d \le 3$, let us consider integrals with $0 < \theta < 1$. Let us denote $\chi_{(\tau_N,\tau_N+\theta)}(t) = 1$, if $\tau_N \le t \le \tau_N + \theta$, and otherwise, $\chi_{(\tau_N,\tau_N+\theta)}(t) = 0$, we observe that

$$
\begin{aligned}
&\int_{\tau_N}^{\tau_N+\theta} 1\cdot \|(\mathbf{u}^{(N)}(K,r))^s\|_{L^2(\mathcal{O})}\|\nabla \mathbf{u}^{(N)}(K,r)\|_{L^2(\mathcal{O})}dr\\
&\quad = \int_0^T \chi_{(\tau_N,\tau_N+\theta)}(r)\|(\mathbf{u}^{(N)}(K,r))^s\|_{L^2(\mathcal{O})}\|\nabla \mathbf{u}^{(N)}(K,r)\|_{L^2(\mathcal{O})}dr\\
&\quad \le \|\chi_{(\tau_N,\tau_N+\theta)}(r)\|_{L^6((0,T))}\|(\mathbf{u}^{(N)}(K,t))^s\|_{L^3(0,T;L^2(\mathcal{O}))}\|\nabla \mathbf{u}^{(N)}(K,t)\|_{L^2(0,T;L^2(\mathcal{O}))}\\
&\quad = \|1\|_{L^6((\tau_N,\tau_N+\theta))}\|(\mathbf{u}^{(N)}(K,t))^s\|_{L^3(0,T;L^2(\mathcal{O}))}\|\nabla \mathbf{u}^{(N)}(K,t)\|_{L^2(0,T;L^2(\mathcal{O}))}\\
&\quad \le \theta^{\frac{1}{6}}\|(\mathbf{u}^{(N)}(K,t))^s\|_{L^3(0,T;L^2(\mathcal{O}))}\|\nabla \mathbf{u}^{(N)}(K,t)\|_{L^2(0,T;L^2(\mathcal{O}))},
\end{aligned}
$$

and for $J_1^{(N)}(t)$, if $i=j$,

$$
\begin{aligned}
&\mathbb{E}\left|\int_{\tau_N}^{\tau_N+\theta}\langle A_{ij}(K,\mathbf{u}^{(N)}(K,r))\nabla u_j^{(N)}(K,r),\nabla \Pi_N\phi_i\rangle dr\right|\\
&\le \mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\|A_{ij}(K,\mathbf{u}^{(N)}(K,r))\|_{L^2(\mathcal{O})}\|\nabla u_j^{(N)}(K,r)\|_{L^2(\mathcal{O})}\|\nabla\phi_i\|_{L^\infty(\mathcal{O})}dr\\
&\le C\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}(1+\|(\mathbf{u}^{(N)}(K,r))^s\|_{L^2(\mathcal{O})})\|\nabla \mathbf{u}^{(N)}(K,r)\|_{L^2(\mathcal{O})}\|\phi_i\|_{H^3(\mathcal{O})}dr\\
&\le C\mathbb{E}\left((\theta^{\frac{1}{2}}+\theta^{\frac{1}{6}}\|(\mathbf{u}^{(N)}(K,t))^s\|_{L^3(0,T;L^2(\mathcal{O}))})\|\nabla \mathbf{u}^{(N)}(K,t)\|_{L^2(0,T;L^2(\mathcal{O}))}\right)\cdot\|\phi_i\|_{H^3(\mathcal{O})}\\
&\le C\left\{\theta^{\frac{1}{6}}\left(\mathbb{E}\left(\int_0^T\|(\mathbf{u}^{(N)}(K,r))^s\|^3_{L^2(\mathcal{O})}dr\right)^{\frac{2}{3}}\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}\int_0^T\|\nabla \mathbf{u}^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\right)^{\frac{1}{2}}\right.\\
&\quad\left.+\theta^{\frac{1}{2}}\mathbb{E}\|\nabla \mathbf{u}^{(N)}(K,t)\|_{L^2(0,T;L^2(\mathcal{O}))}\right\}\cdot\|\phi_i\|_{H^3(\mathcal{O})}\le C\theta^{\frac{1}{6}}\|\phi_i\|_{H^3(\mathcal{O})},
\end{aligned}
$$

if $i\neq j$,

$$
\begin{aligned}
&\mathbb{E}\left|\int_{\tau_N}^{\tau_N+\theta}\langle A_{ij}(K,\mathbf{u}^{(N)}(K,r))\nabla u_j^{(N)}(K,r),\nabla \Pi_N\phi_i\rangle dr\right|\\
&\le C\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\int_{\mathcal{O}}|u_i^{(N)}(K,r)(u_j^{(N)}(K,r))^{s-1}\chi_j(K,\mathbf{u}^{(N)}(K,r))\nabla u_j^{(N)}(K,r)\nabla\phi_i|dxdr\\
&\le C\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\int_{\mathcal{O}}|u_i^{(N)}(K,r)\nabla(u_j^{(N)}(K,r))^s\nabla\phi_i|dxdr\\
&\le C\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\|\mathbf{u}^{(N)}(K,r)\|_{L^2(\mathcal{O})}\|\nabla(\mathbf{u}^{(N)}(K,r))^s\|_{L^2(\mathcal{O})}\|\nabla\phi_i\|_{L^\infty(\mathcal{O})}dr\\
&\le C\mathbb{E}\left\{\left(\int_{\tau_N}^{\tau_N+\theta}\|u^{(N)}(K,r)\|^2_{L^2(\mathcal{O})}dr\right)^{\frac{1}{2}}\left(\int_{\tau_N}^{\tau_N+\theta}\|\nabla(u^{(N)}(K,r))^s\|^2_{L^2(\mathcal{O})}dr\right)^{\frac{1}{2}}\right\}\cdot\|\phi_i\|_{H^3(\mathcal{O})}\\
&\le C\theta^{\frac{1}{2}}\mathbb{E}\left\{\left(\sup_{t\in(0,T)}\|u^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)^{\frac{1}{2}}\cdot\left(\int_{\tau_N}^{\tau_N+\theta}\|\nabla(u^{(N)}(K,r))^s\|^2_{L^2(\mathcal{O})}dr\right)^{\frac{1}{2}}\right\}\cdot\|\phi_i\|_{H^3(\mathcal{O})}\\
&\le C\theta^{\frac{1}{2}}\left(\mathbb{E}\sup_{t\in(0,T)}\|u^{(N)}(K,t)\|^2_{L^2(\mathcal{O})}\right)^{\frac{1}{2}}\cdot\left(\mathbb{E}\int_0^T\|\nabla(u^{(N)}(K,t))^s\|^2_{L^2(\mathcal{O})}dr\right)^{\frac{1}{2}}\cdot\|\phi_i\|_{H^3(\mathcal{O})}\\
&\le C\theta^{\frac{1}{6}}\|\phi_i\|_{H^3(\mathcal{O})}.
\end{aligned}
$$

If we denote $I_2=\left\{\omega\in\Omega:0\le\int_0^T\|\mathbf{u}^{(N)}(K,r)\|^3_{L^2(O)}dr\le 1\right\}$, and $I_2^c=\left\{\omega\in\Omega:\int_0^T\|\mathbf{u}^{(N)}(K,r)\|^3_{L^2(O)}dr>1\right\}$, then

$$\mathbb{E}\left(\int_0^T \|\mathbf{u}^{(N)}(K,r)\|^3_{L^2(O)}dr\right)^{\frac{2}{3}}$$
$$= \int_{I_2\cup I_2^c}\left(\int_0^T \|\mathbf{u}^{(N)}(K,r)\|^3_{L^2(O)}dr\right)^{\frac{2}{3}} P(d\omega) \le 1 + E\int_0^T \|\mathbf{u}^{(N)}(K,r)\|^3_{L^2(O)}dr \le C.$$

For $J_2^{(N)}(t)$, we have

$$\begin{aligned}
&E\left|\int_{\tau_N}^{\tau_N+\theta} \langle \Pi_N(\sigma_{ij}(\mathbf{u}^{(N)}(K,r)))dW_j(r),\phi_i\rangle\right|^2\\
&\le E\left(\int_{\tau_N}^{\tau_N+\theta} \|\sigma(\mathbf{u}^{(N)}(K,r))\|^2_{L(Y;L^2(O))}dr\right)\cdot \|\phi_i\|^2_{L^2(O)}\\
&\le CE\left(\int_{\tau_N}^{\tau_N+\theta} (1+\|\mathbf{u}^{(N)}(K,r)\|^2_{L^2(O)})dr\right)\cdot \|\phi_i\|^2_{L^2(O)}\\
&= C\left(\theta + E\int_0^T \chi_{(\tau_N,\tau_N+\theta)}\|\mathbf{u}^{(N)}(K,r)\|^2_{L^2(O)}dr\right)\cdot \|\phi_i\|^2_{L^2(O)}\\
&\le C\left(\theta + E\|\chi_{(\tau_N,\tau_N+\theta)}\|_{L^3((0,T))}\left(\int_0^T \|\mathbf{u}^{(N)}(K,r)\|^3_{L^2(O)}dr\right)^{\frac{2}{3}}\right)\cdot \|\phi_i\|^2_{L^2(O)}\\
&\le C\left(\theta + \theta^{\frac{1}{3}}E\left(\int_0^T \|\mathbf{u}^{(N)}(K,r)\|^3_{L^2(O)}dr\right)^{\frac{2}{3}}\right)\cdot \|\phi_i\|^2_{L^2(O)} \le C\theta^{\frac{1}{3}}\|\phi_i\|^2_{H^3(O)}.
\end{aligned}$$

Let $\kappa > 0, \varepsilon > 0$, by the Chebyshev inequality, for every ϕ_i, with $\|\phi_i\|_{H^3(O)} = 1$,

$$P\left\{\|J_1^{(N)}(\tau_N+\theta) - J_1^{(N)}(\tau_N)\|_{H^3(O)'} \ge \kappa\right\} \le \frac{1}{\kappa}E\|J_1^{(N)}(\tau_N+\theta) - J_1^{(N)}(\tau_N)\|_{H^3(O)'} \le \frac{C\theta^{\frac{1}{6}}}{\kappa},$$

choose $\delta_1 = (\kappa\varepsilon/C)^6$, we infer that

$$\sup_{N\in N}\sup_{0<\theta<\delta_1} P\left\{\|J_1^{(N)}(\tau_N+\theta) - J_1^{(N)}(\tau_N)\|_{H^3(O)'} \ge \kappa\right\} \le \varepsilon.$$

Still applying the Chebyshev inequality,

$$P\left\{\|J_2^{(N)}(\tau_N+\theta) - J_2^{(N)}(\tau_N)\|_{H^3(O)'} \ge \kappa\right\} \le \frac{1}{\kappa^2}E\|J_2^{(N)}(\tau_N+\theta) - J_2^{(N)}(\tau_N)\|^2_{H^3(O)'} \le \frac{C\theta^{\frac{1}{3}}}{\kappa^2},$$

choose $\delta_2 = (\kappa^2\varepsilon/C)^3$, we infer that

$$\sup_{N\in N}\sup_{0<\theta<\delta_2} P\left\{\|J_2^{(N)}(\tau_N+\theta) - J_2^{(N)}(\tau_N)\|_{H^3(O)'} \ge \kappa\right\} \le \varepsilon.$$

This verifies the Aldous condition for $J_1^{(N)}(t)$ and $J_2^{(N)}(t)$. Then we have shown that the set of measures $\{L(\mathbf{u}^{(N)}(K,t)) : N \in N\}$ is tight on (Z_T, T). □

The Lemma 12 [12] has shown that $Z_T \times C^0([0,T];Y_0)$ satisfies the assumption for Skorokhod-Jakubowski theorem, and we have shown that $\mathbf{u}^{(N)}$ is tight in Z_T in Lemma 6. (Theorem 23 [12], Theorem C.1 [16,20]).

By the Skorokhod-Jakubowski theorem, we can find a probability space $(\widetilde{\Omega}, \widetilde{F}, \widetilde{P})$, and on this space $Z_T \times C^0([0,T];Y_0)$-valued random variables $(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}})$,

$(\overset{(N)}{\widetilde{\mathbf{u}}}(K,t), \overset{(N)}{\widetilde{\mathbf{W}}})$ that $(\overset{(N)}{\widetilde{\mathbf{u}}}(K,t), \overset{(N)}{\widetilde{\mathbf{W}}})$ has the same law as $(\mathbf{u}^{(N)}(K,t), \mathbf{W})$ on $B(Z_T \times C^0([0,T]; Y_0))$, and

$$(\overset{(N)}{\widetilde{\mathbf{u}}}(K,t), \overset{(N)}{\widetilde{\mathbf{W}}}) \to (\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}) \quad \text{in } Z_T \times C^0([0,T]; Y_0),\ \widetilde{P}\text{-a.s. as } N \to \infty.$$

By the definition of Z_T, we have

$$\begin{aligned}
&\overset{(N)}{\widetilde{\mathbf{u}}}(K,t) \to \overset{(K)}{\widetilde{\mathbf{u}}} \ \text{ in } C^0([0,T]; H^3(O)'),\\
&\overset{(N)}{\widetilde{\mathbf{u}}}(K,t) \rightharpoonup \overset{(K)}{\widetilde{\mathbf{u}}} \ \text{ weakly in } L^2(0,T; H^1(O)),\\
&\overset{(N)}{\widetilde{\mathbf{u}}}(K,t) \to \overset{(K)}{\widetilde{\mathbf{u}}} \ \text{ in } C^0([0,T]; L^2_w(O)),\\
&\overset{(N)}{\widetilde{\mathbf{u}}}(K,t) \to \overset{(K)}{\widetilde{\mathbf{u}}} \ \text{ in } L^2(0,T; L^2(O)),\\
&\overset{(N)}{\widetilde{\mathbf{W}}} \to \widetilde{\mathbf{W}} \text{ in } C^0([0,T]; Y_0).
\end{aligned}$$

We have shown that $u_i^{(N)}(K,t)$ is non-negative, P-a.s. Let us show that for a.e. $(x,t) \in O \times (0,T)$, $\widetilde{u}_i^{(N)}(K,t)$ is non-negative, $\widetilde{P}$-a.s. with its limit $\widetilde{u}_i^{(K)}(x,t) \geq 0$, $\widetilde{P}$-a.s. for every $1 \leq i \leq n$.

Lemma 7. *For every $1 \leq i \leq n$, $\widetilde{u}_i^{(K)}(x,t) \geq 0$ in $\mathcal{O}$, a.e. $t \in [0,T]$, $\widetilde{P}$-a.s.*

Proof. Let us denote $Q_T = O \times (0,T)$, and $1 \leq i \leq n$. Since $u_i^{(N)}(K,t) \geq 0$ in Q_T, P-a.s. we have $E\|(u_i^{(N)}(K,t))^-\|_{L^2(0,T;L^2(O))} = 0$, where $z^- = \max\{0, -z\}$. The function $u_i^{(N)}(K,t)$ and its negative part are Z_T-Borel measurable. By the equivalence of laws of $u_i^{(N)}(K,t)$ and $\widetilde{u}_i^{(N)}(K,t)$ in Z_T, once we denote $\mu_i^{(N)}$ and $\widetilde{\mu}_i^{(N)}$ as the laws of $u_i^{(N)}(K,t)$ and $\widetilde{u}_i^{(N)}(K,t)$, respectively, then

$$\begin{aligned}
\widetilde{E}\|(\widetilde{u}_i^{(N)}(K,t))^-\|_{L^2(Q_T)} &= \int_{L^2(Q_T)} \|y^-\|_{L^2(Q_T)} d\widetilde{\mu}_i^{(N)}(y)\\
&= \int_{L^2(Q_T)} \|y^-\|_{L^2(Q_T)} d\mu_i^{(N)}(y) = E\|(u_i^{(N)}(K,t))^-\|_{L^2(Q_T)} = 0.
\end{aligned}$$

This implies that $\widetilde{u}_i^N(K,t) \geq 0$ a.e. in Q_T, $\widetilde{P}$-a.s. The convergence (up to a subsequence) $\overset{(N)}{\widetilde{\mathbf{u}}}(K,t) \to \overset{(K)}{\widetilde{\mathbf{u}}}$ a.e. in Q_T, $\widetilde{P}$-a.s. then indicates that $\widetilde{u}_i^N \geq 0$ in Q_T, $\widetilde{P}$-a.s. □

In Lemma 11, a similar though more complicated (as the index K is involved) proof will be discussed in detail. This allows us to state Lemma 8 without proof.

Lemma 8. *For every $r, t \in (0,T)$ with $r \leq t$, $\overline{\phi}_i \in L^2(O)$ and $\phi_i \in H^3(O)$ satisfying $\nabla\phi_i \cdot \nu = 0$ on ∂O, we have*

$$\lim_{N\to\infty} \widetilde{E} \int_0^T \left(\widetilde{u}_i^N(K,t) - \widetilde{u}_i^K(t), \overline{\phi}_i\right)^2_{L^2(O)} dt = 0,$$

$$\lim_{N\to\infty} \widetilde{E} \left(\widetilde{u}_i^N(0) - \underset{i}{\widetilde{u}}(0), \overline{\phi}_i\right)^2_{L^2(O)} = 0,$$

$$\lim_{N\to\infty} \widetilde{E} \int_0^T \left| \sum_{j=1}^n \int_0^t \langle A_{ij}(K, \overset{(N)}{\widetilde{\mathbf{u}}}(K,r))\nabla \widetilde{u}_j^N(K,r) - A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}})\nabla \widetilde{u}_j^K, \nabla\phi_i\rangle dr \right| dt = 0,$$

$$\lim_{N\to\infty} \widetilde{E} \int_0^T \left| \sum_{j=1}^n \int_0^t \left(\sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(K,r)) d\widetilde{W}_j^N(r) - \sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r)) d\underset{j}{\widetilde{W}}(r), \overline{\phi}_i\right)_{L^2(\mathcal{O})} \right|^2 dt = 0.$$

We consider the stochastic partial differential equations, that for $\mathbf{u} = (u_1, \ldots, u_n)$,

$$du_i - div\left(\sum_{j=1}^n A_{ij}(K,\mathbf{u})\nabla u_j\right) dt = \sum_{j=1}^n \sigma_{ij}(\mathbf{u}) dW_j(t) \quad \text{in } O,\ t>0,\ 1\le i\le n, \tag{38}$$

With

$$\sum_{j=1}^n A_{ij}(K,\mathbf{u})\nabla u_j \cdot \nu = 0 \quad \text{on } \partial O,\ t>0,\ u_i(0) = u_i^0 \quad \text{in } O,\ 1\le i\le n, \tag{39}$$

and derive Lemma 9.

Lemma 9. $\overset{(K)}{\widetilde{\mathbf{u}}}(t)$ *satisfies for every* $\phi_i \in H^1(\mathcal{O})$, $1\le i\le n$,

$$\begin{aligned}\langle \widetilde{u}_i^{(K)}(t), \phi_i\rangle = \langle \widetilde{u}_i(0), \phi_i\rangle + \sum_{j=1}^n \int_0^t \langle div(A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r))\nabla \widetilde{u_j^{(K)}}(r)), \phi_i\rangle dr \\ + \langle \sum_{j=1}^n \int_0^t \sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r)) d\widetilde{W}_j(r), \phi_i\rangle.\end{aligned} \tag{40}$$

Proof. Let us define

$$\begin{aligned}\Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi)(t) = \langle u_i^{(N)}(0), \phi_i\rangle + \langle \sum_{j=1}^n \int_0^t \Pi_N \sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(K,r)) d\widetilde{W}_j^N(r), \phi_i\rangle \\ + \sum_{j=1}^n \int_0^t \langle \Pi_N div(A_{ij}(K, \overset{(N)}{\widetilde{\mathbf{u}}}(K,r))\nabla \widetilde{u}_j^N(K,r)), \phi_i\rangle dr,\end{aligned}$$

And

$$\begin{aligned}\Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi)(t) = \langle \underset{i}{\widetilde{u}}(0), \phi_i\rangle + \langle \sum_{j=1}^n \int_0^t \sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r)) d\underset{j}{\widetilde{W}}(r), \phi_i\rangle \\ + \sum_{j=1}^n \int_0^t \langle div(A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r))\nabla \widetilde{u}_j^K(r)), \phi_i\rangle dr,\end{aligned}$$

for $t\in(0,T)$ and $1\le i\le n$.

Since for every $\overline{\phi}_i \in L^2(\mathcal{O})$, we have

$$\|\langle \widetilde{u}_i^N(K,t), \overline{\phi}_i\rangle - \langle \widetilde{u}_i^K(t), \overline{\phi}_i\rangle\|_{L^2(\widetilde{\Omega}\times(0,T))} = \widetilde{E}\int_0^T \langle \widetilde{u}_i^N(K,t) - \widetilde{u}_i^K(t), \overline{\phi}_i\rangle^2 dt,$$

then by Lemma 8, we can show that

$$\lim_{N\to\infty} \|\langle \widetilde{u}_i^N(K,t) - \widetilde{u}_i^K(t), \overline{\phi}_i\rangle\|_{L^2(\widetilde{\Omega}\times(0,T))} = 0.$$

For every $\phi \in H^3(O)$ satisfying $\nabla\phi\cdot\nu = 0$ on ∂O, we have

$$\begin{aligned}
&\|\Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi) - \Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi)\|_{L^1(\widetilde{\Omega}\times(0,T))} \\
&= \widetilde{E}\int_0^T |\Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}^{(N)}, \phi) - \Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi)|dt \le \widetilde{E}\int_0^T |\langle \widetilde{u}_i^N(0) - \underset{i}{\widetilde{u}}(0), \phi_i\rangle|dt \\
&+ \widetilde{E}\int_0^T \left|\sum_{j=1}^n \int_0^t \langle \sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(K,r))d\widetilde{W}_j^N(r) - \sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))d\widetilde{W}_j(r), \phi_i\rangle\right| dt \\
&+ \widetilde{E}\int_0^T \left|\sum_{j=1}^n \int_0^t \langle A_{ij}(K, \overset{(N)}{\widetilde{\mathbf{u}}}(K,r))\nabla\widetilde{u}_j^N(K,r) - A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r))\nabla\widetilde{u}_j^K(r), \nabla\phi_i\rangle dr\right| dt,
\end{aligned} \tag{41}$$

and Lemma 8 indicates that each term in the right-hand-side of (41) vanishes, as $N \to \infty$, thus

$$\lim_{N\to\infty} \|\Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi) - \Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi)\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0.$$

$\mathbf{u}^{(N)}(K,t)$ is the strong solution (in the probabilistic sense) to (11)–(13). By the definition, $\mathbf{u}^{(N)}(K,t)$ satisfies $\langle u_i^{(N)}(K,t), \phi_i\rangle = \Lambda_i^{(N)}(\mathbf{u}^{(N)}, \mathbf{W}, \phi)(t)$, P-a.s. for every $t \in (0,T)$, $\phi \in H^1(O)$. In particular, we have

$$\int_0^T E|\langle u_i^{(N)}(K,t), \phi_i\rangle - \Lambda_i^{(N)}(\mathbf{u}^{(N)}, \mathbf{W}, \phi)(t)|dt = 0.$$

Since $L(\mathbf{u}^{(N)}, W)$ coincides with $L(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}})$, we have

$$\int_0^T \widetilde{E}|\langle \widetilde{u}_i^{(N)}(K,t), \phi_i\rangle - \Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi)(t)|dt = 0, \quad 1 \le i \le n.$$

Let $N \to \infty$, then $\int_0^T \widetilde{E}|\langle \widetilde{u}_i^K(t), \phi_i\rangle - \Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi)(t)|dt = 0$, for every $\phi \in H^3(O)$ satisfying $\nabla\phi\cdot\nu = 0$ on ∂O. By the density argument, it holds for every $\phi \in H^1(O)$. Then for a.e. $t \in [0,T], \omega \in \widetilde{\Omega}$, we deduce that $\langle \widetilde{u}_i^K(t), \phi_i\rangle - \Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi)(t) = 0, 1 \le i \le n$. By the definition of $\Lambda_i^{(K)}$, we conclude that for a.e. $t \in [0,T], \omega \in \widetilde{\Omega}$, (40) holds, then we complete the proof. □

Let us focus on $\overset{(K)}{\widetilde{\mathbf{u}}}(t)$. We try to show that $\{\overset{(K)}{\widetilde{\mathbf{u}}}(t)\}_{K\in N}$ is tight in (Z_T, T), then to show the limit of $\overset{(K)}{\widetilde{\mathbf{u}}}(t)$ is a martingale solution of (1)–(3), as $K \to \infty$.

We mention that the tightness proof for $\overset{(K)}{\widetilde{\mathbf{u}}}(t)$ with respect to $K \in N$ is similar to the proof when tightness of $\overset{(N)}{\widetilde{\mathbf{u}}}(K,t)$ with respect to $N \in N$ being shown, so we are allowed to give a brief proof to Lemma 10.

Lemma 10. *(i)* For every $T > 0$, there exists a constant $C > 0$, which does not depend on K, such that

$$\sup_{K\in N} \widetilde{E}\left(\sup_{t\in[0,T]} \| \overset{(K)}{\widetilde{\mathbf{u}}}(t)\|^2_{L^2(O)}\right) \le C,$$
$$\sup_{K\in N} \widetilde{E}\left(\int_0^T \|\nabla \overset{(K)}{\widetilde{\mathbf{u}}}(r)\|^2_{L^2(O)} dr\right) \le C,$$
$$\sup_{K\in N} \widetilde{E}\left(\int_0^T \|\nabla(\overset{(K)}{\widetilde{\mathbf{u}}}(r))^s\|^2_{L^2(O)} dr\right) \le C,$$
$$\sup_{K\in N} \widetilde{E}\left(\int_0^T \|(\overset{(K)}{\widetilde{\mathbf{u}}}(r))^s\|^3_{L^2(O)} dr\right) \le C.$$

(ii) Let us denote the law of $\overset{(K)}{\widetilde{\mathbf{u}}}(t)$ as $L(\overset{(K)}{\widetilde{\mathbf{u}}})$, then the set of measures $\{L(\overset{(K)}{\widetilde{\mathbf{u}}}) : K \in N\}$ is tight on (Z_T, T).

Proof. Let us choose $\phi_i = \pi_i \widetilde{u}_i^K$ in (40), and derive that

$$\frac{1}{2}\|P^{\frac{1}{2}} \overset{(K)}{\widetilde{\mathbf{u}}}(t)\|^2_{L^2(O)} - \frac{1}{2}\|P^{\frac{1}{2}}\widetilde{\mathbf{u}}(0)\|^2_{L^2(O)} = \sum_{i,j=1}^n \int_0^t \langle \pi_i\sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r)) d\underset{j}{\widetilde{W}}(r), \widetilde{u}_i^K(r)\rangle$$
$$+ \frac{1}{2}\int_0^t \|P^{\frac{1}{2}}\sigma(\overset{(K)}{\widetilde{\mathbf{u}}}(r))\|^2_{L(Y;L^2(O))} dr - \sum_{i,j=1}^n \int_0^t \langle \pi_i A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r))\nabla \widetilde{u}_j^K(r), \nabla \widetilde{u}_i^K(r)\rangle dr.$$

Since

$$\sum_{i,j=1}^n \langle \pi_i A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r))\nabla \widetilde{u}_j^K(r), \nabla \widetilde{u}_i^K(r)\rangle \ge \alpha_1 \sum_{i=1}^n \int_O |\nabla \widetilde{u}_i^K(r)|^2 dx$$
$$+ \alpha_2 \sum_{i=1}^n \int_O |\widetilde{u}_i^K(r)|^s |\nabla \widetilde{u}_i^K(r)|^2 dx \ge \alpha_1 \|\nabla \overset{(K)}{\widetilde{\mathbf{u}}}(r)\|^2_{L^2(O)} + \alpha_2 \|\nabla(\overset{(K)}{\widetilde{\mathbf{u}}}(r))^{\frac{s+2}{2}}\|^2_{L^2(O)},$$

then for some positive constants $\alpha_1, \alpha_2, \alpha_3, C_0, C_1$,

$$\begin{aligned} &\alpha_3 \widetilde{E}\left(\sup_{t\in[0,T]} \| \overset{(K)}{\widetilde{\mathbf{u}}}(t)\|^2_{L^2(O)}\right) + \alpha_1 \widetilde{E}\int_0^t \|\nabla \overset{(K)}{\widetilde{\mathbf{u}}}(r)\|^2_{L^2(O)} dr \\ &+ \alpha_2 \widetilde{E}\int_0^t \|\nabla(\overset{(K)}{\widetilde{\mathbf{u}}}(r))^{\frac{s+2}{2}}\|^2_{L^2(O)} dr \le C_0 + C_1 \widetilde{E}\int_0^T \left(\sup_{\tau\in[0,r]} \| \overset{(K)}{\widetilde{\mathbf{u}}}(\tau)\|^2_{L^2(O)}\right) dr. \end{aligned} \tag{42}$$

The rest proof can be completely referred to Lemmas 5-6. We rely on (42) to derive (i), and by the method in Lemma 6, we can show that $\overset{(K)}{\widetilde{\mathbf{u}}}(t)$ is tight in (Z_T, T). □

Let us denote the limit of $\overset{(K)}{\widetilde{\mathbf{u}}}(t)$ as $\widetilde{\mathbf{u}}(t)$, by the method in Lemma 7, we can show that $\widetilde{\mathbf{u}}(t) \geq 0$, a.e. $(x,t) \in O \times (0,T)$, $\widetilde{P}$-a.s. It remains to show that $\widetilde{\mathbf{u}}(t)$ is a martingale solution of (1)–(3). Similar procedures have been implemented [12,18,19].

Lemma 11. For every $r, t \in (0,T)$ with $r \leq t$, $\overline{\phi}_i \in L^2(O)$ and $\phi_i \in H^3(O)$ satisfying $\nabla\phi_i \cdot \nu = 0$ on ∂O, we have

$$\lim_{K\to\infty} \widetilde{E} \int_0^T \left(\widetilde{u}_i^K(t) - \underset{i}{\widetilde{u}}(t), \overline{\phi}_i \right)^2_{L^2(O)} dt = 0, \tag{43}$$

$$\lim_{K\to\infty} \widetilde{E} \left(\widetilde{u}_i^K(0) - \underset{i}{\widetilde{u}}(0), \overline{\phi}_i \right)^2_{L^2(O)} = 0, \tag{44}$$

$$\lim_{K\to\infty} \widetilde{E} \int_0^T \left| \sum_{j=1}^n \int_0^t \langle A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r)) \nabla \widetilde{u}_j^K(r) - A_{ij}(\widetilde{\mathbf{u}}(r)) \nabla \underset{j}{\widetilde{u}}(r), \nabla\phi_i \rangle dr \right| dt = 0, \tag{45}$$

$$\lim_{K\to\infty} \widetilde{E} \int_0^T \left| \sum_{j=1}^n \int_0^t \left(\sigma_{ij}(\widetilde{u}^{(K)}(r)) d\widetilde{W}_j(r) - \sigma_{ij}(\widetilde{u}(r)) d\widetilde{W}_j(r), \overline{\phi}_i \right)_{L^2(O)} \right|^2 dt = 0. \tag{46}$$

Proof. We mainly consider (45). For every $\phi = (\phi_1, \ldots, \phi_n) \in H^3(O)$,

$$\begin{aligned} \langle \widetilde{u}_i^K(t), \phi_i \rangle &= \langle \widetilde{u}_i(0), \phi_i \rangle \\ &- \sum_{j=1}^n \int_0^t \langle A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r)) \nabla \widetilde{u}_j^K(r), \nabla\phi_i \rangle dr + \langle \sum_{j=1}^n \int_0^t \sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r)) d\underset{j}{\widetilde{W}}(r), \phi_i \rangle. \end{aligned}$$

We notice that for every $1 \leq i, j, k \leq n$, we have

$$\begin{aligned} &\int_0^t \langle A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r)) \nabla \widetilde{u}_j^K(r) - A_{ij}(\widetilde{\mathbf{u}}(r)) \nabla \underset{j}{\widetilde{u}}(r), \nabla\phi_i \rangle dr \\ &= \int_0^t \int_O \left(A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r)) \nabla \widetilde{u}_j^K(r) - A_{ij}(\widetilde{\mathbf{u}}(r)) \nabla \underset{j}{\widetilde{u}}(r) \right) \nabla\phi_i \, dxdr \\ &\quad + \int_0^t \int_O \left(A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r)) - A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r)) \right) \nabla \widetilde{u}_j^K(r) \nabla\phi_i \, dxdr. \end{aligned} \tag{47}$$

Let us denote

$$\begin{aligned} I_1^{(K)} &= \int_0^t \int_O (\widetilde{u}_j^K(r) - \underset{j}{\widetilde{u}}(r)) \nabla (\widetilde{u}_k^K(r))^s \nabla\phi_i \, dxdr, \\ I_2^{(K)} &= \int_0^t \int_O \underset{j}{\widetilde{u}}(r) \nabla ((\widetilde{u}_k^K(r))^s - \widetilde{u}_k^K(r)) \nabla\phi_i \, dxdr, \\ I_3^{(K)} &= -\int_0^t \int_O ((\widetilde{u}_k^K(r))^s \widetilde{u}_j^K(r) - \widetilde{u}_k^K(r) \underset{j}{\widetilde{u}}(r)) \Delta\phi_i \, dxdr, \\ I_4^{(K)} &= -\int_0^t \int_O (\widetilde{u}_j^K(r) \nabla (\widetilde{u}_k^K(r))^s - \underset{j}{\widetilde{u}}(r) \nabla \widetilde{u}_k^K(r)) \nabla\phi_i \, dxdr, \end{aligned}$$

and we notice that

$$\int_0^t \langle \widetilde{u}_j^K(r)\nabla(\widetilde{u}_k^K(r))^s - \underset{j}{\widetilde{u}}(r)\nabla\widetilde{u}_k^K(r), \nabla\phi_i\rangle dr = I_1^{(K)} + I_2^{(K)},$$
$$\int_0^t \langle (\widetilde{u}_k^K(r))^s\nabla\widetilde{u}_j^K(r) - \widetilde{u}_k^K(r)\nabla \underset{j}{\widetilde{u}}(r), \nabla\phi_i\rangle dr = I_3^{(K)} + I_4^{(K)}.$$

One step further, we have

$$|I_1^{(K)}| \leq \|\widetilde{u}_j^K(r) - \underset{j}{\widetilde{u}}(r)\|_{L^2(0,T;L^2(O))}\|\nabla(\widetilde{u}_k^K(r))^s\|_{L^2(0,T;L^2(O))}\|\phi_i\|_{H^3(O)},$$

and $\overset{(K)}{\widetilde{\mathbf{u}}} \to \widetilde{\mathbf{u}}$ in Z_T implies that as $K \to \infty, I_1^{(K)} \to 0$. For $I_2^{(K)}$, since $(\widetilde{u}_k^K(r))^s$ converges weakly to $\widetilde{u}_k^s(r)$ in $L^2(0,T;H^1(O))$, and $\widetilde{u}_j(r)\nabla\phi_i \in L^2(0,T;L^2(O))$, then as $K \to \infty, I_2^{(K)} \to 0$.

Since $I_4^{(K)} = -I_1^{(K)} - I_2^{(K)}$, then as $K \to \infty, I_4^{(K)} \to 0$. For $I_3^{(K)}$, relying on continuous embeddings $H^3(O) \hookrightarrow W^{2,4}(O)$, $H^1(O) \hookrightarrow L^4(O)$ and $H^3(O) \hookrightarrow W^{1,\infty}(O)$ when $d \leq 3$, we have

$$\begin{aligned}
|I_3^{(K)}| &= \Bigg|\int_0^t \int_O (\widetilde{u}_k^K(r))^s(\widetilde{u}_j^K(r) - \underset{j}{\widetilde{u}}(r))\Delta\phi_i \, dxdr \\
&\quad + \int_0^t \int_O \underset{j}{\widetilde{u}}(r)((\widetilde{u}_k^K(r))^s - \widetilde{u}_k^K(r))\Delta\phi_i \, dxdr\Bigg| \\
&\leq \|\widetilde{u}_j^K(r) - \underset{j}{\widetilde{u}}(r)\|_{L^2(0,T;L^2(O))}\|(\widetilde{u}_k^K(r))^s\|_{L^2(0,T;H^1(O))}\|\phi_i\|_{H^3(O)} \\
&\quad + \left|\int_0^t \int_O \underset{j}{\widetilde{u}}(r)\Delta\phi_i((\widetilde{u}_k^K(r))^s - \widetilde{u}_k^K(r)) \, dxdr\right|,
\end{aligned}$$

Still by the fact that $\overset{(K)}{\widetilde{\mathbf{u}}} \to \widetilde{\mathbf{u}}$ in Z_T, and $\underset{j}{\widetilde{u}}(r)\Delta\phi_i \in L^2(0,T;L^2(O))$, we deduce that as $K \to \infty, I_3^{(K)} \to 0$. By the structural information of $A(\mathbf{u}) = (A_{ij}(\mathbf{u}))$, we can show that

$$\lim_{K\to\infty} \int_0^t \int_O \left(A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))\nabla\widetilde{u}_j^K(r) - A_{ij}(\widetilde{\mathbf{u}}(r))\nabla \underset{j}{\widetilde{u}}(r)\right)\nabla\phi_i \, dxdr = 0, \quad \widetilde{P}\text{-a.s.} \tag{48}$$

One step further, if $i = j$, $A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r)) = A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))$, and if $i \neq j$, $|A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r)) - A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))| = sa_{ij}\widetilde{u}_i^K(\widetilde{u}_j^K)^{s-1}(1 - \chi_j(K, \overset{(K)}{\widetilde{\mathbf{u}}}))$. Thus $A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r)) \neq A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))$ only if $i \neq j$ and $\widetilde{u}_j^K < \frac{1}{K}$.

If $i \neq j$, for every $0 < t < T$, we have

$$\begin{aligned}
&\left|\int_0^t \int_O \left(A_{ij}(K, \overset{(K)}{\widetilde{\mathbf{u}}}(r)) - A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))\right)\nabla\widetilde{u}_j^K(r)\nabla\phi_i \, dxdr\right| \\
&\leq C\int_0^t \int_O \left|(\widetilde{u}_j^K)^{s-1}(1 - \chi_j(K, \overset{(K)}{\widetilde{\mathbf{u}}}))\widetilde{u}_i^K\nabla\widetilde{u}_j^K(r)\nabla\phi_i\right| dxdr \\
&\leq \frac{C}{K^{s-1}}\int_0^t \int_O \left|\widetilde{u}_i^K\nabla\widetilde{u}_j^K(r)\nabla\phi_i\right| dxdr \\
&\leq \frac{C}{K^{s-1}}\|\widetilde{u}_i^K\|_{L^2(0,T;L^2(O))}\|\nabla\widetilde{u}_j^K(r)\|_{L^2(0,T;L^2(O))}\|\phi_i\|_{H^3(O)},
\end{aligned} \tag{49}$$

and by (49) we derive that

$$\lim_{K\to\infty}\int_0^t\int_O\left(A_{ij}(K,\overset{(K)}{\widetilde{\mathbf{u}}}(r))-A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))\right)\nabla\widetilde{u}_j^K(r)\nabla\phi_i\,dxdr=0,\quad \widetilde{P}\text{-a.s.}\tag{50}$$

Also, by the method in Lemma 6, we have

$$\begin{aligned}&\widetilde{E}\left(\left|\int_0^t\int_O A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))\nabla\widetilde{u}_j^K\nabla\phi_i\,dxdr\right|^2\right)\\&\le\widetilde{E}\left(\int_0^t\int_O\left|A_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))\nabla\widetilde{u}_j^K\right|dxdr\right)^2\|\nabla\phi_i\|^2_{L^\infty(O)}\\&\le C\widetilde{E}\left(\int_0^t(1+\|(\overset{(K)}{\widetilde{\mathbf{u}}}(r))^s\|_{L^2(O)})\|\nabla\overset{(K)}{\widetilde{\mathbf{u}}}(r)\|_{L^2(O)}dr\right)^2\|\phi_i\|^2_{H^3(O)}\\&\quad+C\widetilde{E}\left(\int_0^t\|\overset{(K)}{\widetilde{\mathbf{u}}}(r)\|_{L^2(O)}\|\nabla(\overset{(K)}{\widetilde{\mathbf{u}}}(r))^s\|_{L^2(O)}dr\right)^2\|\phi_i\|^2_{H^3(O)}\le C.\end{aligned}\tag{51}$$

Combining (47),(48),(50) and (51), by the Vitali's convergence theorem (A.4. [12]), we can show that (45) holds. For rest three convergences of this lemma, the proof be completely referred to previous results(Lemma 10 [19], Lemma 16 [12], and Lemma 26 [18]), so we state the proof in a rather brief way.

Since $\overset{(K)}{\widetilde{\mathbf{u}}}\to\widetilde{\mathbf{u}}$ in $C^0([0,T];L^2_w(O))$, $\widetilde{P}$-a.s. then for every $\overline{\phi}_i\in L^2(O)$,

$$\lim_{K\to\infty}\left(\widetilde{u}_i^K(t),\overline{\phi}_i\right)_{L^2(O)}=\left(\underset{i}{\widetilde{u}}(t),\overline{\phi}_i\right)_{L^2(O)},\quad \widetilde{P}\text{-a.s.}$$

By the dominated convergence theorem, we conclude that

$$\lim_{K\to\infty}\int_0^T\left(\widetilde{u}_i^K(t)-\underset{i}{\widetilde{u}}(t),\overline{\phi}_i\right)^2_{L^2(O)}dt=0.$$

Since $\overset{(K)}{\widetilde{\mathbf{u}}}\to\widetilde{\mathbf{u}}$ in $C^0([0,T];L^2_w(O))$, $\widetilde{P}$-a.s. and $\widetilde{\mathbf{u}}$ is continuous at $t=0$, we deduce that $\lim_{K\to\infty}\left(\widetilde{u}_i^K(0),\overline{\phi}_i\right)_{L^2(O)}=\left(\underset{i}{\widetilde{u}}(0),\overline{\phi}_i\right)_{L^2(O)}$, $\widetilde{P}$-a.s. Then by the Vitali's convergence theorem, (43)–(44) hold for every $\overline{\phi}_i\in L^2(O)$.

For the limit (16), we mainly rely on

$$\begin{aligned}&\int_0^t\|\left(\sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))-\sigma_{ij}(\widetilde{\mathbf{u}}(r)),\overline{\phi}_i\right)_{L^2(O)}\|^2_{L(Y;L^2(O))}dr\\&\le\int_0^t\|\sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))-\sigma_{ij}(\widetilde{\mathbf{u}}(r))\|^2_{L(Y;L^2(O))}\|\overline{\phi}_i\|^2_{L^2(O)}dr\le C\|\overset{(K)}{\widetilde{\mathbf{u}}}-\widetilde{\mathbf{u}}\|^2_{L^2(0,T;L^2(O))}\|\overline{\phi}_i\|^2_{L^2(O)},\end{aligned}$$

and $\overset{(K)}{\widetilde{\mathbf{u}}}\to\widetilde{\mathbf{u}}$ in $L^2(0,T;L^2(O))$, $\widetilde{P}$-a.s. implies that

$$\lim_{K\to\infty}\int_0^t\|\left(\sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r))-\sigma_{ij}(\widetilde{\mathbf{u}}(r)),\overline{\phi}_i\right)_{L^2(O)}\|^2_{L(Y;L^2(O))}dr=0.$$

By the Vitali's convergence theorem, we obtain that for every $\overline{\phi}_i\in L^2(O)$,

$$\lim_{K\to\infty} \widetilde{E} \int_0^t \| \left(\sigma_{ij}(\overset{(K)}{\widetilde{\mathbf{u}}}(r)) - \sigma_{ij}(\widetilde{\mathbf{u}}(r)), \overline{\phi}_i \right)_{L^2(O)} \|^2_{L(Y;L^2(O))} dr = 0.$$

Then by the Itô isometry, we have

$$\lim_{K\to\infty} \widetilde{\mathbb{E}} \left| \int_0^t \left(\left(\sigma_{ij}(\widetilde{u}]^{(K)}(r)) - \sigma_{ij}(\widetilde{u}(r)) \right) d\widetilde{W}_j(r), \overline{\phi}_i \right)_{L^2(O)} \right|^2 = 0.$$

and by the dominated convergence theorem, (46) holds, then we finish the proof of this lemma. □

The proof of Theorem 1 is similar to the proof of Lemma 9.

Proof. Let us define

$$\Lambda_i(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}, \phi)(t) = \langle \underset{i}{\widetilde{u}}(0), \phi_i \rangle + \sum_{j=1}^n \int_0^t \langle div(A_{ij}(\widetilde{\mathbf{u}}(r)) \nabla \underset{j}{\widetilde{u}}(r)), \phi_i \rangle dr + \langle \sum_{j=1}^n \int_0^t \sigma_{ij}(\widetilde{\mathbf{u}}(r)) d\underset{j}{\widetilde{W}}(r), \phi_i \rangle,$$

for $t \in (0,T)$ and $1 \le i \le n$.

By the method in Lemma 9, for $\overline{\phi}_i \in L^2(O)$, $\phi \in H^3(O)$ with $\nabla\phi \cdot \nu = 0$ on ∂O, we have

$$\lim_{K\to\infty} \| \langle \widetilde{u}_i^K(t) - \underset{i}{\widetilde{u}}(t), \overline{\phi}_i \rangle \|_{L^2(\widetilde{\Omega}\times(0,T))} = 0,$$

$$\lim_{K\to\infty} \| \Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi) - \Lambda_i(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}, \phi) \|_{L^1(\widetilde{\Omega}\times(0,T))} = 0.$$

$\overset{(K)}{\widetilde{\mathbf{u}}}(t)$ satisfies that $\langle \widetilde{u}_i^K(t), \phi_i \rangle = \Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi)(t)$, $\widetilde{P}$-a.s. for every $t \in (0,T)$, $\phi \in H^1(O)$. In particular, for every $\phi \in H^3(O)$ with $\nabla\phi \cdot \nu = 0$ on ∂O,

$$\int_0^T \widetilde{E} |\langle \widetilde{u}_i^{(K)}(t), \phi_i \rangle - \Lambda_i^{(K)}(\overset{(K)}{\widetilde{\mathbf{u}}}, \widetilde{\mathbf{W}}, \phi)(t)| dt = 0.$$

Let $K \to \infty$, then $\int_0^T \widetilde{E} |\langle \underset{i}{\widetilde{u}}(t), \phi_i \rangle - \Lambda_i(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}, \phi)(t)| dt = 0$. By the density argument, it holds for every $\phi \in H^1(O)$. For a.e. $t \in [0,T], \omega \in \widetilde{\Omega}$, we deduce that $\langle \underset{i}{\widetilde{u}}(t), \phi_i \rangle - \Lambda_i(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}, \phi)(t) = 0, 1 \le i \le n$. By the definition of Λ_i, we conclude that for a.e. $t \in [0,T], \omega \in \widetilde{\Omega}$,

$$\langle \widetilde{u}_i(t), \phi_i \rangle = \langle \widetilde{u}_i(0), \phi_i \rangle + \sum_{j=1}^n \int_0^t \langle div(A_{ij}(\widetilde{\mathbf{u}}(r)) \nabla \widetilde{u}_j(r)), \phi_i \rangle dr + \langle \sum_{j=1}^n \int_0^t \sigma_{ij}(\widetilde{\mathbf{u}}(r)) d\widetilde{W}_j(r), \phi_i \rangle.$$

The system $(\widetilde{U}, \widetilde{\mathbf{W}}, \widetilde{\mathbf{u}})$ is a martingale solution of (1)–(3), and then we have shown Theorem 1. □

6. Conclusion

In this work, we have shown that a martingale solution exists to a superlinear stochastic cross-diffusion population system of Shigesada-Kawasaki-Teramoto type.

We consider the case when $1 < s < 2$ in (3), and in this situation, the diffusion matrix does not satisfy the local Lipschitz property. The key idea in showing a martingale solution exists is to regularize the diffusion matrix, and this regularization preserves the essential structure of the model.

We have shown that a martingale solution exists when $s > 2$ in (3). Once $s > 2$, the diffusion matrix itself satisfies the local Lipschitz property. We do not have to regularize the diffusion matrix in order to derive approximated solutions. But when $s > 2$, the uniform estimation can not be derived in a straightforward way. The key idea is to construct an auxiliary sequence to estimate approximated solutions.

The deterministic Shigesada-Kawasaki-Teramoto type population system has been widely studied in literatures, while few works are available concerning the stochastic counterpart. We have mentioned in the introduction that the entropy method is the main tool in the analysis of deterministic cross-diffusion systems. But unfortunately, so far it is very difficult to apply this well-known entropy method for stochastic cross-diffusion systems.

The stochastic Galerkin method has been applied in this work. A regularization of the entropy variable method has been adopted to show a martingale solution exists to a stochastic Shigesada-Kawasaki-Teramoto type population system [18].

In order to show martingale solutions exist to other stochastic cross-diffusion systems, we may have to upgrade existing approximation schemes, which is the central topic for future investigations. One step further, let us state some other future research directions.

We are aware that in this work, we focus on showing a martingale solution exists to a stochastic cross-diffusion system. Whether this martingale solution is unique or not remains unknown, and so far few uniqueness results have been derived for stochastic cross-diffusion systems. The number of martingale solutions for a stochastic cross-diffusion system is another major problem for future research.

In this paper, we have shown some regularity properties of the martingale solution. Concerning potential blow-up issues for this martingale solution, few results have been derived for stochastic cross-diffusion systems. This is another topic of great importance in future investigations.

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