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NUMERICAL BLOW-UP TIME FOR NONLINEAR PARABOLIC PROBLEMS

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Abstract

In this paper, we analyze numerically some of the features of the blow-up phenomena arising from a nonlinear parabolic equation subject to nonlinear boundary conditions. More precisely, we study numerical approximations of solutions of the problem

$$\begin{cases} (\log u(x, t))_t = u_{xx}(x, t) + u^{\beta-1}(x, t), & (x, t) \in (0, 1) \times (0, T), \\ -u_x(0, t) + u^{\alpha}(0, t) = 0, & t > 0, \\ u_x(1, t) + u^{\alpha}(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x) \geq \gamma > 0, & 0 \leq x \leq 1, \end{cases}$$

where $\beta \geq \alpha > 1$. We obtain some conditions under which the solution of the semidiscrete form blows up in a finite time. We estimate its semidiscrete blow-up time and also establish the convergence of the semidiscrete blow-up time to the real one. Finally, we give some numerical experiments to illustrate our analysis.

1. Introduction

Consider the following initial-boundary value problem:

$$\begin{cases} (\log u(x, t))_t = u_{xx}(x, t) + u^{\beta-1}(x, t), & (x, t) \in (0, 1) \times (0, T), \\ -u_x(0, t) + u^{\alpha}(0, t) = 0, & t > 0, \\ u_x(1, t) + u^{\alpha}(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x) \geq \gamma > 0, & 0 \leq x \leq 1, \end{cases} \tag{1.1}$$

where $\beta \geq \alpha > 1$ are parameters. The function u_0 satisfies the compatibility condition

$$\begin{cases} -u_0'(0) + u_0^{\alpha}(0) = 0, \\ u_0'(1) + u_0^{\alpha}(1) = 0. \end{cases}$$

The first equation of (1.1) can be rewritten in the following form:

$$u_t(x, t) = uu_{xx}(x, t) + u^{\beta}(x, t), \quad (x, t) \in (0, 1) \times (0, T).$$

Without loss of generality, we may consider the following:

$$u_t(x, t) = uu_{xx}(x, t) + u^{\beta}(x, t), \quad (x, t) \in (0, 1) \times (0, T), \tag{1.2}$$

$$u_x(0, t) = u^{\alpha}(0, t), \quad t > 0, \tag{1.3}$$

$$u_x(1, t) = -u^{\alpha}(1, t), \quad t > 0, \tag{1.4}$$

$$u(x, 0) = u_0(x) \geq \gamma > 0, \quad 0 \leq x \leq 1, \tag{1.5}$$

where $\beta \geq \alpha > 1$.

Definition 1.1. We say that the solution u of (1.2)-(1.5) *blows up* in a finite time if there exists a finite time T_b such that $\| u(., t) \|_{\infty} < \infty$ for $t \in [0, T_b)$ but

$$\lim_{t \to T_b} \| u(., t) \|_{\infty} = \infty,$$

where $\| u(., t) \|_{\infty} = \max_{0 \leq x \leq 1} | u(x, t) |$. The time T_b is called the *blow-up time* of the solution u.

The theoretical study of solutions of nonlinear parabolic equations with nonlinear boundary conditions which blow up in a finite time has been the subject of investigations of many authors (see [5, 11, 12] and the references cited therein). Concerning the problem (1.1), under certain conditions given, Zhang [5] has shown that if $\beta \geq \alpha > 1$, then the solution must blow up in a finite time T and T satisfies $T \leq \frac{1}{\beta - 1} \gamma^{1-\beta}$. He also shows that if $\beta \leq 1$, then the solution $u(x, t)$ of the problem exists globally. But the numerical study has not been treated. The problem (1.2)-(1.5) belongs to a class of equations that are used in biology. These equations are used in the area of population biology, and ecological interactions, and are characterized by birth and death [2].

In this work, we are interested in the numerical study using a semidiscrete form of (1.2)-(1.5).

Let $I \geq 2$ be an integer. Then we set $h = \frac{1}{I}$ and define the grid $x_i = ih$, for $i = 0, \ldots, I$.

We approximate the solution u of (1.1) by the solution U_h of the semidiscrete equations:

$$\frac{d}{dt}U_i(t) - U_i(t)\delta^2 U_i(t) = U_i^{\beta}(t), \quad 1 \leq i \leq I-1, \quad t \in (0, T_b^h), \tag{1.6}$$

$$\frac{d}{dt}U_0(t) - U_0(t)\delta^2 U_0(t) = -\frac{2U_0^{\alpha+1}(t)}{h} + U_0^{\beta}(t), \quad t \in (0, T_b^h), \tag{1.7}$$

$$\frac{d}{dt}U_I(t) - U_I(t)\delta^2 U_I(t) = -\frac{2U_I^{\alpha+1}(t)}{h} + U_I^{\beta}(t), \quad t \in (0, T_b^h), \tag{1.8}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \leq i \leq I, \tag{1.9}$$

where

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2},$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2},$$

$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

Here, $[0, T_b^h)$ is the maximal time interval on which the solution $U_h(t)$ of (1.6)-(1.9) is finite, where $\| U_h(t) \|_\infty = \max_{0 \le x \le 1} | U_h(t) |$. When T_b^h is infinite, we say that the solution $U_h(t)$ exists *globally*. When T_b^h is finite, we say that the solution $U_h(t)$ *blows up in finite time* and in the last case the time T_b^h is called the *blow-up time* of the solution $U_h(t)$. Our aim in the present work is to give some conditions under which the solution of (1.6)-(1.9) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. Concerning the numerical study of blow-up phenomena, one can also find in [1, 3, 6, 7, 9] some results where the authors have proposed some schemes for the numerical calculation of solutions which present singularities.

The paper is organized as follows: in the next section, we give some properties concerning our semidiscrete scheme. In Section 3, under some conditions, we prove that the solution of the semidiscrete form of (1.2)-(1.5) blows up in a finite time as well as the convergence of the semidiscrete blow-up time. Finally, in the last section, we estimate the numerical blow-up time and give some numerical results to illustrate our study.

2. Properties of the Semidiscrete Scheme

In this section, we give some lemmas which will be used later. The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1. *Let* $a_h(t), b_h(t) \in C^0([0, T], \mathbb{R}^{I+1})$ *and let* $V_h(t) \in C^1([0, T], \mathbb{R}^{I+1})$ *such that*

$$\frac{d}{dt} V_i(t) - b_i(t)\delta^2 V_i(t) + a_i(t) V_i(t) \ge 0, \quad 0 \le i \le I, \quad t \in (0, T), \tag{2.1}$$

$$V_i(0) \ge 0, \quad 0 \le i \le I. \tag{2.2}$$

Then we have

$$V_i(t) \geq 0, \quad 0 \leq i \leq I, \quad t \in [0, T). \tag{2.3}$$

Proof. Let $T_0 < T$ and define the vector $Z_h(t) = e^{\gamma t}V_h(t)$, where γ is sufficiently small such that $(a_i(t) - \gamma) > 0$ for $0 \leq i \leq I$, $t \in [0, T_0]$. Let $m = \min_{0 \leq i \leq I, 0 \leq t \leq T_0} Z_i(t)$. Since, for $i \in \{0, ..., I\}$, $Z_i(t)$ is a continuous function on the compact $[0, T_0]$, there exist $t_0 \in [0, T_0]$ and $i_0 \in \{0, ..., I\}$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{\varepsilon \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - \varepsilon)}{\varepsilon} \leq 0, \quad 0 \leq i_0 \leq I, \tag{2.4}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \geq 0, \quad \text{if } i_0 = 0, \tag{2.5}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \geq 0, \quad 1 \leq i_0 \leq I-1, \tag{2.6}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \geq 0, \quad \text{if } i_0 = I. \tag{2.7}$$

From (2.1), we obtain the following inequality:

$$\frac{dZ_{i_0}(t_0)}{dt} - b_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \gamma)Z_{i_0}(t_0) \geq 0. \tag{2.8}$$

It follows from (2.4)-(2.7) that $(a_{i_0}(t_0) - \gamma)Z_{i_0}(t_0) \geq 0$, which implies that $Z_{i_0}(t_0) \geq 0$ because $(a_{i_0}(t_0 - \gamma)) > 0$. We deduce that $V_h(t) \geq 0$ for $t \in [0, T_0]$ and the proof is complete. □

Lemma 2.2. *Let* $V_h(t), W_h(t) \in C^1([0, T], \mathbb{R}^{I+1})$ *and* $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and* $b_h(t) \geq 0$ *such that*

$$\frac{dV_i(t)}{dt} - b_i(t)\delta^2 V_i(t) + f(V_i(t), t) < \frac{dW_i(t)}{dt} - b_i(t)\delta^2 W_i(t) + f(W_i(t), t),$$

$$0 \le i \le I, \quad t \in (0, T),$$

$$V_i(0) < W_i(0), \quad 0 \le i \le I.$$

Then

$$V_i(t) < W_i(t), \quad 0 \le i \le I, \quad t \in (0, T).$$

Proof. Introduce the vector $Z_h(t) = W_h(t) - V_h(t)$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0[$, $0 \le i \le I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, ..., I\}$. It is not hard to see that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{\varepsilon \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - \varepsilon)}{\varepsilon} \le 0, \quad 0 \le i_0 \le I,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \ge 0, \quad \text{if } i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0, \quad 1 \le i_0 \le I - 1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \ge 0, \quad \text{if } i_0 = I,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - b_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + f(W_{i_0}(t_0), t) - f(V_{i_0}(t_0), t) \le 0.$$

But this inequality contradicts the first strict differential inequality of the lemma and the proof is complete. □

Lemma 2.3. *Let* U_h *be the solution of* (1.6)-(1.9). *Then* $\frac{dU_i(t)}{dt} \ge 0$ *for* $0 \le i \le I$, $t \in [0, T_b^h[$.

3. Blow-up Time of the Semidiscrete Problem

In this section, under some assumptions, we prove that the semidiscrete solution U_h of problem (1.6)-(1.9) blows up in a finite time. Then we estimate the semidiscrete blow-up time, and finally, we prove that this time converges to the real one when the mesh size goes to zero.

Lemma 3.1. *Let* $U_h \in \mathbb{R}^{I+1}$ *such that* $U_h > 0$. *Then we have*

$$\delta^2 U_i^\beta \geq \beta U_i^{\beta-1}\delta^2 U_i \quad \textit{for} \quad 0 \leq i \leq I, \quad \beta > 1.$$

Proof. Using Taylor's expansion, we obtain

$$\delta^2 U_0^\beta = \beta U_0^{\beta-1}\delta^2 U_0 + (U_1 - U_0)^2 \frac{\beta(\beta-1)}{h^2}\theta_0^{\beta-2},$$

$$\delta^2 U_i^\beta = \beta U_i^{\beta-1}\delta^2 U_i + (U_{i+1} - U_i)^2 \frac{\beta(\beta-1)}{2h^2}\theta_i^{\beta-2}$$

$$+ (U_{i-1} - U_i)^2 \frac{\beta(\beta-1)}{2h^2}\xi_i^{\beta-2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_I^\beta = \beta U_I^{\beta-1}\delta^2 U_I + (U_{I-1} - U_I)^2 \frac{\beta(\beta-1)}{2h^2}\theta_i^{\beta-2},$$

where θ_i is an intermediate value between U_i and U_{i+1} and ξ_i is an intermediate value between U_{i-1} and U_i. Using the fact that $U_h > 0$, we have the desired result. □

Theorem 3.1. *Let* U_h *be the solution of problem* (1.6)-(1.9). *Assume that* $\beta - \alpha \geq 1$ *and suppose that there exists a constant* $\lambda > 0$ *such that*

$$\varphi_i\delta^2\varphi_i - \varphi_i^\beta \geq \lambda\varphi_i^\beta, \quad 1 \leq i \leq I-1, \tag{3.1}$$

$$\varphi_0\delta^2\varphi_0 - \varphi_0^\beta + \frac{2}{h}\varphi_0^\alpha \geq \lambda\varphi_0^\beta, \tag{3.2}$$

$$\varphi_I\delta^2\varphi_I - \varphi_I^\beta + \frac{2}{h}\varphi_I^\alpha \geq \lambda\varphi_I^\beta. \tag{3.3}$$

Then the solution U_h *of problem* (1.6)-(1.9) *blows up in a finite time* T_b^h *and we have the following estimate*:

$$T_b^h \leq \frac{1}{\lambda} \frac{\| \varphi_h \|_\infty^{1-\beta}}{\beta - 1}. \tag{3.4}$$

Proof. Let $[0, T_b^h)$ be the maximal time interval on which $\| U_h(t) \|_\infty < \infty$. Our aim is show that T_b^h is finite and satisfies (3.4).

Introducing the vector J_h such that

$$J_i(t) = \frac{d}{dt} U_i(t) - \lambda U_i^\beta(t), \quad 0 \leq i \leq I, \quad t \in [0, T_b^h). \tag{3.5}$$

We shall prove that $J_i(t) \geq 0$, for $0 \leq i \leq I$, $t \geq 0$.

From (3.5), we have

$$\frac{d}{dt} J_i - U_i \delta^2 J_i = \frac{d}{dt}\left(\frac{d}{dt} U_i - \lambda U_i^\beta\right) - U_i \delta^2 \left(\frac{d}{dt} U_i - \lambda U_i^\beta\right), \quad 0 \leq i \leq I.$$

Using Lemma 3.1, we obtain

$$\frac{d}{dt} J_i - U_i \delta^2 J_i \geq \frac{d}{dt}\left(\frac{d}{dt} U_i - U_i \delta^2 U_i\right) - \lambda \beta U_i^{\beta-1}\left(\frac{d}{dt} U_i - U_i \delta^2 U_i\right)$$

$$+ \delta^2 U_i \frac{d}{dt} U_i, \quad 0 \leq i \leq I.$$

By a straightforward computation, we have

$$\frac{d}{dt} J_i - U_i \delta^2 J_i - \beta U_i^{\beta-1} J_i \geq \delta^2 U_i \frac{d}{dt} U_i, \quad 1 \leq i \leq I - 1,$$

$$\frac{d}{dt} J_0 - U_0 \delta^2 J_0 - \beta U_0^{\beta-1} J_0 + \frac{2(\alpha + 1)}{h} U_0^\alpha J_0$$

$$\geq \frac{2\lambda}{h} U_0^{\alpha+\beta} (\beta - \alpha - 1) + \delta^2 U_0 \frac{d}{dt} U_0,$$

$$\frac{d}{dt}J_I - U_I\delta^2 J_I - \beta U_I^{\beta-1}J_I + \frac{2(\alpha+1)}{h}U_I^{\alpha}J_I$$

$$\geq \frac{2\lambda}{h}U_I^{\alpha+\beta}(\beta-\alpha-1) + \delta^2 U_I \frac{d}{dt}U_I.$$

Using the fact that $\beta \geq \alpha > 1$ and the hypotheses of the theorem, we get

$$\frac{d}{dt}J_i - U_i\delta^2 J_i - \beta U_i^{\beta-1}J_i \geq 0,$$

$$\frac{d}{dt}J_0 - U_0\delta^2 J_0 - \left(\beta U_0^{\beta-1} - \frac{2(\alpha+1)}{h}U_0^{\alpha}\right)J_0 \geq 0,$$

$$\frac{d}{dt}J_I - U_I\delta^2 J_I - \left(\beta U_I^{\beta-1} + \frac{2(\alpha+1)}{h}U_I^{\alpha}\right)J_I \geq 0,$$

$$J_i(0) \geq 0.$$

We deduce from Lemma 2.1 that $J_h(t) \geq 0$ for $t \in [0, T_b^h)$, which implies that

$$\frac{d}{dt}U_i(t) \geq \lambda U_i^{\beta}(t), \quad 0 \leq i \leq I.$$

Integrating the above inequality over (t_0, T_b^h), we arrive at

$$T_b^h - t_0 \leq \frac{1}{\lambda}\frac{(U_i(t_0))^{1-\beta}}{\beta-1} \tag{3.6}$$

which implies that

$$T_b^h \leq \frac{1}{\lambda}\frac{\|\varphi_h\|_{\infty}^{1-\beta}}{\beta-1},$$

and the proof is complete. □

Theorem 3.2. *Assume that* (1.2)-(1.5) *has a solution* $u \in C^{4,1}([0, 1] \times [0, T])$ *and the initial condition* φ_h *at* (1.9) *satisfies*:

$$\| \varphi_h - u_h(0) \|_\infty = o(1), \ as \ h \to 0, \tag{3.7}$$

where $u_h(t) = (u(x_0, t), \ldots, u(x_I))^T, \quad t \in [0, T]$. *Then*, *for* h *sufficiently small*, *problem* (1.6)-(1.9) *has a unique solution*

$$U_h \in \mathcal{C}^1([0, T], \mathbb{R}^{I+1})$$

such that

$$\max_{t \in [0, T]} \| U_h(t) - u_h(t) \|_\infty = O(\| \varphi_h - u_h(0) \|_\infty + h^2), \quad h \to 0. \tag{3.8}$$

The proof of the theorem of convergence of the solution U_h is similar to those given in [1, 10], so we omit it here.

Theorem 3.3. *Suppose that the solution of* (1.2)-(1.5) *blows up in a finite time* T_b *such that* $u \in \mathcal{C}^{4,1}([0, 1] \times [0, T], \mathbb{R})$ *and the initial condition at* (1.9) *satisfies*

$$\| \varphi_h - u_h(0) \|_\infty = o(1), \ as \ h \to 0.$$

Then, *under the hypothesis of Theorem* 3.1, *the solution* U_h *of* (1.6)-(1.9) *blows up in a finite time* T_b^h *and we have*

$$\lim_{h \to 0} T_b^h = T_b. \tag{3.9}$$

Proof. Let $\varepsilon > 0$. Then there exists a positive constant *N* such that

$$\frac{1}{\lambda} \frac{y^{1-\beta}}{\beta - 1} \leq \frac{\varepsilon}{2} < \infty \ \text{ for } \ y \in [N, +\infty[. \tag{3.10}$$

Since the solution *u* blows up at the finite time T_b, there exists T_1 such that

$$| T_1 - T_b | \leq \frac{\varepsilon}{2} \ \text{ and } \ \| u(., t) \|_\infty \geq 2N \ \text{ for } \ t \in [T_1, T_b].$$

Let $T_2 = \frac{T_1 + T_b}{2}$. Then $\sup_{t\in[0, T_2]} \| u(., t) \|_\infty < \infty$.

It follows from Theorem 3.2 that $\sup_{t\in[0, T_2]} \| U_h(t) - u_h(t) \|_\infty \leq N$.

Applying the triangular inequality, we get

$$\| U_h(t) \|_\infty \geq \| u_h(t) \|_\infty - \| U_h(t) - u_h(t) \|_\infty,$$

which implies $\| U_h(t) \|_\infty \geq N$ for $t \in [0, T_2]$. From Theorem 3.1, $U_h(t)$ blows up in a finite time T_b^h. We deduce from (3.6) and (3.10) that

$$| T_b - T_b^h | \leq | T_b - T_2 | + | T_2 - T_b^h | \leq \frac{\varepsilon}{2} + \frac{1}{\lambda} \frac{\| U_h(T_2) \|_\infty^{-(\beta+1)}}{\beta + 1} \leq \varepsilon$$

which achieves the proof. □

4. Numerical Experiments

In this section, we estimate the numerical blow-up time and present some numerical results of the problem (1.2)-(1.5). For the numerical computation, we consider the equation on the finite interval $x \in [0, 1]$ and using the standard central difference approximation, we obtain the system of ODEs:

$$\frac{d}{dt} U_i(t) = U_i(t) \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2} + U_i^\beta(t),$$

$$1 \leq i \leq I - 1, \quad t \in (0, T_b^h),$$

$$\frac{d}{dt} U_0(t) = U_0(t) \frac{2U_1(t) - 2U_0(t)}{h^2} - \frac{2U_0^{\alpha+1}(t)}{h} + U_0^\beta(t), \quad t \in (0, T_b^h),$$

$$\frac{d}{dt} U_I(t) = U_I(t) \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} - \frac{2U_I^{\alpha+1}(t)}{h} + U_I^\beta(t), \quad t \in (0, T_b^h),$$

$$U_i(0) = \varphi_i, \quad 0 \leq i \leq I.$$

We apply the algorithm based on the technique of the arc length transformation. The main idea of the method is to transform the ODE into a tractable form by the arc length transformation technique and to generate a linearly convergent sequence to the blow-up time. This sequence is then accelerated by the Aitken Δ^2 method. The present method is applied to the blow-up problems of PDEs by discretising the equations in space and integrating the resulting ODEs by an ODE solver. Solver *DOP*54 is used with the tolerance parameters $Reltol = Abstol = 1.d - 15$ and $Initialstep = 0$ (see [3, 4, 6, 8, 9]).

We report the cases with $\alpha = 2, 3$ and $\beta \geq \alpha$. Several other cases with $\beta \geq \alpha > 1$ were tested as well and the results are similar.

Without loss of generality, the initial value $u_0(x) = 4 * exp(x)$ is set to be superior to zero. The spatial step size *h* varies from $1/16$ to $1/512$. The reason of choosing smaller spatial step sizes is not for the stability of numerical scheme, but for observing the blow-up more accurately. The order *s* of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

In tables further, we list the computed blow-up time T_b^h for various values of parameters α and β. These results show that there is a relationship between the blow-up time and the parameters α and β. Indeed, when α is fixed and β increases, the blow-up time diminishes. This phenomenon is well known from the biological point of view. From these tables, we can also assure the convergence of the blow-up time T_b^h of the solution of (1.2)-(1.5), since the rate of convergence is near 2, which is just the accuracy of the difference approximation in space.

Table 1. Blow-up times, steps and orders of the approximations obtained for $\alpha = 2$, $\beta = 3$

I	T_b^h	Steps	Orders
16	0.02045421	4610	-
32	0.02047033	6951	-
64	0.02047384	11104	2.19
128	0.02047466	18935	2.10
256	0.02047485	34583	2.04
512	0.02047490	100695	2.01

Table 2. Blow-up times, steps and orders of the approximations obtained for $\alpha = 2$, $\beta = 4$

I	T_b^h	Steps	Orders
16	0.00043719	1219	-
32	0.00044224	1682	-
64	0.00044351	2400	1.98
128	0.00044376	2860	2.08
256	0.00044384	4328	2.00
512	0.00044386	7191	2.00

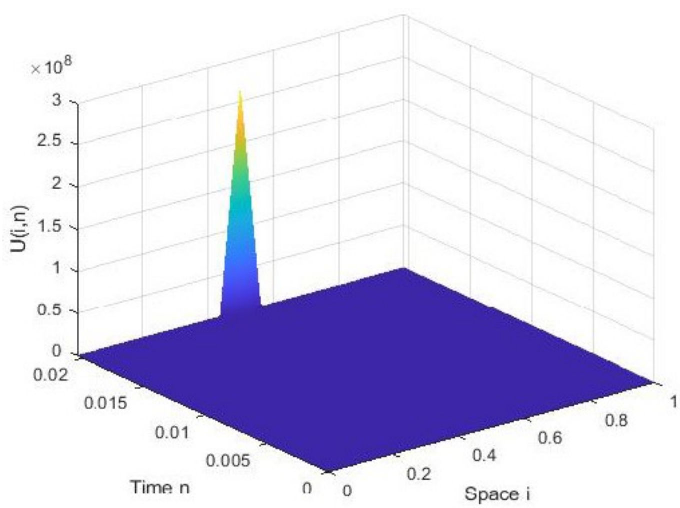
Table 3. Blow-up times, steps and orders of the approximations obtained for $\alpha = 3$, $\beta = 3$

I	T_b^h	Steps	Orders
16	0.0341992	5873	-
32	0.0341074	8791	-
64	0.0340784	13913	1.67
128	0.0340697	23430	1.74
256	0.0340673	44735	1.82
512	0.0340666	146945	1.94

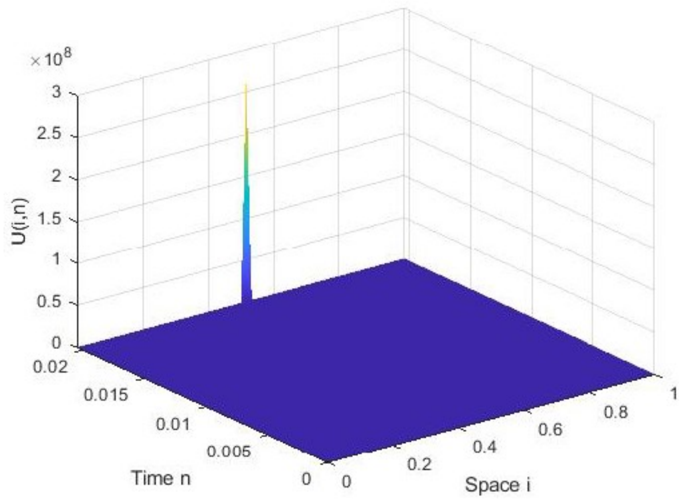
Table 4. Blow-up times, steps and orders of the approximations obtained for $\alpha = 3, \beta = 3.5$

I	T_b^h	Steps	Orders
16	0.00291688	2396	-
32	0.00290369	3228	-
64	0.00289989	4457	1.79
128	0.00289870	6745	1.80
256	0.00289837	10742	1.87
512	0.00289828	21914	1.90

Remark 4.1. In the following figures, we show the evolution of the semidiscrete solution. The parameter value $\alpha = 2$ is adopted together with $\beta = 3$ and $h = 1/I$. The four different stages of the semidiscrete solution at $I = 16, 64, 128$ and 512 are depicted. We can also appreciate that the semidiscrete solution blows up in a finite time at a single node. The contour maps in figures also indicate this conclusion, since the contours get flat outside this node. This result provides some numerical evidence of blowing up as studied in [5].

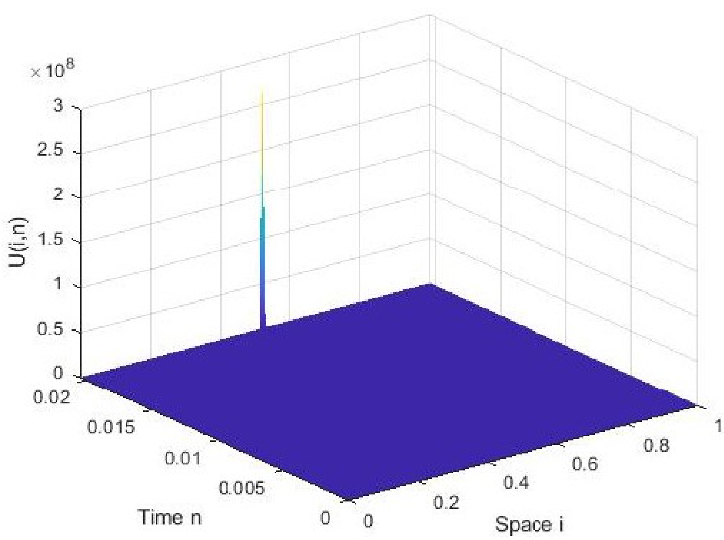


(a)

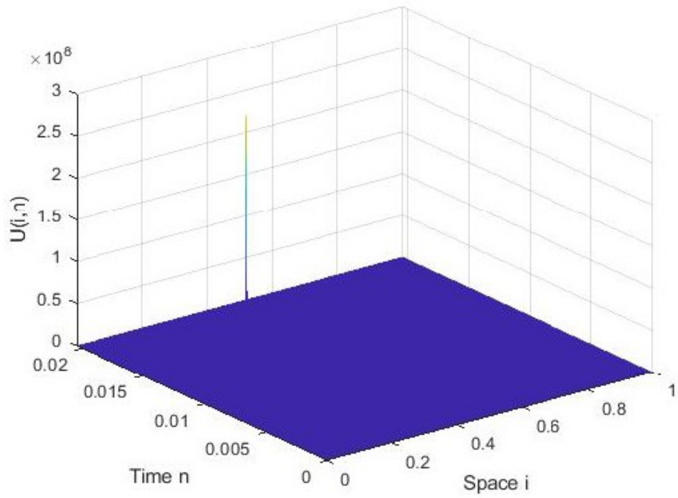


(b)

Figure 1. The evolution profiles of the semidiscrete solution for $\alpha = 2$, $\beta = 3$ with $I = 16$ (in (a)) and $I = 64$ (in (b)).



(a)



(b)

Figure 2. The evolution profiles of the semidiscrete solution for $\alpha = 2$, $\beta = 3$ with $I = 128$ (in (a)) and $I = 512$ (in (b)).

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