Advances in Differential Equations and Control Processes 2025, 32(2), 2712.
https://doi.org/10.59400/adecp2712

Article

Stabilizing Schrödinger—ODE systems with boundary delay for industrial process optimization

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CITATION

Xie YR, Chen YW, Tian CY, et al. Stabilizing Schrödinger—ODE systems with boundary delay for industrial process optimization. Advances in Differential Equations and Control Processes. 2025; 32(2): 2712. https://doi.org/10.59400/adecp2712

ARTICLE INFO

Received: 13 February 2025
Accepted: 10 April 2025
Available online: 24 April 2025

Abstract: In this paper, we focus on the stabilization of a Schrödinger-ODE cascaded system with boundary delayed control. The system is stabilized through the utilization of integral-type feedback control, in which the integral kernel functions serve as parameters. The objective is to identify an appropriate set of kernel functions that ensure exponential stability characteristics of the closed-loop system. The initial step is to select a target system that must be exponentially stable. We propose an auxiliary system for the task at hand. Initially, we need to establish the equivalence between the auxiliary system and the original controlled time-delay system. This stage is primarily concerned with the elimination of the influence of input memory. The second system is leveraged to ascertain the equivalence between stable target system and the auxiliary system. This paper presents a method to choose parameter functions to create an exponentially stable feedback controller.

Keywords: Schrödinger-ODE cascaded systems; exponential stability; integral-type feedback control; kernel function

MSC Classification: 35R15; 31A10; 37L15

1. Introduction

There are various categories of differential equations, among which they can be divided into first-order differential equations, second-order differential equations, high-order differential equations, and the recently popular fractional order differential equations according to the order of derivatives. They can also be divided into ordinary differential equations (ODE) and partial differential equations (PDE) according to the number of independent variables. Cascades often occur in systems described by ordinary differential equations (ODE) with infinite-dimensional actuator dynamics described by partial differential equations (PDE). These cascades have been observed in a variety of contexts, including electromagnetic coupling [1,2], mechanical coupling [3,4], as well as coupled chemical reactions [5,6].

Results on the controllability of coupled ODE-PDE and PDE-PDE systems without delay are presented in [7,8] and as referred to therein. Feedback controllers for PDE-ODE cascaded systems, including the ODE-Reaction diffusion equation, have been designed in [9], ODE-Wave equation in [10] and ODE-Schrödinger equation in [11]. The Schrödinger-ODE cascaded system is a system that has been extensively applied in practice. Its dynamics are controlled by the respective PDE- ODE equations with Neumann boundary control:

$$\begin{cases} \dot{Y}(t) = FY(t) + Gq_x(0,t), \;\; t > 0 \\ q_t(y,t) = -iq_{yy}(y,t), \quad x \in (0,1), \;\; t > 0 \\ q(0,t) = 0, \;\; q_x(1,t) = \varrho(t), \;\; t > 0 \\ q(x,0) = q_0(x), \quad x \in [0,1] \\ X(0) = X_0 \end{cases} \tag{1}$$

where $F \in \mathbb{C}^{n\times n}, G \in \mathbb{C}^{n\times 1}$; $Y(t) \in \mathbb{C}^{n\times 1}$ is the states of ODE; $q(y,t) \in \mathbb{C}$ is the state of Schrödinger equation, $\varrho(t)$ is the boundary control. Y_0 and $q_0(y)$ are the initial values of ODE and PDE, respectively. At the boundary point $x = 0$, the PDE and ODE systems are connected.

Let's refer back to the study on Schrödinger-ODE cascaded systems. The stabilization of the ODE-Schrödinger cascade is using the backstepping approach as outlined in [8].The backstepping method and sliding mode control, as described in [11], provide a means of stabilizing the boundary for a cascade of Schrödinger equation-ODE systems that are subject to unmatched disturbances. This study aims to stabilize a cascaded ODE-Schrödinger system under time-delayed boundary observation. This is achieved by utilizing the observer and predictor systems as outlined in [12]. The stabilization of an ODE-Schrödinger cascade system with an external disturbance is considered in [13]. Using the backstepping approach, the authors developed a new ESO that estimates both the state and the disturbance, as well as a stabilizing control law.

Nevertheless in the context of industrial procedures and other forms of practice implementation, a certain degree of temporal delay is to be expected. It is well established that even the introduction of an arbitrarily small delay in the feedback loop can result in the destabilization of a controlled system (see [14, 15]). Conversely, the incorporation of an appropriate time-delay term in the controller of certain systems has been demonstrated to enhance performance (see [16, 17]). As highlighted in [18], however, it is not the case that all feedback control laws are robust to small time delays. Accordingly, an important area of investigation in contemporary control theory is the design of a stabiliser for a time-delay system.

When discussing how to control distributed parameterized systems, it is essential that the impact of time delays is taken into account. This issue has attracted considerable attention from researchers in the fields of engineering and mathematics in recent years [19–22]. There are numerous methodologies that may be employed. For example, the control method research of the classical Smith predictor in adaptive time-delay compensation [23–26], the spectral analysis approach [27], the control Lyapunov function approach [28], $\frac{1}{2}$-stability criterion [29], Partial state predictor [30], and integral-type feedback control [31]. Integral-type feedback controllers guarantee the stability of closed-loop systems (see [32–34]). In particular, Wang in [34] applies the integral-type controller to stabilize the high-dimensional wave equation, thereby overcoming the limitations of dimensionality. Motivated by works [34–40], the present paper discusses a Schrödinger-ODE cascaded system with boundary-delay the control:

$$\begin{cases} \dot{Y}(t) = FY(t) + Gq_y(0,t), \ \ t > 0 \\ q_t(y,t) = -iq_{yy}(y,t), \quad y \in (0,1), \ \ t > 0 \\ q(0,t) = 0, \ \ q_y(1,t) = \varrho(t-\tau), \ \ t > 0 \\ q(y,0) = q_0(y), \quad y \in [0,1], \\ Y(0) = Y_0, \ \ \varrho(s-\tau) = \rho_0(s), \quad s \in [0,\tau] \end{cases} \tag{2}$$

where τ is the delay time, $\rho_0(s)$ represents the historical behavior of controller.

The remainder of this manuscript is organized in the following way. Our target system of choice is an exponentially stable system in Section 2. In Section 3, we proved that linear transformations and their inverse transformations are bounded linear operators by constructing reversible linear transformations between systems. Subsequently, we proved the equivalence between a closed-loop system governed by a control function and an exponentially stable target system. In Section 4, we prove that the closed-loop system remains exponential stable. Ultimately, the conclusion of the paper is reached in Section 5.

The paper considers an energy Hilbert space $\mathcal{H} = \mathbb{C}^n \times L^2(0,1)$, with inner product

$$\langle h_1, h_2 \rangle = Y_1^T \overline{Y_2} + \int_0^1 k_1(y)\overline{k_2(y)}\mathrm{d}y, \ \ \forall h_i = (Y_i, k_i) \in \mathcal{H}, \ i = 1, 2,$$

and $\mathcal{H}$-norm

$$\|h_i\|_{\mathcal{H}} = (|Y_i|^2_{\mathbb{C}^n} + \|k_i\|^2_{L^2(0,1)})^{\frac{1}{2}}.$$

2. Target system selection

In this section, we are going to choose an exponentially stable system as our target system. In order to facilitate comprehension, we present the equation in an equivalent form.

Based on [41], set

$$\rho(s,t) = \varrho(t+s-\tau), \quad s \in [0,\tau].$$

Subsequently, system (2) is demonstrated to be equivalent to the following partial differential equations:

$$\begin{cases} \rho_t(s,t) = \rho_s(s,t), \quad s \in (0,\tau), \ \ t > 0 \\ \rho(\tau,t) = \varrho(t), \ \ t > 0 \\ \dot{Y}(t) = FY(t) + Gq_y(0,t), \ \ t > 0 \\ q_t(y,t) = -iq_{yy}(y,t), \quad y \in (0,1), \ \ t > 0 \\ q(0,t) = 0, \ \ q_y(1,t) = \rho(0,t), \ \ t > 0 \\ q(y,0) = q_0(y), \quad y \in [0,1] \\ Y(0) = Y_0, \ \ \rho(s,0) = \rho_0(s), \quad s \in [0,\tau] \end{cases} \tag{3}$$

The system (3) is an ODE-PDE cascaded system. The control variable $\varrho(t)$ should be expressed as follows in accordance with [32,33]:

$$\varrho(t) = -\int_0^{\tau} i\hbar(\tau - r, 1)\rho(r,t)dr + \int_0^1 \hbar(\tau, z)q(z,t)dz + \langle \Re(\tau), Y(t)\rangle \tag{4}$$

where $\hbar(s,z)$ and $\Re(s)$ are the parametrization functions. Exponential stability of the closed-loop system is ensured by the controller design.

It is demonstrated that in the context of the control function (4), the closed-loop system corresponding to Equation (3) is defined by the following equation:

$$\begin{cases} \rho_t(s,t) = \rho_s(s,t), \quad s \in (0,\tau), \;\; t > 0 \\ \rho(\tau,t) = -\int_0^{\tau} i\hbar(\tau - r, 1)\rho(r,t)dr + \int_0^1 \hbar(\tau,z)q(z,t)dz + \langle \Re(\tau), Y(t)\rangle \\ \dot{Y}(t) = FY(t) + Gq_y(0,t), \;\; t > 0 \\ q_t(y,t) = -iq_{yy}(y,t), \quad y \in (0,1), \;\; t > 0 \\ q(0,t) = 0, \;\; q_y(1,t) = \rho(0,t), \;\; t > 0 \\ q(y,0) = q_0(y), \quad y \in [0,1] \\ Y(0) = Y_0, \;\; \rho(s,0) = \rho_0(s), \quad s \in [0,\tau] \end{cases} \tag{5}$$

The first step is the elimination of the time delay. The objective is the selection of appropriate equations for $\hbar(s,z)$ and $\Re(s)$ so that Equation (5) is equivalent to:

$$\begin{cases} \phi_t(s,t) = \phi_s(s,t), \quad s \in (0,\tau), \;\; t > 0 \\ \phi(\tau,t) = 0, \;\; t > 0 \\ \dot{Y}(t) = FY(t) + Gq_y(0,t), \;\; t > 0 \\ q_t(y,t) = -iq_{yy}(y,t), \quad y \in (0,1), \;\; t > 0 \\ q(0,t) = 0, \;\; t > 0 \\ q_y(1,t) = \phi(0,t) + \int_0^1 \hbar_0(z)q(z,t)dz + \langle \Re_0, Y(t)\rangle, \;\; t > 0 \\ q(y,0) = q_0(y), \quad y \in [0,1] \\ Y(0) = Y_0, \;\; \phi(s,0) = \phi_0(s), \quad s \in [0,\tau] \end{cases} \tag{6}$$

where $\hbar_0(z) = \hbar(0,z)$ and $\Re_0 = \Re(0)$.

The second step is to choose the right set of initial conditions $(\hbar_0, \Re_0)$ such that Equation (6) is exponentially stable. To achieve this goal, we use the idea from [42] to ensure the equivalence between Equation (6) and the below target system:

$$\begin{cases} \phi_t(s,t) = \phi_s(s,t), \quad s \in (0,\tau), \;\; t > 0 \\ \phi(\tau,t) = 0, \;\; t > 0 \\ \dot{Y}(t) = (A + GK)Y(t) + G\varphi_y(0,t), \;\; t > 0 \\ \varphi_t(y,t) = -i\varphi_{yy}(y,t), \quad y \in (0,1), \;\; t > 0 \\ \varphi(0,t) = 0, \;\; \varphi_y(1,t) = \phi(0,t), \;\; t > 0 \\ \varphi(y,0) = \varphi_0(y), \quad y \in [0,1] \\ Y(0) = Y_0, \;\; \phi(s,0) = \phi_0(s), \quad s \in [0,\tau] \end{cases} \tag{7}$$

Suppose that $(\hbar_0, \Re_0)$ can make system (6) exponentially stable. We construct control predictor equations

$$\begin{cases} \overline{h}_s(s,z) = -i\overline{h}_{zz}(s,z), \quad s\in(0,\tau), \ \ z\in(0,1)\\ \overline{h}(s,0)=\langle i\Re(s),G\rangle\,, \ \ \overline{h}_z(s,1)=0, \quad s\in(0,\tau)\\ \dot{\Re}(s)=F^H\Re(s), \quad s\in(0,\tau)\\ \overline{h}(0,z)=\overline{h}_0(z), \quad z\in(0,1)\\ \Re(0)=\Re_0 \end{cases} \tag{8}$$

Then the control system (5) is exponentially stable under the control law (4).

3. Design for system (ρ, Y, q) to (ϕ, Y, φ)

3.1. The equivalence transformation between Equations (5) and (6)

This section establishes the equivalence between Equations (5) and (6).

In light of the boundary condition pertaining to ρ, as outlined in Equation (5), we will proceed to the construction of a transformation in the following way:

$$\begin{cases} \phi(s,t)=\rho(s,t)+\int_0^s i\overline{h}(s-r,1)\rho(r,t)dr-\int_0^1\overline{h}(s,z)q(z,t)dz-\langle\Re(s),Y(t)\rangle\\ Y(t)=Y(t)\\ q(y,t)=q(y,t) \end{cases} \tag{9}$$

We are going to choose appropriate parameter functions $\overline{h}(s,z), \Re(s)$ such that $(\phi(s,t), Y(t), q(y,t))$ satisfy Equation (6).

Theorem 1. *Consider the closed system defined by Equation* (5)*, and assume that the solution is given by the expression* (9)*. Then, the solution defined by Equation* (9) *is also a solution of the system defined by Equation* (6)*.*

Proof. Take $(\rho(s,t), Y(t), q(y,t))$ as a solution of Equation (5). In line with Equation (9), we have showing how the constraints conditions in Equation (5) were used,

$$\begin{aligned} \phi_t(s,t) &= \rho_t(s,t)+\int_0^s i\overline{h}(s-r,1)\rho_t(r,t)dr-\int_0^1\overline{h}(s,z)q_t(z,t)dz\\ &\quad -\left\langle\Re(s),\dot{Y}(t)\right\rangle\\ &= \rho_t(s,t)+\int_0^s i\overline{h}(r,1)\rho_t(s-r,t)dr+\int_0^1 i\overline{h}(s,z)q_{zz}(z,t)dz\\ &\quad -\langle\Re(s),FY(t)+Gq_y(0,t)\rangle\\ &= \rho_t(s,t)+\int_0^s i\overline{h}(r,1)\rho_t(s-r,t)dr+i\overline{h}(s,1)q_z(1,t)-i\overline{h}(s,0)q_z(0,t)\\ &\quad -i\overline{h}_z(s,1)q(1,t)+\int_0^1 i\overline{h}_{zz}(s,z)q(z,t)dz-\left\langle F^H\Re(s),Y(t)\right\rangle\\ &\quad -\langle\Re(s),G\rangle\,q_y(0,t)\\ &= \rho_s(s,t)+\int_0^s i\overline{h}(r,1)\rho_s(s-r,t)dr+i\overline{h}(s,1)\rho(0,t)-i\overline{h}_z(s,1)q(1,t)\\ &\quad +\int_0^1 i\overline{h}_{zz}(s,z)q(z,t)dz-\left\langle F^H\Re(s),Y(t)\right\rangle\\ &\quad -\left[i\overline{h}(s,0)+\langle\Re(s),G\rangle\right]q_y(0,t), \end{aligned}$$

and

$$\phi_s(s,t) = \rho_s(s,t)+i\overline{h}(s,1)\rho(0,t)+\int_0^s i\overline{h}(r,1)\rho_s(s-r,t)dr$$

$$-\int_0^1 \hbar_s(s,z)q(z,t)dz - \left\langle \dot{\Re}(s), Y(t) \right\rangle,$$

using the differential equations in Equation (8), from these, we obtain that $\phi_t(s,t) = \phi_s(s,t)$ for every $s \in (0,\tau)$ and $t > 0$.

Obviously,

$$\begin{aligned}\phi(\tau,t) &= \rho(\tau,t) + \int_0^\tau i\hbar(\tau - r,1)\rho(r,t)dr - \int_0^1 \hbar(\tau,z)q(z,t)dz - \langle \Re(\tau), Y(t)\rangle \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\phi(0,t) &= \rho(0,t) - \int_0^1 \hbar(0,z)q(z,t)dz - \langle \Re(0), Y(t)\rangle \\ &= q_y(1,t) - \int_0^1 \hbar_0(z)q(z,t)dz - \langle \Re_0, Y(t)\rangle .\end{aligned}$$

Hence,

$$q_y(1,t) = \phi(0,t) + \int_0^1 \hbar_0(z)q(z,t)dz + \langle \Re_0, Y(t)\rangle .$$

Especially, we have

$$\begin{aligned}\phi(s,0) &= \rho(s,0) + \int_0^s i\hbar(s-r,1)\rho(r,0)dr - \int_0^1 \hbar(s,z)q(z,0)dz - \langle \Re(s), Y(0)\rangle \\ &= \rho_0(s) + \int_0^s i\hbar(s-r,1)\rho_0(r)dr - \int_0^1 \hbar(s,z)q_0(z)dz - \langle \Re(s), Y_0\rangle .\end{aligned}$$

Therefore, $(\phi(s,t), Y(t), q(y,t))$ satisfy (6).

If we rewrite the $\phi(s,t)$ Equation (9) into the following form:

$$\rho(s,t) + \int_0^s i\hbar(s-r,1)\rho(r,t)dr = \phi(s,t) + \int_0^1 \hbar(s,z)q(z,t)dz + \langle \Re(s), Y(t)\rangle .$$

It is a convolution type of the form $\rho(s,t)$. This makes us think about the reverse transformation from Equation (6) to Equation (5):

$$\begin{cases}\rho(s,t) = \phi(s,t) + \int_0^s i\tilde{\hbar}(s-r,1)\phi(r,t)dr - \int_0^1 \tilde{\hbar}(s,z)q(z,t)dz - \left\langle \tilde{\Re}(s), Y(t)\right\rangle \\ Y(t) = Y(t) \\ q(y,t) = q(y,t)\end{cases} \tag{10}$$

□

Theorem 2. *Take* $(\phi(s,t), Y(t), q(y,t))$ *be the solution of Equation* (6) *and let* $(\rho(s,t), Y(t), q(y,t))$ *be defined as Equation* (10). *If* $\tilde{\hbar}(s,z)$ *and* $\tilde{\Re}(s)$ *satisfy the following equations*

$$\begin{cases} \tilde{\overline{h}}_s(s,z) = -i\tilde{\overline{h}}_{zz}(s,z) - i\tilde{\overline{h}}(s,1)\overline{h}_0(z), \quad s \in (0,\tau), \ \ z \in (0,1) \\ \tilde{\overline{h}}(s,0) = \left\langle i\tilde{\Re}(s), G)\right\rangle, \ \ \tilde{\overline{h}}_z(s,1) = 0, \quad s \in (0,\tau) \\ \dot{\tilde{\Re}}(s) = F^H\tilde{\Re}(s) - i\tilde{\overline{h}}(s,1)\Re_0, \quad s \in (0,\tau) \\ \tilde{\overline{h}}(0,z) = -\overline{h}_0(z), \quad z \in (0,1) \\ \tilde{\Re}(0) = -\Re_0 \end{cases} \tag{11}$$

and

$$\begin{cases} \tilde{\overline{h}}(s,z) = -\int_0^s i\overline{h}(s-r,1)\tilde{\overline{h}}(r,z)dr - \overline{h}(s,z) \\ \tilde{\Re}(s) = -\int_0^s i\overline{h}(s-r,1)\tilde{\Re}(r)dr - \Re(s) \end{cases} \tag{12}$$

then $(\rho(s,t), Y(t), q(y,t))$ *is a solution of Equation* (5).

Proof. We just have to check the constraint condition over $\rho(s,t)$ and $q_y(y,t)$.

The first thing to do is to check the boundary condition $q_y(y,t)$ with $y = 1$. Course

$$\rho(s,t) = \phi(s,t) + \int_0^s i\tilde{\overline{h}}(s-r,1)\phi(r,t)dr - \int_0^1 \tilde{\overline{h}}(s,z)q(z,t)dz - \left\langle \tilde{\Re}(s), Y(t)\right\rangle,$$

and the constraint from Equation (6) and Equation (11) gives us

$$\begin{aligned} q_y(1,t) &= \phi(0,t) + \int_0^1 \overline{h}_0(z)q(z,t)dz + \langle \Re_0, Y(t)\rangle \\ &= \phi(0,t) - \int_0^1 \tilde{\overline{h}}(0,z)q(z,t)dz - \left\langle \tilde{\Re}(0), Y(t)\right\rangle \\ &= \rho(0,t), \end{aligned}$$

in which we have taken the initial condition of $(\tilde{\overline{h}}(s,z), \tilde{\Re}(s))$.

Then, we check the differential equation for $\rho(s,t)$, and our calculation method is this:

$$\begin{aligned} \rho_t(s,t) &= \phi_t(s,t) + \int_0^s i\tilde{\overline{h}}(s-r,1)\phi_t(r,t)dr - \int_0^1 \tilde{\overline{h}}(s,z)q_t(z,t)dz - \left\langle \tilde{\Re}(s), \dot{Y}(t)\right\rangle \\ &= \phi_t(s,t) + \int_0^s i\tilde{\overline{h}}(r,1)\phi_t(s-r,t)dr + \int_0^1 i\tilde{\overline{h}}(s,z)q_{zz}(z,t)dz \\ &\quad - \left\langle \tilde{\Re}(s), FY(t) + Gq_y(0,t)\right\rangle \\ &= \phi_t(s,t) + \int_0^s i\tilde{\overline{h}}(r,1)\phi_t(s-r,t)dr + i\tilde{\overline{h}}(s,1)q_z(1,t) - i\tilde{\overline{h}}(s,0)q_z(0,t) \\ &\quad -i\tilde{\overline{h}}_z(s,1)q(1,t) + \int_0^1 i\tilde{\overline{h}}_{zz}(s,z)q(z,t)dz - \left\langle F^H\tilde{\Re}(s), Y(t)\right\rangle \\ &\quad - \left\langle \tilde{\Re}(s), G\right\rangle q_y(0,t) \\ &= \phi_s(s,t) + \int_0^s i\tilde{\overline{h}}(r,1)\phi_s(s-r,t)dr + i\tilde{\overline{h}}(s,1)\phi(0,t) - i\tilde{\overline{h}}_z(s,1)q(1,t) \\ &\quad + \int_0^1 \left[i\tilde{\overline{h}}(s,1)\overline{h}_0(z) + i\tilde{\overline{h}}_{zz}(s,z)\right] q(z,t)dz \\ &\quad + \left\langle i\tilde{\overline{h}}(s,1)\Re_0 - F^H\tilde{\Re}(s), Y(t)\right\rangle - \left[i\tilde{\overline{h}}(s,0) + \left\langle \tilde{\Re}(s), G\right\rangle\right] q_y(0,t), \end{aligned}$$

and we have used the equations in Equation (11), and

$$\begin{aligned}\rho_s(s,t) &= \phi_s(s,t)+i\tilde{\overline{h}}(s,1)\phi(0,t)+\int_0^s i\tilde{\overline{h}}(r,1)\phi_s(s-r,t)dr\\ &\quad -\int_0^1 \tilde{\overline{h}}_s(s,z)q(z,t)dz-\left\langle \dot{\tilde{\Re}}(s),Y(t)\right\rangle .\end{aligned}$$

Based on the above expression, we can see that $\rho_t(s,t)=\rho_s(s,t)$ for each $s\in(0,\tau)$ and $t>0$.

At last, we check the boundary conditions for $\rho(s,t)$. In accordance with the definition in Equation (10), we know that

$$\begin{aligned}\rho(\tau,t) &= \phi(\tau,t)+\int_0^\tau i\tilde{\overline{h}}(\tau-r,1)\phi(r,t)dr-\int_0^1 \tilde{\overline{h}}(\tau,z)q(z,t)dz-\left\langle \tilde{\Re}(\tau),Y(t)\right\rangle\\ &= \int_0^\tau i\tilde{\overline{h}}(\tau-r,1)\phi(r,t)dr-\int_0^1 \tilde{\overline{h}}(\tau,z)q(z,t)dz-\left\langle \tilde{\Re}(\tau),Y(t)\right\rangle .\end{aligned}$$

In order to show $\rho(\tau,t)$ satisfies the boundary condition in Equation (5), we need to prove

$$\rho(\tau,t)=-\int_0^\tau i\overline{h}(\tau-r,1)\rho(r,t)dr+\int_0^1 \overline{h}(\tau,z)q(z,t)dz+\left\langle \Re(\tau),Y(t)\right\rangle ,$$

where $q(y,t)$ and $Y(t)$ correspond to Equation (6).

For

$$\phi(s,t)+\int_0^s i\tilde{\overline{h}}(s-r,1)\phi(r,t)dr=\rho(s,t)+\int_0^1 \tilde{\overline{h}}(s,z)q(z,t)dz+\left\langle \tilde{\Re}(s),Y(t)\right\rangle ,$$

using equality Equation (12) we calculate as follows:

$$\begin{aligned} &-\int_0^s i\tilde{\overline{h}}(s-r,1)\phi(r,t)dr\\ &= i\int_0^s\left[\int_0^{s-r} i\overline{h}(s-r-r',1)\tilde{\overline{h}}(r',1)dr'+\tilde{\overline{h}}(s-r,1)\right]\phi(r,t)dr\\ &= i\int_0^s \overline{h}(s-r',1)\left[\int_0^{r'} i\tilde{\overline{h}}(r'-r,1)\phi(r,t)dr+\phi(r',t)\right]dr'\\ &= i\int_0^s \overline{h}(s-r',1)\left[\rho(r',t)+\int_0^1 \tilde{\overline{h}}(r',z)q(z,t)dz+\left\langle \tilde{\Re}(r'),Y(t)\right\rangle\right]dr'\\ &= \int_0^s i\overline{h}(s-r',1)\rho(r',t)dr'+\int_0^1\left[\int_0^s i\overline{h}(s-r',1)\tilde{\overline{h}}(r',z)dr'\right]q(z,t)dz\\ &\quad +\left\langle \int_0^s i\overline{h}(s-r',1)\tilde{\Re}(r')dr',Y(t)\right\rangle\\ &= \int_0^s i\overline{h}(s-r',1)\rho(r',t)dr'-\int_0^1\left[\overline{h}(s,z)+\tilde{\overline{h}}(s,z)\right]q(z,t)dz\\ &\quad -\left\langle \Re(s)+\tilde{\Re}(s),Y(t)\right\rangle .\end{aligned}$$

From that we can see that for all $s \in (0, \tau)$ holds

$$\begin{aligned} & -\int_0^s i\tilde{\overline{h}}(s-r,1)\phi(r,t)dr + \int_0^1 \tilde{\overline{h}}(s,z)q(z,t)dz + \left\langle \tilde{\Re}(s), Y(t) \right\rangle \\ = & \int_0^s i\overline{h}(s-r',1)\rho(r',t)dr' - \int_0^1 \overline{h}(s,z)q(z,t)dz - \left\langle \Re(s), Y(t) \right\rangle. \end{aligned}$$

Taking $s = \tau$ yields

$$\begin{aligned} -\rho(\tau,t) &= -\int_0^\tau i\tilde{\overline{h}}(\tau-r,1)\phi(r,t)dr + \int_0^1 \tilde{\overline{h}}(\tau,z)q(z,t)dz + \left\langle \tilde{\Re}(\tau), Y(t) \right\rangle \\ &= \int_0^\tau i\overline{h}(\tau-r',1)\rho(r',t)dr' - \int_0^1 \overline{h}(\tau,z)q(z,t)dz - \left\langle \Re(\tau), Y(t) \right\rangle, \end{aligned}$$

i.e.,

$$\rho(\tau,t) = -\int_0^\tau i\overline{h}(\tau-r,1)\rho(r,t)dr + \int_0^1 \overline{h}(\tau,z)q(z,t)dz + \left\langle \Re(\tau), Y(t) \right\rangle,$$

so $(\rho(s,t), Y(t), q(y,t))$ also satisfy Equation (5). In particular, we have

$$\rho(s,0) = \phi_0(s) + \int_0^s i\tilde{\overline{h}}(s-r,1)\phi_0(r)dr - \int_0^1 \tilde{\overline{h}}(s,z)q_0(z)dz - \left\langle \tilde{\Re}(s), Y_0 \right\rangle.$$

The desired result will follow. □

To obtain the solvability of kernel Equations (8) and (11), we need the following concept.

Definition 1. *Let $\mathbb{Y}$ be a Banach space and $\Im(t)$ is a C_0 semigroup on $\mathbb{Y}$ with generator F. An operator $M \in \mathcal{B}(D(F), \mathbb{Y})$ is called a Miyadera-Voigt perturbation for F, when there are such $\tau > 0$ and $\xi_\tau \in (0,1)$, then*

$$\int_0^\tau \|M\Im(s)y\|_Y \, ds \le \xi_\tau \|y\|, \ \ \forall y \in D(F).$$

We will refer to this class of perturbations as $\mathcal{J}_{\mathbb{Y}}^{MV}(F)$.

Lemma 1. *(Miyadera-Voigt perturbation Theorem) Let $\mathbb{Y}$ be a Banach space, and F be the generator of a C_0 semigroup $\Im(t)$ on $\mathbb{Y}$. If $M \in \mathcal{J}_{\mathbb{Y}}^{MV}(F)$, then $F + M$ generates a C_0 semigroup $\Im^m(t)$* [43].

Theorem 3. *The kernel Equation* (8) *has a unique solution $(\overline{h}(s,z), \Re(s))$ in the space $H^2[0,1] \times \mathbb{C}^n$.*

Proof. We define two operator M_0 and G_0 in $H^2[0,1] \times \mathbb{C}^n$ by

$$\begin{aligned} & M_0(h,k)^T = (-ih'', 0)^T, \quad D(M_0) = \left\{(h,k) \in H^2[0,1] \times \mathbb{C}^n | h(0) = h'(1) = 0\right\}, \\ & G_0(h,k)^T = (-\delta'(z-0)k^T\overline{G}, 0)^T, \quad D(G_0) = \{(h,k) \in H^2[0,1] \times \mathbb{C}^n\}. \end{aligned}$$

Then Equation (8) can be rewritten as

$$\begin{cases} \frac{dZ(S)}{ds} = M_0 Z(S) + G_0 Z(s), s > 0, \\ Z(0) = Z_0, \end{cases}$$

where $Z(S) = (\hbar(s,z), \Re(s))^T$, and $Z_0 = (\hbar_0(z), \Re_0)^T$. From the definition of the operators we see that M_0 is the generator of a C_0 semigroup $\Im_0(t)$ and G_0 is a bounded linear operator on $H^2[0,1] \times \mathbb{C}^n$. From the perturbation theorem of the semigroup of bounded linear operators M_0, we konw that $M_0 + G_0$ generates a C_0 semigroup on $H^2[0,1] \times \mathbb{C}^n$. □

Theorem 4. *In the space* $H^2[0,1] \times \mathbb{C}^n$*,* $(\tilde{\hbar}(s,z), \tilde{\Re}(s))$ *is the unique solution of Equation* (11) .

Proof. Since Equation (11) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \tilde{\hbar} \\ \tilde{\Re} \end{pmatrix} = \begin{pmatrix} -i\partial zz & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \tilde{\hbar} \\ \tilde{\Re} \end{pmatrix} + \begin{pmatrix} 0 & \langle \cdot, -\delta'(z-0)G\rangle \\ 0 & F^H \end{pmatrix}\begin{pmatrix} \tilde{\hbar} \\ \tilde{\Re} \end{pmatrix} + \begin{pmatrix} -i\hbar_0(z) \\ -i\Re_0 \end{pmatrix}\tilde{\hbar}(s,1)$$

M_0 is defined as in the proof of Theorem 3, and define two operators as follows:

$$G_1(h,k)^T = (-\delta'(z-0)k^T\overline{G}, F^H k)^T, \quad D(G_1) = \{(h,k) \in H^2[0,1] \times \mathbb{C}^n\},$$
$$M_1(h,k)^T = (-\hbar_0(z)h'(1), -\Re_0 h'(1))^T, \quad D(M_1) = \left\{(h,k) \in H^2[0,1] \times \mathbb{C}^n\right\}.$$

Then Equation (11) can be written as the evolutionary equation in $H^2[0,1] \times \mathbb{C}^n$

$$\begin{cases} \frac{dN(S)}{ds} = M_0 N(S) + G_1 N(s) + M_1 N(s), \\ N(0) = N_0, \end{cases}$$

where $N(S) = (\tilde{\hbar}(s,z), \tilde{\Re}(s))^T$, and $N_0 = -(\hbar_0(z), \Re_0)^T$.

M_0 is the generator of a C_0 semigroup $\Im_0(t)$ on $H^2[0,1] \times \mathbb{C}^n$ and G is a bounded linear operator, so we can get that $M = M_0 + G$ generates a C_0 semigroup $\Im_1(t)$ on $H^2[0,1] \times \mathbb{C}^n$.

By calculation, we can obtain that M_1 is the Miyadera-Voigt disturbance for M_0 and thus for M. Application of the Miyadera-Voigt perturbation Theorem that introduced in Lemma 1, $M + M_1$ generates a C_0 semigroup $\Im_1^M(t)$ on $H^2[0,1] \times \mathbb{C}^n$, then there is a unique solution for Equation (11) in $H^2[0,1] \times \mathbb{C}^n$. □

Remark 1. *In practice, the problem is,* $\rho_0(s) = \varrho(s-\tau), s \in (0,\tau)$ *is known, but we do not konw* $\phi_0(s)$*. When we are able to determine* $\phi_0(s)$*, then* $\phi(s,t) = \phi_0(s+t)$*. This means that we can directly control the following*

$$\varrho(t) = \int_0^\tau i\tilde{\hbar}(\tau - r, 1)\phi(r,t)dr - \int_0^1 \tilde{\hbar}(\tau, z)q(z,t)dz - \left\langle \tilde{\Re}(\tau), Y(t) \right\rangle \tag{13}$$

Theorem 5. *In space* $L^2[0,\tau] \times \mathcal{H}$*, the transformations Equations* (9) *and* (10) *are bounded.*

Proof. In space $\mathcal{H}$, take $(\hbar(s,z), \Re(s))$ and $(\tilde{\hbar}(s,z), \tilde{\Re}(s))$ as the solutions to Equations (8) and (11), respectively. From the definitions of the transformations, we can get that Equations (8) and (10) are bounded only when

$$\int_0^s i\hbar(s-r,1)\rho(r)dr \in L^2[0,\tau], \ \forall \rho \in L^2[0,\tau]$$

and

$$\int_0^s i\tilde{\hbar}(s-r,1)\phi(r)dr \in L^2[0,\tau], \ \ \forall \phi \in L^2[0,\tau].$$

Note that Equations (4) and (13) associated with the analysis semigroup in [44], and in $s, \hbar(s,1)$ and $\tilde{\hbar}(s,1)$ are continuous. So the above integrals are in $L^2[0,\tau]$. □

3.2. The equivalence between Equations (6) and (7)

In order to simplify the selection of $(\hbar_0, \Re_0)$, we take the backstepping transformation

$$\varphi_{yy}(y,t) = q(y,t) - \int_0^y \ell(y,z)q(z,t)dz - \langle \xi(y), Y(t) \rangle \tag{14}$$

where $(\ell(y,z), \xi(y))$ satisfy the following equations

$$\begin{cases} \ell_{yy}(y,z) = \ell_{zz}(y,z) \\ \xi''(y) = iF^H \xi(y) \\ \ell(y,0) = i\xi(y)^T \overline{G}, \ \ \ell(y,y) = 0 \\ \xi(0) = 0, \ \ \xi'(0) = K^H \end{cases} \tag{15}$$

and the solution of Equation (15) is

$$\begin{cases} \ell(y,z) = i\xi(y-z)^T \overline{G}, \\ \xi(y) = \begin{bmatrix} I & 0 \end{bmatrix} \exp\left\{ \begin{bmatrix} 0 & I \\ iF^H & 0 \end{bmatrix} y \right\} \begin{bmatrix} 0 \\ K^H \end{bmatrix}. \end{cases}$$

Then, we have $\varphi(0,t) = 0$, and

$$\begin{aligned} \varphi_y(1,t) &= q_y(1,t) - \int_0^1 \ell_y(1,z)q(z,t)dz - \left\langle \xi'(1), Y(t) \right\rangle \\ &= \phi(0,t) + \int_0^1 \left[\hbar_0(z) - \ell_y(1,z)\right] q(z,t)dz + \left\langle \Re_0 - \xi'(1), Y(t) \right\rangle. \end{aligned}$$

Obviously, if we take

$$\hbar_0(z) = \ell_y(1,z), \ \ \Re_0 = \xi'(1) \tag{16}$$

then $\varphi(y,t)$ meets the initial and the boundary conditions in Equation (7).

The transformation Equation (14) is invertible, set

$$q(y,t) = \varphi(y,t) + \int_0^y \theta(y,z)\varphi(z,t)dz + \langle \psi(y), Y(t) \rangle, \tag{17}$$

where $(\theta(y,z), \psi(y))$ satisfy the following equations

$$\begin{cases} \theta_{yy}(y,z) = \theta_{zz}(y,z) \\ \psi''(y) = i(F+GK)^H \psi(y) \\ \theta(y,0) = i\psi(y)^T \overline{G}, \ \ \theta(y,y) = 0 \\ \psi(0) = 0, \ \ \psi'(0) = K^H \end{cases} \tag{18}$$

and the solution of Equation (18) is

$$\begin{cases} \theta(y,z) = i\psi(y-z)^T\overline{G}, \\ \psi(y) = \begin{bmatrix} I & 0 \end{bmatrix} \exp\left\{ \begin{bmatrix} 0 & I \\ i(F+GK)^H & 0 \end{bmatrix} y \right\} \begin{bmatrix} 0 \\ K^H \end{bmatrix}. \end{cases}$$

Take $(\phi(s,t), Y(t), q(y,t))$ as the solution to Equation (6), a direct calculation shows that $q(y,t)$ is defined by Equation (17) fulfils $q_t(y,t) = -iq_{yy}(x,t)$.

Then, we think about the boundary conditions of $q(y,t)$ in Equation (6). In line with Equation (17), $q(0,t) = 0$, and

$$\begin{aligned} q_y(1,t) &= \varphi_y(1,t) + \int_0^1 \theta_y(1,z)\varphi(z,t)dz + \left\langle \psi'(1), Y(t) \right\rangle \\ &= \phi(0,t) + \int_0^1 \theta_y(1,z)\varphi(z,t)dz + \left\langle \psi'(1), Y(t) \right\rangle. \end{aligned}$$

On the other side,

$$\begin{aligned} & q_y(1,t) - \phi(0,t) - \int_0^1 \hbar_0(z)q(z,t)dz - \langle \Re_0, Y(t) \rangle \\ = & \int_0^1 \theta_y(1,z)\varphi(z,t)dz - \int_0^1 \hbar_0(z)q(z,t)dz + \left\langle \psi'(1) - \Re_0, Y(t) \right\rangle \\ = & \int_0^1 \theta_y(1,z)\varphi(z,t)dz - \int_0^1 \hbar_0(z) \left[\varphi(z,t) + \int_0^z \theta(z,z')\varphi(z',t)dz' + \psi(y) \right] dz \\ & + \left\langle \psi'(1) - \Re_0, Y(t) \right\rangle \\ = & \int_0^1 \left[\theta_y(1,z) - \hbar_0(z) - \int_z^1 \hbar_0(z)\theta(y,z)dy \right] \varphi(z,t)dz \\ & + \left\langle \psi'(1) - \Re_0 - \int_0^1 \hbar_0(z)\psi(z)dz, Y(t) \right\rangle. \end{aligned}$$

We also need $\theta(y,z)$ and $\psi(y)$. They must fulfill the below non-local requirements

$$\begin{cases} \theta_y(1,z) = \hbar_0(z) + \int_z^1 \hbar_0(z)\theta(y,z)dz \\ \psi'(1) = \Re_0 + \int_0^1 \hbar_0(z)\psi(z)dz \end{cases} \tag{19}$$

under these constraints, $q_y(y,t)$ satisfies the boundary condition at $y = 1$ in Equation (6).

Theorem 6. *The Equation* (18) *with the nonlocal conditions* (19) *is solvable.*

Theorem 7. *Let* $(\ell(y,z), \xi(y))$ *be given by Equation* (15)*, and take* $\hbar_0(z) = \ell_y(1,z)$*,* $\Re_0 = \xi'(1)$*, then the system* (5) *and the system* (7) *are equivalent.*

4. Exponential stability of the controlled system (5)

Consider the next ODE-Schrödinger system

$$\begin{cases} \dot{Y}(t) = (F+GK)Y(t) + G\varphi_y(0,t), \quad t > 0 \\ \varphi_t(y,t) = -i\varphi_{yy}(y,t), \quad y \in (0,1), \quad t > 0 \\ \varphi(0,t) = 0, \quad \varphi_y(1,t) = 0, \quad t > 0 \\ Y(0) = Y_0, \quad \varphi(y,0) = \varphi_0(y), \quad y \in [0,1] \end{cases} \tag{20}$$

and it can be regarded as the evolution equation in $\mathcal{H}$

$$\begin{cases} \frac{dQ(t)}{dt} = A_0 Q(t) \\ Q(0) = Q_0 \end{cases} \tag{21}$$

where $Q(t) = (Y(t), \varphi(y,t)),\ Q_0 = (Y_0, \varphi_0(y))$, then define a linear operator in $\mathcal{H}$ as enumerated below

$$\begin{cases} A_0(Y,h) = ((F+GK)Y + Gh(0), -ih''),\ \forall (Y,h) \in D(A_0) \\ D(A_0) = \left\{(Y,h) \in \mathbb{C}^n \times H^2(0,1) | h(0) = h'(1) = 0\right\} \end{cases} \tag{22}$$

Lemma 2. *Let* (F,G) *in* $\mathbb{C}^n$ *be controllable,* A_0 *be given by Equation* (22)*, and* K *be chosen such that* $(F+GK)$ *is a Hurwitz matrix. Then, the following satatements are true:*

1) A_0 *generates a* C_0*-semigroup* $\Im(t)$ *on* $\mathcal{H}$*;*
2) *Two positive constants* O *and* λ *exist such that* $\|\Im(t)\| \leq Oe^{-\lambda t}$*, for all* $t \geq 0$.

Theorem 8. *In terms of the norm* $\mathcal{H}$*, the solution of Equation* (7) *decays exponentially.*

Proof. Seeing that the ϕ-part in Equation (7) has a solution $\phi(s,t) = \phi_0(t+s)$, i.e. zero as $t+s > \tau$, then we just have to think over the Y-part and q-part in Equation (7),

$$\begin{cases} \dot{Y}(t) = (F+GK)Y(t) + G\varphi_y(0,t),\ \ t > 0 \\ \varphi_t(y,t) = -i\varphi_{yy}(y,t),\quad y \in (0,1),\ \ t > 0 \\ \varphi(0,t) = 0,\ \ \varphi_y(1,t) = v_0(t),\ \ t > 0 \\ Y(0) = Y_0,\ \ \varphi(y,0) = \varphi_0(y),\quad y \in [0,1] \end{cases} \tag{23}$$

and it is equivalent to an evolution equation in $\mathcal{H}$

$$\begin{cases} \frac{dQ(t)}{dt} = A_0 Q(t) + B_0 \phi_0(t) \\ Q(0) = Q_0 \end{cases} \tag{24}$$

where A_0 is defined by Equation (22) and $B_0 = (0, -i\delta(y-1))^T$.

Then the solution of (7) is as enumerated below:

$$\begin{cases} \phi(s,t) = \phi_0(s+t),\quad s \in [0,\tau],\ \ t \geq 0 \\ Q(t) = \Im(t)Q_0 + \int_0^t \Im(t-r) B_0 \phi_0(r) dr \end{cases} \tag{25}$$

Hence, a constant O_t exists such

$$\left\| \int_0^t \Im(t-r) B_0 \phi_0(r) dr \right\|_{\mathcal{H}}^2 \leq O_t^2 \left\| \phi_0 \right\|^2,$$

so, for $0 < t < \tau$, we have

$$\begin{aligned} & \|(\phi(t), Q(t))\|^2 = \|\phi(t)\|^2 + \|Q(t)\|^2 \\ \leq\ & \int_0^\tau |\phi(s,t)|^2 ds + 2\left\|\Im(t)Q_0\right\|^2 + 2\left\| \int_0^t \Im(t-r) B_0 \phi_0(r) dr \right\|_{\mathcal{H}}^2 \\ \leq\ & \int_0^\tau |\phi_0(r)|^2 dr + 2O^2 \left\|(Y_0, \varphi_0)\right\|^2 + 2O_\tau^2 \left\|\phi_0\right\|^2 \\ \leq\ & O_1^2 \left\|(\phi_0, Y_0, \varphi_0)\right\|^2 \end{aligned}$$

where $O_1^2 = \max\left\{1 + 2O_\tau^2, 2O^2\right\}$.

As $t > \tau$, it holds that for $t > 0$, we have

$$\begin{aligned} &\|(\phi(t), Q(t))\|^2 = \|\Im(t-\tau)Q(\tau)\|^2 \leq \|\Im(t-\tau)\|^2 \|Q(\tau)\|^2 \\ \leq\ & O^2 e^{-2\lambda(t-\tau)} \|Q(\tau)\|^2 \leq O^2 e^{-2\lambda(t-\tau)} O_1 \|(\phi_0, Y_0, \varphi_0)\|^2 . \end{aligned}$$

Thus, system (7) is exponentially stable. □

Theorem 9. *Provided that all the conditions in Lemma 2 are satisfied, let* $(\ell(s,z), \xi(s))$ *be given by Equation* (15)*, take* $\overline{h}_0(z) = \ell_y(1,z), \Re_0 = \xi'(1)$*, and let* $(\overline{h}(s,z), \Re(s))$ *be a solution of Equation* (8)*. Under the feedback control*

$$\varrho(t) = -\int_0^\tau i\overline{h}(\tau - r)\rho(r,t)dr + \int_0^1 \overline{h}(\tau, z)q(z,t)dz + \langle \Re(\tau), Y(t)\rangle ,$$

then the closed-loop system (5) *is exponentially stable and has a unique solution.*

Proof. By the assumptions, applying the results of Theorem 7, we can claim the system (5) and the system (7) are equivalent. Due to Theorem 8, Equation (5) is also exponentially stable. □

5. Conclusion

In this work, we address the uniform stabilization challenge for a coupled Schrödinger-ODE system featuring boundary control with time delays. We provide a detailed demonstration that the system in question can be transformed into a target system through feedback equivalence by determining the appropriate kernel functions. Since the target system demonstrates exponential stability, it follows that the controlled system also achieves exponential stability. This method of designing the controller guarantees the stability of the closed-loop system, thereby avoiding the necessity for intricate stability analysis.

Author contributions: Conceptualization, YRX and YWC; methodology, LZ; writing—original draft preparation, YWC; writing—review and editing, CYT; visualization, RQG; supervision, YRX; project administration, YRX; funding acquisition, YRX. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by [the National Natural Science Foundation of China] grant number [NSFC 12301579,12201462] and [the Fundamental Research Funds for the Central Universities of Civil Aviation University of China] grant number [G. No. 3122024PT23].

Conflict of interest: The authors declare no conflict of interest.

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