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Article

A parameter-free series solution approach for differential equations in fluid flow

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Abstract: Viscous incompressible flow across a converging/diverging channel generally known as Jeffery-Hamel flow is an important form of flow in the fluid dynamics sector that exists in a variety of engineering systems, rivers, and in the biological world. This paper proposes a novel Maclaurin Series Method (MSM) to investigate Hamel's fractal flow pattern in a wedge-shaped region. The fundamental partial differential equations are changed by suitable transformation into the dimensionless non-linear ordinary differential equation. The resulting equation is solved through MSM. The Maclaurin series method obtains the solution of the two-dimensional incompressible viscous flow in the converging / diverging channels according to initial condition. The MSM provides an efficient and accurate alternative to traditional solution techniques. To validate the Maclaurin series method, error analysis of the solution is calculated and presented in tabular form, demonstrating excellent agreement with benchmark results. Furthermore, the MSM solution is plotted for various β values. The comparison between MSM approximate and exact solutions confirms the reliability and effectiveness of the method. Overall, the results indicate that the suggested approach is an effective and reliable tool for solving fluid flow problems.

Keywords: converging/diverging channel; Maclaurin series method; boundary value problem; fractal calculus; numerical solution

1. Introduction

Significant advancements in both the theoretical development and practical application of fractal differential equations have been made in recent years [1, 2]. Such equations [3–11] are steadily being utilized to display issues in regions of study as differing as chaos, chaos synchronization, mind-boggling forms, mechanical structures, guidelines, nonstop time arbitrary walk, peculiar diffusive and sub-diffusive procedures, dispersion assembly and wave proliferation marvels and others.

Jeffery [12] and Hamel [13] pioneered the mathematical analysis of the movement of a viscous incompressible bidimensional fluid in a wedge-shaped channel with a sink or source at the vertex. Axford [14] described the results of an artificially imposed magnetic field and the general solution produced by Rosenhead [15] comprising elliptic functions. The equations controlling the Jeffery-Hamel model are emphatically non-direct and accordingly don't give an observational arrangement in a shut nature. Nonlinear complications [16–22] will, in value, be overwhelmed by any of a wide assortment of computational methodologies. Numerical methodologies often fail to provide practical insight into the effects of various parameters involved in complex fluid flow problems. Accordingly, the latest investigations of diverging and converging channels have concentrated on the utilization of Jeffery-Hamel flow equations as a test and test instrument for the precision, unwavering quality, and fervor of new nonlinear equations goals techniques. This problem has since been extensively investigated through a variety of approaches [23–28].

Recently, intelligent methods such as fractional sub-equation neural networks (fSENNs) [29] and neurosymbolic reasoning algorithms [30] have been developed to solve fractional differential equations. While these approaches offer powerful capabilities, they often involve high computational complexity.

Moreover, many physical procedures are inherently nonlinear, a wide range of theoretical and computational strategies have been created by various researchers to determine the non-linearity of these issues. Most of these sophisticated methods have drawbacks such as restricted convergence, divergent outcomes, linearization, flexibility, unreasonable conclusions, and non-compatibility with physical problems [16–22]. In contrast, the proposed Maclaurin Series Method (MSM) provides a simple, analytical alternative that is both efficient and accurate for fractal Jeffery-Hamel flow problems.

The primary objective of this article is to introduce a computational technique, namely the Maclaurin Series Method (MSM), for analyzing the fractal behavior of Jeffery–Hamel flow in a wedge-shaped region [31, 32], along with residual error analysis following the guideline presented by Marinca et al. [33]. The findings of the recommended strategy imply that the method is simple to deal with and is computationally incredibly brilliant. This study offers a very valuable source of knowledge for researchers in this area. While there are several computational and theoretical approaches for solving these problems, this paper proposes an easy yet efficient way of utilizing the Maclaurin series approach to solve the third-order ordinary differential equations. To the best of our knowledge, the first documented series solution for fractal Jeffrey-Hamel flow is presented here.

2. Fractal formulation of viscous flow model

Viscous incompressible flow across a converging/diverging channel [31, 32] governing equations are given below:

$$\frac{\partial(rq_r)}{\partial r} = 0 \tag{1}$$

$$\rho q_r \frac{\partial q_r}{\partial r} = -\frac{\partial p}{\partial r} + \frac{\mu}{r^2}\frac{\partial^2 q_r}{\partial \theta^2} \tag{2}$$

$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{2\mu}{r^2}\frac{\partial g_r}{\partial \theta} \tag{3}$$

From Eq.(1), which implies that rq_r is independent of r, one can define the substitution:

$$rq_\tau = \gamma G(\theta) \tag{4}$$

Putting the expression for q_r into Eq.(2) and Eq.(3), one obtains:

$$-\rho\gamma^2\frac{G^2}{r^3}=-\frac{\partial p}{\partial r}+\frac{\rho\gamma^2}{r^3}\frac{d^2G}{d\theta^2} \tag{5}$$

$$0=-\frac{\partial p}{\partial r}+\frac{2\rho\gamma^2}{r^2}\frac{dG}{d\theta} \tag{6}$$

Removing the pressure p from Eq. (5) and Eq.(6), one can obtain:

$$\frac{d^3G}{d\theta^3}+2G\frac{dG}{d\theta}+4\frac{dG}{d\theta}=0 \tag{7}$$

With boundary conditions

$$\begin{aligned}\theta=\xi:&\quad q_r=0\\ \theta=0:&\quad \partial q_r/\partial\theta=0\end{aligned}$$

Based on Eq.(2)

$$\theta=\xi:-\frac{\partial p}{\partial r}+\frac{\mu}{r^2}\frac{\partial^2 q_r}{\partial\theta^2}=0 \tag{8}$$

By applying Eq. (4), Eq. (8) can be rewritten in terms of $G(\theta)$ as shown below:

$$\begin{aligned}\theta=\xi:&\quad G(\xi)=0,\ d^2G(\xi)/d\theta^2\quad=-A\\ \theta=0:&\quad dG(0)/d\theta=0\end{aligned} \tag{9}$$

Introducing $\psi=\xi-\theta$ in Eq.(7), one can obtain:

$$\frac{d^3G}{d\psi^3}+(2G+4)\frac{dG}{d\psi}=0 \tag{10}$$

subject to the boundary conditions:

$$\begin{aligned}\psi=0:&\quad G(0)=0,\ d^2G(0)/d\psi^2\quad=-A\\ \psi=\xi:&\quad dG(\xi)/d\psi=0\end{aligned} \tag{11}$$

The fractal form of Eq. (10) is given by

$$\frac{d}{d\psi^{2\beta}}\left(\frac{dG}{d\psi^\beta}\right)+(2G+4)\frac{dG}{d\psi^\beta}=0$$

with boundary conditions:

$$\begin{aligned}\psi=0:&\quad G(0)=0,\ \frac{d}{d\psi^\beta}\left(\frac{dG(0)}{d\psi^\beta}\right)=-A\\ \psi=\xi:&\quad \frac{dG(\xi)}{d\psi^\beta}=0\end{aligned} \tag{12}$$

Where $\frac{dG}{d\psi^\beta}$ is the fractal derivative defined as below [1,2]:

$$\frac{dG}{d\psi^\beta}(\psi_0)=\Gamma(1+\beta)\lim_{\substack{\psi-\psi_0\to\Delta\psi\\ \Delta\psi\neq 0}}\frac{G(\psi)-G(\psi_0)}{(\psi-\psi_0)^\beta} \tag{13}$$

Using the two-scale transformation [34,35]:

$$s=\psi^{\beta} \tag{14}$$

Eq. (12) can be written into the following one:

$$\frac{d^{3}G(s)}{ds^{3}}+(2G(s)+4)\frac{dG(s)}{ds}=0 \tag{15}$$

with boundary condition

$$\begin{aligned} &s=0: \quad G(0)=0, \quad \frac{d^{2}G(0)}{ds^{2}}=-A \\ &s=\xi: \quad \frac{dG(\xi)}{ds}=0 \end{aligned} \tag{16}$$

One can obtain the value

$$G'(0)=\alpha \tag{17}$$

the Hamel's flow equation has been changed into an initial value problem; where α denotes an as-yet-undetermined constant.

Setting $s=0$ in eq. (15) results in

$$\frac{d^{3}G(0)}{ds^{3}}+(2G(0)+4)\frac{dG(0)}{ds}=0 \tag{18}$$

subject to initial condition:

$$\begin{aligned} &G(0)=0, \quad d^{2}G(0)/ds^{2}=-A \\ &G'(0)=\alpha \end{aligned} \tag{19}$$

Using eq. (19) in eq. (18), we get

$$G'''(0)=-4\alpha \tag{20}$$

One can yield the following equation by differentiating Eq. (15)

$$\frac{d^{4}G(s)}{ds^{4}}+2\left(G'(s)\right)^{2}+4G''(s)+2G(s)G''(s)=0 \tag{21}$$

Employing $s=0$ in eq. (21) yields

$$\frac{d^{4}G(0)}{ds^{4}}+2\left(G'(0)\right)^{2}+4G''(0)+2G(0)G''(0)=0 \tag{22}$$

From Eq. (19) and Eq. (20), one can get

$$G^{iv}(0)=2\left(-\alpha^{2}+2A\right) \tag{23}$$

Through an analogous procedure using Mathematica, one can acquire

$$\begin{aligned} G^{v}(0)&=2(8\alpha+3\alpha A) \\ G^{vi}(0)&=2\left(-20\alpha^{2}+8A+3A^{2}\right) \\ G^{vii}(0)&=4\left(-16\alpha+5\alpha^{3}-36\alpha A\right) \end{aligned} \tag{24}$$

The Maclaurin series solution is given by

$$G(s) = G(0) + G'(0)\frac{s}{1!} + G''(0)\frac{s^2}{2!} + G'''(0)\frac{s^3}{3!} + G^{iv}(0)\frac{s^4}{4!} + G^{v}(0)\frac{s^5}{5!} + G^{vi}(0)\frac{s^6}{6!} + \dots, \tag{25}$$

$$\begin{aligned} G(s) = \alpha s - \frac{As^2}{2} - \frac{2\alpha s^3}{3} + \frac{1}{12}\left(-\alpha^2 + 2A\right)s^4 + \frac{1}{60}(8\alpha + 3\alpha A)s^5 - \frac{1}{360}\left(-20\alpha^2 + 8A + 3A^2\right)s^6 \\ + \frac{1}{1260}\left(-16\alpha + 5\alpha^3 - 36\alpha A\right)s^7 + s^6 + \frac{1}{10080}\left(-168\alpha^2 + 16A - 33\alpha^2 A + 36A^2\right)s^8 \\ - \frac{1}{22680}\left(-16\alpha + 75\alpha^3 - 162\alpha A - 21\alpha A^2\right)s^9 - \frac{1}{453600}\begin{pmatrix} -1360\alpha^2 + 75\alpha^4 \\ +32A - 1110\alpha^2 A \\ +324A^2 + 42A^3 \end{pmatrix} s^{10} \end{aligned} \tag{26}$$

Therefore, the solution (26) in fractal form:

$$\begin{aligned} G(\psi) = \alpha\psi^{\beta} - \frac{A\psi^{2\beta}}{2} - \frac{2\alpha\psi^{3\beta}}{3} + \frac{1}{12}\left(-\alpha^2 + 2A\right)\psi^{4\beta} + \frac{1}{60}(8\alpha + 3\alpha A)\psi^{5\beta} \\ - \frac{1}{360}\left(-20\alpha^2 + 8A + 3A^2\right)\psi^{6\beta} + \frac{1}{1260}(-16\alpha + 5\alpha^3 - 36\alpha A)\psi^{7\beta} \\ + \frac{1}{10080}\left(-168\alpha^2 + 16A - 33\alpha^2 A + 36A^2\right)\psi^{8\beta} \\ - \frac{1}{22680}\left(-16\alpha + 75\alpha^3 - 162\alpha A - 21\alpha A^2\right)\psi^{9\beta} \\ - \frac{1}{453600}\begin{pmatrix} -1360\alpha^2 + 75\alpha^4 + 32A \\ -1110\alpha^2 A + 324A^2 + 42A^3 \end{pmatrix}\psi^{10\beta} \end{aligned} \tag{27}$$

3. Results and discussion

Using the Pade approximant on boundary condition $dG(\xi)/ds = 0$ and $A = -0.10$, one can obtain the Maclaurin series simulation in the following form (**Table 1**).

Table 1. Error analysis for different values of ψ.

ψ	MSM solution	Error
0.0	0.000	0.00000000000000000693889
0.01	0.0000828831	0.0000000000000000138778
0.02	0.000155733	0.0000000000000000416334
0.03	0.000218522	0.000000000000000773687
0.04	0.000271223	0.00000000000000756686
0.05	0.000313816	0.0000000000000444662
0.06	0.000346284	0.000000000000188651
0.07	0.000368614	0.000000000000638786
0.08	0.000380797	0.00000000000183372

The relationship between the exact and the approximate solutions for the function $G(\psi)$ and $G''(\psi)$ are illustrated in **Figures 1** and **2**. The residuals at various points are presented in **Table 1**, demonstrating remarkable agreement with existing findings [24,25]. **Figure 3** shows the approximate solution $G(\psi)$ for different values of β. As β increases, the approximate solution approaches the exact solution; showing the method is accurate and effective, and its accuracy can be further improved if a solution for a higher-order series is solved.

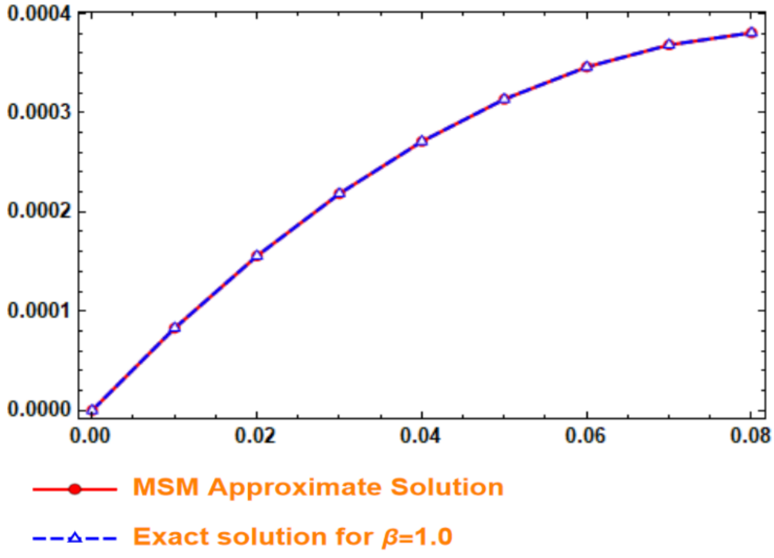

Figure 1. Comparison of MSM approximate solution and exact solution.

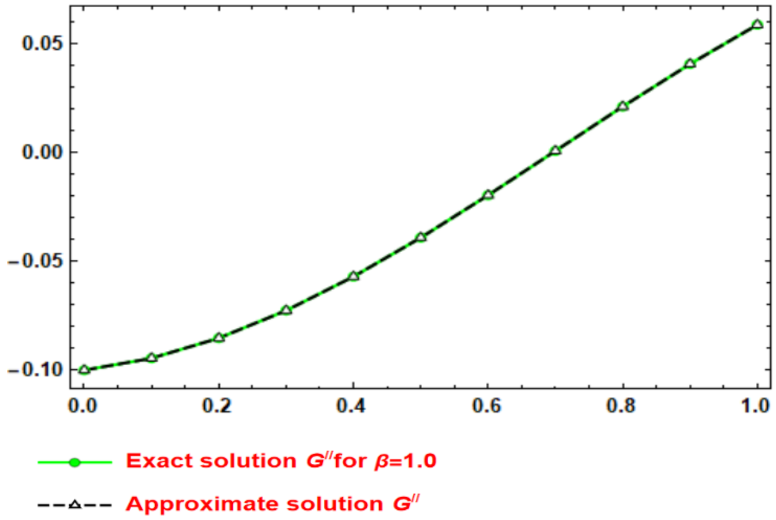

Figure 2. Comparison of exact and MSM approximate solutions for G''.

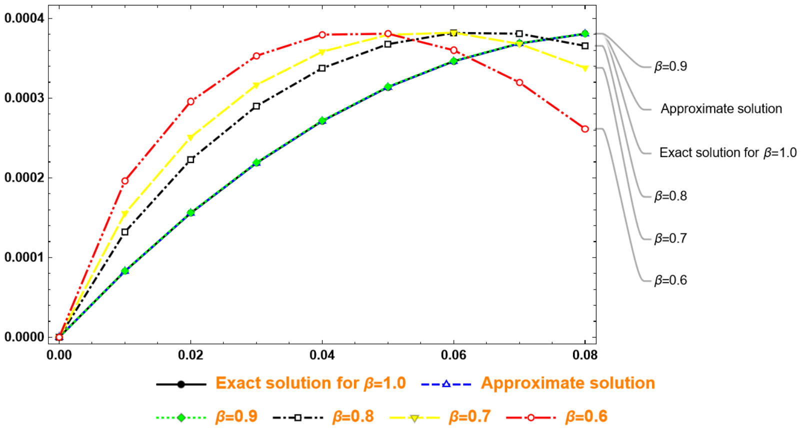

Figure 3. MSM approximate solution for various values of β.

4. Conclusion

In this analysis, in the converging/diverging path, a two-dimensional steady motion of a viscous fluid in fractal form term known as the flow of fractal Hamel is studied. Calculating the approximate solution is using a newly developed Maclaurin series method (MSM). The method is noted to be clear, accurate, and reliable. This functions well on both linear and non-linear problems, and the low error is very impressive. This avoids the huge computational work and difficulties that other numerical methods, such as perturbation, discretization, and linearization.

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