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Article

# Bielecki–Hyers–Ulam stability of non-linear fractional Volterra Fredholm Hammerstein integro-delay dynamic systems with instantaneous impulses on time scale

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**Abstract:** In this paper, we study the existence and uniqueness of solutions, Bielecki–Hyers–Ulam stability, and Bielecki–Hyers–Ulam–Rassias stability for non-linear fractional Volterra Fredholm Hammerstein integro-delay dynamic systems with instantaneous impulses on time scale. Such systems provide a unified framework that encompasses both continuous and discrete models, making them highly appropriate for describing complex real-world phenomena involving memory effects, hereditary properties, and sudden perturbations. Sufficient conditions are established for the existence and uniqueness of solutions to the considered systems. In particular, the Picard operator and the Banach fixed point theorem are utilized to prove the existence and uniqueness of solutions. Moreover, we analyze the qualitative behavior of solutions by proving Bielecki–Hyers–Ulam stability and Bielecki–Hyers–Ulam–Rassias stability. To obtain these stability results, Grönwall's inequality on time scales is used as the main analytical tool. For our results, some suitable assumptions are imposed along with appropriate Lipschitz conditions on the nonlinear terms. By constructing appropriate contractive mappings in a suitably defined Bielecki-type normed space, we develop a unified and systematic framework to handle the combined effects of integral operators, fractional dynamics, delay arguments, and impulsive perturbations. Finally, an illustrative example is provided to demonstrate the effectiveness and applicability of the theoretical findings.

**Keywords:** time scale; Volterra Fredholm Hammerstein integral; Bielecki–Hyers–Ulam stability; impulses; integro-dynamic system

## 1. Introduction

Fractional-order differential equations (DEs) extend the concept of classical integer-order DEs. Fractional calculus has seen obtained significant advancement due to its wide-ranging applications in fields such as physical sciences, electrochemistry, economics, electromagnetics, medicine etc. DEs of fractional order hold significant importance in fields like signal processing, statistical physics, astronomy, control, defence, viscoelasticity, optics, electrical circuits etc. Notable literature provides theoretical tools for qualitative analysis, distinction, and the interconnection among integral models, classical DEs, and fractional-order DEs in this field. In recent years, the integration of fractional calculus into integro-dynamic systems has proven effective in capturing the inherent memory and non-local characteristics of such systems [1]. By utilizing fractional-order derivatives and integrals, these models can more accurately reflect the complex and long-range dependencies found in diverse domains such as physics, biology, and engineering. This approach not only enhances the ability to model dynamic behavior but also improves control and optimization strategies [2]. Additionally, fractional calculus serves as a valuable framework for system identification, parameter tuning, and performance enhancement, thereby increasing the reliability and efficiency of real-world applications [3]. Ali et al. [4, 5] studied toppled and coupled systems of nonlinear implicit fractional DEs. Different fractional Langevin equations were investigated [6–8]. For further details, see researchers's previous studies [9–12].

The integration of impulses with DEs has become essential in mathematical modelling. Daily life encounters numerous processes and phenomena that experience sudden, instantaneous changes at specific moments. These events are characterized by impulse effects, which indicate temporary disruptions within a system. Many scholars have thoroughly investigated the relationship between DEs and impulses, making significant progress in that area of study. Scholars interested in this field can delve into the valuable contributions provided by various papers [13–16]. When systems are influenced either by impulsive effects alone or by time delays, a broad spectrum of dynamic patterns can arise, especially in cases where both impulses and delays are present together. An impulsive delay DE is used to describe situations where sudden changes and time delays happen. For further insights into impulsive delay DEs, readers are referred to exploring the relevant literature [17, 18].

Stability analysis [19,20] represents a significant branch of mathematical sciences, incorporating a range of methodological approaches. Among the various types in this field, Hyers-Ulam (HU) stability stands out as particularly fascinating and important. The concept of HU stability was initially introduced by Ulam [21,22], with Hyers [23] making notable advancements toward its partial resolution, especially in the context of Banach spaces. In 1978, Rassias expanded and generalized the concept, introducing what is now known as HU-Rassias (HUR) stability [24]. Then, one of the earliest contributions in this direction is due to Obloza [25, 26]. Since that time, substantial research has been devoted to HU and HUR stability. Wang et al. [27–30] examined the Ulam's stability of impulsive equations. For further details, see the literature [31–35].

In 1988, Hilger [36] introduced the calculus of measure chains to bridge the gap between continuous and discrete analysis. Aulbach, who supervised Hilger' PhD thesis [37], highlighted three primary objectives of this new calculus: unification, extension, and discretization. Time scale (TS) theory serves as a valuable framework, allowing for the unified study of both continuous-time and discrete-time systems. Typically, continuous and discrete dynamical systems are examined independently, necessitating separate proofs for each scenario through continuous or discrete analysis. However, TS theory merges these analyses into a single formulation. Consequently, proving a result once under this theory applies it universally to both continuous and discrete cases. The foundational work on TS calculus was done by Bohner and Peterson [38, 39]. Impulsive dynamical systems on TSs were studied [40, 41]. More details are provided elsewhere [42–46].

Several publications have delved into the qualitative properties of fractional integro-dynamic systems on TSs featuring impulses. Pervaiz et al. [47] contributed by deriving existence, uniqueness and stability results for fractional impulsive integro causal evolution systems on TSs domain. Pervaiz et al. [48] also carried out an analysis of the controllability as well as stability aspects of certain dynamical fractional systems with delay on TSs, taking into account the effects of impulses. Shah [49] obtained the Bielecki-HU (BHU) and Bielecki-HUR (BHUR) stabilities of fractional non-linear impulsive Hammerstein integro-dynamic system on TS with delay. Shah et al. [50] investigated nonlinear Volterra Fredholm Hammerstein impulsive integro-dynamic systems on TS with delay and successfully derived results concerning their stability and controllability. Within the TS domain framework, the stability properties in the sense of HU and HUR for the considered model and its corresponding fractional-order system have been successfully demonstrated. Nisar et al. [51] derived existence and stability results of a nonlinear fractional neutral dynamic equation with initial conditions on TSs. Morsy et al. [52,53] examined the existence and uniqueness of the solutions for different fractional DEs on TSs.

Inspired by the results by Shah et al. [49,50] and the work outlined above, this paper proves the existence and uniqueness of solutions, along with BHU and BHUR stabilities, for a nonlinear fractional Volterra Fredholm Hammerstein integro-dynamic system with delay and instantaneous impulses on TS, given by:

$$\begin{cases} {}^{c,\mathrm{T}_{\mathbb{S}}}D^{\eta}\Phi(\zeta) = \mathcal{M}(\zeta)\Phi(\zeta) + \int_{\zeta_0}^{\zeta} \mathbb{H}(s,\Phi(s),\Phi(q(s)))\Delta s \\ + \int_{a}^{b} \mathbb{H}(s,\Phi(s),\Phi(q(s)))\Delta s \\ + \mathbb{G}(\zeta,\Phi(\zeta),\Phi(q(\zeta))) \int_{\zeta_0}^{\zeta} g(\zeta,s)\mathbb{H}(s,\Phi(s),\Phi(q(s)))\Delta s, \quad \zeta \in \mathrm{T}_{\mathbb{S}}', \\ \Phi(\zeta_k^+) - \Phi(\zeta_k^-) = \Upsilon_k(\Phi(\zeta_k^-)), \quad k = \overline{1,\, m}, \\ \Phi(\zeta) = \alpha(\zeta), \quad \zeta \in [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \\ \Phi(\zeta_0) = \alpha(\zeta_0) = \Phi_0, \end{cases} \tag{1}$$

where ${}^{c,\mathrm{T}_{\mathbb{S}}}D^{\eta}$ is the classical Caputo derivative [54] of fractional order $\eta$ on time scale $\mathrm{T}_{\mathbb{S}}$, $\lambda_p > 0$, $\overline{1,\, m}$ denotes $1,2,3,4,\cdots,m$, the $m \times m$ matrix $\mathcal{M}(\zeta)$ is regressive and piecewise continuous on ${\mathrm{T}_{\mathbb{S}}}^0 := [\zeta_0, \zeta_f]_{\mathrm{T}_{\mathbb{S}}}$, $\zeta_f > s > \zeta_0 = s_0 \geq 0$ and ${\mathrm{T}_{\mathbb{S}}}' := {\mathrm{T}_{\mathbb{S}}}^0 \backslash \{\zeta_1, \zeta_2, \cdots, \zeta_m\}$. The functions $\mathbb{G} : {\mathrm{T}_{\mathbb{S}}}^0 \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^n$, $\mathbb{H} : {\mathrm{T}_{\mathbb{S}}}^0 \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^n$, $\Upsilon_k : \mathrm{R}^n \to \mathrm{R}^n$, $\alpha : [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}} \to \mathrm{R}$, $g : {\mathrm{T}_{\mathbb{S}}}^0 \times {\mathrm{T}_{\mathbb{S}}}^0 \to \mathrm{R}^n$ are continuous, where $g$ represents the kernel function. $\Phi(\zeta_k^+) := \lim_{\tau_p \to 0^+} \Phi(\zeta_k + \tau_p)$ and $\Phi(\zeta_k^-) := \lim_{\tau_p \to 0^-} \Phi(\zeta_k - \tau_p)$ of $\Phi(\zeta)$ at $\zeta_k$ satisfies

$$\zeta_0 < \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4 < \cdots < \zeta_m < \zeta_{m+1} = \zeta_f < +\infty.$$

Moreover, $q : {\mathrm{T}_{\mathbb{S}}}^0 \to {\mathrm{T}_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}$ with $q(\zeta) \leq \zeta$, is a continuous delay function. In this paper, we consider a TS that does not belong to the set of integers. We also assume that impulses occurring at isolated points are zero, and that the TS is only the continuous subset of real numbers.

## 2. Preliminaries

The fundamental definitions related to TS calculus are taken from Bohner and Peterson [38,39].

TS is any arbitrary, non-empty and closed subset of real numbers represented by

$\mathrm{T}_{\mathbb{S}}$. The forward and backward jump operators, both having domain and range in $\mathrm{T}_{\mathbb{S}}$, are defined respectively (resp.) as follows:

$$\sigma(s) := \inf\{\zeta \in \mathrm{T}_{\mathbb{S}} : \zeta > s\}, \omega(s) := \sup\{\zeta \in \mathrm{T}_{\mathbb{S}} : \zeta < s\}.$$

The graininess function $\mu : \mathrm{T}_{\mathbb{S}} \to [0, \infty)$ is $\mu(\zeta) = \sigma(\zeta) - \zeta$. The TSs derived form, represented by ${\mathrm{T}_{\mathbb{S}}}^z$, is:

$${\mathrm{T}_{\mathbb{S}}}^z := \begin{cases} \mathrm{T}_{\mathbb{S}} \backslash (\omega(\sup \mathrm{T}_{\mathbb{S}}), \sup \mathrm{T}_{\mathbb{S}}], & \text{if } \sup \mathrm{T}_{\mathbb{S}} < \infty, \\ \mathrm{T}_{\mathbb{S}}, & \text{if } \sup \mathrm{T}_{\mathbb{S}} = \infty. \end{cases}$$

A function $\mathcal{W} : \mathrm{T}_{\mathbb{S}} \to \mathrm{R}$ is said to be right-dense continuous if it remains continuous at each right-dense point in $\mathrm{T}_{\mathbb{S}}$ and has a well-defined left-sided limit at every left-dense point of $\mathrm{T}_{\mathbb{S}}$. Furthermore, $\mathcal{W} : \mathrm{T}_{\mathbb{S}} \to \mathrm{R}$ is termed regressive (resp. positively regressive) if the condition $1 + \mu(\zeta)\mathcal{W}(\zeta) \neq 0$ (resp. $1 + \mu(\zeta)\mathcal{W}(\zeta) > 0$) holds for all $\zeta \in {\mathrm{T}_{\mathbb{S}}}^z$. The collection of all right-dense continuous regressive functions is represented by $\mathcal{R}_{\mathcal{G}}(\mathrm{T}_{\mathbb{S}})$, and the set of right-dense continuous positively regressive functions is denoted by $\mathcal{R}_{\mathcal{G}}(\mathrm{T}_{\mathbb{S}})^+$. The delta derivative as well as $\Delta-$integral of $H : \mathrm{T}_{\mathbb{S}} \to \mathrm{R}$ respectively are:

$$H^{\Delta}(\zeta) := \lim_{s \to \zeta,\, s \neq \sigma(\zeta)} \frac{H(\sigma(\zeta)) - H(s)}{\sigma(\zeta) - s},\ \zeta \in {\mathrm{T}_{\mathbb{S}}}^z,$$
$$\int_{x_1}^{x_2} H(\zeta)\Delta\zeta := h(x_2) - h(x_1),\ \forall\, x_1, x_2 \in \mathrm{T}_{\mathbb{S}},$$

where $h^{\Delta} = H$ on ${\mathrm{T}_{\mathbb{S}}}^z$. The matrix dynamic system $\Phi^{\Delta}(\zeta) = \mathcal{M}(\zeta)\Phi(\zeta),\ \Phi(\zeta_0) = \Phi_0,\ \zeta \in {\mathrm{T}_{\mathbb{S}}}^0$, has the solution known as fundamental matrix, denoted by $\Psi_{\mathcal{M}}(\zeta, \zeta_0)$.

Let $\mathbb{C}({\mathrm{T}_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$ (resp. $P_F^{\mathbb{C}}({\mathrm{T}_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$) be the Banach space of those functions that are continuous (resp. the Banach space of those functions that are piecewise continuous) with $\|\Phi\| := \sup_{\zeta \in {\mathrm{T}_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}} \|\Phi(\zeta)\|$ and Bielecki norm $\|\Phi\|_B := \sup_{\zeta \in {\mathrm{T}_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}} \|\Phi(\zeta)\| e_{-\theta}(\zeta, \zeta_0)$, $\theta > 0$ with $-\theta \in \mathcal{R}_{\mathcal{G}}(\mathrm{T}_{\mathbb{S}})^+$. Also, we define Banach space $P_F^{\mathbb{C}^1}({\mathrm{T}_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n) := \{\Phi \in P_F^{\mathbb{C}}({\mathrm{T}_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n) : \Phi^{\Delta} \in P_F^{\mathbb{C}}({\mathrm{T}_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)\}$ with $\|\Phi\|_1 := \max\{\|\Phi\|, \|\Phi^{\Delta}\|\}$.

## 3. Main results

Consider the inequalities below:

$$\begin{cases} \Bigg\| {}^{c,\mathrm{T}_{\mathbb{S}}}D^{\eta}\Omega(s) - \mathcal{M}(s)\Omega(s) - \int_{s_0}^{s} \mathbb{H}(\zeta, \Omega(\zeta), \Omega(q(\zeta)))\Delta\zeta \\ \quad - \int_{a}^{b} \mathbb{H}(\zeta, \Omega(\zeta), \Omega(q(\zeta)))\Delta\zeta - \mathbb{G}(s, \Omega(s), \Omega(q(s))) \\ \quad \times \int_{s_0}^{s} g(s, \zeta)\mathbb{H}(\zeta, \Omega(\zeta), \Omega(q(\zeta)))\Delta\zeta \Bigg\| \leq \epsilon;\ s \in {\mathrm{T}_{\mathbb{S}}}', \\ \Bigg\| \Omega(s_k^+) - \Omega(s_k^-) - \Upsilon_k(\Omega(s_k^-)) \Bigg\| \leq \epsilon,\ k = \overline{1,\ m}, \end{cases} \tag{2}$$

$$\begin{cases} \Big\| {}^{c,\mathrm{T}_{\mathbb{S}}}D^{\eta}\Omega(s) - \mathcal{M}(s)\Omega(s) - \int_{s_0}^{s} \mathbb{H}(\zeta, \Omega(\zeta), \Omega(q(\zeta)))\Delta\zeta \\ \quad - \int_{a}^{b} \mathbb{H}(\zeta, \Omega(\zeta), \Omega(q(\zeta)))\Delta\zeta - \mathbb{G}(s, \Omega(s), \Omega(q(s))) \\ \quad \times \int_{s_0}^{s} g(s,\zeta)\mathbb{H}(\zeta, \Omega(\zeta), \Omega(q(\zeta)))\Delta\zeta \Big\| \le \varphi(s);\ s \in \mathrm{T}_{\mathbb{S}}{}', \\ \Big\| \Omega(s_k^+) - \Omega(s_k^-) - \Upsilon_k(\Omega(s_k^-)) \Big\| \le \kappa_p,\ k = \overline{1,\ m}, \end{cases} \tag{3}$$

where $\epsilon > 0$, $\kappa_p \ge 0$ and $\varphi : \mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}} \to \mathrm{R}^+$ is continuous and increasing.

**Definition 1.** *Equation* (1) *exhibits stability in the BHU sense on* $\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}$ *if for any* $\epsilon > 0$ *and each* $\Omega \in P_F^{\mathbb{C}^1}(\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$ *satisfying Equation* (2)*, there exists* $\Phi \in P_F^{\mathbb{C}^1}(\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$ *of Equation* (1) *such that*

$$\|\Phi(\zeta) - \Omega(\zeta)\| e_{-\theta}(\zeta, \zeta_0) \le C\epsilon,\ \forall\, \zeta \in \mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}$$

*holds true with some constant* $C > 0$ *independent of* $\epsilon$*.*

**Definition 2.** *Equation* (1) *exhibits stability in the BHUR sense on* $\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}$ *if for any* $\kappa_p \ge 0$*, continuous and increasing function* $\varphi : \mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}} \to \mathrm{R}^+$ *and each* $\Omega \in P_F^{\mathbb{C}^1}(\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$ *satisfying Equation* (3)*, there exists* $\Phi \in P_F^{\mathbb{C}^1}(\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$ *of Equation* (1) *such that*

$$\|\Phi(\zeta) - \Omega(\zeta)\| e_{-\theta}(\zeta, \zeta_0) \le C\varphi(\zeta),\ \forall\, \zeta \in \mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}$$

*holds true with some constant* $C > 0$ *independent of* $\varphi(\zeta)$*.*

We recall the Mittag-Leffler function as

$$E_{\eta,\alpha}(\zeta) := \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(k\eta + \alpha)},\ \text{ for } \eta, \alpha > 0.$$

We let $E_\eta(\zeta) := E_{\eta,1}(\zeta)$ for $\alpha = 1$. Therefore,

$$E_{\eta,1}(\alpha\zeta^\eta) = E_\eta(\alpha\zeta^\eta) = \sum_{k=0}^{\infty} \frac{\alpha^k \zeta^{k\eta}}{\Gamma(\eta k + 1)},\ \alpha, \zeta \in C,$$

possesses a noteworthy characteristic that ${}^cD_{0+}^{\eta} E_\eta(\alpha\zeta^\eta) = \alpha E_\eta(\alpha\zeta^\eta)$. $E_\eta(\mathcal{M}\zeta^\eta)$ represents the matrix extension of the Mittag-Leffler function described above, and is expressed as follows:

$$E_\eta(\mathcal{M}\zeta^\eta) = \sum_{k=0}^{\infty} \frac{\mathcal{M}^k \zeta^{k\eta}}{\Gamma(1 + k\eta)}.$$

**Remark 1.** *A function* $\Omega \in P_F^{\mathbb{C}^1}(\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$ *satisfies Equation* (2) $\Leftrightarrow$ $\exists\ f \in P_F^{\mathbb{C}}(\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$ *and finite sequence* $f_k$ *such that* $\|f(\zeta)\| \le \epsilon$*,*

$\forall\, \zeta \in \mathrm{T_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T_{\mathbb{S}}}}$, $\|f_k\| \le \epsilon$, $\forall\, k = \overline{1,\, m}$ *[50],*

$$\begin{cases} {}^{c,\mathrm{T_{\mathbb{S}}}}D^{\eta}\Omega(\zeta) = \mathcal{M}(\zeta)\Omega(\zeta) + \displaystyle\int_{\zeta_0}^{\zeta} \mathbb{H}(s, \Omega(s), \Omega(q(s)))\Delta s \\ + \displaystyle\int_{a}^{b} \mathbb{H}(s, \Omega(s), \Omega(q(s)))\Delta s + \mathbb{G}(\zeta, \Omega(\zeta), \Omega(q(\zeta))) \\ \times \displaystyle\int_{\zeta_0}^{\zeta} g(\zeta, s)\mathbb{H}(s, \Omega(s), \Omega(q(s)))\Delta s + f(\zeta), \\ \Omega(\zeta_0) = \Omega_0,\ \zeta \in \mathrm{T_{\mathbb{S}}}', \\ \Omega(\zeta_k^+) - \Omega(\zeta_k^-) = \Upsilon_k(\Omega(\zeta_k^-)) + f_k. \end{cases} \tag{4}$$

**Lemma 1.** *Every solution* $\Omega \in P_F^{\mathbb{C}^1}(\mathrm{T_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T_{\mathbb{S}}}}, \mathrm{R}^n)$ *of Equation* (2) *also satisfies*

$$\begin{cases} \Bigg\| \Omega(\zeta) - \Omega(\zeta_0) - E_\eta(\mathcal{M}\zeta^\eta)\Omega_0 - \displaystyle\sum_{j=1}^{m} \Upsilon_j(\Omega(\zeta_j^-)) \\ - \displaystyle\int_{\zeta_0}^{\zeta} (\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^\eta) \int_{s_0}^{s} \mathbb{H}(\xi, \Omega(\xi), \Omega(q(\xi)))\Delta\xi\Delta s \\ - \displaystyle\int_{\zeta_0}^{\zeta} (\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^\eta) \int_{a}^{b} \mathbb{H}(\xi, \Omega(\xi), \Omega(q(\xi)))\Delta\xi\Delta s \\ - \displaystyle\int_{\zeta_0}^{\zeta} (\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^\eta)\mathbb{G}(s, \Omega(s), \Omega(q(s))) \\ \times \displaystyle\int_{s_0}^{s} g(s, \xi)\mathbb{H}(\xi, \Omega(\xi), \Omega(q(\xi)))\Delta\xi\Delta s \Bigg\| \\ \le (m + C_1 C_2(\zeta_f - \zeta_0))\epsilon, \end{cases}$$

*for* $\zeta \in (\zeta_k, \zeta_{k+1}]_{\mathrm{T_{\mathbb{S}}}} \subset \mathrm{T_{\mathbb{S}}}^0$, $\|(\zeta - s)^{\eta-1}\| \le C_1$, $\|E_{\eta,\eta}(\mathcal{M}(\zeta - s)^\eta)\| \le C_2$ *[50].*

The same conclusion holds true for solutions of Equation (3). Next, we introduce the following assumptions.

**Assumption 1.** *The function* $\mathbb{H} : \mathrm{T_{\mathbb{S}}}^0 \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^n$ *is Lipschitz continuous with respect to its second and third variables, namely, there exists* $L > 0$ *such that*

$$\big\|\mathbb{H}(\zeta, \zeta_1, \zeta_2) - \mathbb{H}(\zeta, o_1, o_2)\big\| \le \sum_{i=1}^{2} L\|\zeta_i - o_i\|, \forall \zeta \in \mathrm{T_{\mathbb{S}}}^0, \zeta_i, o_i \in \mathrm{R}^n, i \in \{1, 2\}.$$

*In addition, there exists* $\varrho > 0$ *such that*

$$\|\mathbb{H}(\xi, \Phi(\xi), \Phi(q(\xi)))\| \le \varrho, \forall \xi \in \mathrm{T}_{\mathbb{S}}^0.$$

**Assumption 2.** *For all* $k \in \{1, 2, 3, 4, \ldots, m\}$*, the function* $\Upsilon_k : \mathrm{R}^n \to \mathrm{R}^n$ *is Lipschitz continuous, namely, there exists* $M_k > 0$ *such that*

$$\|\Upsilon_k(\zeta_1) - \Upsilon_k(\zeta_2)\| \le M_k\|\zeta_1 - \zeta_2\|, \forall \zeta_1, \zeta_2 \in \mathrm{R}^n.$$

*In addition, there exists* $C_P > 0$ *such that*

$$\|\Upsilon_k(\Phi(\zeta_k^-))\| \le C_P, \forall k \in \{1, 2, 3, 4, \ldots, m\}.$$

**Assumption 3.** *There exist $C_1, C_2, \varpi, \tau > 0$ such that*

$$\begin{aligned}\|(\zeta-s)^{\eta-1}\| \leq & C_1,\\ \|E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\| \leq & C_2,\\ \|\mathbb{G}(s,\Phi_i(s),\Phi_i(q(s)))\| \leq & \varpi, \forall i\in\{1,2\},\\ \|\mathbb{G}(s,\Phi(s),\Phi(q(s)))\| \leq & \varpi,\\ \|g(\zeta,s)\| \leq & \tau,\end{aligned}$$

*they hold true for every $\zeta, s \in \mathrm{T}_{\mathbb{S}}{}^0$, where $\mathbb{G} : \mathrm{T}_{\mathbb{S}}{}^0 \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^n$ and $g : \mathrm{T}_{\mathbb{S}}{}^0 \times \mathrm{T}_{\mathbb{S}}{}^0 \to \mathrm{R}^n$ are continuous functions.*

**Assumption 4.** $\sum\limits_{j=1}^{m} M_j + 2C_1C_2L\frac{1}{\theta^2} + 2C_1C_2L\frac{e^{\theta(\zeta_f-b)}}{\theta}\zeta_f + 2C_1C_2\varpi\tau L\frac{1}{\theta^2} < 1.$

**Assumption 5.** *The function $\varphi : \mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T}_{\mathbb{S}}} \to \mathrm{R}^+$ is increasing and satisfies*

$$\int_{\zeta_0}^{\zeta} \varphi(\xi)\Delta\xi \leq \rho\varphi(\zeta),\ \rho > 0.$$

The following theorem is about the well-posedness of Equation (1) and is the first result obtained in this paper.

**Theorem 1.** *Equation* (1) *has one and only one solution in $P_F^{\mathbb{C}^1}(\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n)$, if Assumptions 1–4 hold.*

**Proof.** Take the following Banach space $\mathcal{U} := \{\Phi \in P_F^{\mathbb{C}^1}(\mathrm{T}_{\mathbb{S}}{}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \mathrm{R}^n), \|\Phi\| \leq \vartheta$, where $\vartheta = \|\alpha(\zeta_0)\| + \sum_{j=1}^{i} C_P + C_2\|\Phi_0\| + C_1C_2\frac{\zeta_f^2}{2}\varrho + C_1C_2b\varrho\zeta_f + C_1C_2\varpi\frac{\zeta_f^2}{2}\tau\varrho\}$ and define an operator $\Lambda : \mathcal{U} \to \mathcal{U}$ by

$$(\Lambda\Phi)(\zeta) := \begin{cases} \alpha(\zeta),\ \zeta \in [\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T}_{\mathbb{S}}}, \\ \alpha(\zeta_0) + E_\eta(\mathcal{M}\zeta^\eta)\Phi_0 \\ \quad + \int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{s_0}^{s}\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s \\ \quad + \int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{a}^{b}\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s \\ \quad + \int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\mathbb{G}(s,\Phi(s),\Phi(q(s))) \\ \quad \times \int_{s_0}^{s} g(s,\xi)\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s,\ \zeta\in(\zeta_0,\zeta_1]_{\mathrm{T}_{\mathbb{S}}}, \\ \alpha(\zeta_0) + \sum\limits_{j=1}^{i}\Upsilon_j(\Phi(\zeta_j^-)) + E_\eta(\mathcal{M}\zeta^\eta)\Phi_0 \\ \quad + \int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{s_0}^{s}\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s \\ \quad + \int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{a}^{b}\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s \\ \quad + \int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\mathbb{G}(s,\Phi(s),\Phi(q(s))) \\ \quad \times \int_{s_0}^{s} g(s,\xi)\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s, \\ \zeta\in(\zeta_i,\zeta_{i+1}]_{\mathrm{T}_{\mathbb{S}}},\ i=\overline{1,\ m}. \end{cases} \tag{5}$$

It has been proved by Shah et al. [50] that $\Lambda : \mathcal{U} \to \mathcal{U}$. It can now be observed that for any $\Phi_1, \Phi_2 \in \mathcal{U}$ and for all $\zeta \in [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T}_{\mathbb{S}}}$, we have $\|(\Lambda\Phi_1)(\zeta) - (\Lambda\Phi_2)(\zeta)\| = 0$. For $\zeta \in (\zeta_m, \zeta_{m+1}]_{\mathrm{T}_{\mathbb{S}}}$, simple calculation shows that

$$
\begin{aligned}
&\Big\|(\Lambda\Phi_1)(\zeta) - (\Lambda\Phi_2)(\zeta)\Big\| \\
&\le \sum_{j=1}^{m} \Big\|\Upsilon_j(\Phi_1(\zeta_j^-)) - \Upsilon_j(\Phi_2(\zeta_j^-))\Big\| \\
&\quad + \int_{\zeta_0}^{\zeta} \|(\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^{\eta})\| \int_{s_0}^{s} \Big\|\mathbb{H}(\xi, \Phi_1(\xi), \Phi_1(q(\xi))) \\
&\quad - \mathbb{H}(\xi, \Phi_2(\xi), \Phi_2(q(\xi)))\Big\| \Delta\xi\Delta s \\
&\quad + \int_{\zeta_0}^{\zeta} \|(\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^{\eta})\| \int_{a}^{b} \Big\|\mathbb{H}(\xi, \Phi_1(\xi), \Phi_1(q(\xi))) \\
&\quad - \mathbb{H}(\xi, \Phi_2(\xi), \Phi_2(q(\xi)))\Big\| \Delta\xi\Delta s \\
&\quad + \int_{\zeta_0}^{\zeta} \|(\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^{\eta})\| \|\mathbb{G}(s, \Phi_1(s), \Phi_1(q(s))) \\
&\quad \times \| \int_{s_0}^{s} \|g(s, \xi)\| \|\mathbb{H}(\xi, \Phi_1(\xi), \Phi_1(q(\xi)))\| \Delta\xi\Delta s \\
&\quad - \int_{\zeta_0}^{\zeta} \|(\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^{\eta})\| \|\mathbb{G}(s, \Phi_2(s), \Phi_2(q(s)))\| \\
&\quad \times \int_{s_0}^{s} \|g(s, \xi)\| \|\mathbb{H}(\xi, \Phi_2(\xi), \Phi_2(q(\xi)))\| \Delta\xi\Delta s \\
&\le \sum_{j=1}^{m} \Big\|\Upsilon_j(\Phi_1(\zeta_j^-)) - \Upsilon_j(\Phi_2(\zeta_j^-))\Big\| \\
&\quad + \int_{\zeta_0}^{\zeta} \|(\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^{\eta})\| \int_{s_0}^{s} \Big\|\mathbb{H}(\xi, \Phi_1(\xi), \Phi_1(q(\xi))) \\
&\quad - \mathbb{H}(\xi, \Phi_2(\xi), \Phi_2(q(\xi)))\Big\| \Delta\xi\Delta s \\
&\quad + \int_{\zeta_0}^{\zeta} \|(\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^{\eta})\| \int_{a}^{b} \Big\|\mathbb{H}(\xi, \Phi_1(\xi), \Phi_1(q(\xi))) \\
&\quad - \mathbb{H}(\xi, \Phi_2(\xi), \Phi_2(q(\xi)))\Big\| \Delta\xi\Delta s \\
&\quad + \int_{\zeta_0}^{\zeta} C_1 C_2 \varpi \int_{s_0}^{s} \tau \|\mathbb{H}(\xi, \Phi_1(\xi), \Phi_1(q(\xi)))\| \Delta\xi\Delta s \\
&\quad - \int_{\zeta_0}^{\zeta} C_1 C_2 \varpi \int_{s_0}^{s} \tau \|\mathbb{H}(\xi, \Phi_2(\xi), \Phi_2(q(\xi)))\| \Delta\xi\Delta s \\
&\le \sum_{j=1}^{m} \Big\|\Upsilon_j(\Phi_1(\zeta_j^-)) - \Upsilon_j(\Phi_2(\zeta_j^-))\Big\| \\
&\quad + \int_{\zeta_0}^{\zeta} \|(\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^{\eta})\| \int_{s_0}^{s} \Big\|\mathbb{H}(\xi, \Phi_1(\xi), \Phi_1(q(\xi))) \\
&\quad - \mathbb{H}(\xi, \Phi_2(\xi), \Phi_2(q(\xi)))\Big\| \Delta\xi\Delta s \\
&\quad + \int_{\zeta_0}^{\zeta} \|(\zeta - s)^{\eta-1} E_{\eta,\eta}(\mathcal{M}(\zeta - s)^{\eta})\| \int_{a}^{b} \Big\|\mathbb{H}(\xi, \Phi_1(\xi), \Phi_1(q(\xi))) \\
&\quad - \mathbb{H}(\xi, \Phi_2(\xi), \Phi_2(q(\xi)))\Big\| \Delta\xi\Delta s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\zeta_0}^{\zeta} C_1C_2\varpi \int_{s_0}^{s} \tau\Big(\|\mathbb{H}(\xi,\Phi_1(\xi),\Phi_1(q(\xi)))\| \\
&-\|\mathbb{H}(\xi,\Phi_2(\xi),\Phi_2(q(\xi)))\|\Big)\Delta\xi\Delta s \\
\le&\sum_{j=1}^{m}\left\|\Upsilon_j(\Phi_1(\zeta_j^-))-\Upsilon_j(\Phi_2(\zeta_j^-))\right\| \\
&+\int_{\zeta_0}^{\zeta}\|(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\|\int_{s_0}^{s}\Big\|\mathbb{H}(\xi,\Phi_1(\xi),\Phi_1(q(\xi))) \\
&-\mathbb{H}(\xi,\Phi_2(\xi),\Phi_2(q(\xi)))\Big\|\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta}\|(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\|\int_{a}^{b}\Big\|\mathbb{H}(\xi,\Phi_1(\xi),\Phi_1(q(\xi))) \\
&-\mathbb{H}(\xi,\Phi_2(\xi),\Phi_2(q(\xi)))\Big\|\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\varpi \int_{s_0}^{s}\tau\Big\|\mathbb{H}(\xi,\Phi_1(\xi),\Phi_1(q(\xi)))-\mathbb{H}(\xi,\Phi_2(\xi),\Phi_2(q(\xi)))\Big\|\Delta\xi\Delta s \\
\le&\sum_{j=1}^{m}M_j\|\Phi_1(\zeta_j^-)-\Phi_2(\zeta_j^-)\| \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L\|\Phi_1(\xi)-\Phi_2(\xi)\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L\|\Phi_1(q(\xi))-\Phi_2(q(\xi))\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L\|\Phi_1(\xi)-\Phi_2(\xi)\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L\|\Phi_1(q(\xi))-\Phi_2(q(\xi))\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\varpi\int_{s_0}^{s} \tau L\|\Phi_1(\xi)-\Phi_2(\xi)\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\varpi\int_{s_0}^{s} \tau L\|\Phi_1(q(\xi))-\Phi_2(q(\xi))\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
\le&\sum_{j=1}^{m}M_j\|\Phi_1(\zeta_j^-)-\Phi_2(\zeta_j^-)\| \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L\sup_{\xi\in\mathrm{T}_{\mathbb{S}}{}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T}_{\mathbb{S}}}}\|\Phi_1(\xi)-\Phi_2(\xi)\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L\sup_{\xi\in\mathrm{T}_{\mathbb{S}}{}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T}_{\mathbb{S}}}}\|\Phi_1(q(\xi))-\Phi_2(q(\xi))\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L\sup_{\xi\in\mathrm{T}_{\mathbb{S}}{}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T}_{\mathbb{S}}}}\|\Phi_1(\xi)-\Phi_2(\xi)\| \\
&e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L\sup_{\xi\in\mathrm{T}_{\mathbb{S}}{}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T}_{\mathbb{S}}}}\|\Phi_1(q(\xi))-\Phi_2(q(\xi))\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\zeta_0}^{\zeta} C_1 C_2 \varpi \int_{s_0}^{s} \tau L \sup_{\xi\in \mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}} \|\Phi_1(\xi)-\Phi_2(\xi)\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1 C_2 \varpi \int_{s_0}^{s} \tau L \sup_{\xi\in \mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}} \|\Phi_1(q(\xi))-\Phi_2(q(\xi))\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
\le& \sum_{j=1}^{m} M_j\|\Phi_1(\zeta_j^-)-\Phi_2(\zeta_j^-)\| + 2\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L\|\Phi_1-\Phi_2\|_B e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L\|\Phi_1-\Phi_2\|_B e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\varpi\int_{s_0}^{s} \tau L\|\Phi_1-\Phi_2\|_B e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
\le& \sum_{j=1}^{m} M_j\|\Phi_1(\zeta_j^-)-\Phi_2(\zeta_j^-)\| + 2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{s_0}^{s} C_1C_2 L e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{a}^{b} C_1C_2 L e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{s_0}^{s} C_1C_2 \varpi\tau L e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s.
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&\Big\|(\Lambda\Phi_1)(\zeta)-(\Lambda\Phi_2)(\zeta)\Big\| e_{-\theta}(\zeta,\zeta_0) \\
\le& \sum_{j=1}^{m} M_j\|\Phi_1(\zeta_j^-)-\Phi_2(\zeta_j^-)\| e_{-\theta}(\zeta,\zeta_0) \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{s_0}^{s} C_1C_2 L e_{-\theta}(\zeta,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{a}^{b} C_1C_2 L e_{-\theta}(\zeta,\zeta_0)e_{-\theta}(\zeta_0,\xi)\Delta\xi\Delta s \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{s_0}^{s} C_1C_2 e_{-\theta}(\zeta,\zeta_0)e_{-\theta}(\zeta_0,\xi)\varpi\tau L\Delta\xi\Delta s \\
\le& \sum_{j=1}^{m} M_j \sup_{\zeta\in \mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}} \|\Phi_1(\zeta_j^-)-\Phi_2(\zeta_j^-)\| e_{-\theta}(\zeta,\zeta_0) \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{s_0}^{s} C_1C_2 L e_{-\theta}(\zeta,\xi)\Delta\xi\Delta s \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{a}^{b} C_1C_2 L e_{-\theta}(\zeta,\xi)\Delta\xi\Delta s \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{s_0}^{s} C_1C_2 e_{-\theta}(\zeta,\xi)\varpi\tau L\Delta\xi\Delta s \\
\le& \sum_{j=1}^{m} M_j\|\Phi_1-\Phi_2\|_B + 2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{s_0}^{s} C_1C_2 L e^{-\theta(\zeta-\xi)}\Delta\xi\Delta s \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{a}^{b} C_1C_2 L e^{-\theta(\zeta-\xi)}\Delta\xi\Delta s \\
&+2\|\Phi_1-\Phi_2\|_B\int_{\zeta_0}^{\zeta}\int_{s_0}^{s} C_1C_2 e^{-\theta(\zeta-\xi)}\varpi\tau L\Delta\xi\Delta s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{m} M_j \|\Phi_1 - \Phi_2\|_B + 2\|\Phi_1 - \Phi_2\|_B \int_{\zeta_0}^{\zeta} C_1 C_2 L \frac{e^{-\theta(\zeta - s)}}{\theta} \Delta s \\
&\quad + 2\|\Phi_1 - \Phi_2\|_B \int_{\zeta_0}^{\zeta} C_1 C_2 L \frac{e^{-\theta(\zeta - b)}}{\theta} \Delta s \\
&\quad + 2\|\Phi_1 - \Phi_2\|_B \int_{\zeta_0}^{\zeta} C_1 C_2 \varpi \tau L \frac{e^{-\theta(\zeta - s)}}{\theta} \Delta s \\
&\leq \sum_{j=1}^{m} M_j \|\Phi_1 - \Phi_2\|_B + 2\|\Phi_1 - \Phi_2\|_B C_1 C_2 L \frac{1}{\theta^2} \\
&\quad + 2\|\Phi_1 - \Phi_2\|_B C_1 C_2 L \frac{e^{-\theta(\zeta - b)}}{\theta} \zeta + 2\|\Phi_1 - \Phi_2\|_B C_1 C_2 \varpi \tau L \frac{1}{\theta^2} \\
&\leq \sum_{j=1}^{m} M_j \|\Phi_1 - \Phi_2\|_B + 2\|\Phi_1 - \Phi_2\|_B C_1 C_2 L \frac{1}{\theta^2} \\
&\quad + 2\|\Phi_1 - \Phi_2\|_B C_1 C_2 L \frac{e^{\theta(\zeta_f - b)}}{\theta} \zeta_f + 2\|\Phi_1 - \Phi_2\|_B C_1 C_2 \varpi \tau L \frac{1}{\theta^2}.
\end{aligned}
$$

Therefore, it holds that

$$
\begin{aligned}
\|\Lambda\Phi_1 - \Lambda\Phi_2\|_B \leq \|\Phi_1 - \Phi_2\|_B \Bigg( &\sum_{j=1}^{m} M_j + 2C_1 C_2 L \frac{1}{\theta^2} + 2C_1 C_2 L \frac{e^{\theta(\zeta_f - b)}}{\theta} \zeta_f \\
&+ 2C_1 C_2 \varpi \tau L \frac{1}{\theta^2} \Bigg).
\end{aligned}
$$

By Assumption 4, $\Lambda$ is a contractive mapping, it serves as a Picard operator on $\mathcal{U}$. As a result, it possesses a unique fixed point, which corresponds to the unique solution of Equation (1) (from Equation (5); see the study by Balachandran et al. [9]) in $P_F^{\mathbb{C}^1}(\mathrm{T_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T_{\mathbb{S}}}}, \mathrm{R}^n)$. $\square$

The following theorem is concerned with the BHU stability of Equation (1) and is the second result obtained in this paper.

**Theorem 2.** *Equation* (1) *has BHU stability on* $\mathrm{T_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T_{\mathbb{S}}}}$*, if Assumptions 1–3 hold.*

**Proof.** If $\Omega \in P_F^{\mathbb{C}^1}(\mathrm{T_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T_{\mathbb{S}}}}, \mathrm{R}^n)$ satisfies Equation (2), the unique solution $\Phi \in P_F^{\mathbb{C}^1}(\mathrm{T_{\mathbb{S}}}^0 \cup [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T_{\mathbb{S}}}}, \mathrm{R}^n)$ of

$$
\begin{cases}
{}^{c,\mathrm{T_{\mathbb{S}}}}D^{\eta}\Phi(\zeta) = \mathcal{M}(\zeta)\Phi(\zeta) + \displaystyle\int_{\zeta_0}^{\zeta} \mathbb{H}(s, \Phi(s), \Phi(q(s)))\Delta s \\
\quad + \displaystyle\int_{a}^{b} \mathbb{H}(s, \Phi(s), \Phi(q(s)))\Delta s + \mathbb{G}(\zeta, \Phi(\zeta), \Phi(q(\zeta))) \\
\quad \times \displaystyle\int_{\zeta_0}^{\zeta} g(\zeta, s)\mathbb{H}(s, \Phi(s), \Phi(q(s)))\Delta s, \\
\zeta \in \mathrm{T_{\mathbb{S}}}' = \mathrm{T_{\mathbb{S}}}^0 \backslash \{\zeta_1, \zeta_2, \cdots, \zeta_m\}, \\
\Phi(\zeta_k^+) - \Phi(\zeta_k^-) = \Upsilon_k(\Phi(\zeta_k^-)), \quad k = \overline{1, m}, \\
\Phi(\zeta) = \Omega(\zeta), \quad \zeta \in [\zeta_0 - \lambda_p, \zeta_0]_{\mathrm{T_{\mathbb{S}}}}, \\
\Phi(\zeta_0) = \Omega(\zeta_0) = \Phi_0,
\end{cases}
$$

is

$$\Phi(\zeta)=\begin{cases}\Omega(\zeta),\ \zeta\in[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T}_{\mathbb{S}}},\\ \Omega(\zeta_0)+E_\eta(\mathcal{M}\zeta^\eta)\Phi_0\\ +\displaystyle\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{s_0}^{s}\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s\\ +\displaystyle\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{a}^{b}\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s\\ +\displaystyle\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\mathbb{G}(s,\Phi(s),\Phi(q(s)))\\ \times\displaystyle\int_{s_0}^{s}g(s,\xi)\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s,\ \zeta\in(\zeta_0,\zeta_1]_{\mathrm{T}_{\mathbb{S}}},\\ \Omega(\zeta_0)+\displaystyle\sum_{j=1}^{i}\Upsilon_j(\Phi(\zeta_j^-))+E_\eta(\mathcal{M}\zeta^\eta)\Phi_0\\ +\displaystyle\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{s_0}^{s}\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s\\ +\displaystyle\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{a}^{b}\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s\\ +\displaystyle\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\mathbb{G}(s,\Phi(s),\Phi(q(s)))\\ \times\displaystyle\int_{s_0}^{s}g(s,\xi)\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s,\\ \zeta\in(\zeta_i,\zeta_{i+1}]_{\mathrm{T}_{\mathbb{S}}},\ i=\overline{1,\ m}.\end{cases}$$

Since for all $\zeta\in[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T}_{\mathbb{S}}}$, $\|\Omega(\zeta)-\Phi(\zeta)\|=0$. For $\zeta\in(\zeta_m,\zeta_{m+1}]_{\mathrm{T}_{\mathbb{S}}}$, using Lemma 1, we obtain

$$\begin{aligned}&\|\Omega(\zeta)-\Phi(\zeta)\|e_{-\theta}(\zeta,\zeta_0)\\ \leq&\Bigg\|\Omega(\zeta)-\Omega(\zeta_0)-E_\eta(\mathcal{M}\zeta^\eta)\Omega_0-\sum_{j=1}^{m}\Upsilon_j(\Omega(\zeta_j^-))\\ &-\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{s_0}^{s}\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))\Delta\xi\Delta s\\ &-\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{a}^{b}\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))\Delta\xi\Delta s\\ &-\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\mathbb{G}(s,\Omega(s),\Omega(q(s)))\\ &\times\int_{s_0}^{s}g(s,\xi)\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))\Delta\xi\Delta s\Bigg\|e_{-\theta}(\zeta,\zeta_0)\\ &+\sum_{j=1}^{m}\Bigg\|\Upsilon_j(\Omega(\zeta_j^-))-\Upsilon_j(\Phi(\zeta_j^-))\Bigg\|e_{-\theta}(\zeta,\zeta_0)\\ &+\Bigg\|\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\\ &\times\int_{s_0}^{s}\Big(\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))-\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Big)\Delta\xi\Delta s\Bigg\|e_{-\theta}(\zeta,\zeta_0)\\ &+\Bigg\|\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^\eta)\int_{a}^{b}\Big(\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))\\ &-\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Big)\Delta\xi\Delta s\Bigg\|e_{-\theta}(\zeta,\zeta_0)\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\mathbb{G}(s,\Omega(s),\Omega(q(s)))\right.\\
&\left.\times\int_{s_0}^{s}g(s,\xi)\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))\Delta\xi\Delta s\right\|e_{-\theta}(\zeta,\zeta_0)\\
&-\left\|\int_{\zeta_0}^{\zeta}(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\mathbb{G}(s,\Phi(s),\Phi(q(s)))\right.\\
&\left.\times\int_{s_0}^{s}g(s,\xi)\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Delta\xi\Delta s\right\|e_{-\theta}(\zeta,\zeta_0)\\
\leq&(m+C_1C_2(\zeta_f-\zeta_0))\epsilon e_{-\theta}(\zeta,\zeta_0)+\sum_{j=1}^{m}M_j\|\Omega(\zeta_j^-)-\Phi(\zeta_j^-)\|e_{-\theta}(\zeta,\zeta_0)\\
&+\int_{\zeta_0}^{\zeta}\|(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\|\int_{s_0}^{s}\left\|\Big(\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))\right.\\
&\left.-\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Big)\right\|e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
&+\int_{\zeta_0}^{\zeta}\|(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\|\int_{a}^{b}\left\|\Big(\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))\right.\\
&\left.-\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\Big)\right\|e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
&+\int_{\zeta_0}^{\zeta}\|(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\|\|\mathbb{G}(s,\Omega(s),\Omega(q(s)))\\
&\times\int_{s_0}^{s}\|g(s,\xi)\|\|\mathbb{H}(\xi,\Omega(\xi),\Omega(q(\xi)))\|e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
&-\int_{\zeta_0}^{\zeta}\|(\zeta-s)^{\eta-1}E_{\eta,\eta}(\mathcal{M}(\zeta-s)^{\eta})\|\|\mathbb{G}(s,\Phi(s),\Phi(q(s)))\\
&\times\int_{s_0}^{s}\|g(s,\xi)\|\|\mathbb{H}(\xi,\Phi(\xi),\Phi(q(\xi)))\|e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
\leq&(m+C_1C_2(\zeta_f-\zeta_0))\epsilon e_{-\theta}(\zeta,\zeta_0)+\sum_{j=1}^{m}M_j\|\Omega(\zeta_j^-)-\Phi(\zeta_j^-)\|e_{-\theta}(\zeta,\zeta_0)\\
&+\int_{\zeta_0}^{\zeta}C_1C_2\int_{s_0}^{s}L\|\Omega(\xi)-\Phi(\xi)\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
&+\int_{\zeta_0}^{\zeta}C_1C_2\int_{s_0}^{s}L\|\Omega(q(\xi))-\Phi(q(\xi))\|\\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
&+\int_{\zeta_0}^{\zeta}C_1C_2\int_{a}^{b}L\|\Omega(\xi)-\Phi(\xi)\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
&+\int_{\zeta_0}^{\zeta}C_1C_2\int_{a}^{b}L\|\Omega(q(\xi))-\Phi(q(\xi))\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
&+\int_{\zeta_0}^{\zeta}C_1C_2\varpi\int_{s_0}^{s}\tau L\|\Omega(\xi)-\Phi(\xi)\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
&+\int_{\zeta_0}^{\zeta}C_1C_2\varpi\int_{s_0}^{s}\tau L\|\Omega(q(\xi))-\Phi(q(\xi))\|e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s\\
\leq&(m+C_1C_2(\zeta_f-\zeta_0))\epsilon e^{-\theta(\zeta-\zeta_0)}\\
&+\sum_{j=1}^{m}M_j\sup_{\zeta\in\mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}}\|\Omega(\zeta_j^-)-\Phi(\zeta_j^-)\|e_{-\theta}(\zeta,\zeta_0)\\
&+\int_{\zeta_0}^{\zeta}C_1C_2\int_{s_0}^{s}L\sup_{\xi\in\mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}}\|\Omega(\xi)-\Phi(\xi)\|\\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L \sup_{\xi\in \mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}} \|\Omega(q(\xi))-\Phi(q(\xi))\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L \sup_{\xi\in \mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}} \|\Omega(\xi)-\Phi(\xi)\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L \sup_{\xi\in \mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}} \|\Omega(q(\xi))-\Phi(q(\xi))\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\varpi\int_{s_0}^{s} \tau L \sup_{\xi\in \mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}} \|\Omega(\xi)-\Phi(\xi)\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s \\
&+\int_{\zeta_0}^{\zeta} C_1C_2\varpi\int_{s_0}^{s} \tau L \sup_{\xi\in \mathrm{T_{\mathbb{S}}}^0\cup[\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}} \|\Omega(q(\xi))-\Phi(q(\xi))\| \\
&\times e_{-\theta}(\xi,\zeta_0)e_{-\theta}(\zeta_0,\xi)e_{-\theta}(\zeta,\zeta_0)\Delta\xi\Delta s \\
\le&(m+C_1C_2(\zeta_f-\zeta_0))\epsilon e^{\theta(\zeta-\zeta_0)}+\sum_{j=1}^{m}M_j\|\Omega-\Phi\|_B \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L\|\Omega-\Phi\|_B e_{-\theta}(\zeta,\xi)\Delta\xi\Delta s \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L\|\Omega-\Phi\|_B e_{-\theta}(\zeta,\xi)\Delta\xi\Delta s \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\varpi\int_{s_0}^{s} \tau L\|\Omega-\Phi\|_B e_{-\theta}(\zeta,\xi)\Delta\xi\Delta s \\
\le&(m+C_1C_2(\zeta_f-\zeta_0))\epsilon e^{\theta(\zeta_f-\zeta_0)}+\sum_{j=1}^{m}M_j\|\Omega-\Phi\|_B \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L\|\Omega-\Phi\|_B e^{-\theta(\zeta-\xi)}\Delta\xi\Delta s \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L\|\Omega-\Phi\|_B e^{-\theta(\zeta-\xi)}\Delta\xi\Delta s \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\varpi\int_{s_0}^{s} \tau L\|\Omega-\Phi\|_B e^{-\theta(\zeta-\xi)}\Delta\xi\Delta s \\
\le&(m+C_1C_2(\zeta_f-\zeta_0))\epsilon e^{\theta(\zeta_f-\zeta_0)}+\sum_{j=1}^{m}M_j\|\Omega-\Phi\|_B \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\int_{s_0}^{s} L\|\Omega-\Phi\|_B e^{\theta(\zeta_f)}\Delta\xi\Delta s \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\int_{a}^{b} L\|\Omega-\Phi\|_B e^{\theta(\zeta_f)}\Delta\xi\Delta s \\
&+2\int_{\zeta_0}^{\zeta} C_1C_2\varpi\int_{s_0}^{s} \tau L\|\Omega-\Phi\|_B e^{\theta(\zeta_f)}\Delta\xi\Delta s \\
\le&(m+C_1C_2(\zeta_f-\zeta_0))\epsilon e^{\theta(\zeta_f-\zeta_0)}+\sum_{j=1}^{m}M_j\|\Omega-\Phi\|_B \\
&+\int_{\zeta_0}^{\zeta}\left(2C_1C_2\int_{s_0}^{s} Le^{\theta(\zeta_f)}+2C_1C_2\int_{a}^{b} Le^{\theta(\zeta_f)}\right. \\
&\left.+2C_1C_2\varpi\int_{s_0}^{s}\tau Le^{\theta(\zeta_f)}\right)\|\Omega-\Phi\|_B\Delta\xi\Delta s.
\end{aligned}
$$

By applying the inequality [40] in Lemma 1, we arrive at the following result:

$$\|\Omega-\Phi\|_B \leq (m+C_1C_2(\zeta_f-\zeta_0))\epsilon e^{\theta(\zeta_f-\zeta_0)} \prod_{\zeta_0<\zeta_j<\zeta}(1+M_j)e_P(\zeta,\zeta_0),$$

where $P = \big(2C_1C_2\int_{s_0}^{s} Le^{\theta(\zeta_f)} + 2C_1C_2\int_a^b Le^{\theta(\zeta_f)} + 2C_1C_2\varpi\int_{s_0}^{s}\tau Le^{\theta(\zeta_f)}\big)\Delta\xi$. Hence Equation (1) has BHU stability on $\mathrm{T_{\mathbb{S}}}^0 \cup [\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}$. □

The next theorem, which is concerned with the BHUR stability of Equation (1), is the third result obtained in this paper. Since it can be proved in the same way as for Theorem 2, we omit the details of the proof.

**Theorem 3.** *Equation* (1) *has BHUR stability on* $\mathrm{T_{\mathbb{S}}}^0 \cup [\zeta_0-\lambda_p,\zeta_0]_{\mathrm{T_{\mathbb{S}}}}$*, if Assumptions 1–3 and 5 hold.*

## 4. Example

In this section, we provide an example to illustrate the stability results obtained in this paper.

**Example 1.** *Consider*

$$\begin{cases} {}^{c,\mathrm{T_{\mathbb{S}}}}D^\eta\Phi(\zeta)=\dfrac{1}{\zeta-2}\Phi(\zeta)+\displaystyle\int_{\zeta_0}^{\zeta}\frac{(s+\sin(\Phi(s)))}{200}\Delta s+\int_0^4\frac{(s+\sin(\Phi(s)))}{200}\Delta s\\ +\displaystyle\int_{\zeta_0}^{\zeta}e^{-15}\frac{(s+\sin(\Phi(s)))}{200}\Delta s,\ \ \zeta\in[0,4]_{\mathrm{T_{\mathbb{S}}}}\backslash\{2\},\\ \Phi(0)=1,\\ \Upsilon_k(\Phi(\zeta_k^-))=\Phi(\zeta_k^+)-\Phi(\zeta_k^-)=\dfrac{2+\sin(\Phi(\zeta_1^-))}{10},\ k=1,\ \zeta_1^-=2, \end{cases} \tag{6}$$

*and its associated inequality*

$$\begin{cases} \big|{}^{c,\mathrm{T_{\mathbb{S}}}}D^\eta\Omega(\zeta)-\dfrac{1}{\zeta-2}\Omega(\zeta)-\displaystyle\int_{\zeta_0}^{\zeta}\frac{(s+\sin(\Omega(s)))}{200}\Delta s-\int_0^4\frac{(s+\sin(\Omega(s)))}{200}\Delta s\\ -\displaystyle\int_{\zeta_0}^{\zeta}e^{-15}\frac{(s+\sin(\Omega(s)))}{200}\Delta s\big|\leq 1.77,\ \zeta\in[0,4]_{\mathrm{T_{\mathbb{S}}}}\backslash\{2\},\\ \big|\Upsilon_k(\Omega(\zeta_k^-))-\Omega(\zeta_k^+)+\Omega(\zeta_k^-)\big|\leq 1.77,\ k=1. \end{cases} \tag{7}$$

*Let* $\mathrm{T_{\mathbb{S}}}'=[0,4]_{\mathrm{T_{\mathbb{S}}}}\backslash\{2\},\ \zeta_0=0,\ \zeta_f=4,\ p(\zeta)=\frac{1}{\zeta-2},\ \mathbb{G}(\zeta,\Phi(\zeta),\Phi(q(\zeta)))=e^{-15}$, $g(\zeta,s)=1,\ \mathbb{H}(\zeta,\Phi(\zeta),\Phi(q(\zeta)))=\frac{\zeta+\sin(\Phi(\zeta))}{200}$ *for* $\zeta\in\mathrm{T_{\mathbb{S}}}'$ *and* $\epsilon=1.77$*. If* $\Omega\in P_F^{\mathbb{C}^1}([0,4]_{\mathrm{T_{\mathbb{S}}}},\mathrm{R})$ *satisfies Equation* (7)*, then there exist* $f\in P_F^{\mathbb{C}}([0,4]_{\mathrm{T_{\mathbb{S}}}},\mathrm{R})$ *and* $f_0\in\mathrm{R}$ *such that* $|f(\zeta)|\leq 1.77$ *for* $\zeta\in\mathrm{T_{\mathbb{S}}}'$ *and* $|f_0|\leq 1.77$*. So*

$$\begin{cases} {}^{c,\mathrm{T_{\mathbb{S}}}}D^\eta\Omega(\zeta)=\dfrac{1}{\zeta-2}\Omega(\zeta)+\displaystyle\int_{\zeta_0}^{\zeta}\frac{(s+\sin(\Omega(s)))}{200}\Delta s+\int_0^4\frac{(s+\sin(\Omega(s)))}{200}\Delta s\\ +\displaystyle\int_{\zeta_0}^{\zeta}e^{-15}\frac{(s+\sin(\Omega(s)))}{200}\Delta s+f(\zeta),\quad \zeta\in\mathrm{T_{\mathbb{S}}}',\ \Omega(0)=1,\\ \Upsilon_k(\Omega(\zeta_k^-))=\Omega(\zeta_k^+)-\Omega(\zeta_k^-)+f_0,\ k=1, \end{cases}$$

*and the solution of Equation* (6) *is*

$$\begin{cases}\Phi(\zeta) = E_\eta(p\zeta^\eta) + \Upsilon_1(\Phi(\zeta_1^-)) \\ + \int_{\zeta_0}^{\zeta} (\zeta - s)^{\eta-1} E_{\eta,\eta}(p(\zeta - s)^\eta) \int_{\zeta_0}^{s} \frac{(u + \sin(\Phi(u)))}{200} \Delta u \Delta s \\ + \int_{\zeta_0}^{\zeta} (\zeta - s)^{\eta-1} E_{\eta,\eta}(p(\zeta - s)^\eta) \int_{0}^{4} \frac{(u + \sin(\Phi(u)))}{200} \Delta u \Delta s \\ + \int_{\zeta_0}^{\zeta} (\zeta - s)^{\eta-1} E_{\eta,\eta}(p(\zeta - s)^\eta) \int_{\zeta_0}^{s} e^{-15} \frac{(u + \sin(\Phi(u)))}{200} \Delta u \Delta s. \end{cases}$$

*The Assumptions 1–3 are obvious and through straightforward calculations, we obtain*

$$|\Lambda(\Phi) - \Lambda(\Omega)| \leq \left(\frac{1}{10} + \frac{8}{200} + \frac{16}{200} + \frac{8}{200} e^{-15}\right) |\Phi - \Omega|.$$

*Obviously,* $\left(\frac{1}{10} + \frac{8}{200} + \frac{16}{200} + \frac{8}{200} e^{-15}\right) < 1$*, so Assumption 4 holds for Equation* (6)*. Hence Assumptions 1–4 hold for Equation* (6)*, so Equation* (6) *has only one solution in* $P_F^{\mathbb{C}^1}([0,4]_{\mathrm{T}_\mathbb{S}}, \mathrm{R})$ *and is BHU stable on* $\mathrm{T}_\mathbb{S}'$*. Furthermore, if we take* $\varphi(\zeta) = e^\zeta$ *in Equation* (7)*, then* $\int_{\zeta_0}^{\zeta} e^s \Delta s = e^\zeta - e^{\zeta_0} \leq e^\zeta < 30 e^\zeta$ *for* $\rho = 30$*, i.e.,* $\int_{\zeta_0}^{\zeta} e^s \Delta s < 30 e^\zeta$ *and thus Assumption 5 holds for Equation* (6)*. Hence, Equation* (6) *is BHUR stable on* $\mathrm{T}_\mathbb{S}'$

## 5. Conclusion

This study establishes the existence and uniqueness results for Equation (1), along with demonstrating its BHU and BHUR stabilities. The primary tools employed to derive our results include the Picard operator, Grönwall's inequality and Banach fixed point theorem. To facilitate the analysis and overcome potential challenges, specific assumptions are introduced, and the findings are further supported through illustrative examples. Stability refers to a system's capacity to regain equilibrium or follow a desired path after being subjected to external disturbances. Moreover, based on the derived results, it is possible to conclude that the fixed point method proves highly effective in establishing the existence as well as uniqueness of solutions for integro-delay dynamic systems across various TS domains.

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