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Article

Global martingale solutions for stochastic superquadratic cross-diffusion population systems of Shigesada-Kawasaki-Teramoto type

Xi Lin

Department of Mathematics and Physics, Guangzhou Maritime University, Guangzhou 510725, China; linxi@gzmtu.edu.cn

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Abstract: For a stochastic cross-diffusion population system with superquadratic transition rate, we show that a global martingale solution exists. The existence of a global martingale solution proof for a stochastic population system is quite different from the existence of a weak solution proof for a deterministic population system. For deterministic population systems, we apply the entropy method to show a weak solution exists. For stochastic population systems, we rely on the Galerkin approximation scheme to derive the sequence of approximated solutions. We apply the Itô formula to derive uniform estimates. After the tightness property be proved based on the estimation, a space changing result then be used to confirm the limit is a martingale solution of the cross-diffusion system. In the uniform estimation process, we notice that we have to estimate stochastic processes that are not in the finite dimensional space, otherwise we can not derive strong enough estimation for the tightness proof. We are not able to apply the Itô formula to stochastic processes that are not finitely dimensional processes. In this situation, we have to introduce an auxiliary sequence. The estimation of the approximated sequence has to be derived on the estimation of an auxiliary sequence, which is the key idea of this work. The nonnegative property for approximated solutions has been shown by the standard Stampacchia method.

Keywords: global martingale solutions; existence of solutions; Galerkin approximation; auxiliary sequence; tightness criterion; stochastic Shiegesada-Kawasaki-Teramoto (SKT) system; superquadratic cross-diffusion; population dynamics

1. Introduction

The dynamics and motions of interacting population species can largely be described by the cross-diffusion system. A well known example is the Shigesada-Kawasaki-Teramoto population system [1]. Generalized cross-diffusion models have also been derived when the dependence of the transition rates on the densities is nonlinear. In this work, we take the random influence of the environment into consideration.

Let us denote $\mathbf{u} = (u_1, ..., u_n)$, and consider

$$du_i - \mathrm{div}(\sum_{j=1}^{n} A_{ij}(\mathbf{u})\nabla u_j)dt = \sum_{j=1}^{n} \sigma_{ij}(\mathbf{u})dW_j(t) \quad \text{in} \quad \mathcal{O},\ t > 0,\ 1 \le i \le n, \tag{1}$$

with no-flux boundary and initial conditions

$$\sum_{j=1}^{n} A_{ij}(\mathbf{u})\nabla u_j \cdot \nu = 0 \quad \text{on } \partial\mathcal{O},\ t > 0,\ u_i(0) = u_i^0 \ \text{ in } \mathcal{O},\ 1 \leq i \leq n, \tag{2}$$

where $O \subset \mathrm{R}^d$, $d \leq 3$ is a bounded domain with Lipschitz boundary, ν is the exterior unit normal vector to ∂O and u_i^0 is a possibly random initial datum. The concentrations $u_i(\omega,x,t)$ are defined on $\Omega \times O \times [0,T]$, where $\omega \in \Omega$ represents the stochastic factor, $x \in O$ the spatial variable, and $t \in [0,T]$ the time index. $A(\mathbf{u}) = (A_{ij}(\mathbf{u}))$ is the diffusion matrix, and $\sigma(\mathbf{u}) = (\sigma_{ij}(\mathbf{u}))$ is the multiplicative noise term.

The diffusion matrix $A(\mathbf{u}) = (A_{ij}(\mathbf{u}))$ is

$$A_{ii}(\mathbf{u}) = a_{i0} + (s+1)a_{ii}u_i^s + \sum_{k=1,k\neq i}^{n} a_{ik}u_k^s, \text{ and } A_{ij}(\mathbf{u}) = s a_{ij} u_i u_j^{s-1}, \text{ if } i \neq j, \tag{3}$$

where $a_{i0} > 0$, $a_{ik} > 0$ and $s > 0$.

If $s = 2$, the existence of a global martingale solution to (1)−(3) has been shown [2]. In this work, we show that if $s > 2$, a global martingale solution also exists to (1)−(3).

We declare in the following discussion, C, C_i, α_i, β_i are positive constants independent of variables, with values subject to change. For each vector $\mathbf{u} = (u_1, ..., u_n)$, we adopt the notation $\mathbf{v} = \mathbf{u}^p$, $p > 0$, $\mathbf{v} = (v_1, ..., v_n)$, such that $v_i = u_i^p$.

The approximated solutions are derived by the Galerkin approximation [2]. A different approximation method has been applied, by trying to regularize the entropy variable [3,4]. In this work, we still consider the Galerkin approximation scheme. We fix an orthonormal basis $(e_k)_{k\geq 1}$ of $L^2(O)$ and a number $N \in N$ such that the space $H_N = \text{span}\ \{e_1, ..., e_N\}$ satisfies $H_N \subset H^1(O) \cap L^\infty(O)$. We introduce the projection operator $\Pi_N : L^2(O) \to H_N$, with

$$\Pi_N(\mathbf{v}) = \sum_{i=1}^{N} \langle \mathbf{v}, e_i \rangle e_i, \quad \mathbf{v} \in L^2(O).$$

The approximated solutions $\mathbf{u}^{(N)}$ are derived by the existence and uniqueness theorem of stochastic differential equations ([5], Theorem 3.1.1, original edition [6]). We apply the Itô formula to $\|\mathbf{u}^{(N)}(t)\|_{L^2(O)}$ to estimate $\mathbf{u}^{(N)}$. The major assumption for us to apply the Itô formula in the inner product computation is $\mathbf{u}^{(N)} \in H_N$. Then we show the approximated sequence $(\mathbf{u}^{(N)})_{N\in N}$ is tight in a topological space.

Once $s > 2$, in order to derive strong enough estimations for a tightness proof, we have to apply the Itô formula to $\|(\mathbf{u}^{(N)}(t))^{\frac{s}{2}}\|_{L^2(O)}$. But unfortunately, $(\mathbf{u}^{(N)}(t))^{\frac{s}{2}} \notin H_N$ in general. Original editions [3,4] adopt the Galerkin approximation scheme. Though $u_i^{(N)} \in H_N$, $\log u_i^{(N)} \notin H_N$ in general. The Galerkin approximation scheme has to be replaced by a regularization of the entropy variable method in these references. In the corrigendum [4], a detailed comparison between these two approximation methods has been provided.

How to estimate $(\mathbf{u}^{(N)}(t))^{\frac{s}{2}}$ is the novelty of this work. In the Lemma 6 and

Lemma 8, we will introduce an auxiliary sequence to estimate $(\mathbf{u}^{(N)}(t))^{\frac{s}{2}}$.

When $s=2$, the domain for each element of the diffusion matrix is all real numbers. The existence and uniqueness theorem can be directly applied to derive the approximated solutions of (1)–(3). But for general $s>2$ case, the domain for each element of the diffusion matrix may not be all real numbers, but only non-negative real numbers. The assumption for the existence and uniqueness theorem may not be satisfied. How to derive approximated solutions despite these difficulties is another novelty of this work.

The existence of a weak solution to a deterministic Shigesada-Kawasaki-Teramoto type population system has been investigated [7–12]. A weak solution has been shown exists to a Maxwell-Stefan type cross-diffusion system [13, 14]. The cross-diffusion system with the degeneration phenomenon has been considered and a weak solution also exists [15,16]. These papers mainly rely on a so-called entropy method.

The stochastic Shigesada-Kawasaki-Teramoto type cross-diffusion population system has been taken into consideration [2]. It has been proved that a martingale solution exists to (1)–(3) once $s=2$ in (3). The stochastic Galerkin method is the main tool in the existence proof, which has also been applied [17,18].

A martingale solution has also been proved exists to (1)–(3) once $s=1$ in (3) [3]. The approximated solutions have been derived by a regularization of the entropy variable method [3]. A global martingale solution has been shown exists to a stochastic cross-diffusion system with the volume-filling effect [4]. The idea is also to derive approximated solutions by this regularization of the entropy variable method.

2. Notations, stochastic background, assumptions and main results

Let (Ω,F,P) be a probability space with a complete right continuous filtration $F=(\mathrm{F}_t)_{t\geq 0}$. The space $L^2(O)$ is the vector space of all square integrable functions $\mathbf{u}:O\to R$ with the inner product $\langle\cdot,\cdot\rangle$, i.e., $\langle\mathbf{u},\mathbf{v}\rangle=\int_O \mathbf{u}\mathbf{v}dx$ if $\mathbf{u},\mathbf{v}\in L^2(O)$. The space $H^1(O)$ consists of every $\mathbf{u}\in L^2(O)$ such that the distributional derivatives $\partial\mathbf{u}/\partial x_1,\ldots,\partial\mathbf{u}/\partial x_n$ belong to $L^2(O)$. Let H be a Hilbert space, $L^2(\Omega;H)$ represents the space consisting of all H-valued random variables $\mathbf{u}$ with $E\|\mathbf{u}\|_H^2=\int\|\mathbf{u}(\omega)\|_H^2P(d)<\infty$.

In subsequent sections, H often refers to a space with variables time and space involved. The $L^2(O)$ norm of a vector-valued random variable $\mathbf{u}=(u_1,\ldots,u_n)$ is understood as $\|\mathbf{u}\|_{L^2(O)}^2=\sum_{i=1}^n\|u_i\|_{L^2(O)}^2$. We fix a Hilbert basis $(e_k)_{k\in N}$ of $L^2(O)$, and choose an orthonormal basis of any separable Hilbert space Y with orthonormal basis $(\eta_k)_{k\in N}$.

Let us denote

$$\mathcal{L}(Y;L^2(\mathcal{O}))=\{L:Y\to L^2(\mathcal{O})\text{ linear continuous: }\sum_{k=1}^{\infty}\|L\eta_k\|_{L^2(\mathcal{O})}^2<\infty\}$$

as the space of Hilbert-Schmidt operators from Y to $L^2(O)$ endowed with the norm $\|L\|_{L(Y;L^2(O))}^2=\sum_{k=1}^{\infty}\|L\eta_k\|_{L^2(O)}^2$.

Let $(\beta_{jk})_{1\le j\le n,k\in\mathbb{N}}$ be a sequence of independent one-dimensional Brownian motions and for $1 \le j \le n$,

$$W_j(x,t,\omega) = \sum_{k\in\mathbb{N}} \eta_k(x)\beta_{jk}(t,\omega), \quad \sigma_{ij}(\mathbf{u})dW_j(t) = \sum_{k,l\in\mathbb{N}} \sigma_{ij}^{kl}(\mathbf{u})e_l d\beta_{jk}(t),$$
$$\text{with } \sigma_{ij}^{kl}(\mathbf{u}) = (\sigma_{ij}(\mathbf{u})\eta_k, e_l)_{L^2(\mathcal{O})}, \tag{4}$$

and $\mathbf{W} = (W_1, ...,W_n)$ takes values in another separable Hilbert space Y_0 that $Y\subset Y_0$.

Let us give assumptions on multiplicative noise terms $\sigma = \sigma_{ij}(\mathbf{u},t,\omega) : L^2(O) \times[0,T]\times\to L(Y;L^2(O))$. Noise terms $\sigma = \sigma_{ij}(\mathbf{u},t,\omega)$ are assumed to be $B(L^2(O) \bigotimes[0,T]\bigotimes F;\ B(L(Y;L^2(O))))$- measurable and F-adapted with the property that for every $\mathbf{u},\mathbf{v}\in L^2(O)$ and $1 \le i,\ j \le n$,

$$\|\sigma_{ij}(\mathbf{u})\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))} \le C(1 + \|\mathbf{u}\|^2_{L^2(\mathcal{O})}), \quad \|u_i^{\frac{s}{2}-1}\sigma_{ij}(\mathbf{u})\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))} \le C(1 + \|\mathbf{u}^{\frac{s}{2}}\|^2_{L^2(\mathcal{O})}),$$
$$\|\sigma_{ij}(\mathbf{u}) - \sigma_{ij}(\mathbf{v})\|_{\mathcal{L}(Y;L^2(\mathcal{O}))} \le C\|\mathbf{u} - \mathbf{v}\|_{L^2(\mathcal{O})}, \quad \sum_{j=1}^{n} \|\sigma_{ij}(\mathbf{u})\|_{\mathcal{L}(Y;L^2(\mathcal{O}))} \le C\|u_i\|_{L^2(\mathcal{O})}. \tag{5}$$

Let us give the definition of the solution to (1)−(3).

Definition 1. *Let $T > 0$ be an arbitrary positive number, the system $(\widetilde{U},\widetilde{\mathbf{W}},\widetilde{\mathbf{u}})$ is a global martingale solution to (1)−(3) if $\widetilde{U} = (\widetilde{\Omega}, \widetilde{F}, \widetilde{P}, \widetilde{F})$ is a stochastic basis with filtration $\widetilde{F} = (\widetilde{F}_t)_{t\in(0,T)}$, $\widetilde{\mathbf{W}}$ is a cylindrical Wiener process, and $\widetilde{\mathbf{u}}(t) = (\widetilde{u}_1(t), ...,\widetilde{u}_n(t))$ is an $\widetilde{F}_t$-adapted stochastic process for every $t\in (0,T)$ such that for $1 \le i \le n$,*

$$\widetilde{u_i}\in L^2(\widetilde{\Omega};C^0([0,T];\ L^2_\omega(O))) \cap L^2(\widetilde{\Omega};\ L^2(0,T;\ H^1(O))).$$

The law of $\widetilde{u_i}(0)$ is the same as for u_i^0, and $\widetilde{\mathbf{u}}$ satisfies that for every $\phi_i\in H^1(O)$ and $1 \le i \le n$,

$$\langle\widetilde{u_i}(t), \phi_i\rangle = \langle\widetilde{u_i}(0), \phi_i\rangle - \sum_{j=1}^{n}\int_0^t \langle A_{ij}(\widetilde{\mathbf{u}}(r))\nabla\widetilde{u_j}(r), \nabla\phi_i\rangle dr + \langle\sum_{j=1}^{n}\int_0^t \sigma_{ij}(\widetilde{\mathbf{u}}(r))d\widetilde{W_j}(r), \phi_i\rangle.$$

The topological space C^0 ([0,T]; $L^2_\omega(O)$) represents all weakly continuous functions $\mathbf{u}$: $[0,T]\to L^2(O)$ having the property $sup_{0<t<T}\|\mathbf{u}(t)\|_{L^2(O)}< \infty$. Other notations of spaces and respective topologies have detailed explanation [2, 17], so we will present them directly.

The assumption (6) is a key factor in this existence analysis. There exists a sequence of positive real numbers $\pi = (\pi_1,...,\pi_n)$ such that

$$\pi_i a_{ij} = \pi_j a_{ji}, \quad (s + 1)a_{ii} > \frac{s^2}{4}\sum_{j=1,j\ne i}^{n} a_{ij}, \ \ for\ every\ \ 1 \le i, j \le n. \tag{6}$$

For the initial data and dimension d, we require that

$$u_i^0 \ge 0\ a.e.\text{ in } \mathcal{O},\ \mathbb{P}\text{-}a.s. \quad \mathbb{E}\|(\mathbf{u}^0)^{\frac{s}{2}}\|^p_{L^2(\mathcal{O})} < \infty, \quad p = \frac{24}{4-d}, \quad d \le 3. \tag{7}$$

After these preparations, let us give the main result Theorem 1.

Theorem 1. *(Existence of a global martingale solution) Let $T > 0$ be an arbitrary positive number, and let $\sigma = (\sigma_{ij})_{i,j=1}^{n}$ with $\sigma_{ij}: L^2(O)\times[0,T]\times\Omega\to L(Y;L^2(O))$. If (5), (6) and (7) hold, $s \ge 2$, then there exists a global non-negative martingale solution to (1)−(3), P-a.s.*

Firstly, we derive a sequence of approximated solutions through the Galerkin approximation. Let us consider

$$Pd\mathbf{u} = a(\mathbf{u})dt + b(\mathbf{u})dW(t), \quad t > 0, \quad u_i(0) = |\Pi_N(u_i^0)|, \quad 1 \le i \le n, \tag{8}$$

where $a = (a_1, ..., a_n) : H_N \to R^n$, $b = (b_{ij})_{1\le i,j\le n}$, and b_{ij}: $H_N \to L(Y; H_N)$,

$$a_i(\mathbf{u}) = \Pi_N div(\sum_{j=1}^{n} \pi_i M_{ij}(\mathbf{u})\nabla u_j), \quad b_{ij}(\mathbf{u}) = \pi_i \Pi_N \sigma_{ij}(\mathbf{u}), \tag{9}$$

with $P = diag(\pi_1, ..., \pi_n)$ a diagonal matrix. The diffusion matrix $M(\mathbf{u}) = (M_{ij}(\mathbf{u}))$ is

$$M_{ii}(\mathbf{u}) = a_{i0} + (s+1)a_{ii}|u_i|^s + \sum_{k=1,k\neq i}^{n} a_{ik}|u_k|^s, \text{ and } M_{ij}(\mathbf{u}) = s a_{ij} u_i \cdot |u_j|^{s-1}, \text{ if } i \neq j. \tag{10}$$

We remark that $\Pi_N(u_i^0)$ may have no sign, so we instead consider the initial value $u_i(0) = |\Pi_N(u_i^0)|$, $1 \le i \le n$, and the Hilbert-Schmidt operator $\sigma_{ij}(\mathbf{u})$, $1 \le i, j \le n$ has been projected.

Once a non-negative P-$a.s.$ strong solution(in the probabilistic sense) denoted as $\mathbf{u}^{(N)} = (u_1^{(N)}, ..., u_n^{(N)})$ exists to (8)−(10), i.e.,

$$\pi_i u_i^{(N)}(t) = \pi_i u_i^{(N)}(0) + \int_0^t a_i(\mathbf{u}^{(N)}(r))dr + \int_0^t \sum_{j=1}^{n} b_{ij}(\mathbf{u}^{(N)}(r))dW_j(r), \quad 1 \le i \le n,$$

then $u^{(N)}$ is also a strong solution(in the probabilistic sense) to

$$du_i - \Pi_N div(\sum_{j=1}^{n} A_{ij}(\mathbf{u})\nabla u_j)dt = \sum_{j=1}^{n} \Pi_N(\sigma_{ij}(\mathbf{u}))dW_j(t), \ u_i(0) = |\Pi_N(u_i^0)|, \ 1 \le i \le n. \tag{11}$$

We rely on the existence and uniqueness theorem [5,6] to derive the strong solution (in the probabilistic sense) of (8)−(10). By a standard Stampacchia type argument, we can show that $\mathbf{u}^{(N)}$ is non-negative P-a.s.

The second step is to provide uniform estimates of those approximated solutions. Then we show that the approximated sequence is tight in a certain topological space.

The third step is, by applying some fundamental tools in stochastic analysis [2–4, 19], we find another sequence possessing same laws to the existing one, and show that this sequence indeed converges to a martingale solution of (1)−(3).

We rely on the Itô formula to derive uniform estimates of approximated solutions in the second step. In the end of this section, we present the Itô formula we use in this work [20,21].

Lemma 1. *Let $V \subset H \subset V'$, and let U be a separable Hilbert space, $\mathbf{X}_0 \in L^2(\Omega; H)$. Let $a \in L^2(\Omega\times(0,T), V')$, $b \in L^2(\Omega\times(0,T), L(U,H))$ be progressively measurable. Let us define the stochastic process.*

$$\mathbf{X}(t) = \mathbf{X}_0 + \int_0^t a(r)dr + \int_0^t b(r)d\mathbf{W}(r),$$

then for every 0 < t < T, we have

$$\frac{1}{2}\|\mathbf{X}(t)\|_H^2 = \frac{1}{2}\|\mathbf{X}_0\|_H^2 + \int_0^t \langle a(r), \mathbf{X}\rangle_{V',V} dr + \frac{1}{2}\int_0^t \|b(r)\|_{\mathcal{L}(U,H)}^2 dr + \int_0^t (\mathbf{X}, b(r)d\mathbf{W}(r))_H.$$

3. Matrix analysis

We mention that for every $1 \le i, j \le n$, the relation $\pi_i a_{ij} = \pi_j a_{ji}$ holds for matrix analysis Lemmas 2–3, with the application of this condition implicitly contained in computations. Also, $s \ge 2$.

Lemma 2. *For every $z = (z_1, z_2, ..., z_n) \in R^n$ and $u = (u_1, u_2, ..., u_n) \in R^n$, there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ such that*

$$\sum_{i,j=1}^n \pi_i M_{ij}(\mathbf{u}) z_i z_j \ge \alpha_1 \sum_{i=1}^n z_i^2 + \alpha_2 \sum_{i=1}^n |u_i|^s z_i^2,$$

Proof. Let us define a matrix $\bar{M}(\mathbf{u}) = (\bar{M}_{ij}(\mathbf{u}))$, with

$\bar{M}_{ii}(\mathbf{u}) = \frac{s^2}{4}\sum_{k=1,k\neq i}^n a_{ik}|u_i|^s + \sum_{k=1,k\neq i}^n a_{ik}|u_k|^s = \sum_{k=1,k\neq i}^n (\frac{s^2}{4} a_{ik}|u_i|^s + a_{ik}|u_k|^s)$, and $\bar{M}_{ij}(\mathbf{u}) = M_{ij}(\mathbf{u})\ if\ i \neq j$.

By assumption (6), $(s+1)a_{ii} > \frac{s^2}{4}\sum_{k=1,k\neq i}^n a_{ik}$, there exist positive constants $\{\beta_i\}_{i=1,...,n}$, such that for every $\pi_i > 0$, $\pi_i M_{ii}(\mathbf{u}) - \pi_i \bar{M}_{ii}(\mathbf{u}) \ge \pi_i a_{i0} + \beta_i |u_i|^s$, and

$$\sum_{i,j=1}^n \pi_i M_{ij}(\mathbf{u}) z_i z_j \ge \sum_{i,j=1}^n \pi_i \bar{M}_{ij}(\mathbf{u}) z_i z_j + \sum_{i=1}^n \pi_i a_{i0} z_i^2 + \sum_{i=1}^n \beta_i |u_i|^s z_i^2. \tag{12}$$

Provided that $\sum_{i,j=1}^n \pi_i \bar{M}_{ij}(\mathbf{u}) z_i z_j \ \ge\ 0$, we have

$$\sum_{i,j=1}^n \pi_i M_{ij}(\mathbf{u}) z_i z_j \ \ge \sum_{i=1}^n \pi_i a_{i0} z_i^2 + \sum_{i=1}^n \beta_i |u_i|^s z_i^2,$$

choose $\alpha_1 = \min\{\pi_i a_{i0}: 1 \le i \le n\}$, $\alpha_2 = \min\{\beta_i: 1 \le i \le n\}$, we can show this lemma.

We are left to show that $\sum_{i,j=1}^n \pi_i \bar{M}_{ij}(\mathbf{u}) z_i z_j \ge 0$. Let us denote $\bar{M}_{ii}^k(\mathbf{u}) = \frac{s^2}{4} a_{ik}|u_i|^s + a_{ik}|u_k|^s$, $k \neq i$, then $\bar{M}_{ii}(\mathbf{u}) = \sum_{k=1,k\neq i}^n \bar{M}_{ii}^k(\mathbf{u})$, and therefore,

$$\sum_{i,j=1}^{n}\pi_i\bar{M}_{ij}(\mathbf{u})z_iz_j=\sum_{i=1}^{n}\pi_i\bar{M}_{ii}(\mathbf{u})z_i^2+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\pi_i\bar{M}_{ij}(\mathbf{u})z_iz_j$$
$$=\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}(\pi_i\bar{M}_{ii}^{j}(\mathbf{u})z_i^2+\pi_i\bar{M}_{ij}(\mathbf{u})z_iz_j)$$
$$=\sum_{i=1}^{n}\sum_{j=1,j<i}^{n}(\pi_i\bar{M}_{ii}^{j}(\mathbf{u})z_i^2+\pi_i\bar{M}_{ij}(\mathbf{u})z_iz_j)+\sum_{i=1}^{n}\sum_{j=1,j>i}^{n}(\pi_i\bar{M}_{ii}^{j}(\mathbf{u})z_i^2+\pi_i\bar{M}_{ij}(\mathbf{u})z_iz_j)$$
$$=\sum_{i=1}^{n}\sum_{j=1,j<i}^{n}(\pi_i\bar{M}_{ii}^{j}(\mathbf{u})z_i^2+\pi_i\bar{M}_{ij}(\mathbf{u})z_iz_j)+\sum_{i=1}^{n}\sum_{j=1,j<i}^{n}(\pi_j\bar{M}_{jj}^{i}(\mathbf{u})z_j^2+\pi_j\bar{M}_{ji}(\mathbf{u})z_jz_i)$$
$$=\sum_{i=1}^{n}\sum_{j=1,j<i}^{n}\Big[\pi_i\bar{M}_{ii}^{j}(\mathbf{u})z_i^2+(\pi_i\bar{M}_{ij}(\mathbf{u})+\pi_j\bar{M}_{ji}(\mathbf{u}))z_iz_j+\pi_j\bar{M}_{jj}^{i}(\mathbf{u})z_j^2\Big].$$

Let us show that if $i\neq j$, either $u_i\neq 0$ or $u_j\neq 0$, then $\pi_i\bar{M}_{ii}^{j}(\mathbf{u})z_i^2+(\pi_i\bar{M}_{ij}(\mathbf{u})+\pi_j\bar{M}_{ji}(\mathbf{u}))z_iz_j+\pi_j\bar{M}_{jj}^{i}(\mathbf{u})z_j^2\geq 0$, which is equivalent to show

$$(\pi_i\bar{M}_{ij}(\mathbf{u})+\pi_j\bar{M}_{ji}(\mathbf{u}))^2\leq 4\pi_i\bar{M}_{ii}^{j}(\mathbf{u})\pi_j\bar{M}_{jj}^{i}(\mathbf{u}).\tag{13}$$

We check that

$$\pi_i\bar{M}_{ij}(\mathbf{u})+\pi_j\bar{M}_{ji}(\mathbf{u})=s\pi_ia_{ij}(u_i|u_j|^{s-1}+u_j|u_i|^{s-1}),$$

since $u_iu_j|u_i|^{s-1}|u_j|^{s-1}\leq|u_i|^s|u_j|^s$, so

$$(\pi_i\bar{M}_{ij}(\mathbf{u})+\pi_j\bar{M}_{ji}(\mathbf{u}))^2\leq\pi_i^2a_{ij}^2(s^2|u_i|^{2s-2}u_j^2+2s^2|u_i|^s|u_j|^s+s^2u_i^2|u_j|^{2s-2}).$$

Also,

$$4\pi_i\bar{M}_{ii}^{j}(\mathbf{u})\pi_j\bar{M}_{jj}^{i}(\mathbf{u})=\pi_i^2a_{ij}^2(s^2|u_i|^{2s}+(\frac{s^4}{4}+4)|u_i|^s|u_j|^s+s^2|u_j|^{2s}).$$

So long as

$$|u_i|^{2s}+|u_j|^{2s}-|u_i|^{2s-2}u_j^2-u_i^2|u_j|^{2s-2}=(|u_i|^{2s-2}-|u_j|^{2s-2})(u_i^2-u_j^2),$$

$s>2$, thus

$$|u_i|^{2s}+|u_j|^{2s}-|u_i|^{2s-2}u_j^2-u_i^2|u_j|^{2s-2}\geq 0,$$

so

$$s^2|u_i|^{2s}+s^2|u_j|^{2s}\geq s^2|u_i|^{2s-2}u_j^2+s^2u_i^2|u_j|^{2s-2}.$$

The fact that $\frac{s^4}{4}+4\geq 2s^2$ indicates that $(\frac{s^4}{4}+4)|u_i|^s|u_j|^s\geq 2s^2|u_i|^s|u_j|^s$, and we conclude that (13) holds. If $i\neq j$, $u_i=u_j=0$, then

$$\pi_i\bar{M}_{ii}^{j}(\mathbf{u})z_i^2+(\pi_i\bar{M}_{ij}(\mathbf{u})+\pi_j\bar{M}_{ji}(\mathbf{u}))z_iz_j+\pi_j\bar{M}_{jj}^{i}(\mathbf{u})z_j^2=0,$$

and we finish the proof of this lemma. □

In the next section, we will consider an auxiliary system. Its diffusion matrix $A^H(\mathbf{v})=(A_{ij}^H(\mathbf{v}))$ is $A_{ij}^H(\mathbf{v})=a_{i0}+(s+1)a_{ii}v_i^2+\sum_{k\neq i}^{n}a_{ik}v_k^2$, if $i=j$, and

$A^H_{ij}(\mathbf{v}) = sa_{ij}v_iv_j, \, if \; i \neq j.$

Lemma 3. *For every* $\boldsymbol{z} = (z_1,z_2, ...,z_n) \in R^n$ *and* $\boldsymbol{v} = (v_1,v_2, ...,v_n)$, $v_i \geq 0$, $1 \leq i \leq n$, *there exist constants* $\alpha_1 > 0$, $\alpha_2 > 0$ *such that*

$$\sum_{i,j=1}^{n} \pi_i A^H_{ij}(\mathbf{v}) z_i z_j \geq \alpha_1 \sum_{i=1}^{n} z_i^2 + \alpha_2 \sum_{i=1}^{n} v_i^2 z_i^2.$$

Proof. We define a new matrix $\bar{A}^H(\mathbf{v}) = (\bar{A}^H_{ij}(\mathbf{v}))$, with

$$\bar{A}^H_{ii}(\mathbf{v}) = \frac{s^2}{4}\sum\nolimits_{k=1,k\neq i}^{n} a_{ik}v_i^2 + \sum\nolimits_{k=1,k\neq i}^{n} a_{ik}v_k^2 = \sum\nolimits_{k=1,k\neq i}^{n} (\frac{s^2}{4}a_{ik}v_i^2 + a_{ik}v_k^2), \text{ and } \bar{A}^H_{ij}(\mathbf{v}) = A^H_{ij}(\mathbf{v}) \; if \; i\neq j.$$

Based on the assumption $(s+1)a_{ii} > \frac{s^2}{4}\sum_{k=1,k\neq i}^{n} a_{ik}$, there exist positive constants $\{\beta_i\}_{i=1,...,n}$, such that for every $\pi_i > 0$, $\pi_i A^H_{ii}(\mathbf{v}) - \pi_i \bar{A}^H_{ii}(\mathbf{v}) \geq \pi_i a_{i0} + \beta_i v_i^2$, and

$$\sum_{i,j=1}^{n} \pi_i A^H_{ij}(\mathbf{v}) z_i z_j \geq \sum_{i,j=1}^{n} \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j + \sum_{i=1}^{n} \pi_i a_{i0} z_i^2 + \sum_{i=1}^{n} \beta_i v_i^2 \, z_i^2. \tag{14}$$

Provided that $\sum_{i,j=1}^{n} \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j \geq 0$, we have $\sum_{i,j=1}^{n} \pi_i A^H_{ij}(\mathbf{v}) z_i z_j \geq \sum_{i=1}^{n} \pi_i a_{i0} z_i^2 + \sum_{i=1}^{n} \beta_i {v_i}^2 z_i^2$. Choose $\alpha_1 = \min\{\pi_i a_{i0}: 1 \leq i \leq n\}$, $\alpha_2 = \min\{\beta_i: 1 \leq i \leq n\}$, we can show this lemma.

We are left to prove that $\sum_{i,j=1}^{n} \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j \geq 0$. Let us denote $(\bar{A}^H_{ii})^k(\mathbf{v}) = \frac{s^2}{4} a_{ik} v_i^2 + a_{ik} v_k^2$, $k \neq i$, then $\bar{A}^H_{ii}(\mathbf{v}) = \sum_{k=1,k\neq i}^{n} (\bar{A}^H_{ii})^k(\mathbf{v})$, and therefore,

$$\begin{aligned}
\sum_{i,j=1}^{n} \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j &= \sum_{i=1}^{n} \pi_i \bar{A}^H_{ii}(\mathbf{v}) z_i^2 + \sum_{i=1}^{n} \sum\nolimits_{j=1,j\neq i}^{n} \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j \\
&= \sum_{i=1}^{n} \sum_{j=1,j\neq i}^{n} (\pi_i (\bar{A}^H_{ii})^j(\mathbf{v}) z_i^2 + \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j) \\
&= \sum_{i=1}^{n} \sum_{j=1,j<i}^{n} (\pi_i (\bar{A}^H_{ii})^j(\mathbf{v}) z_i^2 + \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j) + \sum_{i=1}^{n} \sum_{j=1,j>i}^{n} (\pi_i (\bar{A}^H_{ii})^j(\mathbf{v}) z_i^2 + \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j) \\
&= \sum_{i=1}^{n} \sum_{j=1,j<i}^{n} (\pi_i (\bar{A}^H_{ii})^j(\mathbf{v}) z_i^2 + \pi_i \bar{A}^H_{ij}(\mathbf{v}) z_i z_j) + \sum_{i=1}^{n} \sum_{j=1,j<i}^{n} (\pi_j (\bar{A}^H_{jj})^i(\mathbf{v}) z_j^2 + \pi_j \bar{A}^H_{ji}(\mathbf{v}) z_j z_i) \\
&= \sum_{i=1}^{n} \sum_{j=1,j<i}^{n} \Big[\pi_i (\bar{A}^H_{ii})^j(\mathbf{v}) z_i^2 + (\pi_i \bar{A}^H_{ij}(\mathbf{v}) + \pi_j \bar{A}^H_{ji}(\mathbf{v})) z_i z_j + \pi_j (\bar{A}^H_{jj})^i(\mathbf{v}) z_j^2\Big].
\end{aligned}$$

Let us show that if $i \neq j$, either $v_i > 0$ or $v_j > 0$, then

$$\pi_i (\bar{A}^H_{ii})^j(\mathbf{v}) z_i^2 + (\pi_i \bar{A}^H_{ij}(\mathbf{v}) + \pi_j \bar{A}^H_{ji}(\mathbf{v})) z_i z_j + \pi_j (\bar{A}^H_{jj})^i(\mathbf{v}) z_j^2 \geq 0,$$

which is equivalent to show

$$(\pi_i \bar{A}^H_{ij}(\mathbf{v}) + \pi_j \bar{A}^H_{ji}(\mathbf{v}))^2 \leq 4\pi_i (\bar{A}^H_{ii})^j(\mathbf{v}) \pi_j (\bar{A}^H_{jj})^i(\mathbf{v}). \tag{15}$$

Since

$$(\frac{s^2}{4} v_i^2 + v_j^2)(v_i^2 + \frac{s^2}{4} v_j^2) \geq (s v_i v_j) \cdot (s v_i v_j) = s^2 v_i^2 v_j^2,$$

and $\pi_i\bar{A}_{ij}^H(\mathbf{v})+\pi_j\bar{A}_{ji}^H(\mathbf{v})=2s\pi_i a_{ij}v_iv_j,\ 4\pi_i(\bar{A}_{ii}^H)^j(\mathbf{v})\pi_j(\bar{A}_{jj}^H)^i(\mathbf{v})=4\pi_i^2a_{ij}^2(\frac{s^2}{4}v_i^2+v_j^2)(v_i^2+\frac{s^2}{4}v_j^2)$, we directly show that (15) holds.

If $i\neq j,\quad v_i=v_j=0,$ then $\pi_i(\bar{A}_{ii}^H)^j(v)z_i^2+(\pi_i\bar{A}_{ij}^H(v)+\pi_j\bar{A}_{ji}^H(v))z_iz_j+\pi_j(\bar{A}_{jj}^H)^i(v)z_j^2=0$, and we finish the proof of this lemma. □

4. Stochastic Galerkin approximation

In the Lemma 4, we show that a strong solution $\mathbf{u}^{(N)}(t)$ exists to (8)−(10). In the Lemma 5, we show that $\mathbf{u}^{(N)}(t)$ is non-negative, P-a.s. thus $\mathbf{u}^{(N)}(t)$ is also a strong solution of (11). In the Lemma 6, we consider the property of $(\mathbf{u}^{(N)})^{\frac{s}{2}}$, as $N\to\infty$. Though $(\mathbf{u}^{(N)})^{\frac{s}{2}}\notin H_N$, we can still estimate $(\mathbf{u}^{(N)})^{\frac{s}{2}}$ in the Lemma 8, and the Lemma 6 provides the main tool.

Lemma 4. *For every $T>0$, there exists a unique strong (in the probabilistic sense) solution $\mathbf{u}^{(N)}(t)\in H_N$, $0<t<T$ to (8)−(10), $P-a.s.$*

Proof. Let $R>0,\ T>0,\ \omega\in\Omega$ *and let* $\mathbf{y}=(y_1,...,y_n)\in R^n,\ \mathbf{z}=(z_1,...,z_n)\in R^n$, and $\mathbf{y},\mathbf{z}\in H_N$ with $\|\mathbf{y}\|_{H_N},\|\mathbf{z}\|_{H_N}\leq R$. We notice that

$$M_{ij}(\mathbf{y})-M_{ij}(\mathbf{z})=(s+1)a_{ii}(|y_i|^s-|z_i|^s)+\sum_{k=1,k\neq i}^{n}a_{ik}(|y_k|^s-|z_k|^s),\quad i=j,$$

$$M_{ij}(\mathbf{y})-M_{ij}(\mathbf{z})=sa_{ij}(y_i|y_j|^{s-1}-z_i|z_j|^{s-1})=sa_{ij}(y_i-z_i)|y_j|^{s-1}+sa_{ij}(|y_j|^{s-1}-|z_j|^{s-1})z_i,i\neq j,$$

plus in finite dimensional space H_N, the assumption that $\|\mathbf{y}\|_{H_N},\|\mathbf{z}\|_{H_N}\leq R$ is equivalent to the fact that for a.e. $x\in O$, $|y_i|,|z_i|$ are bounded uniformly by a positive constant.

There exists $\zeta_i(x),\ \lambda_i(x)\geq 0$, such that for $x\in O$, $\min\{|y_i(x)|,|z_i(x)|\}\leq\zeta_i(x)$, $\lambda_i(x)\leq\max\{|y_i(x)|,|z_i(x)|\}$, with

$$\||y_i|^s-|z_i|^s\|_{L^2(\mathcal{O})}=\|s(|y_i|-|z_i|)\zeta_i^{s-1}\|_{L^2(\mathcal{O})}\&\leq\|s\zeta_i^{s-1}\|_{L^\infty(\mathcal{O})}\||y_i|-|z_i|\|_{L^2(\mathcal{O})}\leq C\|y_i-z_i\|_{L^2(\mathcal{O})},$$

and

$$\||y_i|^{s-1}-|z_i|^{s-1}\|_{L^2(\mathcal{O})}=\|(s-1)(|y_i|-|z_i|)\lambda_i^{s-2}\|_{L^2(\mathcal{O})}$$
$$\leq\|(s-1)\lambda_i^{s-2}\|_{L^\infty(\mathcal{O})}\||y_i|-|z_i|\|_{L^2(\mathcal{O})}\leq C\|y_i-z_i\|_{L^2(\mathcal{O})}.$$

If $i=j$, then

$$\|M_{ij}(\mathbf{y})-M_{ij}(\mathbf{z})\|_{L^2(\mathcal{O})}^2=\|(s+1)a_{ii}(|y_i|^s-|z_i|^s)+\sum_{k=1,k\neq i}^{n}a_{ik}(|y_k|^s-|z_k|^s)\|_{L^2(\mathcal{O})}^2$$
$$\leq C(\||y_i|^s-|z_i|^s\|_{L^2(\mathcal{O})}^2+\sum_{k=1,k\neq i}^{n}\||y_k|^s-|z_k|^s\|_{L^2(\mathcal{O})}^2)\leq C\sum_{i=1}^{n}\|y_i-z_i\|_{L^2(\mathcal{O})}^2=C\|\mathbf{y}-\mathbf{z}\|_{L^2(\mathcal{O})}^2,$$

and if i ≠ j, then

$$\|M_{ij}(\mathbf{y})-M_{ij}(\mathbf{z})\|_{L^2(\mathcal{O})}^2=\|sa_{ij}(y_i-z_i)|y_j|^{s-1}+sa_{ij}(|y_j|^{s-1}-|z_j|^{s-1})z_i\|_{L^2(\mathcal{O})}^2$$
$$\leq C(\|(y_i-z_i)|y_j|^{s-1}\|_{L^2(\mathcal{O})}^2+\|(|y_j|^{s-1}-|z_j|^{s-1})z_i\|_{L^2(\mathcal{O})}^2)\leq$$
$$C\||y_j|^{s-1}\|_{L^\infty(\mathcal{O})}^2\|y_i-z_i\|_{L^2(\mathcal{O})}^2+C\|z_i\|_{L^\infty(\mathcal{O})}^2\||y_j|^{s-1}-|z_j|^{s-1}\|_{L^2(\mathcal{O})}^2\leq$$
$$C\sum_{i=1}^{n}\|y_i-z_i\|_{L^2(\mathcal{O})}^2=C\|\mathbf{y}-\mathbf{z}\|_{L^2(\mathcal{O})}^2.$$

By the Lemma 2, if for every $1 \le i \le n$, $u_i \ge 0$, then $PA(\mathbf{u})$ is positive semi-definite, thus $PM(\mathbf{y})$, $\mathbf{y} \in \mathrm{R}^n$ is positive semi-definite. Also by the fact that norms are equivalent in finite dimensional spaces, such like H_N in this case, it follows that $\|\nabla(y_i - z_i)\|_{L^2(O)} \le \|y_i - z_i\|_{H^1(O)} \le C\|y_i - z_i\|_{H_N} \le C\|\mathbf{y} - \mathbf{z}\|_{H_N}$, thus

$$\langle a(\mathbf{y}) - a(\mathbf{z}), \mathbf{y} - \mathbf{z}\rangle = -\sum_{i,j=1}^{n} \int_{\mathcal{O}} \pi_i M_{ij}(\mathbf{y}) \nabla(y_i - z_i) \cdot \nabla(y_j - z_j) dx$$

$$+ \sum_{i,j=1}^{n} \int_{\mathcal{O}} \pi_i (M_{ij}(\mathbf{z}) - M_{ij}(\mathbf{y})) \nabla(y_i - z_i) \cdot \nabla z_j dx$$

$$\le C \sum_{i,j=1}^{n} \|M_{ij}(\mathbf{y}) - M_{ij}(\mathbf{z})\|_{L^2(\mathcal{O})} \|\nabla(y_i - z_i)\|_{L^2(\mathcal{O})} \|\nabla z_j\|_{L^\infty(\mathcal{O})} \le C\|\mathbf{y} - \mathbf{z}\|_{H_N}^2,$$

and $\|b(\mathbf{y}) - b(\mathbf{z})\|_{L(Y;H_N)}^2 \le C\|\sigma(\mathbf{y}) - \sigma(\mathbf{z})\|_{L(Y;H_N)}^2 \le C\|\mathbf{y} - \mathbf{z}\|_{H_N}^2$. To verify the weak coercivity condition, we take $\mathbf{y} \in H_N with \|\mathbf{y}\|_{H_N} \le R$, and

$$\langle a(\mathbf{y}), \mathbf{y}\rangle + \|b(\mathbf{y})\|_{\mathcal{L}(Y;H_N)}^2 = -\sum_{i,j=1}^{n} \int_{\mathcal{O}} \pi_i M_{ij}(\mathbf{y}) \nabla y_i \cdot \nabla y_j dx + \|P\sigma(\mathbf{y})\|_{\mathcal{L}(Y;H_N)}^2 \le C(1 + \|\mathbf{y}\|_{H_N}^2).$$

The existence and uniqueness result indicates that for every $N \in N$, a unique strong solution (in the probabilistic sense) $\mathbf{u}^{(N)}$ to (8)–(10) exists. □

Let us replace the diffusion matrix $M(\mathbf{u}) = (M_{ij}(\mathbf{u}))$ in (10) by

$$\begin{gathered} M_{ij}^{+}(\mathbf{u}) = a_{i0} + (s+1)a_{ii}|u_i|^s + \sum_{k=1, k\neq i}^{n} a_{ik}|u_k|^s,\ if\ i = j, \\ M_{ij}^{+}(\mathbf{u}) = s a_{ij} u_i^{+} \cdot |u_j|^{s-1},\ if\ i \neq j, \end{gathered} \tag{16}$$

where $z^+ = \max\{0, z\}$ is the positive part of $z \in R$. We try to apply the Stampacchia truncation method. This truncation method has also been applied [2,22].

Lemma 5. *For every* $1 \le i \le n$, $u_i^{(N)}(t) \ge 0$, P*-a.s.* and $\boldsymbol{u}^{(N)}(t)$ *is also a strong solution (in the probabilistic sense) to (11).*

Proof. The idea of the proof is to approximate the test function $f(z) = z^- = \max\{0, -z\}$, for $z \in R$ and to use Itô formula. We define the following functions: for $\varepsilon > 0$,

$$f_\varepsilon(z) = -z, if\ z \le -\varepsilon,$$

and

$$f_\varepsilon(z) = -3(\frac{z}{\varepsilon})^4 z - 8(\frac{z}{\varepsilon})^3 z - 6(\frac{z}{\varepsilon})^2 z, if -\varepsilon \le z \le 0,$$

and

$$f_\varepsilon(z) = 0, if\ z \ge 0.$$

Then f_ε has at most linear growth, i.e. $|f_\varepsilon(z)| \le C|z|$ for every $z \in R$. $f_\varepsilon^{'}$ and $\psi_\varepsilon = f_\varepsilon f_\varepsilon^{''} + (f_\varepsilon^{'})^2$ are bounded in R.

We set $F_\varepsilon(v) = \int_O f_\varepsilon(v(x))^2 dx$, for square-integrable functions $v: O \to R$. We replace the diffusion coefficients $M_{ij}(\mathbf{u}^{(N)})$ by the modified coefficients in (16). Observe that generally, $M_{ij}^{+}(\mathbf{u}) \neq M_{ij}(\mathbf{u})$, but if $u_i \ge 0$ for every $1 \le i \le n$, then we obtain that $M_{ij}^{+}(\mathbf{u}) = M_{ij}(\mathbf{u})$.

By the Itô formula, we have

$$F_\varepsilon(u_i^{(N)}(t)) = F_\varepsilon(u_i^{(N)}(0)) + 2\int_0^t \int_{\mathcal{O}} f_\varepsilon(u_i^{(N)}) f_\varepsilon'(u_i^{(N)}) \Pi_N(\sum_{j=1}^n \sigma_{ij}(\mathbf{u}^{(N)}))dxdW_j(r) - \\ 2\int_0^t \int_{\mathcal{O}} \psi_\varepsilon(u_i^{(N)}) \sum_{j=1}^n M_{ij}^{+}(\mathbf{u}^{(N)}) \nabla u_i^{(N)} \nabla u_j^{(N)} dxdr + \\ \int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty \psi_\varepsilon(u_i^{(N)}) e_k e_l \sigma_{ij}^{mk}(\mathbf{u}^{(N)}) \sigma_{ij}^{ml}(\mathbf{u}^{(N)}) dxdr = I_{\varepsilon,0}^{(N)} + I_{\varepsilon,1}^{(N)} + I_{\varepsilon,2}^{(N)} + I_{\varepsilon,3}^{(N)}, \tag{17}$$

with notations in $I_{\varepsilon,3}^{(N)}$ refer to (4).

Let us show that the integral $I_{\varepsilon,2}^{(N)}$ is non-positive. Indeed, we write

$$I_{\varepsilon,2}^{(N)} = -2\int_0^t \int_{\mathcal{O}} \psi_\varepsilon(u_i^{(N)}) M_{ii}^{+}(\mathbf{u}^{(N)}) |\nabla u_i^{(N)}|^2 dxdr \\ -2\int_0^t \int_{\mathcal{O}} \psi_\varepsilon(u_i^{(N)}) \sum_{j\neq i} M_{ij}^{+}(\mathbf{u}^{(N)}) \nabla u_i^{(N)} \nabla u_j^{(N)} dxdr. \tag{18}$$

The first term on the right-hand-side of (18) is clearly non-positive, the second term vanishes since $\psi_\varepsilon(u_i^{(N)}) = 0$ if $u_i^{(N)} \geq 0$, and $M_{ij}^{+}(\mathbf{u}^{(N)}) = 0$ if $u_i^{(N)} \leq 0$. This shows that $I_{\varepsilon,2}^{(N)} \leq 0$. Taking the mathematical expectation in (17), the stochastic integral vanishes, and

$$\mathbb{E}F_\varepsilon(u_i^{(N)}(t)) \leq \mathbb{E}F_\varepsilon(u_i^{(N)}(0)) + \mathbb{E}\int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty \psi_\varepsilon(u_i^{(N)}) e_k e_l \sigma_{ij}^{mk}(\mathbf{u}^{(N)}) \sigma_{ij}^{ml}(\mathbf{u}^{(N)}) dxdr. \tag{19}$$

It is shown [22] that as $\varepsilon \to 0$, P-*a.s.*

$$\mathbb{E}F_\varepsilon(u_i^{(N)}(t)) \to \mathbb{E}\|(u_i^{(N)}(t))^-\|_{L^2(\mathcal{O})}^2, \quad \mathbb{E}F_\varepsilon(u_i^{(N)}(0)) \\ \to \mathbb{E}\|(u_i^{(N)}(0))^-\|_{L^2(\mathcal{O})}^2, \mathbb{E}\int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty \psi_\varepsilon(u_i^{(N)}) e_k e_l \sigma_{ij}^{mk}(\mathbf{u}^{(N)}) \sigma_{ij}^{ml}(\mathbf{u}^{(N)}) dxdr \\ \to \mathbb{E}\int_0^t \int_{\mathcal{O}} \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty e_k e_l \sigma_{ij}^{mk}(-(\mathbf{u}^{(N)})^-) \sigma_{ij}^{ml}(-(\mathbf{u}^{(N)})^-) dxdr.$$

Thus, the limit $\varepsilon \to 0$ in (19) gives

$$\mathbb{E}\|(u_i^{(N)}(t))^-\|_{L^2(\mathcal{O})}^2 \\ \leq \mathbb{E}\|(u_i^{(N)}(0))^-\|_{L^2(\mathcal{O})}^2 + \mathbb{E}\int_0^t \sum_{j=1}^n \sum_{k,l=1}^N \sum_{m=1}^\infty e_k e_l \sigma_{ij}^{mk}(-(\mathbf{u}^{(N)})^-) \sigma_{ij}^{ml}(-(\mathbf{u}^{(N)})^-) dxdr \\ \leq \mathbb{E}\|(u_i^{(N)}(0))^-\|_{L^2(\mathcal{O})}^2 + \mathbb{E}\int_0^t \sum_{j=1}^n \|\sigma_{ij}(-(\mathbf{u}^{(N)})^-)\|_{\mathcal{L}(Y;L^2(\mathcal{O}))}^2 dr. \tag{20}$$

The first term on the right-hand-side of (20) vanishes since $u_i^{(N)}(0) = |_N(u_i^0)| \geq 0$. For the second term, by the linear growth condition of σ_{ij} in (5), we derive that

$$\mathbb{E}\|(u_i^{(N)}(t))^-\|_{L^2(\mathcal{O})}^2 \leq C\mathbb{E}\int_0^t \|(u_i^{(N)}(r))^-\|_{L^2(\mathcal{O})}^2 dr.$$

The Gronwall lemma implies that $E\|(u_i^{(N)}(t))^-\|^2_{L^2(O)}=0$ for $t\in(0,T)$, and consequently for every $1\le i\le n$, $u_i^{(N)}(t)\ge 0$ in O, P-$a.s.$ for a.e. $t\in[0,T]$. Then we finish the proof of this lemma. □

In the introduction part, we have mentioned that in order to estimate $(\mathbf{u}^{(N)}(t))^{\frac{s}{2}}$, an auxiliary system has to be considered. This auxiliary system is given by

$$\partial_t\mathbf{v}=div(A^H(\mathbf{v})\nabla\mathbf{v})-G(\mathbf{v})A^H(\mathbf{v})\nabla\mathbf{v}+H^{-1}(\mathbf{v})\sigma(\mathbf{v}^{\frac{2}{s}})d\mathbf{W}(t)/dt. \tag{21}$$

$H(\mathbf{v})=(H_{ij}(\mathbf{v}))$, $G(\mathbf{v})=(G_{ij}(\mathbf{v}))$ and $A^H(\mathbf{v})=(A^H_{ij}(\mathbf{v}))$ are matrices, with

$$H_{ij}(\mathbf{v})=\tfrac{2}{s}v_i^{\frac{2}{s}-1},\ \text{if } i=j,\quad H_{ij}(\mathbf{v})=0,\ \text{if } i\neq j,$$
$$G_{ij}(\mathbf{v})=(1-\tfrac{2}{s})\tfrac{\nabla v_i}{v_i},\ \text{if } i=j,\quad G_{ij}(\mathbf{v})=0,\ \text{if } i\neq j,$$
$$A^H_{ij}(\mathbf{v})=a_{i0}+(s+1)a_{ii}v_i^2+\sum_{k\neq i}^n a_{ik}v_k^2,\quad \text{if } i=j,\ \ A^H_{ij}(\mathbf{v})=sa_{ij}v_iv_j,\ \ \text{if } i\neq j.$$

Once for every $1\le i\le n$, $u_i\ge 0$, and $v_i=u_i^{\frac{s}{2}}$, we have

$$\left(H^{-1}(v)A(u)H(v)\right)_{ij}=\textstyle\sum_{k,l=1}^n H^{-1}_{ik}(v)A_{kl}(u)H_{ij}(v)=$$
$$H^{-1}_{ii}(v)A_{ij}(u)H_{jj}(v){=}v_i^{1-\frac{2}{s}}v_j^{\frac{2}{s}-1}A_{ij}(u)\ (v)A_{ij}(u)H_{jj}(v)v_i^{1-\frac{2}{s}}v_j^{\frac{2}{s}-1}A_{ij}(u)=A^H_{ij}(v),$$

and

$$\left(\nabla H^{-1}(v)A(u)H(v)\right)_{ij}=\sum_{k,l=1}^n\nabla H^{-1}_{ik}(v)A_{kl}(u)H_{lj}(v)=\nabla H^{-1}_{ii}(v)A_{ij}(u)H_{jj}(v)=$$
$$\left(1-\tfrac{2}{s}\right)\tfrac{\nabla v_i}{v_i}\cdot v_i^{1-\frac{2}{s}}v_j^{\frac{2}{s}-1}A_{ij}(u)=\left(G(v)A^H(v)\right)_{ij},$$

then

$$\begin{aligned}&H^{-1}(\mathbf{v})div(A(\mathbf{u})\nabla\mathbf{u})+H^{-1}(\mathbf{v})\sigma(\mathbf{u})d\mathbf{W}(t)/dt\\&=div(H^{-1}(\mathbf{v})A(\mathbf{u})\nabla\mathbf{u})-\nabla H^{-1}(\mathbf{v})A(\mathbf{u})\nabla\mathbf{u}+H^{-1}(\mathbf{v})\sigma(\mathbf{u})d\mathbf{W}(t)/dt\\&=div(H^{-1}(\mathbf{v})A(\mathbf{u})H(\mathbf{v})\nabla\mathbf{v})-\nabla H^{-1}(\mathbf{v})A(\mathbf{u})H(\mathbf{v})\nabla\mathbf{v}+H^{-1}(\mathbf{v})\sigma(\mathbf{u})d\mathbf{W}(t)/dt\\&=div(A^H(\mathbf{v})\nabla\mathbf{v})-G(\mathbf{v})A^H(\mathbf{v})\nabla\mathbf{v}+H^{-1}(\mathbf{v})\sigma(\mathbf{v}^{\frac{2}{s}})d\mathbf{W}(t)/dt.\end{aligned}\tag{22}$$

Let us consider

$$\partial_t\mathbf{v}=\Pi_N(div(A^H(\mathbf{v})\nabla\mathbf{v})-G(\mathbf{v})A^H(\mathbf{v})\nabla\mathbf{v})+\Pi_N(H^{-1}(\mathbf{v})\sigma(\mathbf{v}^{\frac{2}{s}}))d\mathbf{W}(t)/dt, \tag{23}$$

and try to show that as $N\to\infty$, $(\mathbf{u}^{(N)})^{\frac{s}{2}}$ approximates to the strong solution of (23).

Lemma 6. *If for a sub-sequence of $N\in N$ we do not relabel, that a unique strong solution (in the probabilistic sense) denoted as $\mathbf{v}^{(N)}$ exists to (23), $\mathbf{v}^{(N)}$ is non-negative, P-$a.s.$ then as $N\to\infty$, $(\mathbf{u}^{(N)})^{\frac{s}{2}}-\mathbf{v}^{(N)}\to 0$ for a.e. $(x,t)\in O\times(0,T)$, P-$a.s.$*

Proof. Let us demote

$$I_1^{(N)} = div(A^H((\mathbf{u}^{(N)})^{\frac{s}{2}})\nabla(\mathbf{u}^{(N)})^{\frac{s}{2}}) - G((\mathbf{u}^{(N)})^{\frac{s}{2}})A^H((\mathbf{u}^{(N)})^{\frac{s}{2}})\nabla(\mathbf{u}^{(N)})^{\frac{s}{2}}$$
$$+H^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}})\sigma(\mathbf{u}^{(N)})d\mathbf{W}(t)/dt - \Pi_N(div(A^H((\mathbf{u}^{(N)})^{\frac{s}{2}})\nabla(\mathbf{u}^{(N)})^{\frac{s}{2}})$$
$$-G((\mathbf{u}^{(N)})^{\frac{s}{2}})A^H((\mathbf{u}^{(N)})^{\frac{s}{2}})\nabla(\mathbf{u}^{(N)})^{\frac{s}{2}}) - \Pi_N(H^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}})\sigma(\mathbf{u}^{(N)}))d\mathbf{W}(t)/dt, I_2^{(N)}$$
$$= \Pi_N(div(A^H((\mathbf{u}^{(N)})^{\frac{s}{2}})\nabla(\mathbf{u}^{(N)})^{\frac{s}{2}}) - G((\mathbf{u}^{(N)})^{\frac{s}{2}})A^H((\mathbf{u}^{(N)})^{\frac{s}{2}})\nabla(\mathbf{u}^{(N)})^{\frac{s}{2}})+$$
$$\Pi_N(H^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}})\sigma(\mathbf{u}^{(N)}))d\mathbf{W}(t)/dt, I_3^{(N)}$$
$$= H^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}})(div(A(\mathbf{u}^{(N)})\nabla\mathbf{u}^{(N)}) - \Pi_N div(A(\mathbf{u}^{(N)})\nabla\mathbf{u}^{(N)}) + (\sigma(\mathbf{u}^{(N)})$$
$$-\Pi_N\sigma(\mathbf{u}^{(N)}))d\mathbf{W}(t)/dt).$$

We have $H_{ij}^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}}) = \frac{s}{2}(u_i^{(N)})^{\frac{s}{2}-1}$, if $i=j$, and $H_{ij}^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}}) = 0$, if $i \neq j$. Then

$$\partial_t(\mathbf{u}^{(N)})^{\frac{s}{2}} = H^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}})(\Pi_N(div(A(\mathbf{u}^{(N)})\nabla\mathbf{u}^{(N)})) + \Pi_N(\sigma(\mathbf{u}^{(N)}))d\mathbf{W}(t)/dt), \tag{24}$$

and by (22),

$$\begin{aligned} div(A^H((\mathbf{u}^{(N)})^{\frac{s}{2}})\nabla(\mathbf{u}^{(N)})^{\frac{s}{2}}) - G((\mathbf{u}^{(N)})^{\frac{s}{2}})A^H((\mathbf{u}^{(N)})^{\frac{s}{2}})\nabla(\mathbf{u}^{(N)})^{\frac{s}{2}} + H^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}}) \cdot \sigma(\mathbf{u}^{(N)})d\mathbf{W}(t)/dt = \\ H^{-1}((\mathbf{u}^{(N)})^{\frac{s}{2}})(div(A(\mathbf{u}^{(N)})\nabla\mathbf{u}^{(N)}) + \sigma(\mathbf{u}^{(N)})d\mathbf{W}(t)/dt). \end{aligned} \tag{25}$$

By (24)−(25), we denote that

$$\partial_t(\mathbf{u}^{(N)})^{\frac{s}{2}} = I_1^{(N)} - I_3^{(N)} + I_2^{(N)}. \tag{26}$$

By the definition of $I_1^{(N)}$ and $I_3^{(N)}$, for every $N \in N$,

$$\Pi_N I_1^{(N)} = 0, \quad \Pi_N(H((\mathbf{u}^{(N)})^{\frac{s}{2}})I_3^{(N)}) = 0,$$

and as $N \to \infty$, $I_1^{(N)} - {}_N I_1^{(N)}$, $H((\mathbf{u}^{(N)})^{\frac{s}{2}})I_3^{(N)} - {}_N(H((\mathbf{u}^{(N)})^{\frac{s}{2}})I_3^{(N)}) \to 0$. We derive that as $N \to \infty$, $I_1^{(N)}$, $H((\mathbf{u}^{(N)})^{\frac{s}{2}})I_3^{(N)} \to 0$, for a.e. $(x,t) \in O \times (0,T)$, P-$a.s.$

Let us denote $I_3^{(N)} = (I_{31}^{(N)}, ..., I_{3n}^{(N)})$, then for every $1 \leq i \leq n$, we have $(u_i^{(N)})^{1-\frac{s}{2}} I_{3i}^{(N)} \to 0$, as $N \to \infty$. Either $(u_i^{(N)})^{1-\frac{s}{2}} \to 0$ or $I_{3i}^{(N)} \to 0$, as $N \to \infty$. As $1 - \frac{s}{2} < 0$, and $u_i^{(N)} \in H_N$, we deduce that $I_{3i}^{(N)} \to 0$, as $N \to \infty$. We conclude that as $N \to \infty$, $I_1^{(N)}, I_3^{(N)} \to 0$.

By the definition of $\mathbf{v}^{(N)}$, P-$a.s.$ we have for a.e.$(x,t) \in O \times (0,T)$,

$$\begin{aligned} \partial_t\mathbf{v}^{(N)} = \Pi_N(div(A^H(\mathbf{v}^{(N)})\nabla\mathbf{v}^{(N)}) - G(\mathbf{v}^{(N)})A^H(\mathbf{v}^{(N)}) \\ \nabla\mathbf{v}^{(N)}) + \Pi_N(H^{-1}(\mathbf{v}^{(N)})\sigma((\mathbf{v}^{(N)})^{\frac{2}{s}}))d\mathbf{W}(t)/dt. \end{aligned} \tag{27}$$

Then by (26)–(27), we derive that as $N \to \infty$, $(\mathbf{u}^{(N)})^{\frac{s}{2}} - \mathbf{v}^{(N)} \to 0$ for a.e. $(x,t) \in O \times (0,T)$, P-$a.s.$ and we finish the proof of this lemma. □

5. Energy estimates of approximated solutions $\mathbf{u}^{(N)}$

We divide the energy estimates of $\mathbf{u}^{(N)}$ into two lemmas. For the first estimation Lemma 7, we apply the Itô formula to $\mathbf{u}^{(N)}(t)$. For the second estimation Lemma 8, we apply the Itô formula to $\mathbf{v}^{(N)}(t)$. Then by the Lemma 6, we derive estimations for $(\mathbf{u}^{(N)}(t))^{\frac{s}{2}}$.

Lemma 7. *For every $T > 0$, there exists a constant $C > 0$, which does not depend on N, such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E}(\sup_{t\in(0,T)} \|\mathbf{u}^{(N)}\|^2_{L^2(\mathcal{O})}) \leq C, \tag{28}$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}(\int_0^T \|\nabla \mathbf{u}^{(N)}\|^2_{L^2(\mathcal{O})} dr) \leq C. \tag{29}$$

Proof. Let us apply the Itô formula to the process $\mathbf{u}^{(N)}(t)$, where $\mathbf{u}^{(N)}(t)$ is a strong solution (in the probabilistic sense) to (11), $P^{\frac{1}{2}} = diag(\pi_1^{\frac{1}{2}}, ..., \pi_n^{\frac{1}{2}})$, $t\in [0,T]$, and

$$\begin{aligned}\frac{1}{2}\|P^{\frac{1}{2}}\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})} - \frac{1}{2}\|\Pi_N(P^{\frac{1}{2}}\mathbf{u}^0)\|^2_{L^2(\mathcal{O})} &= \frac{1}{2}\int_0^t \|\Pi_N(P^{\frac{1}{2}}\sigma(\mathbf{u}^{(N)}(r)))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))} dr \\ &+ \sum_{i,j=1}^n \int_0^t \langle \Pi_N div(\pi_i A_{ij}(\mathbf{u}^{(N)}(r))\nabla u_j^{(N)}(r)), u_i^{(N)}(r)\rangle dr \\ &+ \sum_{i,j=1}^n \int_0^t \langle \Pi_N(\pi_i\sigma_{ij}(\mathbf{u}^{(N)}(r)))dW_j(r), u_i^{(N)}(r)\rangle,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{1}{2}\|P^{\frac{1}{2}}\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})} - \frac{1}{2}\|\Pi_N(P^{\frac{1}{2}}\mathbf{u}^0)\|^2_{L^2(\mathcal{O})} = \frac{1}{2}\int_0^t \|\Pi_N(P^{\frac{1}{2}}\sigma(\mathbf{u}^{(N)}(r)))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))} dr \\ - \sum_{i,j=1}^n \int_0^t \langle \pi_i A_{ij}(\mathbf{u}^{(N)}(r))\nabla u_j^{(N)}(r), \nabla u_i^{(N)}(r)\rangle dr + \sum_{i,j=1}^n \int_0^t \langle \pi_i\sigma_{ij}(\mathbf{u}^{(N)}(r))dW_j(r), u_i^{(N)}(r)\rangle.\end{aligned} \tag{30}$$

The second term on the right-hand-side of (30) can be estimated by Lemma 2. For $z_i,z_j\in R$, there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$, with values of α_1, α_2 changing from line to line, such that if u is non-negative, P-$a.s.$ then $\sum_{i,j=1}^n \pi_i A_{ij}(\mathbf{u})z_i z_j \geq \alpha_1 \sum_{i=1}^n z_i^2 + \alpha_2 \sum_{i=1}^n u_i^s z_i^2$. Since we have already shown $\mathbf{u}^{(N)}(t)$ are non-negative P-$a.s.$

$$\begin{aligned}\sum_{i,j=1}^n \langle \pi_i A_{ij}(\mathbf{u}^{(N)}(r))\nabla u_j^{(N)}(r), \nabla u_i^{(N)}(r)\rangle \geq \alpha_1 \sum_{i=1}^n \int_{\mathcal{O}} |\nabla u_i^{(N)}|^2 dx + \\ \alpha_2 \sum_{i=1}^n \int_{\mathcal{O}} |u_i^{(N)}|^s |\nabla u_i^{(N)}|^2 dx \geq \alpha_1 \|\nabla \mathbf{u}^{(N)}\|^2_{L^2(\mathcal{O})} + \alpha_2 \|\nabla (\mathbf{u}^{(N)})^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{1}{2}\|P^{\frac{1}{2}}\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})} + \alpha_1 \int_0^t \|\nabla \mathbf{u}^{(N)}(r)\|^2_{L^2(\mathcal{O})} dr + \alpha_2 \int_0^t \|\nabla (\mathbf{u}^{(N)}(r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})} dr \\ \leq \frac{1}{2}\|P^{\frac{1}{2}}\mathbf{u}^0\|^2_{L^2(\mathcal{O})} + \frac{1}{2}\int_0^t \|P^{\frac{1}{2}}\sigma(\mathbf{u}^{(N)}(r))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))} dr \\ + \sum_{i,j=1}^n \int_0^t \langle \pi_i\sigma_{ij}(\mathbf{u}^{(N)}(r))dW_j(r), u_i^{(N)}(r)\rangle,\end{aligned} \tag{31}$$

and for the second term on the right-hand-side of (31), by assumption (5),

$$\frac{1}{2}\int_0^t \|P^{\frac{1}{2}}\sigma(\mathbf{u}^{(N)}(r))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr \leq C\int_0^T (1+\|\mathbf{u}^{(N)}(r)\|^2_{L^2(\mathcal{O})})dr.$$

For the third term on the right-hand-side of (31), the process

$$\mu^{(N)}(t) = \sum_{i,j=1}^n \int_0^t \langle \pi_i\sigma_{ij}(\mathbf{u}^{(N)}(r))dW_j(r), u_i^{(N)}(r)\rangle, \quad t\in[0,T],$$

is an F_t-martingale. By the Burkholder-Davis-Gundy inequality, which states that

$$E(\sup_{0\leq t\leq T}|\mu^{(N)}(t)|) \leq CE(\langle\mu^{(N)}(T)\rangle^{\frac{1}{2}}),$$

with the notation $\langle\mu^{(N)}(T)\rangle$ represents the quadratic variation of $\mu^{(N)}(T)$. Thus

$$\begin{gathered}\mathbb{E}(\sup_{t\in[0,T]}|\sum_{i,j=1}^n \int_0^t \langle \pi_i\sigma_{ij}(\mathbf{u}^{(N)}(r))dW_j(r), u_i^{(N)}(r)\rangle|) \leq \\ C\mathbb{E}\left(\sum_{i,j=1}^n \int_0^T \left(\int_{\mathcal{O}} \pi_i\sigma_{ij}\left(u^{(N)}(r)\right)u_i^{(N)}(r)dx\right)^2\right)^{\frac{1}{2}} \leq \\ C\mathbb{E}\left(\sum_{i,j=1}^n \int_0^T \left(\int_{\mathcal{O}} \sigma_{ij}\left(u^{(N)}(r)\right)u_i^{(N)}(r)dx\right)^2\right)^{\frac{1}{2}} \leq \\ C\mathbb{E}\left(\int_0^T \left(\int_{\mathcal{O}}\sum_{i,j=1}^n \sigma_{ij}^2(u^{(N)}(r))dx\right)\left(\int_{\mathcal{O}}\sum_{i=1}^n \left(u_i^{(N)}(r)\right)^2\right)dr\right)^{\frac{1}{2}} \leq \\ C\mathbb{E}\left(\int_0^T ||u^{(N)}(r)||^2_{L^2(\mathcal{O})}||\sigma(u^{(N)}(r))||^2_{\mathcal{L}(Y,L^2(\mathcal{O}))}\right)^{\frac{1}{2}}\end{gathered}$$

One step further, by the assumption on σ, H¨older inequality and the Young inequality, for every positive constant ε_0,

$$\begin{gathered}\mathbb{E}\left(\sup_{t\in[0,T]}|\sum_{i,j=1}^n \int_0^t \left\langle \pi_i\sigma_{ij}\left(\mathbf{u}^{(N)}(r)\right)dW_j(r), u_i^{(N)}(r)\right\rangle\Big|\right) \leq \\ C\mathbb{E}\left(\int_0^T ||u^{(N)}(r)||^2_{L^2(\mathcal{O})}||\sigma(u^{(N)}(r))||^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}\right)^{\frac{1}{2}} \leq \\ C\mathbb{E}\left(\left(\sup_{t\in(0,T)}||u^{(N)}(t)||^2_{L^2(\mathcal{O})}\right)^{\frac{1}{2}}\left(\int_0^T (1+||u^{(N)}(r)||^2_{L^2(\mathcal{O})})dr\right)^{\frac{1}{2}}\right) \leq \\ C\varepsilon_0\mathbb{E}\left(\sup_{t\in[0,T]}\big|\big|u^{(N)}(t)\big|\big|^2_{L^2(\mathcal{O})}\right) + \frac{C}{4\varepsilon_0}\left(T+\mathbb{E}\int_0^T \big|\big|u^{(N)}(r)\big|\big|^2_{L^2(\mathcal{O})}dr\right).\end{gathered} \tag{32}$$

Combining (31) and (32), for every positive constant ε_0, and $t\in[0,T]$,

$$\begin{gathered}\tfrac{1}{2}\mathbb{E}(\sup_{t\in[0,T]}\|P^{\frac{1}{2}}\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})}) + \alpha_1\mathbb{E}\int_0^t \|\nabla\mathbf{u}^{(N)}(r)\|^2_{L^2(\mathcal{O})}dr \\ +\alpha_2\mathbb{E}\int_0^t \|\nabla(\mathbf{u}^{(N)}(r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr \leq \tfrac{1}{2}\mathbb{E}\|P^{\frac{1}{2}}\mathbf{u}^0\|^2_{L^2(\mathcal{O})} + C\mathbb{E}\int_0^T(1+\|\mathbf{u}^{(N)}(r)\|^2_{L^2(\mathcal{O})})dr \\ +C\varepsilon_0\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})}) + \tfrac{C}{4\varepsilon_0}\cdot(T+\mathbb{E}\int_0^T\|\mathbf{u}^{(N)}(r)\|^2_{L^2(\mathcal{O})}dr) \\ \leq C + C(1+\tfrac{1}{4\varepsilon_0})\mathbb{E}\int_0^T\|\mathbf{u}^{(N)}(r)\|^2_{L^2(\mathcal{O})}dr + C\varepsilon_0\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})}),\end{gathered}$$

thus we conclude that there exist positive constants α_1, α_2, α_3, with values subject to change, such that

$$\alpha_3\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})}) + \alpha_1\mathbb{E}\int_0^t \|\nabla\mathbf{u}^{(N)}(r)\|^2_{L^2(\mathcal{O})}dr$$
$$+\alpha_2\mathbb{E}\int_0^t \|\nabla(\mathbf{u}^{(N)}(r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr \leq C + C(1+\tfrac{1}{4\varepsilon_0})\mathbb{E}\int_0^T (\sup_{\tau\in[0,r]}\|\mathbf{u}^{(N)}(\tau)\|^2_{L^2(\mathcal{O})})dr$$
$$+C\varepsilon_0\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})}).$$

Let us choose this ε_0 such that $C\varepsilon_0 < \alpha_3$, then for some positive constants C_0,C_1, α_1, α_2, α_3, with their values different from the above equation,

$$\alpha_3\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})}) + \alpha_1\mathbb{E}\int_0^t \|\nabla\mathbf{u}^{(N)}(r)\|^2_{L^2(\mathcal{O})}dr$$
$$+\alpha_2\mathbb{E}\int_0^t \|\nabla(\mathbf{u}^{(N)}(r))^{\frac{s+2}{2}}\|^2_{L^2(\mathcal{O})}dr \leq C_0 + C_1\mathbb{E}\int_0^T (\sup_{\tau\in[0,r]}\|\mathbf{u}^{(N)}(\tau)\|^2_{L^2(\mathcal{O})})dr, \tag{33}$$

and by the Gronwall lemma,

$$\mathbb{E}(\sup_{t\in(0,T)}\|\mathbf{u}^{(N)}(t)\|^2_{L^2(\mathcal{O})}) \leq C, \tag{34}$$

with $C>0$ independent of N, and we have shown that (28) holds. By (28) and (33), we deduce that (29) holds. □

Lemma 8. *For every $T>0$, there exists a constant $C>0$, which does not depend on N, such that*

$$\sup_{N\in\mathbb{N}}\mathbb{E}(\int_0^T \|\nabla(\mathbf{u}^{(N)})^s\|^2_{L^2(\mathcal{O})}dr) \leq C, \tag{35}$$

$$\sup_{N\in\mathbb{N}}\mathbb{E}(\int_0^T \|(\mathbf{u}^{(N)})^s\|^3_{L^2(\mathcal{O})}dr) \leq C. \tag{36}$$

Proof. Let us firstly show that if for a subsequence of $N\in N$, $\mathbf{v}^{(N)}$ is a strong solution (in the probabilistic sense) to (23), and $\mathbf{v}^{(N)}$ is non-negative, P-$a.s.$ then $\mathbf{v}^{(N)}$ satisfies

$$\sup_{N\in\mathbb{N}}\mathbb{E}(\int_0^T \|\nabla(\mathbf{v}^{(N)})^2\|^2_{L^2(\mathcal{O})}dr) \leq C, \tag{37}$$

$$\sup_{N\in\mathbb{N}}\mathbb{E}(\int_0^T \|(\mathbf{v}^{(N)})^2\|^3_{L^2(\mathcal{O})}dr) \leq C. \tag{38}$$

Let us apply the Itô formula to $\mathbf{v}^{(N)}(t)$, and we have

$$\tfrac{1}{2}\|P^{\frac{1}{2}}\mathbf{v}^{(N)}(t)\|^2_{L^2(\mathcal{O})}$$
$$= \tfrac{1}{2}\|P^{\frac{1}{2}}\mathbf{v}^{(N)}(0)\|^2_{L^2(\mathcal{O})} + \sum_{i,j=1}^{n}\int_0^t \langle \Pi_N div(\pi_i A^H_{ij}(\mathbf{v}^{(N)}(r))\nabla v^{(N)}_j), v^{(N)}_i\rangle dr$$
$$-\sum_{i,j=1}^{n}\int_0^t \langle \pi_i\Pi_N(A^H_{ij}(\mathbf{v}^{(N)}(r))G_{ii}(\mathbf{v}^{(N)}(r))\nabla v^{(N)}_j, v^{(N)}_i\rangle dr$$
$$+\sum_{i,j=1}^{n}\int_0^t\langle \Pi_N(\pi_i H^{-1}_{ii}(\mathbf{v}^{(N)}(r))\sigma_{ij}((\mathbf{v}^{(N)}(r))^{\frac{2}{s}}))dW_j(r), v^{(N)}_i\rangle dr$$
$$+\tfrac{1}{2}\int_0^t \|\Pi_N(P^{\frac{1}{2}}H^{-1}(\mathbf{v}^{(N)}(r))\sigma((\mathbf{v}^{(N)}(r))^{\frac{2}{s}}))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr,$$

which follows

$$\frac{1}{2}\|P^{\frac{1}{2}}\mathbf{v}^{(N)}(t)\&\|^2_{L^2(\mathcal{O})} = \frac{1}{2}\|P^{\frac{1}{2}}\mathbf{v}^{(N)}(0)\|^2_{L^2(\mathcal{O})} - \sum_{i,j=1}^n \int_0^t \int_{\mathcal{O}} \pi_i A^H{}_{ij}(\mathbf{v}^{(N)}(r))\nabla v_i^{(N)} \nabla v_j^{(N)} dxdr$$
$$-(1-\frac{2}{s})\sum_{i,j=1}^n \int_0^t \int_{\mathcal{O}} \pi_i A^H{}_{ij}(\mathbf{v}^{(N)}(r))\frac{\nabla v_i^{(N)}}{v_i^{(N)}}\nabla v_j^{(N)} \cdot v_i^{(N)} dxdr$$
$$+\frac{s}{2}\sum_{i,j=1}^n \int_0^t \langle \pi_i \sigma_{ij}((\mathbf{v}^{(N)}(r))^{\frac{2}{s}}) dW_j(r), (v_i^{(N)})^{2-\frac{2}{s}}\rangle dr$$
$$+\frac{1}{2}\int_0^t \|\Pi_N(P^{\frac{1}{2}}H^{-1}(\mathbf{v}^{(N)}(r))\sigma((\mathbf{v}^{(N)}(r))^{\frac{2}{s}}))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr,$$

then

$$\begin{aligned}\frac{1}{2}\|P^{\frac{1}{2}}\mathbf{v}^{(N)}(t)\|^2_{L^2(\mathcal{O})} &= \frac{1}{2}\|P^{\frac{1}{2}}\mathbf{v}^{(N)}(0)\|^2_{L^2(\mathcal{O})} \\ &+(\frac{2}{s}-2)\sum_{i,j=1}^n \int_0^t \int_{\mathcal{O}} \pi_i A^H{}_{ij}(\mathbf{v}^{(N)}(r))\nabla v_i^{(N)}\nabla v_j^{(N)} dxdr \\ &+\frac{1}{2}\int_0^t \|\Pi_N(P^{\frac{1}{2}}H^{-1}(\mathbf{v}^{(N)}(r))\sigma((\mathbf{v}^{(N)}(r))^{\frac{2}{s}}))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr \\ &+\frac{s}{2}\sum_{i,j=1}^n \int_0^t \langle (v_i^{(N)}(r))^{2-\frac{2}{s}}, \Pi_N(\pi_i\sigma_{ij}((\mathbf{v}^{(N)}(r))^{\frac{2}{s}}))dW_j(r)\rangle.\end{aligned} \tag{39}$$

For the third term on the right-hand-side of (39), by assumption (5),

$$\frac{1}{2}\int_0^t \|P^{\frac{1}{2}}H^{-1}(\mathbf{v}^{(N)}(r))\sigma((\mathbf{v}^{(N)}(r))^{\frac{2}{s}})\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr$$
$$= \frac{1}{2}\sum_{i,j=1}^n \int_0^t \|\frac{\sqrt{\pi_i}s}{2}(v_i^{(N)}(r))^{1-\frac{2}{s}}\sigma_{ij}((\mathbf{v}^{(N)}(r))^{\frac{2}{s}})\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr \le$$
$$C\int_0^T (1+\|\mathbf{v}^{(N)}(r)\|^2_{L^2(\mathcal{O})})dr.$$

For the fourth term on the right-hand-side of (39), the process

$$\mu^{(N)}(t) = \sum_{i,j=1}^n \int_0^t \langle \pi_i\sigma_{ij}((\mathbf{v}^{(N)}(r))^{\frac{2}{s}})dW_j(r), (v_i^{(N)}(r))^{2-\frac{2}{s}}\rangle, \quad t\in[0,T],$$

is an F_t-martingale. Still by the Burkholder-Davis-Gundy inequality, plus the assumption on σ, H\"older inequality and the Young inequality, we can show that for every positive constant ε_0,

$$\mathbb{E}\left(\sup_{t\in[0,T]}|\sum_{i,j=1}^n \pi_i\left\langle v_i^{(N)}(r), \left(v_i^{(N)}\right)^{1-\frac{2}{s}}\sigma_{ij}((v^{(N)}(r))^{\frac{s}{2}})dW_j(r)\right\rangle\Big|\right) \le$$
$$C\mathbb{E}\left(\int_0^T ||v^{(N)}(r)||^2_{L^2(\mathcal{O})}||\left(v^{(N)}(r)\right)^{1-\frac{2}{s}}\sigma((v^{(N)}(r))^{\frac{2}{s}})||^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}\right)^{\frac{1}{2}} \le$$
$$C\mathbb{E}\left\{\left(\sup_{t\in(0,T)}||v^{(N)}(t)||^2_{L^2(\mathcal{O})}\right)^{\frac{1}{2}}\left(\int_0^T\left(1+||v^{(N)}(r)||^2_{L^2(\mathcal{O})}\right)dr\right)^{\frac{1}{2}}\right\} \le$$
$$C\varepsilon_0\mathbb{E}\left(\sup_{t\in[0,T]}\left|\left|v^{(N)}(t)\right|\right|^2_{L^2(\mathcal{O})}\right) + \frac{C}{4\varepsilon_0}\left(T+\mathbb{E}\int_0^T\left|\left|v^{(N)}(r)\right|\right|^2_{L^2(\mathcal{O})}dr\right)$$

We deduce that

$$\frac{1}{2}\mathbb{E}(\sup_{t\in[0,T]}\|P^{\frac{1}{2}}\mathbf{v}^{(N)}(t)\|^2_{L^2(\mathcal{O})})+(2-\frac{2}{s})\sum_{i,j=1}^{n}\mathbb{E}\int_0^t\int_{\mathcal{O}}\pi_i A^H{}_{ij}(\mathbf{v}^{(N)}(r))\nabla v_i^{(N)}\nabla v_j^{(N)}dxdr$$
$$\leq \frac{1}{2}\mathbb{E}\|P^{\frac{1}{2}}\mathbf{v}^{(N)}(0)\|^2_{L^2(\mathcal{O})}+C\mathbb{E}\int_0^T(1+\|\mathbf{v}^{(N)}(r)\|^2_{L^2(\mathcal{O})})dr$$
$$+C\cdot\frac{\varepsilon_0 s}{2}\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{v}^{(N)}(t)\|^2_{L^2(\mathcal{O})})+C\cdot\frac{s}{8\varepsilon_0}(T+\mathbb{E}\int_0^T\|\mathbf{v}^{(N)}(r)\|^2_{L^2(\mathcal{O})}dr).$$

The Lemma 3 indicates that for $z_i, z_j\in R$, there exist constants $\alpha_1>0,\ \alpha_2>0$, such that if $\mathbf{v}$ is non-negative, P-$a.s.$ then $\sum_{i,j=1}^n\pi_i A^H{}_{ij}(\mathbf{v})z_iz_j\geq\alpha_1\sum_{i=1}^n z_i^2+\alpha_2\sum_{i=1}^n v_i^2z_i^2.\mathbf{v}^{(N)}$ is non-negative, P-$a.s.$ thus

$$\alpha_3\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{v}^{(N)}(t)\|^2_{L^2(\mathcal{O})})+\alpha_1\mathbb{E}\int_0^t\|\nabla\mathbf{v}^{(N)}(r)\|^2_{L^2(\mathcal{O})}dr+\alpha_2\mathbb{E}\int_0^t\|\nabla(\mathbf{v}^{(N)}(r))^2\|^2_{L^2(\mathcal{O})}dr$$
$$\leq C+C(1+\frac{s}{8\varepsilon_0})\int_0^T\mathbb{E}(\sup_{\tau\in[0,r]}\|\mathbf{v}^{(N)}(\tau)\|^2_{L^2(\mathcal{O})})dr+C\cdot\frac{\varepsilon_0 s}{2}\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{v}^{(N)}(t)\|^2_{L^2(\mathcal{O})}).$$

Choose this ε_0 such that $C\varepsilon_0 s<2\alpha_3$, then for some positive constants C_0, $C_1,\alpha_1,\alpha_2,\alpha_3$, with their values different from the above equation,

$$\alpha_3\mathbb{E}(\sup_{t\in[0,T]}\|\mathbf{v}^{(N)}(t)\|^2_{L^2(\mathcal{O})})+\alpha_1\mathbb{E}\int_0^t\|\nabla\mathbf{v}^{(N)}(r)\|^2_{L^2(\mathcal{O})}dr+$$
$$\alpha_2\mathbb{E}\int_0^t\|\nabla(\mathbf{v}^{(N)}(r))^2\|^2_{L^2(\mathcal{O})}dr\leq C_0+C_1\int_0^T\mathbb{E}(\sup_{\tau\in[0,r]}\|\mathbf{v}^{(N)}(\tau)\|^2_{L^2(\mathcal{O})})dr, \tag{40}$$

and by the Gronwall lemma,

$$\mathbb{E}(\sup_{t\in(0,T)}\|\mathbf{v}^{(N)}(t)\|^2_{L^2(\mathcal{O})})\leq C, \tag{41}$$

with the constant $C>0$ in (41) independent of N. By (40)−(41), we derive that (37) holds.

In order to show (38), we need a higher order moment estimate, and we list this estimation as a direct result, which is (42). The proof of (42) relies on (41), with (41) already been proved. Detailed proof of this result [2]. Let $p=\frac{24}{4-d}$ and by assumption (7), $E\|\mathbf{v}^0\|^p_{L^2(O)}<\infty$ (please refer to Remark 2, Remark 18 [2] for a further understanding of this initial condition), then we have

$$\sup_{N\in\mathbb{N}}\mathbb{E}(\sup_{t\in(0,T)}\|\mathbf{v}^{(N)}(t)\|^p_{L^2(\mathcal{O})})\leq C. \tag{42}$$

By the Gagliardo-Nirenberg inequality and $H\backslash"older$ inequality, we choose $\theta=\frac{d}{d+2}$, then

$$\mathbb{E}\int_0^T||(v^{(N)}(r))^2||^2_{L^2(\mathcal{O})}dr\leq C\mathbb{E}\int_0^T\left|\left|\nabla\left(v^{(N)}(r)\right)^2\right|\right|^{2\theta}_{L^2(\mathcal{O})}\left|\left|\left(v^{(N)}(r)\right)^2\right|\right|^{2(1-\theta)}_{L^1(\mathcal{O})}dr=$$
$$C\mathbb{E}\int_0^T\left|\left|\nabla\left(v^{(N)}(r)\right)^2\right|\right|^{\frac{2d}{d+2}}_{L^2(\mathcal{O})}\left|\left|\left(v^{(N)}(r)\right)^2\right|\right|^{\frac{4}{d+2}}_{L^1(\mathcal{O})}dr\leq$$
$$C\mathbb{E}\left\{\left(\int_0^T||\nabla v^{(N)}(r)||^2_{L^2(\mathcal{O})}dr\right)^{\frac{d}{d+2}}\left(\int_0^T||v^{(N)}(r)||^4_{L^2(\mathcal{O})}dr\right)^{\frac{2}{d+2}}\right\}$$

and $\int_0^T\|\mathbf{v}^{(N)}(r)\|^4_{L^2(O)}dr\leq T\underset{t\in(0,T)}{sup}\|\mathbf{v}^{(N)}(t)\|^4_{L^2(O)}$, thus

$$\mathbb{E}\int_0^T ||(v^{(N)}(r))^2||_{L^2(\mathcal{O})}^2 dr \leq$$
$$CT^{\frac{2}{d+2}}\mathbb{E}\left\{\left(\sup_{t\in(0,T)}||v^{(N)}(t)||_{L^2(\mathcal{O})}^2\right)^{\frac{2}{d+2}}\left(\int_0^T||\nabla(v^{(N)}(r))^2||_{L^2(\mathcal{O})}^2\right)^{\frac{d}{d+2}}\right\}\leq \tag{43}$$
$$CT^{\frac{2}{d+2}}\left(\mathbb{E}\sup_{t\in(0,T)}||v^{(N)}(t)||_{L^2(\mathcal{O})}^4\right)^{\frac{2}{d+2}}\left(\mathbb{E}\int_0^T||\nabla(v^{(N)}(r))^2||_{L^2(\mathcal{O})}^2\right)^{\frac{d}{d+2}}\leq C,$$

Since $d \leq 3$, we have

$$\mathbb{E}\int_0^T ||(v^{(N)})^2||_{L^2(\mathcal{O})}^3 dr \leq C\mathbb{E}\int_0^T ||(v^{(N)})^2||_{H^1(\mathcal{O})}^{\frac{3d}{2+d}}||(v^{(N)})^2||_{L^1(\mathcal{O})}^{\frac{6}{2+d}} dr \leq$$
$$C\mathbb{E}\left(\sup_{t\in(0,T)}||v^{(N)}||_{L^2(\mathcal{O})}^{\frac{12}{2+d}}\int_0^T||(v^{(N)})^2||_{H^1(\mathcal{O})}^{\frac{3d}{2+d}}dr\right)\leq$$
$$C\left(\mathbb{E}\left(\sup_{t\in(0,T)}||v^{(N)}||_{L^2(\mathcal{O})}^{\frac{24}{4-d}}\right)\right)^{\frac{4-d}{2(2+d)}}\left(\mathbb{E}\int_0^T||(v^{(N)})^2||_{H^1(\mathcal{O})}^2 dr\right)^{\frac{3d}{2(2+d)}}$$

by (37), (42) and (43), we can show that (38) holds.

After we have shown (37)–(38), let us consider (35)–(36). By (37) and (43), there exists a subsequence of $(\mathbf{v}^{(N)})^2$ weakly converging to a function w in $L^2(\Omega;L^2(0,T;H^1(O)))$. By the Lemma 6, we deduce that $(\mathbf{u}^{(N)})^s$ converges to w weakly in $L^2(\Omega;\, L^2(0,T;\, H^1(O)))$. Then we can show that (35) holds.

By (38), there exists a subsequence of $(\mathbf{v}^{(N)})^2$ weakly converging to a function w' in $L^3(\Omega;\, L^3(0,T;\, L^2(O)))$. By the Lemma 6, we deduce that $(\mathbf{u}^{(N)})^s$ converges to w' weakly in $L^3(\Omega;\, L^3(0,T;\, L^2(O)))$. We also show that (36) holds, and finish the proof of this lemma. □

6. Proof of the main theorem

We introduce those topological spaces [2]

$$Z_T = C^0([0,T];\, H^3(O)^{'}) \cap L^2_\omega(0,T;\, H^1(O)) \cap L^2(0,T;\, L^2(O)) \cap C^0([0,T];\, L^2_\omega(O)),$$

endowed with the topology T, with T the maximum one of above topological spaces. On this space, we can formulate a compactness criterion. The origins and proving details for this criterion are [17, 18, 23, 24]. A detailed explanation of the Aldous condition [17]. For some preparation materials to have a better knowledge of those tightness criterions we use in following lemmas [23].

Lemma 9. *Let us denote the set of laws of* $\mathbf{u}^{(N)}$ *as* $L(\mathbf{u}^{(N)})$*, then the set of measures* $\{L(\mathbf{u}^{(N)}) : N \in N\}$ *is tight on* (Z_T,T).

Proof. Theorem 10 [2] provides criterions for the tightness of the approximated sequence $(\mathbf{u}^{(N)})_{N\in N}$ in Z_T, and we have verified two of those criterions in the Lemma 7, which are

$$\sup_{N\in\mathbb{N}}\mathbb{E}(\sup_{t\in(0,T)}\|\mathbf{u}^{(N)}\|^2_{L^2(\mathcal{O})})\leq C,\quad \sup_{N\in\mathbb{N}}\mathbb{E}(\int_0^T\|\mathbf{u}^{(N)}\|^2_{H^1(\mathcal{O})}dr)\leq C.$$

By the fact that embeddings $H^3(O)\hookrightarrow H^1(O)\hookrightarrow L^2(O)$ are dense and continuous and the embedding $H^1(O)\hookrightarrow L^2(O)$ is compact, when $d\leq 3$, it remains to show that $(\mathbf{u}^{(N)})_{N\in N}$ satisfies the Aldous condition in $H^3(O)'$. It is sufficient to show that for $(\tau_N)_{N\in N}$ a sequence of F-stopping times with $0\leq\tau_N\leq T$, for every $\varepsilon>0$, $\kappa>0$, there exists $\delta>0$, such that

$$\sup_{N\in\mathbb{N}}\sup_{0<\theta<\delta}\mathbb{P}(\|\mathbf{u}^{(N)}(\tau_N+\theta)-\mathbf{u}^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\geq\kappa)\leq\varepsilon.$$

Let $t\in(0,T)$ and $\phi\in H^3(O)$, then

$$\langle u_i^{(N)}(t),\phi_i\rangle=\langle u_i^{(N)}(0),\phi_i\rangle-\sum_{j=1}^n\int_0^t\langle A_{ij}(\mathbf{u}^{(N)})\nabla u_j^{(N)},\nabla\Pi_N\phi_i\rangle dr$$
$$+\sum_{j=1}^n\int_0^t\langle\Pi_N(\sigma_{ij}(\mathbf{u}^{(N)}(r)))dW_j(r),\phi_i\rangle=J_0^{(N)}+J_1^{(N)}(t)+J_2^{(N)}(t),$$

where $\langle\ \cdot\ ,\ \cdot\ \rangle$ is the dual pairing between $H^3(O)'$ and $H^3(O)$. Denote $I_1=\{\omega\in\Omega:0\leq\int_0^T\|(\mathbf{u}^{(N)}(r))^s\|^3_{L^2(O)}dr\leq 1\}$, with the complement of I_1 given by $\bar{I}_1=\{\omega\in\Omega:\int_0^T\|(\mathbf{u}^{(N)}(r))^s\|^3_{L^2(O)}dr>1\}$. Thus

$$\mathbb{E}(\|(\mathbf{u}^{(N)}(t))^s\|^2_{L^3(0,T;L^2(\mathcal{O}))})=\mathbb{E}(\int_0^T\|(\mathbf{u}^{(N)}(r))^s\|^3_{L^2(\mathcal{O})}dr)^{\frac{2}{3}}$$
$$=\int_{I_1\cup\bar{I}_1}(\int_0^T\|(\mathbf{u}^{(N)}(r))^s\|^3_{L^2(\mathcal{O})}dr)^{\frac{2}{3}}\mathbb{P}(d\omega)\leq 1+\mathbb{E}\int_0^T\|(\mathbf{u}^{(N)}(r))^s\|^3_{L^2(\mathcal{O})}dr\leq C.$$

By the H\"older inequality, we have

$$(\mathbb{E}(\|(\mathbf{u}^{(N)})^s\|_{L^3(0,T;L^2(\mathcal{O}))}\|\nabla\mathbf{u}^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))}))^2$$
$$\leq\mathbb{E}(\|(\mathbf{u}^{(N)})^s\|^2_{L^3(0,T;L^2(\mathcal{O}))})\mathbb{E}(\|\nabla\mathbf{u}^{(N)}\|^2_{L^2(0,T;L^2(\mathcal{O}))})\leq C.$$

Using the continuous embedding of $H^3(O)\hookrightarrow W^{1,\infty}(O)$, when $d\leq 3$, let us consider integrals with $0<\theta<1$. Let us denote $\chi_{(\tau_N,\tau_N+\theta)}(t)=1$, if $\tau_N\leq t\leq\tau_N+\theta$, and otherwise, $\chi_{(\tau_N,\tau_N+\theta)}(t)=0$, we observe that

$$\int_{\tau_N}^{\tau_N+\theta}1\cdot\|(\mathbf{u}^{(N)})^s\|_{L^2(\mathcal{O})}\|\nabla\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}dr=\int_0^T\chi_{(\tau_N,\tau_N+\theta)}(r)\|(\mathbf{u}^{(N)})^s\|_{L^2(\mathcal{O})}\|\nabla\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}dr$$
$$\leq\|\chi_{(\tau_N,\tau_N+\theta)}(r)\|_{L^6((0,T))}\|(\mathbf{u}^{(N)})^s\|_{L^3(0,T;L^2(\mathcal{O}))}\|\nabla\mathbf{u}^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))}$$
$$=\|1\|_{L^6((\tau_N,\tau_N+\theta))}\|(\mathbf{u}^{(N)})^s\|_{L^3(0,T;L^2(\mathcal{O}))}\|\nabla\mathbf{u}^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))}$$
$$\leq\theta^{\frac{1}{6}}\|(\mathbf{u}^{(N)})^s\|_{L^3(0,T;L^2(\mathcal{O}))}\|\nabla\mathbf{u}^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))},$$

and for $J_1^{(N)}(t)$, if $i=j$,

$$\begin{aligned}
&\mathbb{E}|\int_{\tau_N}^{\tau_N+\theta}\langle A_{ij}(\mathbf{u}^{(N)})\nabla u_j^{(N)},\nabla\Pi_N\phi_i\rangle dr|\\
&\le \mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\|A_{ij}(\mathbf{u}^{(N)})\|_{L^2(\mathcal{O})}\|\nabla u_j^{(N)}\|_{L^2(\mathcal{O})}\|\nabla\phi_i\|_{L^\infty(\mathcal{O})}dr\\
&\le C\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}(1+\|(\mathbf{u}^{(N)})^s\|_{L^2(\mathcal{O})})\|\nabla\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}\|\phi\|_{H^3(\mathcal{O})}dr\\
&\le C\mathbb{E}((\theta^{\frac12}+\theta^{\frac16}\|(\mathbf{u}^{(N)})^s\|_{L^3(0,T;L^2(\mathcal{O}))})\|\nabla\mathbf{u}^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))})\cdot\|\phi\|_{H^3(\mathcal{O})}\\
&\le C\{\theta^{\frac16}(\mathbb{E}(\int_0^T\|(\mathbf{u}^{(N)})^s\|_{L^2(\mathcal{O})}^3dr)^{\frac23})^{\frac12}\cdot(\mathbb{E}\int_0^T\|\nabla\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}^2dr)^{\frac12}\\
&+\theta^{\frac12}\mathbb{E}\|\nabla\mathbf{u}^{(N)}\|_{L^2(0,T;L^2(\mathcal{O}))}\}\cdot\|\phi\|_{H^3(\mathcal{O})}\le C\theta^{\frac16}\|\phi\|_{H^3(\mathcal{O})},
\end{aligned}$$

if $i\neq j$,

$$\begin{aligned}
&\mathbb{E}|\int_{\tau_N}^{\tau_N+\theta}\langle A_{ij}(\mathbf{u}^{(N)})\nabla u_j^{(N)},\nabla\Pi_N\phi_i\rangle dr|\\
&\le \mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\int_{\mathcal{O}}|sa_{ij}u_i^{(N)}(u_j^{(N)})^{s-1}\nabla u_j^{(N)}\nabla\phi_i|dxdr\\
&\le C\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\int_{\mathcal{O}}|u_i^{(N)}\nabla(u_j^{(N)})^s\nabla\phi_i|dxdr\\
&\le C\mathbb{E}\int_{\tau_N}^{\tau_N+\theta}\|\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}\|\nabla(\mathbf{u}^{(N)})^s\|_{L^2(\mathcal{O})}\|\nabla\phi\|_{L^\infty(\mathcal{O})}dr\\
&\le C\mathbb{E}\left\{\left(\int_{\tau_N}^{\tau_N+\theta}||u^{(N)}||_{L^2(\mathcal{O})}dr\right)^{\frac12}\left(\int_{\tau_N}^{\tau_N+\theta}||\nabla(u^{(N)})^s||_{L^2(\mathcal{O})}^2dr\right)^{\frac12}\right\}\\
&\cdot||\phi||_{H^3(\mathcal{O})}\le C\theta^{\frac12}\left(\mathbb{E}\sup_{t\varepsilon(0,T)}||u^{(N)}||_{L^2(\mathcal{O})}^2\right)^{\frac12}\left(\mathbb{E}\int_0^T||\nabla(u^{(N)})^s||_{L^2(\mathcal{O})}^2\right)^{\frac12}\\
&\cdot||\phi||_{H^3(\mathcal{O})}\le C\theta^{\frac16}||\phi||_{H^3(\mathcal{O})}
\end{aligned}$$

Denote $I_2=\{\omega\in\Omega:0\le\int_0^T\|\mathbf{u}^{(N)}(r)\|_{L^2(O)}^3dr\le1\}$, with the complement of I_2 given by $\bar{I}_2=\{\omega\in\Omega:\int_0^T\|\mathbf{u}^{(N)}(r)\|_{L^2(O)}^3dr>1\}$, then

$$\begin{aligned}
\mathbb{E}(\int_0^T\|\mathbf{u}^{(N)}(r)\|_{L^2(\mathcal{O})}^3dr)^{\frac23}&=\int_{I_2\cup\bar{I}_2}(\int_0^T\|\mathbf{u}^{(N)}(r)\|_{L^2(\mathcal{O})}^3dr)^{\frac23}\mathbb{P}(d\omega)\\
&\le1+\mathbb{E}\int_0^T\|\mathbf{u}^{(N)}(r)\|_{L^2(\mathcal{O})}^3dr\le C,
\end{aligned}$$

then for $J_2^{(N)}(t)$, we have

$$\begin{aligned}
&\mathbb{E}|\int_{\tau_N}^{\tau_N+\theta}\langle\Pi_N(\sigma_{ij}(\mathbf{u}^{(N)}(r))dW_j(r),\phi_i\rangle|^2\le\mathbb{E}(\int_{\tau_N}^{\tau_N+\theta}\|\sigma(\mathbf{u}^{(N)})\|_{\mathcal{L}(Y;L^2(\mathcal{O}))}^2dr)\cdot\|\phi\|_{L^2(\mathcal{O})}^2\\
&\le C\mathbb{E}(\int_{\tau_N}^{\tau_N+\theta}(1+\|\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}^2)dr)\cdot\|\phi\|_{L^2(\mathcal{O})}^2\\
&=C(\theta+\mathbb{E}\int_0^T\chi_{(\tau_N,\tau_N+\theta)}\|\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}^2dr)\cdot\|\phi\|_{L^2(\mathcal{O})}^2\\
&\le C(\theta+\mathbb{E}\|\chi_{(\tau_N,\tau_N+\theta)}\|_{L^3((0,T))}(\int_0^T\|\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}^3dr)^{\frac23})\cdot\|\phi\|_{L^2(\mathcal{O})}^2\\
&\le C(\theta+\theta^{\frac13}\mathbb{E}(\int_0^T\|\mathbf{u}^{(N)}\|_{L^2(\mathcal{O})}^3dr)^{\frac23})\cdot\|\phi\|_{L^2(\mathcal{O})}^2\le C\theta^{\frac13}\|\phi\|_{H^3(\mathcal{O})}^2.
\end{aligned}$$

Let $\kappa>0$, $\varepsilon>0$, by the definition of $H^3(O)'$ norm and Chebyshev inequality, for every ϕ_i, with $\|\phi_i\|_{H^3(O)}=1$,

$$\mathbb{P}\{\|J_1^{(N)}(\tau_N+\theta)-J_1^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\ge\kappa\}\le\frac{1}{\kappa}\mathbb{E}\|J_1^{(N)}(\tau_N+\theta)-J_1^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'}\le\frac{C\theta^{\frac16}}{\kappa},$$

choose $\delta_1 = (\kappa\varepsilon/C)^6$, we infer that

$$\sup_{N\in\mathbb{N}} \sup_{0<\theta<\delta_1} \mathbb{P}\{\|J_1^{(N)}(\tau_N+\theta) - J_1^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'} \geq \kappa\} \leq \varepsilon.$$

Still applying the Chebyshev inequality,

$$\mathbb{P}\{\|J_2^{(N)}(\tau_N+\theta) - J_2^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'} \geq \kappa\} \leq \frac{1}{\kappa^2}\mathbb{E}\|J_2^{(N)}(\tau_N+\theta) - J_2^{(N)}(\tau_N)\|^2_{H^3(\mathcal{O})'} \leq \frac{C\theta^{\frac{1}{3}}}{\kappa^2},$$

choose $\delta_2 = (\kappa^2\varepsilon/C)^3$, we infer that

$$\sup_{N\in\mathbb{N}} \sup_{0<\theta<\delta_2} \mathbb{P}\{\|J_2^{(N)}(\tau_N+\theta) - J_2^{(N)}(\tau_N)\|_{H^3(\mathcal{O})'} \geq \kappa\} \leq \varepsilon.$$

This verifies the Aldous condition for $J_1^{(N)}(t)$ and $J_2^{(N)}(t)$, and $J_0^{(N)}$ is automatically satisfied since no parameter t is involved for $J_0^{(N)}$. The set of measures $\{L(\mathbf{u}^{(N)}(t)) : N\in N\}$ is tight on (Z_T,T). □

The Lemma 12 [2] has shown that $Z_T\times C^0([0,T];Y_0)$ satisfies the assumption for Skorohod-Jakubowski theorem, and we have shown that $u^{(N)}$ is tight in Z_T in the Lemma 9. Readers are referred to the Theorem 23 [2], Theorem C.1 [18, 19] for this famous Skorokhod-Jakubowski theorem.

By the Skorokhod-Jakubowski theorem, we are able to find a probability space $(\widetilde{\Omega}, \widetilde{\mathrm{F}}, \widetilde{\mathrm{P}})$, and on this space $Z_T\times C^0([0,T];Y_0)$-valued random variables $(\widetilde{\mathbf{u}},\widetilde{\mathbf{W}})$, $(\overset{(N)}{\widetilde{\mathbf{u}}},\overset{(N)}{\widetilde{\mathbf{W}}})$ that $(\overset{(N)}{\widetilde{\mathbf{u}}},\overset{(N)}{\widetilde{\mathbf{W}}})$ has the same law as $(\mathbf{u}^{(N)},\mathbf{W})$ on $B(Z_T\times C^0([0,T];Y_0))$ and

$$(\overset{(N)}{\widetilde{\mathbf{u}}},\overset{(N)}{\widetilde{\mathbf{W}}})\rightarrow(\widetilde{\mathbf{u}},\widetilde{\mathbf{W}}) \quad \text{in } Z_T\times C^0([0,T];Y_0), \widetilde{\mathrm{P}}\text{-}a.s. \text{ as } N\rightarrow\infty.$$

By the definition of Z_T, we have

$$\begin{gathered}\overset{(N)}{\widetilde{\mathbf{u}}} \rightarrow \widetilde{\mathbf{u}} \text{ in } C^0([0,T];H^3(\mathcal{O})'), \quad \overset{(N)}{\widetilde{\mathbf{u}}} \rightharpoonup \widetilde{\mathbf{u}} \text{ weakly in } L^2(0,T;H^1(\mathcal{O})),\\ \overset{(N)}{\widetilde{\mathbf{u}}} \rightarrow \widetilde{\mathbf{u}} \text{ in } C^0([0,T];L^2_w(\mathcal{O})), \quad \overset{(N)}{\widetilde{\mathbf{u}}} \rightarrow \widetilde{\mathbf{u}} \text{ in } L^2([0,T];L^2(\mathcal{O})),\\ \overset{(N)}{\widetilde{\mathbf{W}}} \rightarrow \widetilde{\mathbf{W}} \text{ in } C^0([0,T];Y_0).\end{gathered} \tag{44}$$

We have shown that $u_i^{(N)}(t)$ is non-negative, P-$a.s.$ then by the method of Lemma 25 [3], we can show that for a.e. $(x,t) \in O\times(0,T)$, $\widetilde{u}_i^N(t)$ is non-negative, $\widetilde{\mathrm{P}}$-$a.s.$ with its limit $\underset{i}{\widetilde{u}}(x,t) \geq 0$, $\widetilde{\mathrm{P}}$-$a.s.$ for every $1 \leq i \leq n$.

Lemma 10. *For every* $1 \leq i \leq n$, $\widetilde{u}_i(x,t) \geq 0$ *in* O, *a.e.*$t \in [0,T]$, $\widetilde{P}-a.s.$

Proof. We for convenience denote $Q_T = O\times(0,T)$, and let $1 \leq i \leq n$. Since $u_i^{(N)}(t) \geq 0$ in Q_T, P-$a.s.$ we have $E\|(u_i^{(N)}(t))^-\|_{L^2(0,T;L^2(O))} = 0$, where $z^- = \max\{0,-z\}$. The function $u_i^{(N)}(t)$ is Z_T-Borel measurable and so does its negative part. Therefore, using the equivalence of laws of $u_i^{(N)}(t)$ and $\widetilde{u}_i^N(t)$ in Z_T and writing $\mu_i^{(N)}$ and $\widetilde{\mu}_i^N$

for the laws of $u_i^{(N)}(t)$ and $\widetilde{u}_i^N(t)$, respectively, we obtain

$$\widetilde{\mathbb{E}}\|(\widetilde{u}_i^N(t))^-\|_{L^2(Q_T)} = \int_{L^2(Q_T)} \|y^-\|_{L^2(Q_T)} d\widetilde{\mu}_i^N(y)$$
$$= \int_{L^2(Q_T)} \|y^-\|_{L^2(Q_T)} d\mu_i^{(N)}(y) = \mathbb{E}\|(u_i^{(N)}(t))^-\|_{L^2(Q_T)} = 0.$$

This shows that $\widetilde{u}_i^N(t) \geq 0$ $a.e.$ in Q_T, $\widetilde{\mathrm{P}}$-$a.s.$ The convergence (up to a subsequence) $\overset{(N)}{\widetilde{\mathbf{u}}}(t) \to \widetilde{\mathbf{u}} a.e.$ in Q_T, $\widetilde{\mathrm{P}}$-$a.s.$ then implies that $\widetilde{u}_i \geq 0$ in Q_T, $\widetilde{\mathrm{P}}$-$a.s.$ □

Let us state some important convergence results in the Lemma 11. This lemma is a key tool in the proof of the main theorem.

Lemma 11. *For every* $r, t \in (0,T)$ *with* $r \leq t$, $\varphi_i \in L^2(\mathcal{O})$ *and* $\phi_i \in H^3(O)$ *satisfying* $\nabla\phi_i \cdot \nu = 0$ *on* ∂O, *we have*

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T \langle \widetilde{u}_i^N(t) - \underset{i}{\widetilde{u}}(t), \varphi_i \rangle^2 dt = 0, \tag{45}$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \langle \widetilde{u}_i^N(0) - \underset{i}{\widetilde{u}}(0), \varphi_i \rangle^2 = 0, \tag{46}$$

$$\lim_{N\to\infty} \widetilde{\mathbb{E}} \int_0^T |\sum_{j=1}^n \int_0^t \langle A_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r)) \nabla \widetilde{u}_j^N(r) - A_{ij}(\widetilde{\mathbf{u}}(r)) \nabla \underset{j}{\widetilde{u}}(r), \nabla\phi_i \rangle dr| dt = 0, \tag{47}$$

$$\lim_{N\to\infty} \widetilde{E} \int_0^t \| \langle \sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r)) dW_j^{(N)}(r) - \sigma_{ij}(\widetilde{\mathbf{u}}(r)) dW_j(r), \varphi_i \rangle|^2 = 0 \tag{48}$$

Proof. The proof largely relies on the Lemma 10 [4], Lemma 16 [1] and Lemma 26 [3]. The verification for the Vitali's convergence theorem (A.4. [2]) be completely referred to the Lemma 16 [2], so we make an omittance. Some convergence results and embedding theorems for test functions are also referred to the Lemma 16 [2].

Since $\overset{(N)}{\widetilde{\mathbf{u}}} \to \widetilde{\mathbf{u}}$ in $C^0([0,T];\ L_w^2(O))$, $\widetilde{\mathrm{P}}$-$a.s.$ then for every $\varphi_i \in L^2(O)$,

$$\lim_{N\to\infty} \langle \widetilde{u}_i^N(t), \varphi_i \rangle = \langle \underset{i}{\widetilde{u}}(t), \varphi_i \rangle, \quad \widetilde{\mathrm{P}}\text{-}a.s.$$

By the dominated convergence theorem, we conclude that

$$\lim_{N\to\infty} \int_0^T \langle \widetilde{u}_i^N(t) - \underset{i}{\widetilde{u}}(t), \varphi_i \rangle^2 dt = 0.$$

Since $\overset{(N)}{\widetilde{\mathbf{u}}} \to \widetilde{\mathbf{u}}$ in $C^0([0,T];\ L_w^2(O))$, $\widetilde{\mathrm{P}}$-$a.s.$ and $\widetilde{\mathbf{u}}$ is continuous at $t = 0$, we deduce that $\lim_{N\to\infty} \langle \widetilde{u}_i^N(0), \varphi_i \rangle = \langle \underset{i}{\widetilde{u}}(0), \varphi_i \rangle$, $\widetilde{\mathrm{P}}$-$a.s.$ Then by the Vitali's convergence theorem, we can show (45)–(46).

By the structural information of the diffusion matrix $A(\mathbf{u}) = (A_{ij}(\mathbf{u}))$, in order to show (47), we have to show that for every $1 \leq j$, $k \leq n$,

$$\lim_{N\to\infty} \int_0^t \langle \nabla \widetilde{u}_j^N(r) - \nabla \underset{j}{\widetilde{u}}(r), \phi_i \rangle dr = 0, \tag{49}$$

$$\lim_{N\to\infty}\int_0^t\langle \widetilde{u}_j^N(r)\nabla(\widetilde{u}_k^N(r))^s-\underset{j}{\widetilde{u}}(r)\nabla\widetilde{u}_k^N(r),\nabla\phi_i\rangle dr=0, \tag{50}$$

$$\lim_{N\to\infty}\int_0^t\langle (\widetilde{u}_k^N(r))^s\nabla\widetilde{u}_j^N(r)-\widetilde{u}_k^N(r)\nabla\underset{j}{\widetilde{u}}(r),\nabla\phi_i\rangle dr=0. \tag{51}$$

By (44), we can show that (49) holds for every $\phi_i\in H^1(O)$. Let us denote

$$I_1^{(N)}=\int_0^t\int_{\mathcal{O}}(\widetilde{u}_j^N(r)-\underset{j}{\widetilde{u}}(r))\nabla(\widetilde{u}_k^N(r))^s\nabla\phi_i dxdr,$$

$$I_2^{(N)}=\int_0^t\int_{\mathcal{O}}\widetilde{u}_j(r)\nabla((\widetilde{u}_k^N(r))^s-\widetilde{u}_k^N(r))\nabla\phi_i dxdr,$$

$$I_3^{(N)}=-\int_0^t\int_{\mathcal{O}}((\widetilde{u}_k^N(r))^s\widetilde{u}_j^N(r)-\widetilde{u}_k^N(r)\underset{j}{\widetilde{u}}(r))\Delta\phi_i dxdr,$$

$$I_4^{(N)}=-\int_0^t\int_{\mathcal{O}}(\widetilde{u}_j^N(r)\nabla(\widetilde{u}_k^N(r))^s-\underset{j}{\widetilde{u}}(r)\nabla\widetilde{u}_k^N(r))\nabla\phi_i dxdr,$$

and we notice that

$$\int_0^t\langle \widetilde{u}_j^N(r)\nabla(\widetilde{u}_k^N(r))^s-\underset{j}{\widetilde{u}}(r)\nabla\widetilde{u}_k^N(r),\nabla\phi_i\rangle dr=I_1^{(N)}+I_2^{(N)},$$

$$\int_0^t\langle (\widetilde{u}_k^N(r))^s\nabla\widetilde{u}_j^N(r)-\widetilde{u}_k^N(r)\nabla\underset{j}{\widetilde{u}}(r),\nabla\phi_i\rangle dr=I_3^{(N)}+I_4^{(N)}.$$

One step further, we have

$$|I_1^{(N)}|\ \le\ \|\widetilde{u}_j^N(r)-\underset{j}{\widetilde{u}}(r)\|_{L^2(0,T;L^2(\mathcal{O}))}\|\nabla(\widetilde{u}_k^N(r))^s\|_{L^2(0,T;L^2(\mathcal{O}))}\|\phi_i\|_{H^3(\mathcal{O})},$$

and by (44), we deduce that as $N\to\infty$, $I_1^{(N)}\to0$. For $I_2^{(N)}$, since $(\widetilde{u}_k^N(r))^s$ converges weakly to $\widetilde{u}_k^s(r)$ in $L^2(0,T;H^1(O))$, and $\underset{j}{\widetilde{u}}(r)\nabla\phi_i\in L^2(0,T;L^2(O))$, then as $N\to\infty$, $I_2^{(N)}\to0$, and (50) holds.

Since $I_4^{(N)}=-I_1^{(N)}-I_2^{(N)}$, then as $N\to\infty$, $I_4^{(N)}\to0$. For $I_3^{(N)}$, we have

$$\begin{aligned}|I_3^{(N)}|&=|\int_0^t\int_{\mathcal{O}}(\widetilde{u}_k^N(r))^s(\widetilde{u}_j^N(r)-\underset{j}{\widetilde{u}}(r))\Delta\phi_i dxdr\\&\quad+\int_0^t\int_{\mathcal{O}}\widetilde{u}_j(r)((\widetilde{u}_k^N(r))^s-\widetilde{u}_k^N(r))\Delta\phi_i dxdr|\\&\le\|\widetilde{u}_j^N(r)-\underset{j}{\widetilde{u}}(r)\|_{L^2(0,T;L^2(\mathcal{O}))}\|(\widetilde{u}_k^N(r))^s\|_{L^2(0,T;H^1(\mathcal{O}))}\|\phi_i\|_{H^3(\mathcal{O})}\\&\quad+|\int_0^t\int_{\mathcal{O}}\widetilde{u}_j(r)\Delta\phi_i((\widetilde{u}_k^N(r))^s-\widetilde{u}_k^N(r)dxdr|,\end{aligned}$$

then by (44) and $\underset{j}{\widetilde{u}}(r)\phi_i\in L^2(0,T;\,L^2(O))$, we deduce that as $N\to\infty$, $I_3^{(N)}\to0$, and (51) holds.

Convergences (49)−(51) indicate that

$$\lim_{N\to\infty}\int_0^t\langle A_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r))\nabla\widetilde{u}_j^N(r)-A_{ij}(\widetilde{\mathbf{u}}(r))\nabla\underset{j}{\widetilde{u}}(r),\nabla\phi_i\rangle dr=0,$$

and by the Vitali's convergence theorem, we can show that (47) holds.

For the final limit (48), we mainly rely on

$$\int_0^t \|\langle\sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r)) - \sigma_{ij}(\widetilde{\mathbf{u}}(r)), \varphi_i\rangle\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}dr$$

$$\leq \int_0^t \|\sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r)) - \sigma_{ij}(\widetilde{\mathbf{u}}(r))\|^2_{\mathcal{L}(Y;L^2(\mathcal{O}))}\|\varphi_i\|^2_{L^2(\mathcal{O})}dr \leq C\|\overset{(N)}{\widetilde{\mathbf{u}}} - \widetilde{\mathbf{u}}\|^2_{L^2(0,T;L^2(\mathcal{O}))}\|\varphi_i\|^2_{L^2(\mathcal{O})},$$

and $\overset{(N)}{\widetilde{\mathbf{u}}} \to \widetilde{\mathbf{u}}$ in $L^2(0,T;\, L^2(O))$, $\widetilde{\mathrm{P}}$-$a.s.$ implies that

$$\lim_{N\to\infty}\int_0^t \|\langle\sigma_{ij}(\overset{(N)}{\widetilde{u}}(r)) - \sigma_{ij}(\widetilde{u}(r)),\varphi_i\rangle\|^2_{L(Y;L^2(O))}dr = 0.$$

By the Vitali's convergence theorem, we obtain that for every $\varphi_i \in L^2(O)$,

$$\lim_{N\to\infty}\widetilde{E}\int_0^t \|\langle\sigma_{ij}(\overset{(N)}{\widetilde{u}}(r)) - \sigma_{ij}(\widetilde{u}(r)),\varphi_i\rangle\|^2_{L(Y;L^2(O))}dr = 0.$$

Then by the I$t\hat{o}$ isometry, we have

$$\lim_{N\to\infty}\widetilde{E}\int_0^t \|\langle\sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r)) - \sigma_{ij}(\widetilde{\mathbf{u}}(r))dW_j(r),\varphi_i\rangle|^2 = 0,$$

and by the dominated convergence theorem, we can show that (48) holds. □

After above preparations, let us give the proof of the main Theorem 1.

Proof. Let us define

$$\Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi)(t) = \langle u_i^{(N)}(0), \phi_i\rangle + \sum_{j=1}^n \int_0^t \langle\Pi_N div(A_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r))\nabla\widetilde{u}_j^N(r)), \phi_i\rangle dr$$

$$+\langle\sum_{j=1}^n \int_0^t \Pi_N\sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r))d\widetilde{W}_j^N(r), \phi_i\rangle,$$

And

$$\Lambda_i(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}, \phi)(t) = \langle\underset{i}{\widetilde{u}}(0), \phi_i\rangle + \sum_{j=1}^n \int_0^t \langle div(A_{ij}(\widetilde{\mathbf{u}}(r))\nabla\underset{j}{\widetilde{u}}(r)), \phi_i\rangle dr + \langle\sum_{j=1}^n \int_0^t \sigma_{ij}(\widetilde{\mathbf{u}}(r))d\underset{j}{\widetilde{W}}(r), \phi_i\rangle,$$

for $t \in (0,T)$ and $1 \leq i \leq n$.

Since for every $\varphi_i \in L^2(O)$, we have

$$\|\langle\widetilde{u}_i^N(t), \varphi_i\rangle - \langle\underset{i}{\widetilde{u}}(t), \varphi_i\rangle\|_{L^2(\widetilde{\Omega}\times(0,T))} = \widetilde{\mathbb{E}}\int_0^T \langle\widetilde{u}_i^N(t) - \underset{i}{\widetilde{u}}(t), \varphi_i\rangle^2 dt,$$

then by Lemma 11, we can show that

$$\lim_{N\to\infty}\|\langle\widetilde{u}_i^N(t) - \underset{i}{\widetilde{u}}(t), \varphi_i\rangle\|_{L^2(\widetilde{\Omega}\times(0,T))} = 0.$$

Let $\phi \in H^3(O)$ satisfying $\nabla\phi\cdot\nu = 0$ on ∂O, we have

$$\begin{aligned} &\|\Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi) - \Lambda_i(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}, \phi)\|_{L^1(\widetilde{\Omega}\times(0,T))} \\ &= \widetilde{\mathbb{E}}\int_0^T |\Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi) - \Lambda_i(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}, \phi)|dt \le \widetilde{\mathbb{E}}\int_0^T |\langle \widetilde{u}_i^N(0) - \underset{i}{\widetilde{u}}(0), \phi_i\rangle|dt \\ &+\widetilde{\mathbb{E}}\int_0^T |\sum_{j=1}^n \int_0^t \langle \sigma_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r))d\widetilde{W}_j^N(r) - \sigma_{ij}(\widetilde{\mathbf{u}}(r))d\underset{j}{\widetilde{W}}(r), \varphi_i\rangle|dt \\ &+\widetilde{\mathbb{E}}\int_0^T |\sum_{j=1}^n \int_0^t \langle A_{ij}(\overset{(N)}{\widetilde{\mathbf{u}}}(r))\nabla\widetilde{u}_j^N(r) - A_{ij}(\widetilde{\mathbf{u}}(r))\nabla\underset{j}{\widetilde{u}}(r), \nabla\phi_i\rangle dr|dt, \end{aligned} \tag{52}$$

and Lemma 11 indicates that each term in the right-hand-side of (52) vanishes, as $N\to\infty$, thus

$$\lim_{N\to\infty} \|\Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi) - \Lambda_i(\widetilde{\mathbf{u}}, \widetilde{\mathbf{W}}, \phi)\|_{L^1(\widetilde{\Omega}\times(0,T))} = 0.$$

$\mathbf{u}^{(N)}(t)$ is the strong solution (in the probabilistic sense) of (11). By the definition, $\mathbf{u}^{(N)}(t)$ satisfies $\langle u_i^{(N)}(t),\phi_i\rangle = \Lambda_i^{(N)}(\mathbf{u}^{(N)},\mathbf{W}, \phi)(t)$, P-$a.s.$ for every $t\in (0,T)$, $\phi \in H^1(O)$. In particular, we have

$$\int_0^T E|\langle u_i^{(N)}(t),\phi_i\rangle - \Lambda_i^{(N)}(\mathbf{u}^{(N)},\mathbf{W}, \phi)(t)|dt = 0.$$

Since L($\mathbf{u}^{(N)}$,$\mathbf{W}$) coincides with L($\overset{(N)}{\widetilde{\mathbf{u}}}$, $\overset{(N)}{\widetilde{\mathbf{W}}}$), we have

$$\int_0^T \widetilde{E}|\langle \widetilde{u}_i^{(N)}(t),\phi_i\rangle - \Lambda_i^{(N)}(\overset{(N)}{\widetilde{\mathbf{u}}}, \overset{(N)}{\widetilde{\mathbf{W}}}, \phi)(t)|dt = 0, \quad 1\le i\le n.$$

Let $N\to\infty$, then $\int_0^T \widetilde{E}|\langle \underset{i}{\widetilde{u}}(t),\phi_i\rangle - \Lambda_i(\widetilde{\mathbf{u}},\widetilde{\mathbf{W}}, \phi)(t)|dt = 0$. This identity holds for every $\phi \in H^3(O)$ satisfying $\nabla\phi\cdot\nu = 0$ on ∂O, and by the density argument, it holds for every $\phi \in H^1(O)$. Hence, for a.e. $t\in [0,T]$, $\omega\in\widetilde{\Omega}$, we deduce that $\langle \underset{i}{\widetilde{u}}(t),\phi_i\rangle$ -$\Lambda_i(\widetilde{\mathbf{u}},\widetilde{\mathbf{W}}, \phi)(t) = 0$, $1\le i\le n$. By the definition of Λ_i, this means that for $a.e.t \in [0,T]$, $\omega\in\widetilde{\Omega}$,

$$\langle \widetilde{u}_i(t), \phi_i\rangle = \langle \widetilde{u}_i(0), \phi_i\rangle + \sum_{j=1}^n \int_0^t \langle div(A_{ij}(\widetilde{\mathbf{u}}(r))\nabla\widetilde{u}_j(r)), \phi_i\rangle dr + \langle \sum_{j=1}^n \int_0^t \sigma_{ij}(\widetilde{\mathbf{u}}(r))d\widetilde{W}_j(r), \phi_i\rangle.$$

We set $\widetilde{U} = (\widetilde{\Omega},\widetilde{\mathrm{F}},\widetilde{\mathrm{P}},\widetilde{F})$, and the system $(\widetilde{U},\widetilde{\mathbf{W}},\widetilde{\mathbf{u}})$ is a martingale solution of (1)−(3). □

7. Conclusion

In this work, we have shown that a martingale solution exists to (1)−(3) when $s > 2$. If $1 < s < 2$, the diffusion matrix does not satisfy the local Lipschitz condition. Without the local Lipschitz condition, we are not able to apply the existence and uniqueness theorem to derive approximated solutions.

In this situation, we have to regularize the diffusion matrix. The key idea is to introduce a sequence of diffusion matrices denoted as $A(K,\mathbf{u})$, $K\in\mathrm{N}$. For every

$K{\in}\mathrm{N}$, $A(K,\mathbf{u})$ satisfies the local Lipschitz condition. Then we derive a sequence of approximated solutions denoted as $\mathbf{u}^{(N)}(K,t)$.

We can show that as $N{\to}\infty,\mathbf{u}^{(N)}(K,t)$ converges, and we denote its limit as $\mathbf{u}^{(K)}$. Then we let $K{\to}\infty$, and $u^{(K)}$ converges to a martingle solution of (1)−(3).

Once $1 < s < 2$, we do not have to construct the auxiliary sequence. Strong enough estimations can be derived once we apply the $It\widehat{o}$ formula to $\|\mathbf{u}^{(N)}(K,t)\|_{L^2(O)}$. As methods are quite different, in this work we focus on $s > 2$ case.

Once $0 < s < 1$, how to regularize the diffusion matrix in order to derive approximated solutions, and how to estimate these approximated solutions are very difficult. We are still investigating how to show a martingale solution exists to (1)−(3) if $0 < s < 1$.

We notice that in order to show a martingale solution exists, the diffusion coefficients (a_{ij}) have to satisfy assumption (6). We rely on this assumption to derive matrix analysis results. How to show a martingale solution exists under a weaker assumption for diffusion coefficients is another main topic in future investigations. We conjecture that if $s> 0$, a martingale solution of (1)−(3) exists if $a_{ij} > 0, 1 \leq i,\ j \leq n.$

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