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Article

Existence and multiplicity of solutions to N-Laplacian equation with discontinuous exponential growth in $\mathbb{R}^N$

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Abstract: This research explores the existence and multiplicity of solutions to N-Laplacian equations with discontinuous exponential nonlinearities in the whole Euclidean space. Through combining symmetric rearrangement techniques and variational methods for non-differentiable functionals, it identifies sufficient conditions for the existence of weak solutions when perturbation parameters are small, and uncovers the rich solution structure caused by discontinuous growth and non-smooth operators.These studies connect critical Sobolev growth and exponential nonlinearities, which is an important link in phase transition models and nonlinear analysis.We have proven the existence and multiplicity of weak solutions for the N-Laplacian equation with discontinuous exponential growth . Notably, when the perturbation parameter is sufficiently small, there exist at least multiple weak solutions, which stem from the interaction between the discontinuous exponential nonlinearity and the N-Laplacian operator. Compared to previous findings, our results extend the existing literature on elliptic equations with critical growth and discontinuous nonlinearities. Additionally, the combination of priori estimates with non-differentiable variational methods constitutes a novel approach, distinct from traditional techniques in earlier studies.

Keywords: trudinger-moser inequality; variational methods; non-smooth analysis; mountain pass theorem; free boundary problems; discontinuous nonlinearities

MSC Classification: 35D30; 35J62; 46E35; 49J20

1. Introduction and main results

In recent years, multivalued elliptic equations have been attracting massive attention. This is because a number of free boundary problems in mathematical physics, these problems are primarily characterized by discontinuous nonlinearities, see for example [1–3]. Several techniques have been developed or applied in some existing literatures, for instance, Chang considered the sub- and super-solutions of a PDE as obstacles, and obtained a rather more general existence result about the BVP of a quasilinear elliptic differential equation, which can refer to Chang [3]. The existence and multiplicity of positive solutions were studied by variational methods for nondifferentiable functionals in Alves, Bertone and Gonçlves [4], Alves and Bertone [5], Alves, Gonçlves and Santos [6], global branching method was used in Ambrosetti [7] and Ambrosetti [8].

Badiale and Tarantello established results on the existence and multiplicity of solutions for elliptic problems with critical growth and discontinuous nonlinearities [9]. Carl and Dietrich introduced the weak upper and lower solution method [10], which is used to construct sub-solutions and super-solutions for elliptic equations involving generalized subdifferentiable perturbations. Carl and Heikkila extended this framework to elliptic equations in $\mathbb{R}^N$[11, 12]. By imposing growth conditions that relax classical Lipschitz continuity, they proved the existence of solutions for equations with discontinuous nonlinearities. Hu, Kourogenis, and Papageorgiou further studied nonlinear elliptic eigenvalue problems with discontinuities [13]: under certain conditions, they demonstrated the existence of at least two nontrivial solutions; meanwhile, they proved the existence of at least one nontrivial solution based on a nonsmooth version of the generalized mountain pass theorem. Clarke's pioneering work on non-smooth analysis [14] laid the groundwork for dealing with non-differentiable functionals. His theory of generalized gradients and subdifferentials facilitated the development of critical point theorems for locally Lipschitz functionals, which became essential in later studies. Motreanu and Varga obtained critical point results for locally Lipschitz functionals [15]. Radulescu established mountain pass theorems for non-differentiable functions [16]. Carl, Le, and Motreanu integrated these ideas in their monograph [17], where they presented nonsmooth variational problems, comparison principles, and their applications. De Souza M, De Medeiros E, and Severo U considered quasilinear elliptic problems with Trudinger-Moser nonlinearities and extended this research to nonhomogeneous elliptic problems with exponential critical growth, which can refer to De Souza et al. [18, 19]. Alves and Santos [20] studied multivalued elliptic equation with exponential critical growth in $\mathbb{R}^2$. For Schrödinger equation with exponential growth and singular term, the A-R condition was weakened [21]. These studies connect critical Sobolev growth and exponential nonlinearities, which is an important link in phase transition models and nonlinear analysis.

In this paper, we consider the following problem

$$-div\left(|\nabla u|^{N-2}\nabla u\right) + V(x)|u|^{N-2}u - \epsilon h(x) \in \partial_t F(x,u) \ \ in \ \ \mathbb{R}^N \qquad (P)$$

where $N \geq 2, V$ is a continuous function verifying some conditions, $\epsilon > 0$ is a positive parameter, $0 \not\equiv h \in \left(W^{1,N}\left(\mathbb{R}^N\right)\right)^*$ and $\partial_t F(x,t)$ is the generalized gradient of $F(x,t)$ with respect to t and $F(x,t) = \int_0^t f(x,s)ds, f(x,t)$ is a discontinuous function with exponential critical growth, more precisely,

$$\partial_t F(x,u) = [\underline{f}(x,u(x)), \bar{f}(x,u(x))] \ a.e. \ in \ \mathbb{R}^N,$$

where $\underline{f}(x,t) = \lim\limits_{r\downarrow 0} essinf\{f(x,s) : |s-t| < r\}, \bar{f}(x,t) = \lim\limits_{r\downarrow 0} esssup\{f(x,s) : |s-t| < r\}$.

When $N = 2$, Alves and Santos [20] employed variational techniques to study the existence and multiplicity of nonnegative solutions.

We assume the following conditions on V :

$(V_1)\, V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying $V(x) \geq V_0 > 0, \forall x \in \mathbb{R}^N$,

$(V_2)\, \frac{1}{V} \in L^{\frac{1}{N-1}}\left(\mathbb{R}^N\right)$.

Motivated by a suitable Trudinger-Moser inequality [22–25], more precisely, we assume that there exist $\alpha_0 > 0, c_1, c_2 > 0$ such that for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

$$(f_1)\ \max\left\{|\xi| : \xi \in \partial_t F(x,t)\right\} \leq c_1|t|^{N-1} + c_2\left[e^{\alpha_0|t|^{\frac{N}{N-1}}} - S_{N-2}\left(\alpha_0, t\right)\right],$$

where

$$S_{N-2}\left(\alpha_0, t\right) = \sum_{k=0}^{N-2} \frac{\alpha^k|t|^{kN/(N-1)}}{k!}.$$

(f_2)

$$f(x,t) = 0\; for\; t < 0\; and\; \forall x \in \mathbb{R}^N$$

and

$$f(x,t) > 0\; for\; t > 0\; and\; \forall x \in \mathbb{R}^N.$$

The key characteristics of the class of problems studied in this paper are that they are defined over the entire space $\mathbb{R}^N$, involve exponential critical growth, and include a multivalued N-Laplacian. In this paper, we apply variational methods to nondifferentiable functionals: one solution is derived by employing Ekeland's variational principle, and the other is obtained by applying the Mountain Pass Theorem within the subspace $E \subset W^{1,N}\left(\mathbb{R}^N\right)$ given by

$$E = \left\{ u \in W^{1,N}\left(\mathbb{R}^N\right) \middle| \int_{\mathbb{R}^N} V(x)\left|u\right|^N dx < \infty \right\},$$

which be equipped with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} |\nabla u|^N + V(x)|u|^N dx\right)^{\frac{1}{N}}$$

then the assumption (V_1) implies E is a reflexive Banach space and (V_2) implies $E \hookrightarrow L^q$ is compact embedding for all $q \geq 1$ (see Lemma 2.4 [26]). We define a eigenvalue by

$$\lambda_1 = \inf_{u \in E\setminus\{0\}} \frac{\|u\|_E^N}{\int_{\mathbb{R}^N} |u|^N dx},$$

it is easy to see that $\lambda_1 > 0$.

$(f_3)\ \underset{t\to 0}{lim\,sup}\, \frac{N\max\{|\xi|:\xi\in\partial_t F(x,t)\}}{|t|^{N-1}} < \lambda_1(N)$ uniformly with respect to $x \in \mathbb{R}^N$.

(f_4) There exist a compact set $K \subset \mathbb{R}^N$, constants $c_3, c_4 > 0$ and $\nu > N$, such that

$$F(x,t) \geq c_3 t^\nu - c_4,\quad for\; t \geq 0\; and\; \forall x \in K.$$

(f_5) There is $\tau > N$ verifying

$$0 \leq \tau F(x,t) \leq \underline{f}(x,t)t,\quad for\; t > 0\;\; and\;\; \forall x \in \mathbb{R}^N,$$

(f_6) There are $p>N$ and $\mu>0$ such that

$$F(x,t)\geq \mu t^{p},\ \ for\ t\geq 0\ \ and\ \ \forall x\in\mathbb{R}^{N}.$$

We denote $L^{\Phi}\left(\mathbb{R}^{N}\right)=\left\{u\in L_{\text{loc}}^{1}\left(\mathbb{R}^{N}\right):\int_{\mathbb{R}^{N}}\left(e^{\alpha_0|t|^{\frac{N}{N-1}}}-S_{N-2}\left(\alpha_0,t\right)\right)<+\infty\right.$ for some $\left.\alpha_0>0\right\}$, $L^{\widetilde{\Phi}}\left(\mathbb{R}^{N}\right)$ is the conjugate function space associated with $L^{\Phi}\left(\mathbb{R}^{N}\right)$ [20].

Definition 1. We say that $u\in E$ is a solution of problem (P) if $\widetilde{\rho}\in L^{\widetilde{\Phi}}\left(\mathbb{R}^{N}\right)$ such that

(i) $\int_{\mathbb{R}^{N}}\left(|\nabla u|^{N-2}\nabla u\nabla v+V(x)|u|^{N-2}uv\right)dx-\int_{\mathbb{R}^{N}}\widetilde{\rho}v dx-\epsilon\int_{\mathbb{R}^{N}}hv dx=0,\ v\in E.$

(ii) $\widetilde{\rho}(x)\in\partial_t F(x,u(x))$ a.e. in $\mathbb{R}^{N}$.

(iii) $|[u>0]|>0$, where $|\cdot|$ is the Lebesgue's measure.

Define the energy functional $I_\epsilon:E\to\mathbb{R}$ by

$$I_\epsilon(u)=\frac{1}{N}\|u\|_E^N-\int_{\mathbb{R}^{N}}F(x,u)dx-\epsilon\int_{\mathbb{R}^{N}}hu dx,\ u\in E$$

Our main results can be stated as follows:

Theorem 1. Suppose $\left(V_1\right),\left(V_2\right),\left(f_1\right)-\left(f_6\right)$ are satisfied, then there exists $\epsilon_0,\mu^*>0$ such that for each $0<\epsilon<\epsilon_0,\mu\geq\mu^*$, problem (P) has a nontrivial weak solution $u_\epsilon\in E$ and $I_\epsilon\left(u_\epsilon\right)=c_\epsilon>0$.

Theorem 2. Suppose $\left(V_1\right),\left(V_2\right),\left(f_1\right)-\left(f_3\right)$ are satisfied, then there exists $\epsilon_0>0$ such that for all $\epsilon\in\left(0,\epsilon_0\right)$, problem (P) has a nontrivial weak solution $v_\epsilon\in E$ and $I_\epsilon\left(v_\epsilon\right)=d_\epsilon<0$.

This paper is structured as follows: Section 2 introduces some results concerning exponential critical growth. Section 3 considers the functionals associated with problem (P). Lastly, Theorem 1 is proven in Section 4, and Theorem 2 is proven in Section 5.

2. Some results about the critical exponential growth

In this section, we will give some preliminaries, the first result is crucial in the study of the Palais-Smale condition for I_ϵ.

Lemma 1. Let $\alpha>0$ and $\left\{u_n\right\}$ be a sequence satisfying

$$\underset{n\to\infty}{lim\,sup}\ \left\|u_n\right\|_E<\left(\frac{\alpha_N}{\alpha}\right)^{\frac{N-1}{N}}$$

where $\alpha_N=N\omega_{N-1}^{1/(N-1)},\omega_{N-1}$ is the measure of the unit sphere in $\mathbb{R}^{N}$, then, there exist constants $t>1,C>0$, independent of n, such that

$$\int_{\mathbb{R}^{N}}\left(e^{\alpha|u_n|^{\frac{N}{N-1}}}-S_{N-2}\left(\alpha,u_n\right)\right)^{t}dx\leq C$$

Proof. Since

$$\mathop{lim\,sup}_{n\to\infty} \|u_n\|_E < \left(\frac{\alpha_N}{\alpha}\right)^{\frac{N-1}{N}}$$

hence, passing to a subsequence, there exists $n_0 \in \mathbb{N}$ and $m \in \mathbb{N}$, we have

$$\|u_n\|_E^{\frac{N}{N-1}} < m < \frac{\alpha_N}{\alpha},\ \forall n \geq n_0$$

Fix $t > 1$, choose $k > t > 1$ and $k\alpha m < \alpha_N$, combine with $\frac{m}{\|u_n\|_E^{N-1}} > 1$, using Lemma 2.1 [26], for each $n \geq n_0$, it holds

$$\begin{aligned}\int_{\mathbb{R}^N} \left(e^{\alpha|u_n|^{\frac{N}{N-1}}} - S_{N-2}(\alpha, u_n)\right)^t dx &\leq \int_{\mathbb{R}^N} \left(e^{t\alpha|u_n|^{\frac{N}{N-1}}} - S_{N-2}(t\alpha, u_n)\right) dx \\ &\leq \int_{\mathbb{R}^N} \left(e^{k\alpha|u_n|^{N-1}} - S_{N-2}(k\alpha, u_n)\right) dx \\ &\leq C \int_{\mathbb{R}^N} \left(e^{k\alpha m\left(\frac{|u_n|}{\|u_n\|_E}\right)^{N-1}} - S_{N-2}\left(k\alpha m, \frac{|u_n|}{\|u_n\|_E}\right)\right) dx\end{aligned}$$

so the result is proved by the Trudinger-Moser inequality (see Lemma 2.2 [27]).

For the next result, we will use the Radial Lemma [28], i.e.

$$|u(x)| \leq |x|^{-1} \left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{N}} \|u\|_{L^N},\ \forall x \neq 0$$

for all $u \in W^{1,N}\left(\mathbb{R}^N\right)$ radially symmetric.

Lemma 2. Let $\beta > 0$ and $\|u\|_E \leq M$ such that $\beta M^{\frac{N}{N-1}} < \left(1 - \frac{\eta}{N}\right)\alpha_N$ and $q > N$, then

$$\int_{\mathbb{R}^N} \left(e^{\beta|u|^{\frac{N}{N-1}}} - S_{N-2}(\beta, u)\right) |u|^q dx \leq C(\beta, N)\|u\|_E^q$$

Proof. Applying the Höder inequality and and continuous embedding result, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \left(e^{\beta|u|^{\frac{N}{N-1}}} - S_{N-2}(\beta, u)\right) |u|^q\, dx &\leq \left(\int_{\mathbb{R}^N} \left(e^{p\beta|u|^{\frac{N}{N-1}}} - S_{N-2}(\beta, u)\right) dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |u|^{qp'}\, dx\right)^{\frac{1}{p'}} \\ &\leq \left(\int_{\mathbb{R}^N} \left(e^{p\beta(M\widetilde{u})^{\frac{N}{N-1}}} - S_{N-2}(\beta, u)\right) dx\right)^{\frac{1}{p}} \|u\|_E^q \\ &\leq C(\beta, N)\|u\|_E^q\end{aligned}$$

where $\widetilde{u} = \frac{u}{\|u\|_E}$ and $p > 1$ is sufficiently close 1 such that $\beta p M^{\frac{N}{N-1}} \leq \alpha_N$, $\frac{1}{p} + \frac{1}{p'} = 1$. The last inequality is a direct consequence of Trudinger-Moser inequality since $\|\widetilde{u}\|_{1,\tau} \leq \|\widetilde{u}\|_E = 1$ for any positive $\tau \leq V_0$.

Next, we give the following results:

Lemma 3. Suppose $(f_1) - (f_2)$ are satisfied, then the functional $\Psi : E \to \mathbb{R}$ given by

$$\Psi(u) = \int_{\mathbb{R}^N} F(x, u) dx$$

is well-defined and $\Psi \in Lip_{\text{loc}}(E, \mathbb{R})$.

Proof. For $\theta = tu + (1-t)v, 0 \leq t \leq 1$, from (f_1) and the Hölder inequality, we have

$$\begin{aligned}|\Psi(u) - \Psi(v)| &\leq \int_{\mathbb{R}^N} \left[c_1|\theta|^{N-1} + c_2\left(e^{\alpha_0|\theta|^{\frac{N}{N-1}}} - S_{N-2}\left(\alpha_0, \theta\right)\right)\right]|u-v|\,dx \\ &\leq \left[c_1\|\theta\|_{L^N}^{N-1} + c_2\int_{\mathbb{R}^N}\left(e^{\alpha_0|\theta|^{N-1}} - S_{N-2}\left(\alpha_0\theta\right)\right)^{\frac{N}{N-1}}\,dx^{\frac{N-1}{N}}\right]\|u-v\|_{L^N}\end{aligned}$$

Using the embedding $E \hookrightarrow L^N\left(\mathbb{R}^N\right)$, we infer that there is a neighborhood $\mathcal{N}$ such that

$$|\Psi(u) - \Psi(v)| \leq C\|u-v\|_E,\ u, v \in \mathcal{N}$$

This completes the proof.

3. Functional

$\mathbf{I}_\epsilon$ and compactness analysis

We will apply a mountain-pass theorem that does not rely on a compactness condition. This specific version of the mountain-pass theorem is derived from Ekeland's variational principle. In the following two lemmas, we verify that the functional I_ϵ meets the geometric conditions of the Mountain Pass Theorem.

Lemma 4. Assume that (V_1), (f_1) and (f_3) hold. Then there exist $\epsilon_0, r, \alpha > 0$ such that for $0 < \epsilon < \epsilon_0$, there exist $r > 0$ such that

$$I_\epsilon(u) \geq \alpha \text{ for } \|u\|_E = r$$

here r is independent of ϵ, but α depends on ϵ.

Proof. From (f_3), there exist $\eta, \delta > 0$, such that if $\|u\|_E \leq \delta$,

$$F(x,u) \leq \frac{\lambda_1 - \eta}{N}|u|^N \tag{1}$$

for all $x \in \mathbb{R}^N$. On the other hand, using (f_1) for each $q > N$, we have

$$\begin{aligned}F(x,u) &\leq \frac{c_1}{N}|u|^N + c_2|u|\left[e^{\alpha|u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha,u)\right] \\ &\leq C|u|^q\left[e^{\alpha|u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha,u)\right]\end{aligned} \tag{2}$$

for $\|u\|_E \geq \delta$ and $x \in \mathbb{R}^N$. Combining (1) and (2), we obtain

$$F(x,u) \leq \frac{\lambda_1 - \eta}{N}|u|^N + c_2|u|^q\left[e^{\alpha|u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha,u)\right],$$

for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, fixed $r > 0$ small enough such that $\alpha r^{\frac{N}{N-1}} < \alpha_N$, then Lemma 2.2 and the continuous embedding $E \hookrightarrow L^N\left(\mathbb{R}^N\right)$ implies

$$\begin{aligned}I_\epsilon(u) \quad &\geq \tfrac{1}{N}\|u\|_E^N - \tfrac{\lambda_1-\eta}{N}\|u\|_N^N - C\|u\|_E^q - \epsilon\|h\|_*\|u\|_E \\ &\geq \tfrac{1}{N}\left(1 - \tfrac{\lambda_1-\eta}{\lambda_1}\right)\|u\|_E^N - C\|u\|_E^q - \epsilon\|h\|_*\|u\|_E\end{aligned}$$

for $\|u\|_E \leq r$. Therefore, by fixing $\epsilon_0 > 0$ and choose suitable r, let

$$\alpha_\epsilon = \frac{1}{2N} r^N - \epsilon \|h\| r > 0, \ \forall \epsilon \in (0, \epsilon_0),$$

we can get

$$I_\epsilon(u) \geq \alpha_\epsilon \text{ for } \|u\|_E = r, \ \forall \epsilon \in (0, \epsilon_0)$$

Lemma 5. Assume that (f_1) – (f_6) hold. Then there exists $e \in B_r^c(0)$ such that

$$I_\epsilon(e) < \inf_{\|u\|_E = r} I_\epsilon(u), \epsilon \in (0, \epsilon_0]$$

where r and ϵ_0 are given in Lemma 3.1.

Proof. Let $u \in C_0^\infty\left(\mathbb{R}^N\right) \setminus \{0\}, u > 0$ with compact support $supp(u) \subset K$, where $K \subset \mathbb{R}^N$ is the compact set in (f_4). Since $\nu > N$, for $t > 0$ we have

$$I_\epsilon(tu) \leq \frac{t^N}{N} \|u\|_E^N - c_3 t^\nu \int_{\mathbb{R}^N} |u|^\nu dx + c_4 |K| - t\epsilon \int_{\mathbb{R}^N} hu dx$$

which implies that $I_\epsilon(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t sufficiently large, the proof of the lemma follows.

From Lemma 3.1 and 3.2, by the mountain-pass theorem of local Lipschitz functional without the (PS) condition(see Theorem 2.1 [4]), we get a sequence u_n which satisfies

$$I_\epsilon(u_n) \to c_\epsilon \ \ in \ \mathbb{R}^N \ \ and \ \ \lambda_\epsilon(u_n) =: max\{\|\xi\|_* : \xi \in \partial I_\epsilon(u_n)\} \to 0 \qquad (3)$$

where

$$c_\epsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\epsilon(\gamma(t))$$

and

$$\Gamma =: \{\gamma \in C([0,1] : E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 6. There exist $r, \delta > 0$ and $v \in E$ with $\|v\|_E = 1$ such that $I_\epsilon(tv) < -\delta$ for all $0 < t < r$. In particular,

$$d_\epsilon =: \inf_{\|u\|_E \leq r} I_\epsilon(u) < -\delta$$

Proof. Let

$$\|v\|_E = 1 \ and \int_{\mathbb{R}^N} hv dx > 0$$

then for each $t > 0$,

$$\begin{aligned} I_\epsilon(tv) &= \tfrac{t^N}{N} - \textstyle\int_{\mathbb{R}^N} F(x, tv) dx - \epsilon t \int_{\mathbb{R}^N} hv dx \\ &< \tfrac{t^N}{N} - \epsilon t \textstyle\int_{\mathbb{R}^N} hv dx := -\delta < 0 \end{aligned}$$

by choosing $t > 0$ small enough.

4. Nontrivial positive solution

In this section, we will prove Theorem 1.1. It suffices to look for nontrivial critical points of the functional I_ϵ in the function space E.

Proof of Theorem. Based on Lemmas 3.1 and 3.2, the functional I_ϵ meets all the assumptions of the mountain-pass theorem, with the possible exception of the Palais-Smale condition. Therefore, we shall utilize the mountain-pass theorem for local Lipschitz functionals that does not require the (PS) condition.(see Theorem 2.1 [4]), there exists a sequence $\{u_n\}$ which satisfies

$$I_\epsilon\left(u_n\right) \rightarrow c_\epsilon \ \ in \ \mathbb{R}^N \ \ and \ \ \lambda_\epsilon\left(u_n\right) =: \left\{\|\xi\|_* / \xi \in \partial I_\epsilon\left(u_n\right)\right\} \rightarrow 0,$$

where c_ϵ is the mountain-pass level of I_ϵ. i.e.

$$\frac{1}{N}\left\|u_n\right\|_E^N-\int_{\mathbb{R}^N} F\left(x, u_n\right) d x-\epsilon \int_{\mathbb{R}^N} h u_n d x \rightarrow c_\epsilon \ \ as \ \ n \rightarrow \infty \tag{4}$$

$$\left|\left\langle\lambda_n, v\right\rangle\right| \leq \tau_n\|v\| \ \ for\ any \ \ v \in E \tag{5}$$

where $\tau_n \rightarrow 0$ as $n \rightarrow \infty$. From Equations (4) and (5) and (f_5), we get

$$\begin{aligned} C+\tau_n\left\|u_n\right\| & \geq\left(\frac{\tau}{N}-1\right)\left\|u_n\right\|_E^N-\int_{\mathbb{R}^N}\left(\tau F\left(x, u_n\right)-\widetilde{\rho}_n(x) u_n\right) d x \\ & \geq\left(\frac{\tau}{N}-1\right)\left\|u_n\right\|_E^N \end{aligned}$$

where $\widetilde{\rho}_n(x) \in \partial_t F\left(x, u_n(x)\right)$ a.e. in $\mathbb{R}^N$, since $\tau > N$, so $\{u_n\}$ is bounded in E, up to subsequences, we have $u_n \rightharpoonup u_\epsilon$ weakly in $E, u_n \rightarrow u_\epsilon$ in $L^q\left(\mathbb{R}^N\right)$ for all $q \geq 1$ and $u_n(x) \rightarrow u(x)$ almost everywhere in $\mathbb{R}^N$. Combining with the property of $\widetilde{\rho}_n(x)$, we have

$$\int_{\mathbb{R}^N} \widetilde{\rho}_n(x) \varphi d x \rightarrow \int_{\mathbb{R}^N} \widetilde{\rho}_0(x) \varphi d x, \ \forall \varphi \in C_0^{\infty}\left(\mathbb{R}^N\right)$$

Therefore by Equation (5) passing to the limit, we have

$$\int_{\mathbb{R}^N}\left(\left|\nabla u_\epsilon\right|^{N-2} \nabla u_\epsilon \nabla \varphi+V(x)\left|u_\epsilon\right|^{N-2} u_\epsilon \varphi\right) d x-\int_{\mathbb{R}^N} \widetilde{\rho}_0 \varphi d x-\epsilon \int_{\mathbb{R}^N} h \varphi d x=0$$

for $\varphi \in C_0^{\infty}\left(\mathbb{R}^N\right)$, since $C_0^{\infty}\left(\mathbb{R}^N\right)$ is dense in E, then u_ϵ is a weak solution of (P). Moreover, $u_\epsilon \neq 0$ because $h \not\equiv 0$.

5. Multiplicity results

In this section, we will prove Theorem 1.2. The result follows by a minimization argument and Ekeland's variational principle.

Proof of Theorem. Fix $r > 0$ such that $\alpha_0 r^{\frac{N}{N-1}} < \alpha_N$. Lemma 1 together with (f_1) and (f_4) implies that $I_\epsilon(u) \geq -C$ for all $u \in \bar{B}_r = \{u \in E : \|u\|_E \leq r\}$. Combining Lemma 3.3 with Ekeland's variational principle, for $\forall n \geq 1$, there is $\{v_n\} \subset \bar{B}_r(0)$ which satisfies

$$\begin{cases} I_\epsilon(v_n) < d_\epsilon + \frac{1}{n} \\ I_\epsilon(v_n) < I_\epsilon(v) + \dfrac{1}{n}\left\|v_n - v\right\|_E, \quad \forall v \neq v_n \end{cases} \tag{6}$$

as $n \to \infty$, we have

$$\begin{cases} I_\epsilon\left(v_n\right) \to d_\epsilon \\ \left\|\lambda_n\right\|_{E^*} := min\left\{\|\xi\|_{E^*} \mid \xi \in \partial I_\epsilon\left(v_n\right)\right\} \to 0 \end{cases} \tag{7}$$

for $\lambda_n \in \partial I_\epsilon(v_n)$ and $\{\tilde{\rho}_n\} \subset L^{\tilde{\Phi}}(\mathbb{R}^N)$, $\langle \lambda_n, v\rangle = \int_{\mathbb{R}^N}(|\nabla v_n|^{N-2}\nabla v_n \nabla v + V(x)|v_n|^{N-2}v_n v)dx - \int_{\mathbb{R}^N} \tilde{\rho}_n v dx - \epsilon \int_{\mathbb{R}^N} h v dx \to 0$, where $v \in E$ and $\tilde{\rho}_n(x) \in \partial_t F(x, v_n(x))$ a.e. in $\mathbb{R}^N$.

Since $\{v_n\}$ is bounded in E, there is $v_\epsilon \in E$ such that $v_n \rightharpoonup v_\epsilon$ in E, so

$$\int_{\mathbb{R}^N}\left(\left|\nabla v_\epsilon\right|^{N-2} \nabla v_\epsilon \nabla v + V(x)\left|v_\epsilon\right|^{N-2} v_\epsilon v\right) dx - \int_{\mathbb{R}^N} \tilde{\rho}_0 v dx - \epsilon \int_{\mathbb{R}^N} h v dx = 0$$

where $v \in E$ and $\tilde{\rho}_0(x) \in L^{\tilde{\Phi}}\left(\mathbb{R}^N\right)$. Furthermore, we have $|[v_\epsilon > 0]| > 0$ because of $v_\epsilon \geq 0$ and $v_\epsilon \neq 0$.

In order to complete the proof that v_ϵ is a solution of (P), we must prove that $\widetilde{\rho}_0(x) \in \partial_t F\left(x, v_\epsilon(x)\right)$ a.e. in $\mathbb{R}^N$. In fact, let $w_n = v_n - v_\epsilon$, then $w_n \rightharpoonup 0$ in E and

$$\|v_n\|_E^N = \|v_\epsilon\|_E^N + \|w_n\|_E^N + o_n$$

On one hand,

$$\begin{aligned} o_n(1) = \langle w_n, v_n\rangle &= \|v_n\|_E^N - \int_{\mathbb{R}^N} \tilde{\rho}_n v_n \, dx - \epsilon \int_{\mathbb{R}^N} h v_n \, dx \\ &\quad - \|v_\epsilon\|_E^N + \int_{\mathbb{R}^N} \tilde{\rho}_0 v_\epsilon \, dx + \epsilon \int_{\mathbb{R}^N} h v_\epsilon \, dx \\ &= \|w_n\|_E^N + \left(\int_{\mathbb{R}^N} \tilde{\rho}_0 v_\epsilon \, dx - \int_{\mathbb{R}^N} \tilde{\rho}_n v_\epsilon \, dx\right) \\ &\quad + \left(\int_{\mathbb{R}^N} \tilde{\rho}_n u_\epsilon \, dx - \int_{\mathbb{R}^N} \tilde{\rho}_n u_n \, dx\right) + o_n(1) \\ &= \|w_n\|_E^N - \int_{\mathbb{R}^N} \tilde{\rho}_n (v_n - v_\epsilon) \, dx + o_n(1) \\ &= \|w_n\|_E^N - \int_{\mathbb{R}^N} \tilde{\rho}_n w_n \, dx + o_n(1) \end{aligned} \tag{8}$$

On the other hand, by (f_1),

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \tilde{\rho_n} w_n \, dx\right| &\leq c_1 \int_{\mathbb{R}^N} |v_n|^{N-1} |w_n| \, dx + c_2 \int_{\mathbb{R}^N} R(\alpha_0, v_n) |w_n| \, dx \\ &\leq c_1 \|v_n\|_{L^N}^{N-1} \|w_n\|_{L^N} + \frac{c_2}{2}\left(\int_{\mathbb{R}^N} R(\alpha_0, 2w_n) |w_n| \, dx + \int_{\mathbb{R}^N} R(\alpha_0, 2v_\epsilon) |w_n| \, dx\right)\end{aligned}$$

combining $\alpha_0 r^{\frac{N}{N-1}} < \alpha_N$ with Lemma 2.2, we have

$$\left|\int_{\mathbb{R}^N} \tilde{\rho_n} w_n dx\right| \leq c_1 \|v_n\|_{L^N}^{N-1} \|w_n\|_{L^N} + c_2 C \|w_n\|_{L^q} + c_2 \|R(\alpha_0, 2v_\epsilon)\|_{L^{p'}} \|w_n\|_{L^p}$$

where $p, q > N, \frac{1}{p} + \frac{1}{p'} = 1.$ Since $w_n \rightharpoonup 0$ in E, using the compact embedding, we have

$$\int_{\mathbb{R}^N} \tilde{\rho_n} w_n dx \to 0 \tag{9}$$

From Equations (8) and (9), $w_n \to 0$ in E, i.e. $v_n \to v_\epsilon$ in E, this implies $\rho_0 \in \partial \Psi(v_\epsilon)$, i.e.,

$$\tilde{\rho}_0(x) \in \partial_t F(x, u_\epsilon(x)) \text{ a.e.in } \mathbb{R}^N$$

so the proof is complete.

6. Discussion

We have established the existence and multiplicity of weak solutions for the N-Laplacian equation with discontinuous exponential growth in $\mathbb{R}^N$. Specifically, when the perturbation parameter is sufficiently small, there exist at least multiple weak solutions. The solutions arise from the interaction between the discontinuous exponential nonlinearity and the N-Laplacian operator. In comparison with previous results, our results extend the existing literature on elliptic equations with critical growth and discontinuous nonlinearities. The use of the priori estimate in combination with non-differentiable variational methods is also a novel approach compared to the traditional methods used in earlier works. The limitation of this study is that we only consider the case when the perturbation parameter is small, the behavior of solutions when the parameter is large remains unexplored.

7. Conclusion

This study examines the existence and multiplicity of solutions to$\mathbb{R}^N$ based N-Laplacian equations with discontinuous exponential nonlinearities. Via symmetric rearrangement techniques, priori estimates, and non-differentiable variational methods, we determine sufficient conditions for weak solutions under small perturbation parameters, with multiple solutions driven by the interaction between discontinuous nonlinearities and the N-Laplacian operator.

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