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LYAPUNOV TYPE INEQUALITIES AND THEIR APPLICATIONS ON AN EIGENVALUE PROBLEM FOR DISCRETE FRACTIONAL ORDER EQUATION WITH A CLASS OF BOUNDARY CONDITIONS

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Abstract

The Lyapunov inequality has its importance in the study of broad applications of solutions to differential and difference equations, such as oscillation theory, disconjugacy and eigenvalue problems. This

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paper is devoted to a new Lyapunov-type inequality for discrete fractional order equations with a class of two-point boundary conditions under the concept of the Riemann-Liouville fractional difference operator. We examine some new results for linear and nonlinear Lyapunov-type inequalities by developing suitable Green's function and determining their corresponding maximum value for discrete fractional equations. The associated eigenvalue problem is also examined. We provide a couple of examples to demonstrate the applicability of the findings.

1. Introduction

Fractional differential calculus has become more popular in recent years. It has been explored by many researchers, resulting in a strong mathematical background and numerous articles are credited to its development. Readers can see [1-10] for additional information. Fractional calculus can be used to model physical phenomena such as control systems [11], mechanics, and viscoelasticity [12]. The Riemann-Lioville and Caputo derivatives [1] are two of the most commonly utilized fractional derivatives.

A new field for researchers in the recent years is fractional difference equations. With the fractional difference operators, real-world phenomena are being studied, one can refer to [14]. Nevertheless, some researchers have recently shown a lot of interest in discrete fractional calculus. The existence and uniqueness of the solutions of the discrete FBVPs have been of great interest for over a decade [15-24].

On the other hand, the study of the so-called Lyapunov inequality was first developed in 1907 [25] as follows:

$$\int_c^d |r(\xi)|d\xi > \frac{4}{d-c},$$

for a nonzero solution $u(\iota)$ of the BVP

$$\begin{cases} u(\iota) + r(\iota)u(\iota) = 0, & \iota \in (c, d), \\ u(c) = 0, & u(d) = 0, \end{cases}$$

where r is a real-valued continuous function.

The Lyapunov-type inequality (LTI), along with several of its generalizations, plays a significant role in the analysis of various properties of differential and difference equations in the theory of oscillation, eigenvalues, disconjugacy and criteria of stability for periodic differential equations [13, 26-32]. A great deal of effort has been made over the past few years to obtain valuable results for FBVPs and discrete FBVPs [33-39]. This paper is motivated by the previous works and its focuses on the LTI for the discrete FBVP:

$$\begin{cases} {}^{RL}\Delta^{\vartheta} u(\iota) + r(\iota + \vartheta - 1) u(\iota + \vartheta - 1) = 0, & \iota \in \mathbb{N}_0^{\ell}, \\ u(\vartheta - 3) = 0,\ \Delta u(\vartheta - 3) = 0, & u(\vartheta + \ell) = 0, \end{cases} \tag{1.1}$$

where $r : \mathbb{N}_{\vartheta-2}^{\vartheta+\ell-2} \to [0, \infty)$, ${}^{RL}\Delta^{\vartheta}$ is R-LFDO of order $\vartheta \in (2, 3]$ and $\ell \in \mathbb{N}_2$. Furthermore, we obtain generalized LTI and discuss the application of eigenvalue problems for our proposed discrete FBVPs.

Section 2 concentrates on the essential definitions and preliminary findings. Section 3 establishes a solution to the discrete FBVP (1.1). Some new results of Lyapunov-type inequalities by developing suitable Green's function and determining the corresponding maximum value are presented in in Section 4. Section 5 provides the applications of these inequalities to eigenvalue problems. In Section 6, we discuss examples that highlight the significance of our findings, and the paper is summed up with a brief conclusion in Section 7.

2. Important Results of Discrete Fractional Calculus

We present some basic properties of discrete fractional calculus which are used in the sequel.

Definition 2.1 [40]**.** The *falling function* is described as follows:

$$\iota^{(\vartheta)} := \frac{\Gamma(\iota + 1)}{\Gamma(\iota - \vartheta + 1)},$$

for any $\iota, \vartheta \in \mathbb{R}$.

Definition 2.2 [40]**.** For $\vartheta > 0$, the *fractional sum* of $\mathcal{F} : \mathbb{N}_a \to \mathbb{R}$ is defined as

$$\Delta^{-\vartheta}\mathcal{F}(\iota) := \frac{1}{\Gamma(\vartheta)}\sum_{\xi=a}^{\iota-\vartheta}(\iota - \rho(\xi))^{(\vartheta-1)}\mathcal{F}(\xi),$$

for $\iota \in \mathbb{N}_{a+\vartheta}$ and $\rho(\xi) = \xi + 1$. Also, we define $\Delta^{-0}\mathcal{F}(\iota) := \mathcal{F}(\iota)$, for $\iota \in \mathbb{N}_a$.

Lemma 2.3 [41, 42]**.** *Assume* $\alpha, \vartheta > 0$. *Then the following properties hold*:

(i) $\Delta\iota^{(\vartheta)} = \vartheta\iota^{(\vartheta-1)}$;

(ii) $\vartheta^{(\vartheta)} = \Gamma(\vartheta + 1)$.

Lemma 2.4 [42]**.** *If* $0 \le \mathcal{N} - 1 < \vartheta \le \mathcal{N}$, *then*

$$\Delta^{-\vartheta\, RL}\Delta^{\vartheta}u(\iota) = u(\iota) + \mathcal{A}_1\iota^{(\vartheta-1)} + \mathcal{A}_2\iota^{(\vartheta-2)} + \cdots + \mathcal{A}_{\mathcal{N}}\iota^{(\vartheta-\mathcal{N})},$$

for each $\mathcal{A}_k \in \mathbb{R}$, *with* $1 \le k \le \mathcal{N}$.

3. Solution of a Class of Discrete FBVPs

In this section, we establish a solution of a class of discrete FBVP (1.1). The following theorem concerns a linear variant of (1.1).

Theorem 3.1. *Let* $\Psi : \mathbb{N}_{\vartheta-3}^{\vartheta+\ell} \to \mathbb{R}$ *be given. A function u is a solution of the linear discrete FBVP*:

$$\begin{cases} {}^{RL}\Delta^{\vartheta}u(\iota) + \Psi(\iota + \vartheta - 1) = 0, & \iota \in \mathbb{N}_0^{\ell}, \\ u(\vartheta - 3) = 0,\ \Delta u(\vartheta - 3) = 0, & u(\vartheta + \ell) = 0, \end{cases} \tag{3.1}$$

where $2 < \vartheta \leq 3$, *if and only if* $u(\iota)$, *for* $\iota \in \mathbb{N}_{\vartheta-3}^{\vartheta+\ell}$ *has the form*

$$u(\iota) = -\frac{1}{\Gamma(\vartheta)} \sum_{\xi=a}^{\iota-\vartheta} (\iota - \rho(\xi))^{(\vartheta-1)} \Psi(\xi + \vartheta - 1)$$

$$+ \frac{\iota^{(\vartheta-1)}}{(\vartheta + \ell)^{(\vartheta-1)} \Gamma(\vartheta)} \sum_{\xi=0}^{\ell} (\vartheta + \ell - \rho(\xi))^{(\vartheta-1)} \Psi(\xi + \vartheta - 1). \qquad (3.2)$$

Proof. By using $\Delta^{-\vartheta}$ with Lemma 2.4 on both the sides of (3.1), we have

$$u(\iota) = -\frac{1}{\Gamma(\vartheta)} \sum_{\xi=0}^{\iota-\vartheta} (\iota - \rho(\xi))^{(\vartheta-1)} \Psi(\xi + \vartheta - 1) + \mathcal{A}_1 \iota^{(\vartheta-1)}$$

$$+ \mathcal{A}_2 \iota^{(\vartheta-2)} + \mathcal{A}_3 \iota^{(\vartheta-3)}, \qquad (3.3)$$

where $\mathcal{A}_k \in \mathbb{R}$, for $k = 1, 2, 3$. As a result of the condition $u(\vartheta - 3) = 0$, we get

$$\frac{-1}{\Gamma(\vartheta)} \sum_{\xi=0}^{-3} (\vartheta - 3 - \rho(\xi))^{(\vartheta-1)} \Psi(\xi + \vartheta - 1) + \mathcal{A}_1 (\vartheta - 3)^{(\vartheta-1)}$$

$$+ \mathcal{A}_2 (\vartheta - 3)^{(\vartheta-2)} + \mathcal{A}_3 (\vartheta - 3)^{(\vartheta-3)} = 0. \qquad (3.4)$$

By using Definition 2.1 in (3.4), we conclude that $\mathcal{A}_3 \Gamma(\vartheta - 2) = 0$. Therefore, $\mathcal{A}_3 = 0$. Taking the operator Δ on both the sides of (3.3) with $\mathcal{A}_3 = 0$, we arrive at

$$\Delta u(\iota) = -\Delta^{-(\vartheta-1)} \Psi(\iota + \vartheta - 1) + \mathcal{A}_1 \Delta \iota^{(\vartheta-1)} + \mathcal{A}_2 \Delta \iota^{(\vartheta-2)}. \qquad (3.5)$$

From Lemma 2.3 of (i), Definition 2.2 gives

$$\Delta \iota^{(\vartheta-1)} = (\vartheta - 1) \iota^{(\vartheta-2)}, \quad \Delta \iota^{(\vartheta-2)} = (\vartheta - 2) \iota^{(\vartheta-3)} \qquad (3.6)$$

and

$$\Delta^{-(\vartheta-1)}\Psi(\iota+\vartheta-1)=\frac{1}{\Gamma(\vartheta-1)}\sum_{\xi=0}^{\iota-\vartheta+1}(\iota-\rho(\xi))^{(\vartheta-2)}\Psi(\xi+\vartheta-1). \quad (3.7)$$

So, using (3.5)-(3.7) together, we deduce that

$$\Delta u(\iota)=-\frac{1}{\Gamma(\vartheta-1)}\sum_{\xi=0}^{\iota-\vartheta+1}(\iota-\rho(\xi))^{(\vartheta-2)}\Psi(\xi+\vartheta-1)$$
$$+\mathcal{A}_1(\vartheta-1)\iota^{(\vartheta-1)}+\mathcal{A}_2(\vartheta-2)\iota^{(\vartheta-3)}. \quad (3.8)$$

By using second boundary condition $\Delta u(\vartheta-3)=0$ in (3.8), it follows that $\mathcal{A}_2=0$. From (3.3) along with $\mathcal{A}_2=0$ and $\mathcal{A}_3=0$, we have

$$u(\iota)=-\frac{1}{\Gamma(\vartheta)}\sum_{\xi=0}^{\iota-\vartheta}(\iota-\rho(\xi))^{(\vartheta-1)}\Psi(\xi+\vartheta-1)+\mathcal{A}_1\iota^{(\vartheta-1)}. \quad (3.9)$$

From $u(\vartheta+\ell)=0$, we get

$$u(\vartheta+\ell)=-\frac{1}{\Gamma(\vartheta)}\sum_{\xi=0}^{\ell}(\vartheta+\ell-\rho(\xi))^{(\vartheta-1)}\Psi(\xi+\vartheta-1)+\mathcal{A}_1(\vartheta+\ell)^{(\vartheta-1)}=0.$$

Direct computation yields

$$\mathcal{A}_1=\frac{1}{(\vartheta+\ell)^{(\vartheta-1)}\Gamma(\vartheta)}\sum_{\xi=0}^{\ell}(\vartheta+\ell-\rho(\xi))^{(\vartheta-1)}\Psi(\xi+\vartheta-1).$$

Substituting the value of $\mathcal{A}_1$ in (3.9), the solution of (3.1) satisfies

$$u(\iota)=-\frac{1}{\Gamma(\vartheta)}\sum_{\xi=0}^{\iota-\vartheta}(\iota-\rho(\xi))^{(\vartheta-1)}\Psi(\xi+\vartheta-1)$$
$$+\frac{\iota^{(\vartheta-1)}}{(\vartheta+\ell)^{(\vartheta-1)}\Gamma(\vartheta)}\sum_{\xi=0}^{\ell}(\vartheta+\ell-\rho(\xi))^{(\vartheta-1)}\Psi(\xi+\vartheta-1), \quad (3.10)$$

for $\iota \in \mathbb{N}_{\vartheta-3}^{\vartheta+\ell}$. Conversely, (3.10) is a solution of (3.1) by direct substitution. The proof is complete.

4. Lyapunov-type Inequalities for Linear and Nonlinear Discrete FBVPs

We use the solution from Theorem 3.1 to establish new results for Green's function and LTI for discrete FBVPs.

4.1. Green's function

George Green discovered the theory of Green's functions which is widely accepted and recognized. Green's functions are mainly used to solve non-homogeneous BVPs. In particular, Green's function techniques are commonly used in applied mathematics, physics, especially in aerodynamics, electrodynamics, quantum field theory, and engineering [43, 44]. In this subsection, Green's function formula is discussed as it relates to a three-point discrete FBVP.

Theorem 4.1. *Let* $2 < \vartheta \leq 3$. *Then the discrete FBVP* (1.1) *has a unique solution*

$$u(\iota) = \frac{1}{\Gamma(\vartheta)} \sum_{\xi=0}^{\ell} \mathcal{G}(\iota, \xi) r(\xi + \vartheta - 1) u(\xi + \vartheta - 1), \tag{4.1}$$

where

$$\mathcal{G}(\iota, \xi) : \mathbb{N}_{\vartheta-3}^{\vartheta+\ell} \times \mathbb{N}_0^{\ell} \to \mathbb{R}$$

is defined by

$$\mathcal{G}(\iota, \xi) := \begin{cases} \dfrac{(\vartheta + \ell - \rho(\xi))^{(\vartheta-1)} \iota^{(\vartheta-1)}}{(\vartheta + \ell)^{(\vartheta-1)}} - (\iota - \rho(\xi))^{(\vartheta-1)}, & \text{for } 0 \leq \xi < \iota - \vartheta + 1 \leq \ell, \\ \dfrac{(\vartheta + \ell - \rho(\xi))^{(\vartheta-1)} \iota^{(\vartheta-1)}}{(\vartheta + \ell)^{(\vartheta-1)}}, & \text{for } 0 \leq \iota - \vartheta + 1 \leq \xi \leq \ell. \end{cases} \tag{4.2}$$

Proof. We arrive at equation (3.10) by continuing the proof of Theorem 3.1. Then it follows that

$$u(\iota) = -\frac{1}{\Gamma(\vartheta)}\sum_{\xi=0}^{\iota-\vartheta}(\iota-\rho(\xi))^{(\vartheta-1)}r(\xi+\vartheta-1)u(\xi+\vartheta-1)$$

$$+\frac{1}{\Gamma(\vartheta)}\sum_{\xi=0}^{\iota-\vartheta}\frac{(\vartheta+\ell-\rho(\xi))^{(\vartheta-1)}\iota^{(\vartheta-1)}}{(\vartheta+\ell)^{(\vartheta-1)}}r(\xi+\vartheta-1)u(\xi+\vartheta-1)$$

$$+\frac{1}{\Gamma(\vartheta)}\sum_{\xi=\iota-\vartheta+1}^{\ell}\frac{(\vartheta+\ell-\rho(\xi))^{(\vartheta-1)}\iota^{(\vartheta-1)}}{(\vartheta+\ell)^{(\vartheta-1)}}r(\xi+\vartheta-1)u(\xi+\vartheta-1),$$

$$=\frac{1}{\Gamma(\vartheta)}\left\{\sum_{\xi=0}^{\iota-\vartheta}\left[\frac{(\vartheta+\ell-\rho(\xi))^{(\vartheta-1)}\iota^{(\vartheta-1)}}{(\vartheta+\ell)^{(\vartheta-1)}}-(\iota-\rho(\xi))^{(\vartheta-1)}\right]\right.$$

$$\left.+\sum_{\xi=\iota-\vartheta+1}^{\ell}\left[\frac{(\vartheta+\ell-\rho(\xi))^{(\vartheta-1)}\iota^{(\vartheta-1)}}{(\vartheta+\ell)^{(\vartheta-1)}}\right]\right\}r(\xi+\vartheta-1)u(\xi+\vartheta-1).$$

As a result, it is clear that

$$u(\iota)=\frac{1}{\Gamma(\vartheta)}\sum_{\xi=0}^{\ell}(\iota-\xi)r(\xi+\vartheta-1)u(\xi+\vartheta-1),$$

where $\mathcal{G}(\iota,\xi)$ is given in (4.2).

Lemma 4.2. *The Green's function* $\mathcal{G}(\iota,\xi)$ *described in* (4.2) *satisfies the following conditions*:

(i) $\mathcal{G}(\iota,\xi)\geq 0$ *for each* $\iota\in\mathbb{N}_{\vartheta-3}^{\vartheta+\ell}$ *and* $\xi\in\mathbb{N}_0^{\ell}$;

(ii) $\max_{\iota\in\mathbb{N}_{\vartheta-3}^{\vartheta+\ell}}\mathcal{G}(\iota,\xi)=\mathcal{G}(\xi+\vartheta-1,\xi)$, *for each* $\xi\in\mathbb{N}_0^{\ell}$.

Proof. The proof of Lemma 4.2 is very similar to that in [43].

Theorem 4.3. *The maximum of the function* $\mathcal{G}(\xi + \vartheta - 1, \xi)$ *is given by*

$$\max_{\xi \in \mathbb{N}_0^{\ell}} \mathcal{G}(\xi + \vartheta - 1, \xi) = \frac{\Gamma(\vartheta)\Gamma(\vartheta + \ell)}{\Gamma(\ell + 1)(\vartheta + \ell)^{(\vartheta - 1)}}.$$

Proof. We use the difference operator on $\mathcal{G}(\xi + \vartheta - 1, \xi)$ to find the maximum of $\mathcal{G}(\xi + \vartheta - 1, \xi)$ over ξ. Especially:

$$\Delta \mathcal{G}(\xi + \vartheta - 1, \xi)$$

$$= \Delta\left[\frac{(\xi + \vartheta - 1)^{(\vartheta - 1)}(\vartheta + \ell - \rho(\xi))^{(\vartheta - 1)}}{(\vartheta + \ell)^{(\vartheta - 1)}}\right]$$

$$= \frac{1}{(\vartheta + \ell)^{(\vartheta - 1)}} \Delta\left[\frac{\Gamma(\xi + \vartheta)}{\Gamma(\xi + 1)} \frac{\Gamma(\vartheta + \ell - \xi)}{\Gamma(\ell - \xi + 1)}\right]$$

$$= \frac{1}{(\vartheta + \ell)^{(\vartheta - 1)}} \left[\frac{\Gamma(\xi + \vartheta + 1)}{\Gamma(\xi + 2)} \frac{\Gamma(\vartheta + \ell - \rho(\xi))}{\Gamma(\ell - \xi)} - \frac{\Gamma(\xi + \vartheta)}{\Gamma(\xi + 1)} \frac{\Gamma(\vartheta + \ell - \xi)}{\Gamma(\ell - \xi + 1)}\right]$$

$$= \frac{1}{(\vartheta + \ell)^{(\vartheta - 1)}} \left[\frac{(\ell - \xi)\Gamma(\xi + \vartheta + 1)\Gamma(\vartheta + \ell - \rho(\xi))}{\Gamma(\xi + 2)\Gamma(\ell - \xi + 1)}\right]$$

$$- \frac{1}{(\vartheta + \ell)^{(\vartheta - 1)}} \left[\frac{(\xi + 1)\Gamma(\xi + \vartheta)\Gamma(\vartheta + \ell - \xi)}{\Gamma(\xi + 2)\Gamma(\ell - \xi + 1)}\right],$$

$$= \frac{[(\vartheta - 1)(\ell - 1) - 2\xi(\vartheta - 1)]}{\Gamma(\vartheta + \ell + 1)} \frac{\Gamma(\ell + 2)\Gamma(\xi + \vartheta)\Gamma(\vartheta + \ell - \rho(\xi))}{\Gamma(\xi + 2)\Gamma(\ell - \xi + 1)}.$$

As a result,

$$\Delta \mathcal{G}(\xi + \vartheta - 1, \xi) = [(\vartheta - 1)(\ell - 1) - 2\xi(\vartheta - 1)] F(\xi),$$

with $F(\xi) = \dfrac{\Gamma(\ell + 2)\Gamma(\xi + \vartheta)\Gamma(\vartheta + \ell - \rho(\xi))}{\Gamma(\vartheta + \ell + 1)\Gamma(\xi + 2)\Gamma(\ell - \xi + 1)} > 0$ for all $\xi \in \mathbb{N}_0^{\ell}$.

Now, if $\xi < \frac{(\ell - 1)}{2}$, then $h(\xi) = (\vartheta - 1)(\ell - 1) - 2\xi(\vartheta - 1)$ increases. On the other hand, if $\xi \geq \frac{(\ell - 1)}{2}$, then $h(\xi) = (\vartheta - 1)(\ell - 1) - 2\xi(\vartheta - 1)$ decreases. Therefore,

$$\max_{\xi \in \mathbb{N}_0^{\ell}} \mathcal{G}(\xi + \vartheta - 1, \xi) = \mathcal{G}(\ell + \vartheta - 1, \ell) = \frac{\Gamma(\vartheta)\Gamma(\vartheta + \ell)}{\Gamma(\ell + 1)(\vartheta + \ell)^{(\vartheta - 1)}}.$$

The proof is complete.

4.2. Lyapunov-type inequality for linear BVP

Theorem 4.4. *Let* $r : \mathbb{N}_{\vartheta-2}^{\vartheta+\ell-2} \to [0, \infty)$ *be a nonzero function. If BVP* (1.1) *has a nontrivial solution given by* (4.1), *then*

$$\sum_{\xi=0}^{\ell} |\, r(\xi + \vartheta - 1)\,| \geq \frac{\Gamma(\ell + 1)}{\Gamma(\vartheta + \ell)} (\vartheta + \ell)^{(\vartheta - 1)}. \tag{4.3}$$

Proof. Due to the fact that the discrete FBVP (1.1) has a nonzero solution, we have

$$u(\iota) = \frac{1}{\Gamma(\vartheta)} \sum_{\xi=0}^{\ell} \mathcal{G}(\iota, \xi) r(\xi + \vartheta - 1) u(\xi + \vartheta - 1),$$

$$\| u \| \leq \frac{1}{\Gamma(\vartheta)} \sum_{\xi=0}^{\ell} |\, \mathcal{G}(\iota, \xi)\,|\,|\, r(\xi + \vartheta - 1)\,|\,|\, u(\xi + \vartheta - 1)\,|,$$

$$1 \leq \frac{1}{\Gamma(\vartheta)} \sum_{\xi=0}^{\ell} \mathcal{G}(\xi + \vartheta - 1)|\, r(\xi + \vartheta - 1)\,|.$$

From this, it follows that

$$\sum_{\xi=0}^{\ell} | r(\xi + \vartheta - 1) | \geq \frac{\Gamma(\vartheta)}{\mathcal{G}(\xi + \vartheta - 1, \xi)}$$

$$= \frac{\Gamma(\vartheta)}{\frac{\Gamma(\vartheta)\Gamma(\vartheta + \ell)}{\Gamma(\ell + 1)(\vartheta + \ell)^{(\vartheta - 1)}}}$$

$$= \frac{\Gamma(\ell + 1)}{\Gamma(\vartheta + \ell)} (\vartheta + \ell)^{(\vartheta - 1)}.$$

This completes the proof.

4.3. Generalized Lyapunov-type inequality for nonlinear BVP

Let $u(\iota + \vartheta - 1) = \mathcal{F}(u(\iota + \vartheta - 1))$ in (1.1). Then we get a discrete FBVP of order $\vartheta \in (2, 3]$:

$$\begin{cases} {}^{RL}\Delta^{\vartheta} u(\iota) + r(\iota + \vartheta - 1)\mathcal{F}(u(\iota + \vartheta - 1)) = 0, & \iota \in \mathbb{N}_0^{\ell}, \\ u(\vartheta - 3) = 0, \, \Delta u(\vartheta - 3) = 0, \, u(\vartheta + \ell) = 0, \end{cases} \tag{4.4}$$

where $\mathcal{F} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ is non-decreasing and $r : \mathbb{N}_{\vartheta-2}^{\vartheta+\ell-2} \to [0, \infty)$ is a nontrivial function and $\ell \in \mathbb{N}_2$.

The following corollary is obtained by applying Theorem 4.4.

Corollary 4.5. *If* (4.4) *has a nontrivial solution*, *then*

$$\sum_{\xi=0}^{\ell} | r(\xi + \vartheta - 1) | \geq \frac{\Gamma(\ell + 1)\eta}{\Gamma(\vartheta + \ell)\mathcal{F}(\eta)} (\vartheta + \ell)^{(\vartheta - 1)}, \tag{4.5}$$

where $\eta = \max_{\mathbb{N}_{\vartheta-2}^{\vartheta+\ell}} u(\xi + \vartheta - 1).$

5. Applications to Eigenvalue Problems

In this section, we provide some applications of the Lyapunov-type inequalities (4.3) and (4.5) to discrete fractional eigenvalue BVPs.

Let $r(\iota + \vartheta - 1) = \lambda$ in (1.1). We now discuss a discrete fractional eigenvalue BVP of order $\vartheta \in (2, 3]$ as follows:

$$\begin{cases} {}^{RL}\Delta^{\vartheta} u(\iota) + \lambda u(\iota + \vartheta - 1) = 0, & \iota \in \mathbb{N}_0^{\ell}, \\ u(\vartheta - 3) = 0,\ \Delta u(\vartheta - 3) = 0,\ u(\vartheta + \ell) = 0, \end{cases} \tag{5.1}$$

where $\lambda \in \mathbb{R}$ and $\ell \in \mathbb{N}_2$.

Theorem 5.1. *For any* $\lambda \in \mathbb{R}$, *if* (5.1) *has a nonzero solution*, *then*

$$|\lambda| \geq \frac{(\vartheta + \ell)}{(\ell + 1)^2}.$$

Proof. Applying Theorem 4.4, we obtain

$$\sum_{\xi=0}^{\ell} |\lambda| \geq \frac{\Gamma(\vartheta)}{\mathcal{G}(\xi + \vartheta - 1, \xi)} = \frac{\Gamma(\vartheta)}{\dfrac{\Gamma(\vartheta)\Gamma(\vartheta + \ell)}{\Gamma(\ell + 1)(\vartheta + \ell)^{(\vartheta - 1)}}}.$$

From this, we obtain

$$|\lambda| \geq \frac{(\vartheta + \ell)}{(\ell + 1)^2},$$

which completes the proof.

Let $r(\iota + \vartheta - 1) = \lambda$ in (4.4). Then we get a discrete fractional eigenvalue BVP of order $\vartheta \in (2, 3]$:

$$\begin{cases} {}^{RL}\Delta^{\vartheta} u(\iota) + \lambda \mathcal{F}(u(\iota + \vartheta - 1)) = 0, & \iota \in \mathbb{N}_0^{\ell}, \\ u(\vartheta - 3) = 0,\ \Delta u(\vartheta - 3) = 0,\ u(\vartheta + \ell) = 0, \end{cases} \tag{5.2}$$

where $\lambda \in \mathbb{R}$, $\mathcal{F} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ is a non-decreasing function and $\ell \in \mathbb{N}_2$. Now, we have the following corollary obtained from Theorem 5.1.

Corollary 5.2. *For any* $\lambda \in \mathbb{R}$, *if* (5.2) *has a nontrivial solution*, *then*

$$|\lambda| \geq \frac{(\vartheta + \ell)\eta}{(\ell + 1)^2 \mathcal{F}(\eta)},$$

where $\eta = \max_{\mathbb{N}_{\vartheta-2}^{\vartheta+\ell}} u(\xi + \vartheta - 1)$.

6. Illustrative Examples

Here we shall provide specific examples to illustrate the significance of our findings.

Example 6.1. Consider the following discrete FBVP (1.1):

$$\begin{cases} {}^{RL}\Delta^{2.5}u(\iota) + \left(\dfrac{2(\iota + 1.5) + 1}{3}\right)u(\iota + 1.5) = 0, & \iota \in \mathbb{N}_0^5, \\ u(-0.5) = 0,\ \Delta u(-0.5) = 0,\ u(7.5) = 0. \end{cases}$$

Here $\vartheta = 2.5$, $\ell \in 5$ and $r(\iota) = \dfrac{2\iota + 1}{3}$. We have $r(\iota + 1.5) : \mathbb{N}_{0.5}^{5.5} \to [0, \infty)$ with

$$\sum_{\xi=0}^{5} |r(\xi + 1.5)| = \sum_{\xi=0}^{5} \left|\frac{2\xi + 4}{3}\right| = 18 > 0.$$

Therefore, from Theorem 4.4, we get

$$\sum_{\xi=0}^{5} \left|\frac{2\xi + 4}{3}\right| = 18 \geq \frac{7.5}{6} \approx 1.25.$$

Example 6.2. Consider the following discrete fractional eigenvalue BVP (5.1):

$$\begin{cases} {}^{RL}\Delta^{2.2}u(\iota) + 4u(\iota + 1.2) = 0, & \iota \in \mathbb{N}_0^3, \\ u(-0.8) = 0,\ \Delta u(-0.8) = 0,\ u(5.2) = 0. \end{cases}$$

Here $\vartheta = 2.2$, $\ell = 3$ and $\lambda = 4$. Therefore, from Theorem 5.1, we get

$$|\lambda| = 4 \geq \frac{5.2}{16} \approx 0.3250.$$

7. Conclusion

In this study, we considered a two-point discrete fractional boundary value problem order $\vartheta \in (2, 3]$ with the help of the Riemann-Liouville fractional difference operator. We carried on an essential analysis of the discrete FBVP (1.1) based on the fundamental discrete fractional calculus. In particular, we developed some new results on Lyapunov-type inequalities for discrete FBVPs (1.1). The main results are demonstrated by employing Green's function and the corresponding maximum value. As far as an application is concerned, we provided an example of the eigenvalue problem. It is exciting and challenging to consider Lyapunov-type inequality for linear and nonlinear discrete FBVPs. At the end of the paper, specific examples consistent with the main findings are provided.

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