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Article

Mathematical modeling of criminal activities: An approach based on homotopy perturbations

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Abstract: The manuscript deals with solving a mathematical model of criminal activities (MCA model) in order to facilitate the understanding of the dynamics of crime and implement certain measures in its prevention and deterrence. For the MCA model, which consists of two coupled ordinary differential equations of the diffusion type, the stability and existence of a unique Cauchy solution are first examined. Thereafter, coupled equations of the MCA model are transformed into a dimensionless system on the unit interval and solved using the homotopy perturbation method (HPM). In this way, recursive sequences of approximate HPM solutions are obtained, and, under certain conditions, their convergence has been proven. In the following, using different initial conditions, several numerical simulations of the proposed HPM technique in solving the MCA model were conducted, as well as a practical application of the proposed procedure in modelling real-world crime rate data.

Keywords: crime modeling; coupled differential equations; homotopy perturbations; convergence; simulation; application

1. Introduction

Crime has always been one of the most important, pressing problems of human society. Therefore, it is not surprising that researchers, in an attempt to understand how crime arises and is shaped, devote a significant amount of time to this topic. The development and analysis of mathematical models of criminal activity (hereinafter referred to as the MCA models) has a significant societal impact, as it allows for a deeper understanding of crime dynamics and potentially helps in shaping preventive strategies. Such models can serve as a tool for decision-making at the security policy level, enabling the simulation of different scenarios and the assessment of the effectiveness of security actions.

However, mathematical modeling of crime has attracted significant attention from researchers only in the last few decades. These models, aimed at understanding and modeling crime, are quite diverse. Thus, for instance, the so-called agent-based models are used in [1,2], while dynamic and epidemiological models, interpreted from the aspect of application in modeling criminal activities, are considered in [3–5]. On the other hand, various stochastic models, mainly based on the usage of self-exciting point processes, have been applied in [6–8]. Recently, some MCA models have been proposed based on the use of geospatial software and data learning techniques [9–11]. Of the many models that have been proposed, perhaps the most successful was the differential equation model of the diffusion type [12–14] that describes the development of the number of participants in criminal acts in a certain area.

Motivated by this issue, here we examine such a deterministic MCA model, composed of two coupled differential equations. This system was first examined in terms of the conditions of stability, existence, and uniqueness of a solution that satisfies certain boundary conditions. Then it was solved, in an approximate form, using the well-known homotopy perturbation method (HPM). The famous HPM, first introduced in the pioneering manuscripts of He [15–19], has long since found significant application in solving nonlinear equations of various types, mainly in the field of physical sciences [20–27], but also in some other different fields [28–32]. It is worth noting that HPM has already been used in (approximately) solving some models related to certain criminal activities [33–35], but without general results on HPM solutions and detailed analysis of their convergence.

Unlike the previously mentioned approaches, where representations of the obtained approximations lack verification of their convergence, this study presents a new approach to the application of HPM in solving MCA models. As already pointed out, the stability, existence, and uniqueness of the solution of the MCA model are first examined here. Then the system of differential equations of the MCA model is transformed into a system of equations defined on the unit interval $(0,1)$, and using an appropriate recursive procedure, the (approximate) solution of the MCA model is obtained by solving a series of linear first-order differential equations. Under certain regulatory conditions, the convergence and validation of thus-obtained HPM approximations to the true solution of the MCA system are formally proven. Finally, the practical application of the proposed HPM procedure is shown by a series of real-world data related to the number of criminal offenses in the Republic of Serbia in the previous ten-year period.

In the following Section 2 we start from a coupled system of differential equations that describe the MCA model. Thereafter, the stability of equilibrium of this model, as well as the existence and uniqueness of its solutions, is examined. Section 3 deals with the transformation of the proposed MCA model to the unit interval $(0,1)$, which, among others, allows efficient convergence of the HPM approximations to the exact solutions. This convergence is also formally confirmed and represents one of the most important advantages of the proposed HPM technique, compared to some previously used approximation methods. The following Section 4 presents a procedure for recursively computing HPM approximate solutions with different boundary conditions. The entire approximation procedure is implemented in Wolfram Mathematica software, and its numerical accuracy is also checked and confirmed. Section 5 considers the application of the proposed HPM technique in modeling the crime dynamics in the Republic of Serbia, while Section 6 contains some concluding remarks. Finally, a list of variables, functions, and parameters used in this study is provided in the Appendix.

2. Mathematical model of criminal activity

In this section, our goal is to obtain a crime model that describes its evolution and exhibits behavior close to real criminal activities. Starting from similar assumptions as in [12,13], we denote with $c(t) \geq 0$ the total number of criminal activities in a certain area at time t. Moreover, as stated in [7,8], the behavior of perpetrators of criminal activities is most often conditioned by the so-called attractiveness of the area, which represents the possibility that a criminal will act at a certain moment. In order to increase the flexibility of the MCA model, it is assumed that attractiveness depends not only on the behavior of active perpetrators but also on other factors such as time, characteristics of the observed area, or the type of crime committed. In this way, attractiveness can be represented as the sum of two components:

- $a(t) \geq 0$ denotes the “internal” part of attractiveness, which does not depend on the behavior of criminals but on the other factors mentioned above.
- $b(t) \geq 0$ represents the “dynamic” part of the attractiveness caused by criminal activity.

Obviously, both components $a(t)$ and $b(t)$ can be viewed as functions of the time (t), with the first, “inner” component often being represented as a constant value. To be more precise, let us point out that, say, changes in attractiveness due to factors not influenced by criminal activity (e.g., the time of day or the month in which it is recorded) would then be explained by the “intrinsic” attractiveness $a(t)$. On the other hand, if knowledge of crimes committed in an area tends to encourage more crimes to occur, or, conversely, if the number of police officers in a particular area changes in accordance with the number of crimes committed, this effect would then be represented by the dynamic component $b(t)$.

Let us now consider the dependence of the total number of criminal activities $c(t)$ on the above-mentioned components $a(t)$ and $b(t)$. Note that at some time moment (t), a certain number of perpetrators of criminal acts are arrested and removed from the system, while others appear in the system (due to release from prison or the appearance of new criminals). Accordingly, it can be assumed that the rate of reduction of criminal activities is proportional to the product of the total attractiveness by the number of criminal activities, i.e., it has the form $\kappa c(t)\big(a(t)+b(t)\big)$, where $\kappa > 0$ is the corresponding constant of proportionality. Conversely, as noted in [12], it is also assumed that there is a constant tendency for criminal activity to increase, which, for simplicity, will be denoted by $\gamma > 0$. In this way, the evolution of the number of crimes over time is described by the following equation:

$$\frac{dc(t)}{dt} = \gamma - \kappa c(t)\big(a(t)+b(t)\big). \tag{1}$$

Further on, in order to mathematically model the dynamic part of the attractiveness of criminal activities, represented by the function $b(t)$, note that each committed criminal act at time (t) increases the value of $b(t)$. Therefore, the dynamics of attractiveness increases with increasing value of $c(t)$ proportional to the total number of criminal acts committed, i.e., has the form $\ell c(t)(a(t)+b(t))$, where $\ell > 0$ is the constant of proportionality. In addition, we further assume that $b(t)$ decreases exponentially in time, which can be represented by the following equation:

$$\frac{db(t)}{dt} = \ell c(t)\big(a(t)+b(t)\big) - \omega b(t) \tag{2}$$

where $\omega > 0$ is the (constant) attractiveness decay rate over time. Let us point out that the coupled equations (l) and (2) are of the diffusion type, with the so-called negative feedback loop between certain functions and their derivatives, which has been noted by most authors dealing with this topic. For instance, when the number of crimes $c(t)$ increases over time, the first term on the right-hand side of equation (2) also increases, causing an increase in the dynamic part of the attraction $b(t)$. On the other hand, it will then cause a decrease in the second term in equation (1), that is, a decrease in the value of the derivative $dc(t)/dt$. Obviously, this is the opposite effect of the initial increase in the value of $c(t)$.

Below, within the system of coupled equations (1) and (2), we introduce the following dimensionless quantities:

$$\hat{t} := t\omega\,, \hat{a} := \frac{\kappa a}{\omega}, \hat{b} := \frac{\kappa b}{\omega}, \hat{c} := \frac{\ell c}{\omega}, \hat{\gamma} := \frac{\ell \gamma}{\omega^2}.$$

With these terms, the MCA model, given by coupled equations (1) and (2), changes into the following dimensionless form:

$$\frac{dc(t)}{dt} = \gamma - c(t)\big(a(t) + b(t)\big), \tag{3}$$

$$\frac{db(t)}{dt} = c(t)\big(a(t) + b(t)\big) - b(t), \tag{4}$$

where, for simplicity, the hat marks are omitted. Further, we assume that the solution of the system of differential equations (3) and (4) (hereinafter referred to as the MCA system) satisfies the initial conditions $b(0) = b_0 \geq 0$ and $\mathrm{c}(0) = \mathrm{c}_0 \geq 0$, which will be analyzed below. For the MCA system thus defined, the conditions of existence and uniqueness of its (non-negative) solution, as well as the equilibrium conditions, can be described by the following statement.

Theorem 1. *Let the MCA model be given by the coupled system of equations* (3) and (4), *where* $a(t) \geq 0$ *is the* $\mathbb{C}^\infty$ *function, defined on* $t \in (0, \infty)$, *and* $\gamma > 0$ *is an arbitrary constant. Then, for any initial conditions* $b(0) \geq 0, c(0) \geq 0$, *the MCA model has a unique solution* $b(t) \geq 0, c(t) \geq 0$. *Furthermore, if* $a = a(t) \geq 0$ *is a constant, the MCA model has an asymptotically stable equilibrium point:*

$$(b^*, c^*) = \left(\gamma, \frac{\gamma}{\gamma + a}\right). \tag{5}$$

Proof. Let us denote the functions of the right side of the system (3) and (4) as:

$$F(b, c) = \big(f_1(b, c), f_2(b, c)\big),$$

where $f_1(b, c) = c(a + b) - b$ and $f_2(b, c) = \gamma - c(a + b)$. Obviously, $F(b, c)$ is differentiable on the entire set $\mathbb{R}^2$, so it is locally Lipschitz-continuous in any neighborhood of an arbitrary point $(b_0, \mathrm{c}_0) = \big(b(0), c(0)\big)$, where $b_0, c_0 \geq 0$. According to the Picard-Lindelöf theorem (see, e.g., [36], pg. 38), the existence and uniqueness of a solution to this Cauchy problem for the given initial condition follows.

Further, it can be shown that for the given initial conditions $b(0) \geq 0$, $\mathrm{c}(0) \geq 0$ the unique solution of the system (3) and (4) is closed into the first quadrant $\mathbb{R}_+^2 := \{(b, c) | b, c \geq 0\}$ of the plane $\mathbb{R}^2$. Indeed, we can observe the following two cases:

- If $c(t_0) = 0$ holds for some $t_0 \geq 0$, then according to equation (3), it follows that $dc(t_0)/dt = \gamma > 0$. Therefore, the function $c(t)$ increases at point t_0 and is positive in some neighborhood of this point.

- Similarly, if $b(t_0) = 0$ holds for some $t_0 \geq 0$, then from equation (4), we get $db(t_0)/dt = a(t_0)c(t_0) \geq 0$. Therefore, the function $b(t)$ is non-decreasing in some neighborhood of the point t_0.

Based on the above, we conclude that the positive parts of the coordinate axes are “repelling edges” for both functions $b(t)$ and $c(t)$, i.e., any solution to the system (3) and (4) cannot have negative values.

Finally, for a constant value $a(t) = a \geq 0$, we can determine the equilibrium point of the MCA model (3) and (4). As is known, this point is the solution, with respect to (b, c), of the coupled equations:

$$\frac{dc(t)}{dt} = \frac{db(t)}{dt} = 0,$$

or, equivalently, $c(a + b) = \gamma = b$. From here, it obviously follows that the only equilibrium point is the one given by equation (5). Moreover, in this case the Jacobian of the MCA system (3) and (4) is given by:

$$J(b, c) = \begin{bmatrix} \dfrac{\partial f_1(b,c)}{\partial b} & \dfrac{\partial f_1(b,c)}{\partial c} \\ \dfrac{\partial f_2(b,c)}{\partial b} & \dfrac{\partial f_2(b,c)}{\partial c} \end{bmatrix} = \begin{bmatrix} c-1 & a+b \\ -c & -(a+b) \end{bmatrix},$$

so at the equilibrium point (b^*, c^*) it is valid:

$$J(b^*, c^*) = \begin{bmatrix} -\dfrac{a}{a+\gamma} & a+\gamma \\ -\dfrac{\gamma}{a+\gamma} & -(a+\gamma) \end{bmatrix}.$$

According to this, it follows that the trace of the Jacobian at the equilibrium point is

$$\tau := \mathrm{tr}\big(J(b^*, c^*)\big) = -\frac{a}{a+\gamma} - (a+\gamma) < 0,$$

while its determinant is

$$\Delta := \det\big(J(b^*, c^*)\big) = a + \gamma > 0.$$

This means (see, e.g., [37], pgs. 61–62) that the equilibrium point (b^*, c^*) is indeed asymptotically stable for all values $a \geq 0$ and $\gamma > 0$. □

Remark 1. *To check the stability, we can also look for the eigenvalues* $\vec{\lambda} = (\lambda_1, \lambda_2)$ *of the Jacobian matrix* $J(b^*, c^*)$ *as solutions to the equation:*

$$\det(J(b^*, c^*) - \lambda I) = \begin{vmatrix} -\dfrac{a}{a+\gamma} - \lambda & a+\gamma \\ -\dfrac{\gamma}{a+\gamma} & -(a+\gamma) - \lambda \end{vmatrix} = 0.$$

After some elementary computation, the following quadratic equation is obtained:

$$\lambda^2 + \frac{(a+\gamma)^2 + a}{a+\gamma}\lambda + a + \gamma = 0,$$

or, equivalently,

$$\lambda^2 - \tau\lambda + \Delta = 0,$$

where we use the notation from the previous proof. According to Vieta's formulas, it follows:

$$\lambda_1 + \lambda_2 = \tau < 0, \lambda_1\lambda_2 = \Delta > 0,$$

so both eigenvalues λ_1, λ_2 *have negative real parts. This also implies that the equilibrium* (b^*, c^*) *is asymptotically stable for all* $a \geq 0$ *and* $\gamma > 0$ *(see, e.g., [38]). Here, the following two cases can be distinguished:*

1) *When the discriminant is* $\tau^2 - 4\Delta \geq 0$, *the eigenvalues are real and negative. Thus, the solutions of the MCA system* (3) *and* (4) *exponentially approach the equilibrium point (**Figure 1**, left).*
2) *When* $\tau^2 - 4\Delta < 0$, *the eigenvalues are complex-conjugated, with a negative real part. Then, the solutions have a "spiral" convergence towards equilibrium (**Figure 1**, right).*

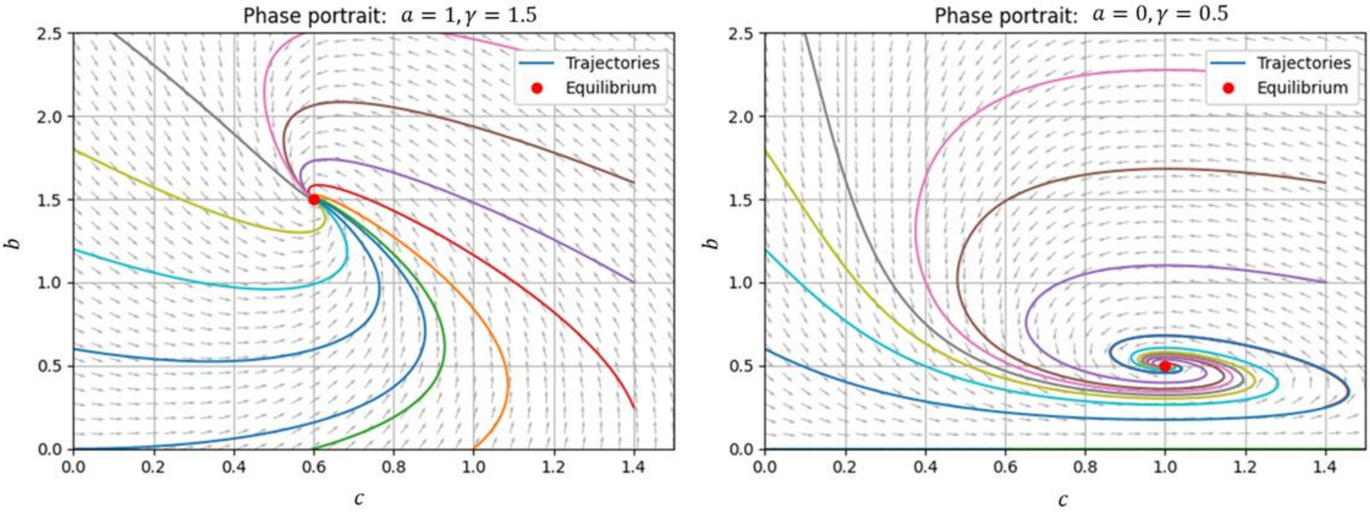

Figure 1. Panel left: Exponential convergence of the solutions $b(t), c(t)$ of the MCA system when $a > 0$. Panel right: Spiral convergence of these solutions when $a = 0$.

Let us notice, as can also be seen in **Figure 1**, that in both cases the equilibrium point (b^*, c^*) is inside the positive quadrant $\mathbb{R}_+^2 \subset \mathbb{R}^2$. Therefore, it represents the attractor of the MCA system (3) and (4), i.e., the point for which is valid:

$$\lim_{t\to\infty} b(t) = b^*, \lim_{t\to\infty} c(t) = c^*.$$

Thus, both functions $b(t)$ and $c(t)$ tend towards the equilibrium point as this system spontaneously approaches time, independent of the initial conditions. Moreover, larger values of $a = a(t) > 0$ give a more pronounced convergence; that is, the parameter enhances the "reaction" of the system and accelerates its stabilization.

Conversely, in the limiting case when $a = 0$, according to equation (5), it follows that the equilibrium point is $(b^*, c^*) = (\gamma, 1)$. The eigenvalues are then (always) complex-conjugated, with a negative real part, and the trace of the Jacobian $\tau = -\gamma < 0$. This value is "closer" to zero than in the previous case, when $a > 0$**,** which means that the rate of convergence decreases, i.e., there is a stable exponential, but slower, spiral attraction towards equilibrium. Finally, note that then, similar to the proof of Theorem l, from $b(t_0) = 0$, for some $t_0 \geq 0$, it follows $db(t_0)/dt = 0$. This means that t_0 is the critical point of the function $\mathrm{b}(t)$, or this function is constant for each $t \geq 0$.

3. Solving the MCA model with HPM

Let us now consider some possibilities of solving the MCA system, given by equations (3)–(4). In general, the coupled system of equations (3)–(4) can be solved numerically or by using some approximate analytical methods, mostly based on some well-known approximation theory or variation calculus methods. In addition, as was pointed out earlier, some of the similar crime models were solved using HPM techniques (e.g., [33–35]) but using only a few iterations and without general results about HPM solutions and proof of their convergence. For these reasons, namely, to obtain the solution of the MCA system (3)–(4) in a more efficient way, and so that the convergence of the solution can be formally verified, a new HPM procedure is considered here.

To this end, as already mentioned, the basic idea is to normalize the infinite time interval $t \in (0, \infty)$ by transforming it into the unit interval $(0,1)$. Therefore, similar to Kevkić and Stojanović [23], we first apply the following transformation of the time argument $t(x) = \ln(1/x)$, i.e., $x(t) = \exp(-t)$, where $x \in (0,1)$. In that way, using the equalities $dt/dx = -1/x$ and $df/dt = -\,xdf/dx$, which hold for any differentiable function $f(t(x))$, the MCA system (3)–(4) can be transformed into the following dimensionless form:

$$x\frac{d\rho(x)}{dx} - \rho(x)\big(\alpha(x) + \beta(x)\big) + \gamma = 0, \tag{6}$$

$$x\frac{d\beta(x)}{dx} + (\rho(x) - 1)\beta(x) + \alpha(x)\rho(x) = 0, \tag{7}$$

where $x \in (0,1)$, $\alpha(x) \coloneqq \alpha(t(x))$, $\beta(x) \coloneqq b(t(x))$, and $\rho(x) \coloneqq c(t(x))$.

In the next step, we apply the HPM technique on the coupled equations (6) and (7); that is, we have introduced the following homotopy functions:

$$R(x,p) = \sum_{j=0}^{\infty} p^j \rho_j(x)\,, B(x,p) = \sum_{k=0}^{\infty} p^k \beta_k(x), \tag{8}$$

where $p \in [0,1]$ is the embedding parameter. To find the solution of the transformed MCA system, given by equations (6) and (7), we have constructed the following homotopy system of partial differential equations:

$$x\frac{\partial R(x,p)}{\partial x} - \alpha(x)R(x,p) + \gamma - pR(x,p)B(x,p) = 0, \tag{9}$$

$$x\frac{\partial B(x,p)}{\partial x} - B(x,p) + \alpha(x)R(x,p) + pR(x,p)B(x,p) = 0, \tag{10}$$

where $R(x,p)$ and $B(x,p)$ are the unknown functions, given by equations (8). According to the above, when $p = 0$, the previous equations (9) and (10) become:

$$x\frac{d\rho_0(x)}{dx} - \alpha(x)\rho_0(x) + \gamma = 0, \tag{11}$$

$$x\frac{d\beta_0(x)}{dx} - \beta_0(x) + \alpha(x)\rho_0(x) = 0. \tag{12}$$

Obviously, they are ordinary first-order differential equations, solving which yields the so-called initial HPM solution:

$$\rho_0(x) = \exp\left(\int \frac{\alpha(x)dx}{x}\right)\left(C_0 - \gamma\int \frac{1}{x}\exp\left(-\int \frac{\alpha(x)dx}{x}\right)dx\right), \tag{13}$$

$$\beta_0(x) = x\left(D_0 - \int \frac{\alpha(x)\rho_0(x)}{x^2}dx\right), \tag{14}$$

where C_0, D_0 are constants, obtained based on appropriate initial (boundary) conditions. On the other hand, when $p = 1$, the system of coupled equations (9) and (10) is obviously equivalent to the transformed MCA system (6) and (7). In this case, according to equations (8), the solution to both these systems will be:

$$\rho(x) = \lim_{p\to 1^-} R(x,p) = \sum_{j=0}^{\infty} p_j(x), \tag{15}$$

$$\beta(x) = \lim_{p\to 1^-} B(x,p) = \sum_{k=0}^{\infty} \beta_k(x), \tag{16}$$

under the condition of convergence of both series. Thus, we say that functions given by equations (15) and (16) represent the HPM solution of the MCA system (6) and (7).

In general, for any $p \in (0,1)$, the HPM solution of coupled equations (9) and (10) can be expressed as a power series in p, as is given in equations (8). To that cause, by substituting equations (8) into equations (9) and (10), and after rearranging some terms in them, one obtains:

$$\sum_{j=0}^{\infty} p^j\left(x\frac{d\rho_j(x)}{dx} - \alpha(x)\rho_j(x)\right) + \gamma - p\left(\sum_{j=0}^{\infty} p^j\,\rho_j(x)\right)\left(\sum_{k=0}^{\infty} p^k\,\beta_k(x)\right) = 0, \tag{17}$$

$$\sum_{k=0}^{\infty} p^k\left(x\frac{d\beta_k(x)}{dx} - \beta_k(x) + \alpha(x)\rho_k(x)\right) + p\left(\sum_{j=0}^{\infty} p^j\,\rho_j(x)\right)\left(\sum_{k=0}^{\infty} p^k\,\beta_k(x)\right) = 0. \tag{18}$$

From here, it is clear that the expressions with powers of p^0 in equations (17) and (18) give the initial HPM solution, as given in equations (13) and (14). Further, equating identical powers p^j, where $j, k = 1,2,\ldots$, leads to the following system of differential equations:

$$x\frac{d\rho_j(x)}{dx}-\alpha(x)\rho_j(x)-\sum_{i=0}^{j-1}\rho_i(x)\,\beta_{j-i-1}(x)=0, \tag{19}$$

$$x\frac{d\beta_k(x)}{dx}-\beta_k(x)+\alpha(x)\rho_k(x)+\sum_{j=0}^{k-1}\rho_j(x)\beta_{k-j-1}(x)=0. \tag{20}$$

Obviously, equations (19) and (20) also represent the ordinary first-order linear differential equations. Therefore, similar to equations (11) and (12), they can be solved recursively by solving alternately on $\rho_{\mathrm{j}}(x)$ and $\beta_k(x)$ for any $j,k=1,2,\dots$ Thus, the following two series of solutions to the system of equations (19) and (20) are obtained:

$$\rho_j(x)=\exp\left(\int\frac{\alpha(x)dx}{x}\right)\left(C_j+\int\frac{q_{j-1}(x)}{x}\exp\left(-\int\frac{\alpha(x)dx}{x}\right)dx\right), \tag{21}$$

$$\beta_k(x)=x\left(D_k-\int\frac{\alpha(x)\rho_k(x)+q_{k-1}(x)}{x^2}dx\right), \tag{22}$$

where C_j, D_k are some constants, and $q_k(x)\coloneqq\sum_{j=0}^{k}\rho_j(x)\beta_{k-j}(x),\ k=0,1,2,\dots$ Now, by substituting the solutions given by equations (13), (14), (21) and (22) into equations (8), we obtain the HPM solution of the homotopy system of coupled equations (9) and (10). Moreover, replacement of the solutions (13), (14), (21) and (22) into series (15) and (16) gives the HPM solution of the MCA system of differential equations (6) and (7). Finally, according to this, for an arbitrary $j,k=0,1,2,\dots$ the following so-called HPM approximations of thus-obtained solution can be defined:

$$\hat{\rho}_j(x)\coloneqq\sum_{i=0}^{j}\rho_i(x)\,,\hat{\beta}_k(x)\coloneqq\sum_{j=0}^{k}\beta_j(x). \tag{23}$$

It is obvious that $\hat{\rho}_j(x)$ and $\hat{\beta}_k(x), j,k=0,1,2,\dots$ are the estimates of unknown functions $\rho(x)$ and $\beta(x)$, respectively, and therefore can be used, under certain conditions, as their approximations. To this end, it should first be noted that the HPM approximations, given by equations (23), as well as the HPM solutions (15) and (16), must satisfy the same boundary conditions, obtained from the initial ones $b(0)=b_0\geq 0$ and $c(0)=c_0\geq 0$ within the basic MCA system (3) and (4). Therefore, by reusing the transformation $x(t)=\exp(-t)$, where $t\geq 0$, the new boundary conditions $\rho(1)=c_0\geq 0$ and $\beta(1)=b_0\geq 0$ are obtained. To ensure that these boundary conditions are satisfied by both the HPM approximations (23) and the HPM solution (15) and (16), the following is assumed below:

1) The constants C_0, D_0 in equations (13) and (14) are chosen so that equalities $\rho_0(1)=c_0$ and $\beta_0(1)=b_0$ hold.
2) The constants $C_j, D_k, j,k=1,2,\dots$ in equations (21) and (22) are chosen so that equalities $\rho_j(1)=\beta_k(1)=0$ hold.

According to the above assumptions, it is obvious that the initial HPM solution, given by equations (13) and (14), as well as the HPM approximations (23), satisfy the same boundary conditions as the true solutions of the MCA system (6) and (7).

Moreover, some sufficient conditions for the convergence of thus-obtained HPM approximations can be given as follows.

Theorem 2. *Let* $\{\rho_j(x)\}_{j=0}^{\infty}$ *and* $\{\beta_k(x)\}_{k=0}^{\infty}$ *be the series of continuous functions, defined by equations (13), (14), (21) and (22) on the unit interval* $(0,1)$. *In addition, suppose that the following conditions are satisfied:*

(i) *The function* $\alpha(x)$ *is bounded on the interval* $(0,1)$, *i.e., there is a constant* $M > 0$ *such that* $|\alpha(x)| \leq M$, *for any* $x \in (0,1)$.

(ii) *The series* $\{C_j\}_{j=0}^{\infty}$ *and* $\{D_k\}_{k=0}^{\infty}$ *are absolutely summable; that is, the inequalities* $C \coloneqq \sum_{j=0}^{+\infty}|C_j| < +\infty$ *and* $D \coloneqq \sum_{k=0}^{+\infty}|D_k| < +\infty$ *hold.*

(iii) *The following inequality is valid*:

$$D + Me \geq (CM + \gamma)e + 2\big(Me(CM + D + \gamma)\big)^{1/2}. \tag{24}$$

Then, the series of HPM approximations $\{\hat{\rho}_j(x)\}_{j=0}^{\infty}$ *and* $\{\hat{\beta}_k(x)\}_{k=0}^{\infty}$, *defined by equations (23), absolutely and uniformly converge on the interval* $(0,1)$. *Moreover, their limits are then given by equations (15) and (16), respectively, i.e., they are solutions of the MCA system given by equations (6) and (7).*

Proof. First, we prove the convergence of the series $\{\hat{\rho}_j(x)\}_{j=0}^{\infty}$ and $\{\hat{\beta}_k(x)\}_{k=0}^{\infty}$ According to the assumptions given in the theorem and equations (13) and (14), for the initial HPM solutions $\rho_0(x)$ and $\beta_0(x)$, for any $x \in (0,1)$ one obtains:

$$|\rho_0(x)| \leq \exp\left(M\int\frac{dx}{x}\right)\left|C_0 - \gamma\int\exp\left(-M\int\frac{dx}{x}\right)\frac{dx}{x}\right| \leq x^M\left(|C_0| + \gamma\left|\int\frac{dx}{x^{M+1}}\right|\right) \leq A_0,$$

$$|\beta_0(x)| \leq x\left|D_0 - \int\frac{\alpha(x)\rho_0(x)}{x^2}dx\right| \leq x\left(|D_0| + \left|\int\frac{\alpha(x)\rho_0(x)}{x^2}dx\right|\right) \leq x|D_0| + M|\rho_0(x)| \leq B_0,$$

where $A_0 \coloneqq |C_0| + \gamma/M$ and $B_0 \coloneqq |D_0| + MA_0 = |C_0|M + |D_0| + \gamma$. Further, according to equation (21), for the function $\rho_1(x)$ we have:

$$|\rho_1(x)| \leq x^M\left(|C_1| + \left|\int\frac{\rho_0(x)\beta_0(x)}{x}dx\right|\right) \leq |C_1| + Q_0 f(x;M). \tag{25}$$

Here, $Q_0 \coloneqq A_0B_0$ and $f(x;\,M) \coloneqq x^M|\ln x| = -x^M\ln x$ is the non-negative function on $x \in (0,1)$, which satisfies:

$$\lim_{x\to 0} f(x;M) = \lim_{x\to 1} f(x;M) = 0.$$

Thus, $f(x;\,M)$ has a maxima at $x_0 \in (0,1)$, for which is valid $\partial f(x;M)/\partial x = 0$, that is, $x_0 = \exp(-1/M)$. Hence, the inequality $f(x;\,M) \leq f(x_0;\,M) = (Me)^{-1}$ holds for any $x \in (0,1)$, and its replacement into equation (25) gives $|\rho_1(x)| \leq A_1$, where $A_1 \coloneqq |C_1| + Q_0(Me)^{-1}$. Similarly, using equations (22) and (25), it follows:

$$|\beta_1(x)| \leq x\left(|D_1| + \left|\int\frac{\alpha(x)\rho_1(x) + \rho_0(x)\beta_0(x)}{x^2}dx\right|\right) \leq x|D_1| + MA_1 + A_0B_0 \leq B_1,$$

where $B_1 \coloneqq |C_1|M + |D_1| + (1 + e^{-1})Q_0$.

In general, by using the induction method, it can be easily shown that for arbitrary $j = 1,2,\ldots$ and $x \in (0,1)$ the equalities $|\rho_j(x)| \leq A_j$ and $|\beta_j(x)| \leq B_j$ hold, where:

$$A_j := |C_j| + \frac{Q_{j-1}}{Me}, B_j := |C_j|M + |D_j| + (1 + e^{-1})Q_{j-1}, Q_j := \sum_{i=0}^{j} A_i B_{j-i}. \tag{26}$$

Now, using the definition of HPM approximations $\hat{\rho}_k(x)$ and $\hat{\beta}_k(x)$, given by equations (23), for an arbitrary $x \in (0,1)$ and $k = 1,2,\dots$ we have:

$$|\hat{\rho}_k(x)| \leq \sum_{j=0}^{k} |\rho_j(x)| \leq \sum_{j=0}^{k} A_j, |\hat{\beta}_k(x)| \leq \sum_{j=0}^{k} |\beta_j(x)| \leq \sum_{j=0}^{k} B_j. \tag{27}$$

Putting $k \to \infty$ into equations (27), the inequalities above imply:

$$\left|\sum_{j=0}^{\infty} \rho_j(x)\right| = \lim_{k\to\infty} |\hat{\rho}_k(x)| \leq A, \left|\sum_{j=0}^{\infty} \beta_j(x)\right| = \lim_{k\to\infty} |\hat{\beta}_k(x)| \leq B. \tag{28}$$

Here, $A := \sum_{j=0}^{\infty} A_j$ and $B := \sum_{j=0}^{\infty} B_j$, whereby according to equations (26), the following equations (by unknowns $A > 0$ and $B > 0$) are valid:

$$C + \frac{\gamma}{M} + \frac{Q}{Me} = A, CM + D + \gamma + (1 + e^{-1})Q = B, \tag{29}$$

where:

$$Q := \sum_{j=0}^{\infty} Q_j = \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j} A_i B_{j-i}\right) = \left(\sum_{i=0}^{\infty} A_i\right)\left(\sum_{j=0}^{\infty} B_j\right) = AB.$$

The system of equations (29), after some calculations, can be presented as follows:

$$A = \frac{CM + \gamma}{M - e^{-1}B}, B^2 + \big((CM + \gamma)e - (D + Me)\big)B + Me(CM + D + \gamma) = 0. \tag{30}$$

Obviously, the system (30) has at least one positive real solution $A > 0, B > 0$ if:

$$\mathrm{D} + Me > (CM + \gamma)e \ \wedge \ (D + Me - (CM + \gamma)e)^2 \geq 4Me(CM + D + \gamma),$$

which is fulfilled based on the assumption given in (24). Thus, according to (28), both sequences $\{\hat{\rho}_k(x)\}_{k=0}^{\infty}$ and $\{\hat{\beta}_k(x)\}_{k=0}^{\infty}$ absolutely and uniformly converge on $(0,1)$.

In the following, we prove the second part of the theorem, i.e., that the sums given by Equations (15) and (16) are solutions of the MCA system (6) and (7). First, notice that according to the assumptions of the theorem and previously obtained results, the series $\{|C_k|\}_{k=0}^{\infty}$, $\{|D_k|\}_{k=0}^{\infty}$ and $\{Q_k\}_{k=0}^{\infty}$ are summable, which implies:

$$\lim_{k\to\infty} |C_k| = \lim_{k\to\infty} |D_k| = \lim_{k\to\infty} Q_k = 0.$$

Further, denote by $r_1(x)$ and $r_2(x)$, respectively, the radii of convergence of the power series $R(x,p)$ and $B(x,p)$ defined by equations (8). Then, by applying the Cauchy-Hadamard theorem, we get:

$$r_1(x) = \left(\limsup_{k\to\infty} |\rho_k(x)|^{\frac{1}{k}}\right)^{-1} \geq \left(\limsup_{k\to\infty}\left(|C_k| + \frac{Q_{k-1}}{Me}\right)^{\frac{1}{k}}\right)^{-1} = 1,$$

$$r_2(x) = \left(\limsup_{k\to\infty}|\beta_k(x)|^{\frac{1}{k}}\right)^{-1} \geq \left(\limsup_{k\to\infty}\left((|C_k|M + |D_k| + (1 + e^{-1})Q_{k-1})^{\frac{1}{k}}\right)\right)^{-1} = 1.$$

According to Abel's theorem (see, e.g., [39]), it follows that functions $R(x,p)$ and $B(x,p)$ are continuous from the left at $p = 1$. Therefore, equations (15) and (16) hold, i.e., the series $\sum_{j=0}^{\infty}\rho_j(x)$ and $\sum_{k=0}^{\infty}\beta_k(x)$ are solutions of the homotopy system of equations (9) and (10), when $p = 1$. Thus, they are also solutions for its equivalent, the MCA system (6) and (7).

Remark 2. *It is worth noting that the previous theory gives only sufficient, but not necessary conditions for the convergence of the HPM approximation to the solution of the MCA model. This means that even if the conditions in Theorem* 2 *do not hold, the HPM approximations can converge. Nevertheless, thanks to the appropriate choices of the constant* $M > 0$, *as well as the initial values* C_0, D_0, *these conditions can be easily fulfilled. For instance, the condition (iii), which ensures the system's stability under bounded attractiveness, when* $M \approx \gamma \to 0$ *becomes* $D \geq 0$, *and it is always fulfilled. Similarly, when* $C_0 \geq 0$ *and using inequality* $\alpha(x) \geq 0$, *which implies* exp $(\int \alpha(x)dx/x) \geq 1$, *for any* $x \in (0,1)$ *one obtains:*

$$\rho_0(x) \geq \left(C_0 - \gamma\int\frac{dx}{x}\right) = C_0 - \gamma\ln x > 0.$$

Therefore, the initial HPM solution $\rho_0(x)$ *is (always) positive. The validation and application of the proposed HPM procedure in solving some practical problems will be seen in the following.*

4. Validation of the HPM procedure

In this section, we examine the quality of the HPM procedure proposed above in more detail. The primary goal is to check, i.e., verify the (fast) convergence of the HPM approximations, given by equations (23), to the solution of the MCA system (6) and (7). At the same time, for any $k = 0,1,2,\ldots$, according to the HPM approximations $\{\hat{\rho}_k(x)\}$ and $\{\hat{\beta}_k(x)\}$, as well as the transformation $x(t) = \exp(-t)$, where $t \in (0,\infty)$, one simply obtains a series of (approximate) solutions $\hat{c}_k \coloneqq \hat{\rho}_k(x(t))$ and $\hat{b}_k \coloneqq \hat{\beta}_k(x(t))$ of the basic MCA system (3) and (4). For simplicity, let us first assume that the internal part of criminal attractiveness is normalized by the MCA system (6) and (7), that is, it is equal to $\alpha(x) \equiv 1$. Thereafter, we consider several different initial (boundary) conditions that can be satisfied by the solutions of the MCA model. Accordingly, we refer to the solutions of equations (11) and (12) and (17) and (18) as HPM series $\{\rho_k(x)\}$, $\{\beta_k(x)\}$ where $k = 0,1,2,\ldots$, and a more detailed description of the procedures for their calculation is given below.

Series A

First, we consider the problem of obtaining the HPM solutions $\rho(x)$ and $\beta(x)$ of the MCA system (6) and (7), with the boundary conditions $\rho(1) = \beta(1) = \gamma > 0$. This means that for the basic solutions $c(t)$ and $b(t)$ of the MCA system (3) and (4), the equalities $c(0) = b(0) = \gamma$ hold, i.e., the initial values of the number of criminal offenses, as well as the dynamic attractiveness of criminal activities, are equal to their (constant) tendency to increase $\gamma > 0$. As already explained in the previous section, these boundary conditions will be met if the HPM series $\{\rho_k(x)\}$ and $\{\beta_k(x)\}$, when $k = 0,1,2,\ldots$, satisfy the equalities $\rho_0(1) = \beta_0(1) = \gamma$ and $\rho_k(1) = \beta_k(1) = 0$, when $k = 1,2,\ldots$ To obtain the initial HPM solutions, we apply their explicit form given by equations (13) and (14). Thus, after some simple computations, one obtains:

$$\rho_0(x) = \exp\left(\int \frac{dx}{x}\right)\left(C_0 - \gamma \int \frac{1}{x} \exp\left(-\int \frac{dx}{x}\right) dx\right) = C_0 x + \gamma,$$

wherein from the condition $\rho_0(1) = \gamma$ it follows $C_0 = 0$, i.e., $\rho_0(x) = \gamma$. Similarly, using the result thus obtained, we get:

$$\beta_0(x) = x\left(D_0 - \gamma \int \frac{dx}{x^2}\right) = D_0 x + \gamma,$$

and in the same way as in the previous one, it follows that $D_0 = 0$, i.e., $\beta_0(x) = \gamma$. Therefore, in this case, both initial HPM solutions $\rho_0(x)$ and $\beta_0(x)$ are constant functions, equal to the parameter $\gamma > 0$.

Further, using the recurrence relations in equations (19) and (20), that is, the explicit form of the HPM series, given by (21) and (22), the functions $\{\rho_k(x)\}$ and $\{\beta_k(x)\}$ can be obtained for an arbitrary k = 1, 2, ... In this case, one obtains:

$$\rho_k(x) = x\left(C_k + \int \frac{q_{k-1}(x)}{x^2} dx\right), \beta_k(x) = x\left(D_k - \int \frac{\rho_k(x) + q_{k-1}(x)}{x^2} dx\right),$$

where $q_k(x) := \sum_{j=0}^{k} \rho_j(x)\beta_{k-j}(x)$, and constants C_k and D_k are easily obtained from the aforementioned boundary conditions $\rho_k(1) = \beta_k(1) = 0$. Hereinafter, the first few terms of HPM series, denoted as Series *A*, are given as follows:

$$\rho_1(x) = \gamma^2(x-1),\ \beta_1(x) = -\gamma^2 x \ln x\,;$$

$$\rho_2(x) = \gamma^3 x\left(\frac{1}{x} - \frac{1}{2}\ln^2 x + \ln x - 1\right), \beta_2(x) = \frac{\gamma^3}{6} x \ln^3 x;$$

$$\rho_3(x) = \frac{\gamma^4}{24}(24(x^2-1) + x\ln^4 x - 4x\ln^3 x + 24x\ln^2 x - 24x(x+1)\ln x),$$

$$\beta_3(x) = \frac{\gamma^4 x}{120}\ (-360(x-1) - \ln^5 x - 20\ln^3 x - 60\ln^2 x + 120(2x+1)\ln x);$$

$$\begin{aligned}\rho_4(x) = \frac{\gamma^5}{720}(&-720(10x^2 - 9x - 1) - x\ln^6 x + 6x\ln^5 x - 90x\ln^4 x + 120x(4x+1)\ln^3 x - 360x(6x+1)\ln^2 x \\ &+ 720x(8x+3)\ln x),\end{aligned}$$

$$\begin{aligned}\beta_4(x) = \frac{\gamma^5 x}{5040}\ (&191520(x-1) + \ln^7 x + 84\ln^5 x + 420\ln^4 x - 6720x\ln^3 x + 5040(8x-1)\ln^2 x \\ &- 10080(13x+6)\ln x), \text{etc.}\end{aligned}$$

According to this, HPM approximations $\{\hat{\rho}_k(x)\}_{k=0}^{\infty}$ and $\{\hat{\beta}_k(x)\}_{k=0}^{\infty}$ are obtained as approximative solutions of the MCA system (6) and (7). Thereafter, applying the transformation $x = \exp(-t)$, where $t \geq 0$, approximate solutions $\{\hat{c}_k(t)\}_{k=0}^{\infty}$ and $\{\hat{b}_k(t)\}_{k=0}^{\infty}$ of the basic MCA system (3) and (4) are simply obtained. The first few HPM approximations $\{\hat{\rho}_k(x)\}$ and $\{\hat{\beta}_k(x)\}$ are shown in the upper panels of **Figure 2**, where one can already see their convergence towards the true solution of the MCA system (6) and (7). Similarly, the bottom panels of **Figure 2** show the convergence of approximate solutions $\{\hat{c}_k(t)\}$ and $\{\hat{b}_k(t)\}$ of the normalized MCA system (3) and (4). In doing so, the value $\gamma = 0.25$ was taken, so the convergence of both approximations towards the equilibrium $(b^*, c^*) = (0.25, 0.20)$ can be clearly observed.

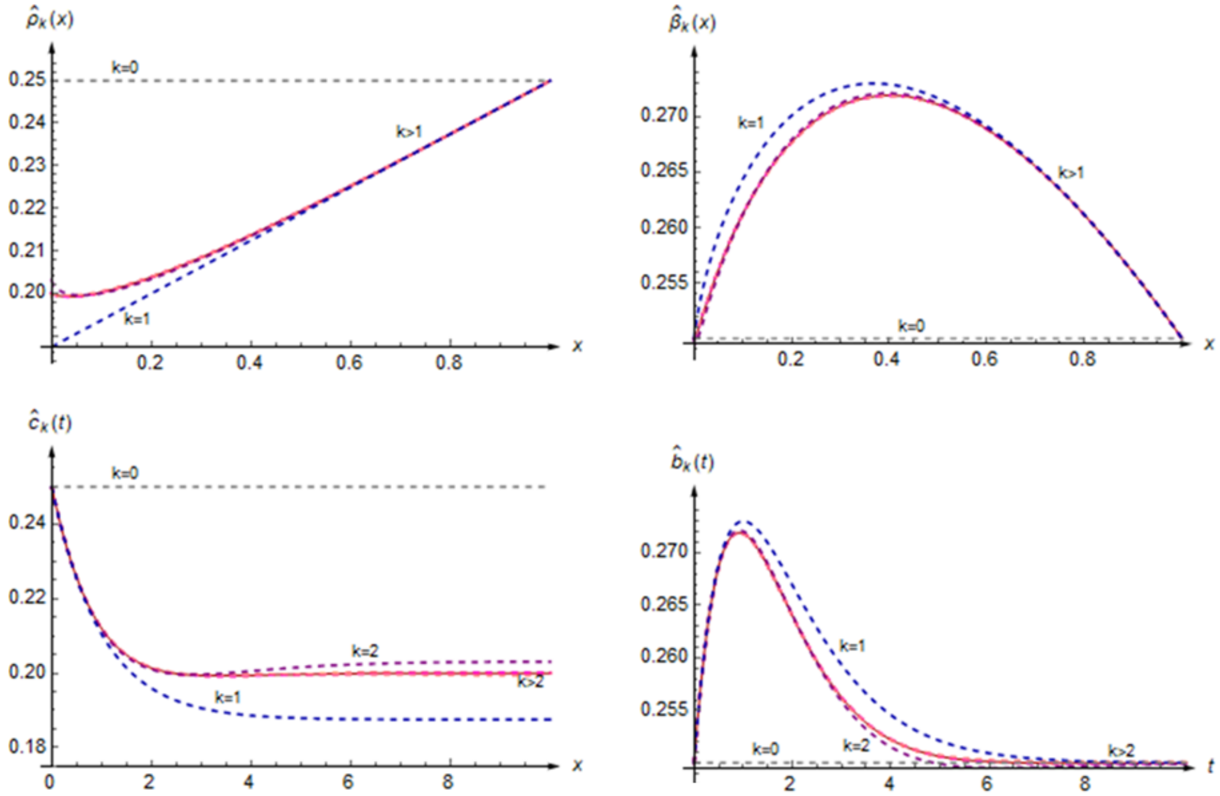

Figure 2. Panels above: HPM approximations of the solution to the MCA system on $x \in (0,1)$. Panels below: Approximations of the solution to MCA system on $t \in (0, \infty)$. (Series A with $\alpha(x) = 1$ and $\gamma = 0.25$.)

Notice that the whole procedure of computing the HPM approximations $\hat{\rho}_k(x)$ and $\hat{\beta}_k(x)$, as well as the functions $\hat{c}_k(t)$ and $\hat{b}_k(t)$, when $k = 0,1,2,\ldots$, has been implemented in the software package MATHEMATICA 11.2. Further on, in order to determine the level of convergence of thus-obtained approximations, as well as their appropriate errors of approximations, the following three different measures have been computed:

(i) The maximum approximation differences (MAD):

$$\mathrm{MAD}(\hat{\rho}_k) := \|\hat{\rho}_k - \hat{\rho}_{k-1}\| = \max_{0<x<1} |\rho_k(x) - \rho_{k-1}(x)|,$$

$$\mathrm{MAD}(\hat{\beta}_k) := \|\hat{\beta}_k - \hat{\beta}_{k-1}\| = \max_{0<x<1} |\beta_k(x) - \beta_{k-1}(x)|,$$

are measures of the “closeness” of adjacent members of the HPM series. Therefore, MAD values are also measures of the convergence of HPM approximations with respect to their order of approximation $k = 1,2, \ldots$

(ii) The maximum absolute errors (MAE):

$$\mathrm{MAE}(\hat{\rho}_k) \coloneqq \max_{0<x<1} \left| x\frac{d\hat{\rho}_k(x)}{dx} - \hat{\rho}_k(x)\left(\alpha(x) + \hat{\beta}_k(x)\right) + \gamma \right|,$$

$$\mathrm{MAE}\big(\hat{\beta}_k\big) \coloneqq \max_{0<x<1} \left| x\frac{d\hat{\beta}_k(x)}{dx} - (\hat{\rho}_k(x) - 1)\hat{\beta}_k(x) + \alpha(x)\hat{\rho}_k(x) \right|,$$

are quantitative indicators of the accuracy of HPM approximations in each iteration $k = 1,2,\ldots$ Note that for the exact solutions $\rho(x)$ and $\beta(x)$ of the MCA system (6) and (7), the equalities $MAE(\rho) = MAE(\beta) = 0$ hold. In this way, MAEs indicate the level of deviation of the HPM approximations from the true solution of the MCA model.

(*iii*) The maximum fractional errors (MFE):

$$\mathrm{MFE}(\hat{\rho}_k) \coloneqq \max_{0<x<1} \left| \frac{xd\hat{\rho}_k(x)/dx - \hat{\rho}_k(x)\left(\alpha(x) + \hat{\beta}_k(x)\right) + \gamma}{\hat{\rho}_k(x)} \right| \times 100\%,$$

$$\mathrm{MFE}\big(\hat{\beta}_k\big) \coloneqq \max_{0<x<1} \left| \frac{xd\hat{\beta}_k(x)/dx - (\hat{\rho}_k(x) - 1)\hat{\beta}_k(x) + \alpha(x)\hat{\rho}_k(x)}{\hat{\beta}_k(x)} \right| \times 100\%,$$

represent the maximum values of the relative errors, defined as the ratio of the MEAs arguments and the values of the HPM approximations in the *k*th iteration.

All the above-mentioned values, obtained for both series of HPM approximations up to the 7th order, are given in **Table 1** above. As can be easily seen, both series of HPM approximations have pronounced convergence, which is somewhat more significantly present with the series $\{\hat{\beta}_k(x)\}$. We assume that this is a consequence of the limited, zero values of these functions at the ends of the unit interval $(0,1)$. Conversely, although the MAD, MAE and MFE values are slightly higher with HPM approximations $\{\hat{\rho}_k(x)\}$, it can be clearly seen that they also tend towards zero and thus confirm the convergence of this series towards the real solution of the MCA system (6) and (7).

Table 1. MAD, MAE, and MFE values of the HPM approximations (Series A when $\alpha(x) = 1$ and $\gamma = 0.25$).

Order	**MAD**		**MAE**		**MFE (%)**	
(k)	$(\hat{\rho}_k)$	$(\hat{\beta}_k)$	$(\hat{\rho}_k)$	$(\hat{\beta}_k)$	$(\hat{\rho}_k)$	$(\hat{\beta}_k)$
1	6.25×10^{-2}	2.30×10^{-2}	1.56×10^{-2}	1.56×10^{-2}	8.333	6.249
2	1.56×10^{-2}	3.50×10^{-3}	3.91×10^{-3}	7.73×10^{-4}	1.923	1.562
3	3.91×10^{-3}	2.97×10^{-4}	9.76×10^{-4}	6.13×10^{-6}	0.1791	0.1381
4	9.77×10^{-4}	2.19×10^{-4}	1.40×10^{-4}	4.69×10^{-8}	0.1208	9.74×10^{-2}
5	9.29×10^{-5}	4.85×10^{-5}	5.11×10^{-5}	2.07×10^{-10}	5.51×10^{-3}	4.21×10^{-3}
6	4.93×10^{-5}	1.75×10^{-5}	5.38×10^{-6}	4.54×10^{-11}	$2.70\times l0^{-3}$	2.11×10^{-3}
7	1.74×10^{-5}	5.96×10^{-6}	3.25×10^{-7}	1.63×10^{-13}	$1.64\times l0^{-4}$	1.25×10^{-4}

Series B

Similar to the previous Series A, here we consider another HPM series obtained by solving the MCA system (6) and (7) using the proposed HPM procedure. This time, for the boundary conditions, we assume that $c(0) = 0$ and $b(0) = \gamma$, which represents the situation that at the initial time $t = 0$ there is no information about the current number of crimes. Hence, $\rho(1) = 0$ and $\beta(1) = \gamma$, so as it is explained earlier, the HPM series $\{\rho_k(x)\}$ and $\{\beta_k(x)\}$ satisfy the equalities $\rho_0(1) = 0$, $\beta_0(1) = \gamma$ and $\rho_k(1) = \beta_k(1) = 0$, when k = 1, 2, ... Then, using equations (13) and (14), the values $C_0 = -\gamma$ and $D_0 = 0$ are easily obtained; that is, the initial HPM solutions:

$$\rho_0(x) = \gamma(1 - x), \beta_0(x) = \gamma(x \ln x + 1).$$

Continuing this procedure, i.e., by applying equations (21) and (22) when $k = 1,2, \ldots$, the other members of the HPM series $\{\rho_k(x)\}$ and $\{\beta_k(x)\}$ are obtained, the first few of which read as follows:

$$\rho_1(x) = \frac{\gamma^2}{2}(x\ln^2 x - 2(x+1)x\ln x + 2x^2 - 2),$$

$$\beta_1(x) = \frac{\gamma^2 x}{6}\left(6(2x+1)\ln x - \ln^3 x - 18(x-1)\right);$$

$$\rho_2(x) = \frac{\gamma^3}{24}(12(3x^3 - 22x^2 + 17x + 2) + 72x(2x+1)\ln x - 12x(x^2 + 6x + 1)\ln^2 x + 4x(4x+1)\ln^3 x - x\ln^4 x),$$

$$\beta_2(x) = \frac{\gamma^3 x}{120}(-15(x-1)(17x - 287) + \ln^5 x - 160x\ln^3 x + 30(x(3x+32) - 4)\ln^2 x - 30(x(x+88) + 46)\ln x);$$

$$\rho_3(x) = \frac{\gamma^4}{720}\Big(90x(22x^3 + 57x^2 - 384x - 243)\ln x - 180x(3x^3 + 5x^2 - 94x + 17)\ln^2 x - 60x(2x^3 + 15x^2 + 56x + 6)\ln^3 x + 60x(5x^2 + 12x + 1)\ln^4 x - 6x(16x+1)\ln^5 x + x\ln^6 x - 30(29x^4 + 513x^3 - 2733x^2 + 2167x + 24)\Big),$$

$$\beta_3(x) = \frac{\gamma^4 x}{45360}(70(1270x^3 + 21627x^2 - 253206x + 230309) - 420(410x^3 + 1053x^2 - 22788x - 13032)\ln x + 315(136x^3 + 45x^2 - 11664x + 2799)\ln^2 x + 630(16x^3 + 165x^2 + 1152x + 138)\ln^3 x - 1890(15x^2 + 64x - 1)\ln^4 x + 378(32x - 1)\ln^5 x - 9\ln^7 x), \text{etc.}$$

The upper panels of **Figure 3** show the plots of the HPM approximations $\{\hat{\rho}_k(x)\}$ and $\{\hat{\beta}_k(x)\}$, while the lower panels of the same figure show the corresponding approximations $\{\hat{c}_k(t)\}$ and $\{\hat{b}_k(t)\}$ of the normalized MCA system (3) and (4), obtained by transformation $x = \exp(-t)$, when $t \geq 0$, and $\gamma = 0.5$. As can be easily seen, and similarly to Series A, there is a fast convergence of the obtained approximate solutions to the exact solution of the MCA model. Moreover, when $t \to \infty$, convergence of approximations $\{\hat{b}_k(t)\}$ and $\{\hat{c}_k(t)\}$ toward the equilibrium $(b^*, c^*) = (1/2, 1/3)$ is obvious.

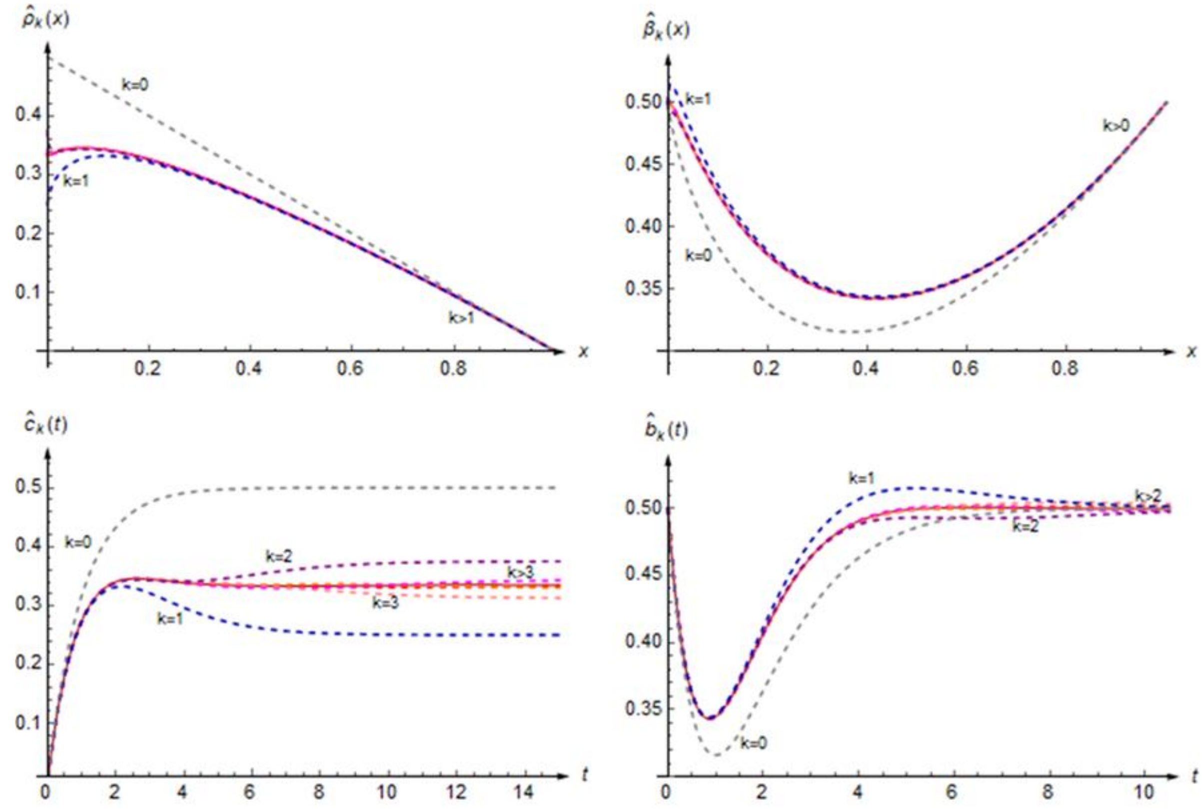

Figure 3. Panels above: HPM approximations of the solutions of the MCA system on $x \in (0,1)$. Panels below: Approximations of the solutions of the MCA system on $t \in (0, \infty)$. (Series B with $\alpha(x) = 1$ and $\gamma = 0.5$.).

The confirmation of these facts, as well as the degree of convergence of the HPM approximations, can also be seen in **Table 2**, where the corresponding values of MAD, MAE, and MFE are shown, defined as before. In this case, in the same way as in the case of Series A, the decrease in these values and their convergence towards zero is clearly visible. Let us note that, similar to the previous one, the series $\{\hat{\beta}_k(x)\}$ has slightly faster convergence than $\{\hat{\rho}_k(x)\}$. Nevertheless, the apparent convergence of both series justifies the application of the proposed HPM procedure in (approximately) solving the MCA model.

Table 2. MAD, MAE, and MFE values of the HPM approximations (Series B when $\alpha(x) = 1$ and $\gamma = 0.5$).

Order	**MAD**		**MAE**		**MFE (%)**	
(k)	($\hat{\rho}_k$)	($\hat{\beta}_k$)	($\hat{\rho}_k$)	($\hat{\beta}_k$)	($\hat{\rho}_k$)	($\hat{\beta}_k$)
1	0.2500	5.07×10^{-2}	0.1250	0.1248	1.919	2.499
2	0.1250	2.19×10^{-2}	6.25×10^{-2}	3.33×10^{-3}	1.788	0.7933
3	6.25×10^{-2}	1.09×10^{-2}	3.12×10^{-3}	3.03×10^{-4}	0.9737	5.92×10^{-2}
4	3.12×10^{-2}	5.80×10^{-3}	1.02×10^{-4}	4.03×10^{-5}	4.17×10^{-2}	7.20×10^{-3}
5	6.20×10^{-3}	1.97×10^{-3}	4.82×10^{-5}	1.11×10^{-6}	4.02×10^{-3}	1.80×10^{-4}
6	5.10×10^{-3}	1.56×10^{-3}	2.37×10^{-7}	5.28×10^{-8}	6.33×10^{-4}	6.40×10^{-5}
7	5.92×10^{-4}	2.45×10^{-4}	1.92×10^{-9}	8.07×10^{-10}	7.12×10^{-5}	1.80×10^{-6}

Series C

At the end of this part, let us consider the problem of finding a (boundary) solution of the MCA system (6) and (7), when $\alpha(x) \equiv a > 0$ and the following initial conditions $\rho'(1) = \beta'(1) = 0$ hold. This means that both functions $\rho(x)$ and $\beta(x)$ have constant tangential values at $x = 1$, and therefore, the solutions $c(t)$ and $b(t)$ of the basic MCA model (3) and (4) satisfy the equalities $c'(0) = b'(0) = 0$. According to the additivity of the derivative, this implies that for the HPM series $\{\rho_k(x)\}$ and $\{\beta_k(x)\}$ the equalities $\rho_k'(1) = \beta_k'(1) = 0$ also hold for each $k = 0,1,2, \ldots$ Using the given boundary conditions and equations (13) and (14), as an initial HPM solution, the constant functions $\rho_0(x) = \gamma/a$ and $\beta_0(x) = \gamma$ are easily obtained. After that, by applying equations (21) and (22) and the inductive method, the rest of the HPM series reads as follows:

$$\rho_k(x) = (-1)^k \left(\frac{\gamma}{a}\right)^{k+1}, \ \beta_k(x) = 0, k = 1,2, \ldots$$

Finally, the (exact) constant solution of the MCA system (6) and (7), and therefore of the normalized MCA system (3) and (4), is given by the equalities:

$$\rho(x) = \sum_{k=0}^{\infty} \rho_k(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{\gamma}{a}\right)^{k+1} = \frac{\gamma}{a+\gamma} \tag{31}$$

$$\beta(x) = \sum_{k=0}^{\infty} \beta_k(x) = \gamma. \tag{32}$$

Obviously, equations (31) and (32) represent the equilibrium (b^*, c^*) of the MCA model, which has already been defined earlier by equation (5).

5. Practical application

We now consider the practical application of the proposed HPM technique in modeling real-world data. For this purpose, depersonalized data on recorded criminal offenses in the territory of the Republic of Serbia in the previous 10 years were used. More precisely, the observed dataset contains, among others, dates, places, and the number of criminal offenses committed from 1 January 2015 to 31 December 2024, which is taken as the basic time series. It should be noted that this dataset was collected according to official reports of the Ministry of Internal Affairs of the Republic of Serbia on a daily level. As an illustration, **Figure 4a** shows the dynamics of the daily number of crimes recorded in this time period, where the emphasized fluctuations are noticeable.

To better describe the deterministic functional dependence and fit these data into the MCA model (3) and (4), they were regrouped quarterly, as can be seen in **Figure 4b**. Thus, a trend of decreasing crime rates over the observed time period is clearly visible. It can be shown that this decline has an exponential trend, also shown in **Figure 4b**, along with the corresponding determination coefficient (R^2). This, among others, indicates the possibility of modeling these data using the previously described MCA model and HPM procedure.

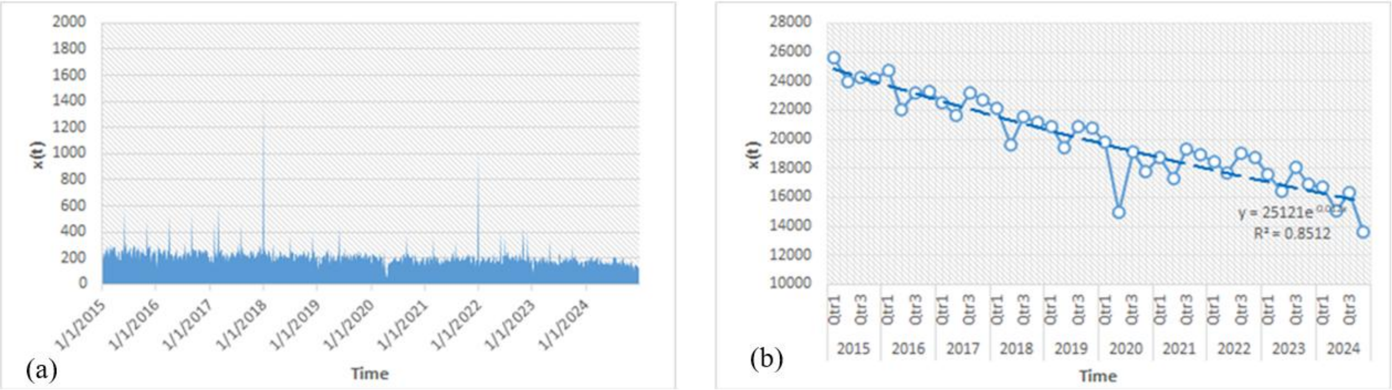

Figure 4. Time series of the number of crimes committed in the Republic of Serbia, observed on a daily level **(a)** and quarterly **(b)**.

In the next step, the observed data series, denoted as $\{x(t)\}$, is taken in the form of percentages and additionally normalized in relation to their maximum value. Therefore, by applying equality:

$$\mathrm{c}(t) := \frac{x(t)}{\max\{x(t)\}}$$

the first time series $\{c(t)\}$ was obtained. It is worth pointing out that, due to the decreasing trend of the series $\{x(t)\}$ the initial value of the series $\{c(t)\}$ is equal to $c(0) = 1$. This allows it to be viewed as a possible realization, with certain random fluctuations, of the deterministic function $c(t)$ that represents the solution of the normalized MCA system (6) and (7).

To obtain the second series $\{b(t)\}$ within each quarter, the highest and lowest monthly values were observed, denoted as $x_{max}(t)$ and $x_{min}(t)$, respectively. Thereafter, the values $b(t)$ are defined as the averages of these extremes, i.e.,

$$\tilde{b}(t) := \frac{x_{max}(t) + x_{min}(t)}{2}.$$

By introducing these values, it is expected that quarters with higher values of $\tilde{b}(t)$ have a greater attractiveness for committing crimes, regardless of the observed time period. Thus, it is assumed that the series $\{\tilde{b}(t)\}$ represents the dynamic attractiveness of crime in each of the observed (quarterly) time periods.

Thereafter, the corresponding values $b(t)$ are computed in a similar way to the previous series $\{c(t)\}$, i.e., by applying equality:

$$\mathrm{b}(t) := \frac{\tilde{b}(t)}{\max\{\tilde{b}(t)\}} + \delta.$$

Notice that, besides normalization with respect to the largest value of the sequence $\{\tilde{b}(t)\}$, here is introduced an additive constant $\delta > 0$ in order to ensure the initial conditions $b(0) = c(0) = \gamma$. In our case, the value $\delta = 0.4$ is taken, which implies the equality $b(0) = 1.00348 \approx 1$. Therefore, in the fitting procedure it can be assumed that $\alpha(x) = \gamma = 1$**;** that is, the HPM procedure identical to the one shown for Series A in the previous section can be applied.

More precisely, using HPM approximations $\{\hat{\rho}_k(x)\}_{k=0}^{\infty}$ and $\{\hat{\beta}_k(x)\}_{k=0}^{\infty}$ defined by equations (23), as well as transformation $x(t) = \exp(-t)$, estimates of the solution of the MCA system (3) and (4) can be obtained as follows:

$$\hat{c}_k(t) := \sum_{j=0}^{k} c_j(t),\ \hat{b}_k(t) := \sum_{j=0}^{k} b_j(t)\,, k = 0,1,2,\ldots$$

Herein is $c_j(t) := \rho_j(\exp(-t))$ and $b_j(t) := \beta_j(\exp(-t))$, when $j = 0, 1, \ldots, k$, where $\{\rho_j\ (x)\}$ and $\{\beta_j\ (x)\}$ are the HPM series defined by equations (13), (14), (21) and (22). Additionally, for the thus-obtained series $\{\hat{c}_k(t)\}$ and $\{\hat{b}_k(x)\}$, the following corresponding estimation errors are defined:

(i) Mean-squared estimation errors (MSEEs) represent the averages of the squared deviations of the estimated and actual values of a given series:

$$\mathrm{MSEE}(\hat{c}_k) := \frac{1}{T}\sum_{t=0}^{T-1}\left(c(t) - \hat{c}_k(t)\right)^2,\ MSEE\left(\hat{b}_k\right) := \frac{1}{T}\sum_{t=0}^{T-1}\left(b(t) - \hat{b}_k(t)\right)^2.$$

(ii) Fractional estimation errors (FEEs) represent the averages of the squared ratios of the deviations defined above and the values of the actual series:

$$\mathrm{FEE}(\hat{c}_k) := \frac{1}{T}\sum_{t=0}^{T-1}\left(\frac{c(t) - \hat{c}_k(t)}{c(t)}\right)^2 \times 100\%,\ FEE\left(\hat{b}_k\right) := \frac{1}{T}\sum_{t=0}^{T-1}\left(\frac{b(t) - \hat{b}_k(t)}{b(t)}\right)^2 \times 100\%.$$

In **Table 3** are shown the first few MSEE and FEE values of the estimates $\hat{c}_k(t)$ and $\hat{b}_k(t)$, obtained according to the aforementioned formulas, where the length of the observed time series is equal to $T = 40$. Thus obtained values clearly indicate the quality of the approximations, because there is a noticeable decrease in the MSEE and FEE values with increasing approximation order $k = 0, 1, 2, \ldots$.

Table 3. Error values of the approximations $\hat{c}_k(t)$ and $\hat{b}_k(t)$ when $k = 0,1,2,3,4,5$.

Order ($\boldsymbol{k}$)	**MSEE**		**FEE%**	
	$\hat{c}_k(t)$	$\hat{b}_k(t)$	$\hat{c}_k(t)$	$\hat{b}_k(t)$
0	0.0620	0.0875	13.904	5.0293
1	2.15×10^{-3}	4.15×10^{-3}	0.5158	0.2468
2	2.13×10^{-3}	3.79×10^{-3}	0.5124	0.2290
3	2.12×10^{-3}	3.47×10^{-3}	0.5088	0.2151
4	2.11×10^{-3}	3.46×10^{-3}	0.4792	0.2146
5	2.08×10^{-3}	3.45×10^{-3}	0.4785	0.2141

In addition, as can be easily seen in **Figure 5**, satisfactory estimates of the series $c(t)$ and $b(t)$ are already obtained for the approximations of the order $k > 2$. At the same time, it should be noted that $c(t)$ and $b(t)$ are time series, which also have certain stochastic components. Therefore, there are deviations from the approximation functions $\hat{c}_k(t)$ and $\hat{b}_k(t)$, which describe the deterministic component of the observed time series. Nevertheless, the result of the proposed approximation procedure represents a basis for further analysis and possibilities for modeling the dynamics of criminality.

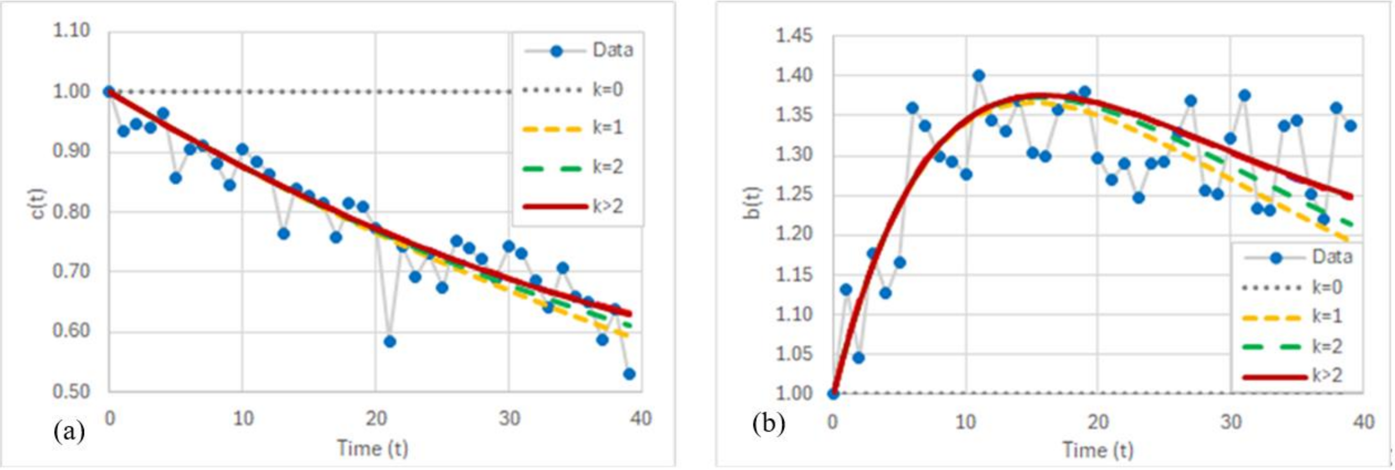

Figure 5. Approximations of the series $c(t)$ (a) and $b(t)$ (b) using the proposed HPM procedure.

6. Conclusion

A mathematical model of criminal activity (MCA model) is described here, which consists of two first-order differential equations. This system is first simplified by introducing appropriate dimensionless variables and thoroughly investigated in terms of its theoretical properties (existence and uniqueness of solutions, equilibria, etc.). After that, by reducing the interval $(0, \infty)$ to a finite one, for the thus transformed MCA system, approximate HPM solutions were obtained, such that satisfied various boundary conditions. The convergence of the HPM approximations to exact solutions of the MCA system is formally proven and further verified by solving it with different boundary conditions. Finally, the practical application of the HPM procedure in approximating real-world data, i.e., modeling crime dynamics in the Republic of Serbia, is provided. The results presented in this way, along with the corresponding numerical simulations, demonstrate the accuracy of the proposed model in predicting actual trends in crime dynamics.

Based on the above, it can be concluded that the proposed HPM procedure is an adequate tool for approximately solving MCA models and therefore can also be used in solving some similar models. Another advantage of the HPM technique is that solutions can be calculated recursively with arbitrary numerical precision, and the entire approximation procedure can be implemented using appropriate software. Nevertheless, it should be pointed out that, in addition to this deterministic approach, appropriate stochastic models of criminal activities are often used in contemporary theory and practice. For these reasons, connecting these two approaches is one of the authors' guidelines for some future research.

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and DJ; project administration, SS and MJ. All authors have read and agreed to the published version of the manuscript.

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Appendix

The following **Table A1** lists the variables, functions and parameters, along with their explanations, that were used in this study.

Table A1. List of variables, functions and parameters used in this study.

Variable (Function, Parameter)	**Explanation**
$a(t)$	The "internal" part of the attractiveness of crime at time t.
$b(t)$	The "dynamic" part of the attractiveness of crime at time t.
$c(t)$	Total number of criminal activities at time t.
κ	Constant ratio of criminal activities to the product of total attractiveness and the number of criminal activities.
γ	Constant of the increasing tendency of criminal activity.
ℓ	Constant ratio of attractiveness dynamics to the total number of crimes committed.
ω	Constant rate of decay in the attractiveness of crime over time.
(b^*, c^*)	Equilibrium point of the MCA system (3) and (4).
$J(b, c)$	The Jacobian of the MCA system (3) and (4).
τ	The trace of the Jacobian.
Δ	The determinant of the Jacobian.
$\vec{\lambda} = (\lambda_1, \lambda_2)$	The eigenvector of the Jacobian matrix $J(b^*, c^*)$
$t(x)$	Transformation of time (t) to unit interval $x \in (0,1)$.
$\alpha(x)$	Transformation of the function $a(t)$ on $x \in (0,1)$.
$\beta(x)$	Transformation of the function $b(t)$ on $x \in (0,1)$.
$\rho(x)$	Transformation of the function $c(t)$ on $x \in (0,1)$.
$\{\rho_k(x)\}$	HPM series of the function $\rho(x)$.
$\{\beta_k(x)\}$	HPM series of the function $\beta(x)$.
$\hat{\rho}_j(x)$	HPM approximation of the function $\rho(x)$.
$\hat{\beta}_k(x)$	HPM approximation of the function $\beta(x)$.
$\{x(t)\}$	Observed real-world data set.