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MULTIPLICITY OF POSITIVE PERIODIC SOLUTIONS FOR A NICHOLSON-TYPE BLOWFLIES MODEL WITH NONLINEAR DECIMATION TERMS

Yidi Zhao, Shaowen Liu, Yuqi Cao, Qing Ma and Yan Yan*

Department of Mathematics
Northeast Forestry University
Harbin 150040, P. R. China
e-mail: 1940509719@qq.com
979368184@qq.com
2350487389@qq.com
979116942@qq.com
yyan90811@nefu.edu.cn

Abstract

This study considers a Nicholson-type blowflies model with nonlinear decimation terms in a periodic environment. The sufficient condition for this model to have at least two positive periodic solutions is elucidated. Our result is obtained by applying the Krasnoselskii fixed

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*Corresponding author

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point theorem. Example and its simulations are given to illustrate our result.

1. Introduction

Flies are familiar creatures to humans. They multiply rapidly and inhabit nearly all parts of the World in large populations. The lifespan of an adult fly is generally considered to be approximately one month, although it depends on the species and environment. Furthermore, a female fly begins to lay eggs approximately four days after emerging from the pupal stage. The laid eggs will hatch into larvae, which molt twice to become the last instar larvae and then turn into pupae when the shell hardens. As such, a fly undergoes a complete metamorphosis from an egg to an adult in approximately two weeks [13, 19]. The population dynamics of flies have been investigated by numerous researchers. In 1980, Gurney et al. [8] proposed the first-order autonomous differential equation

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-qx(t-\tau)} \tag{1.1}$$

to describe the population dynamics of the Australian sheep blowfly. Here, $x(t)$ is the size of the population at time t; $\delta > 0$ is per capital daily adult mortality rate; $p > 0$ is the maximum per capital daily egg production rate; τ is the time required for a blowfly to mature from an egg to an adult; $1/q$ is the size of the blowfly population when the production function ue^{-qu} takes the maximum value. Many studies based on this model were carried out subsequently (for example, see [3, 9, 17]).

In addition, clinical experiments have shown that seasonal fluctuations of the environment greatly affect the population density and internal composition of organisms. Hence, the periodicity hypotheses of parameters were combined into biological population systems to incorporate periodic environmental variation. Equation (1.2) considering a periodic environment was thus modified to read

$$x'(t) = -\delta(t)x(t) + p(t)x(t-\tau(t))e^{-qx(t-\tau(t))}, \tag{1.2}$$

where $\delta,\ p \in C(R^+, (0, \infty))$ and $\tau \in C(R^+, R^+)$ are continuous and periodic with a common period. The existence of positive periodic solutions of equation (1.2) and its generalizations have been studied by several researchers (refer to [6, 10, 15, 21]).

It has been reported that flies can help pollinate and boost the production of some crops and play a role in some medical and forensic fields [5, 20]. However, it cannot be ignored that flies transmit many diseases directly and indirectly to human beings. A disease called *myiasis* is the infestation of the organs or tissues by the larval stages of flies, it usually occurs when a female fleshfly hatches its eggs in its own body and then releases the larvae into the wounds or necrotic areas of humans [1, 16]. Moreover, according to research results, flies are also known to serve as mechanical vectors of human pathogens. Houseflies are important epidemiologic factors for the spread of turkey coronavirus [4]. Synanthropic flies have also been reported to be involved in the mechanical transmission of infantile trachoma virus [14]. Controlling the scale of fly populations is therefore critical to public health. Some studies have investigated the population dynamics of flies with decimation (harvesting) terms. The existence of positive periodic solutions and other qualitative dynamical properties of the Nicholson's blowflies model with decimation (harvesting) terms were analyzed, we refer the reader to [2, 7, 11, 12, 22].

Experimental evidence has suggested that the production rate of flies drops to zero at both low and high density of flies, and saturates at an appropriate size of fly population. The second term

$$p(t)x(t-\tau(t))e^{-qx(t-\tau(t))}$$

in equation (1.2) represents the current density of flies that is affected by the past fly density. It can be seen that production function $f(u) = ue^{-qu}$ in this production term is a unimodal function that reflects the basic properties of the production process of flies as described above. However, the production rate of flies can suddenly increase due to the accumulation of solid waste in concentrated environments and other similar conditions [18]. To deal with this important but often overlooked phenomenon, a more appropriate production function is imperative.

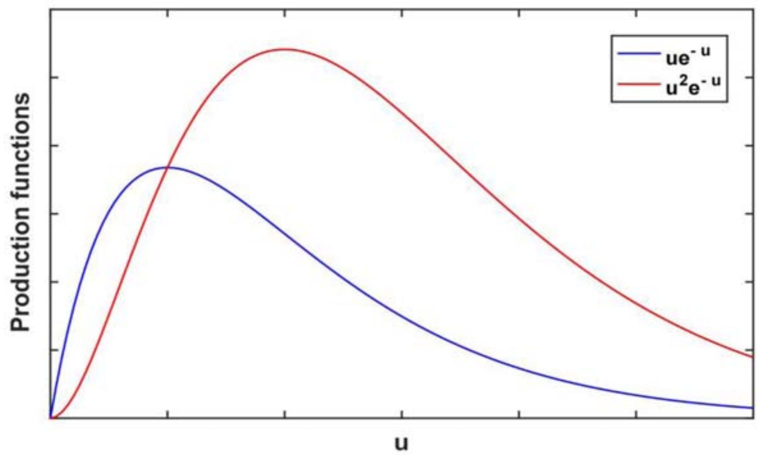

Figure 1. Graphs of production function $f(u) = ue^{-u}$ and $g_i(t, u) = u^2 e^{-u}$ in the case that $q = 1$ and $\gamma = 2$ for all $t \in R$ and $i = 1, 2, ..., m$.

Taking the reason above into account, we consider the generalized Nicholson-type blowflies model

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{m} b_i(t) g_i(t, x(t - \tau_i(t))) - \sum_{j=1}^{n} H_j(t, x(t)), \tag{1.3}$$

where $g_i(t, u) = u^{\gamma} e^{-c_i(t)u}$ for $t \in R$ and $u \in R^+$ with $\gamma > 1$. In this model, the following assumptions will be imposed:

(i) a, b_i and $c_i \in C(R, (0, \infty))$ and $\tau_i \in C(R, R^+)(1 \le i \le m)$ are ω-periodic functions;

(ii) $H_j \in C(R \times R^+, R^+)(1 \le j \le n)$ are nonlinear terms that are ω-periodic functions with respect to t for each fixed $u \in R^+$, i.e.,

$H_j(t, u) = H_j(t + \omega, u)$. Moreover, there exist two constants l and L with $1 \leq l \leq L < \infty$ such that

$$(l - 1)a(t) \leq \frac{\sum_{j=1}^{n} H_j(t, u)}{u} \leq (L - 1)a(t) \tag{1.4}$$

for $t \in R$ and $u \in R^+$.

Let $\bar{\tau} = \max_{1 \leq i \leq m} \{ \max_{0 \leq t \leq \omega} \tau_i(t) \}$. Then we consider equation (1.3) under the initial condition

$$x(t) = \phi(t) > 0 \text{ for } t \in [-\bar{\tau}, 0].$$

The purpose of this paper is to present a sufficient condition which ensures that equation (1.3) has at least two positive ω-periodic solutions. Let $\bar{c} = \max_{1 \leq i \leq m} \{ \sup_{0 \leq t \leq \omega} c_i(t) \}$. Then we define $g(u) = u^{\gamma} / e^{\bar{c}u}$. It can be seen that the function g is strictly increasing on $(0, u_g)$ and strictly decreasing on (u_g, ∞), where $u_g = \gamma / \bar{c}$. For simplicity, we denote

$$\alpha = \frac{1}{\delta^L - 1} \text{ and } \beta = \frac{\delta^l}{\delta^l - 1}, \tag{1.5}$$

in which $\delta = \exp \int_0^{\omega} a(r)dr$. Let $0 < \rho = \alpha / \beta < 1$. Then our main result is as follows:

Theorem 1.1. *Let* $\underline{b}_i = \inf_{0 \leq t \leq \omega} b_i(t)$ *for each* $i = 1, 2, ..., m$. *Suppose that*

$$\alpha\omega \sum_{i=1}^{m} \underline{b}_i g(\rho u_g) > u_g. \tag{1.6}$$

Then equation (1.3) *has at least two positive* ω*-periodic solutions*.

2. Auxiliary Lemmas and Preparations

We begin with the definition of a cone on Banach space. Let X be a Banach space. A closed and nonempty subset $K \subset X$ is said to be a *cone* if

(a) $x \in K,\ y \in K,\ \lambda_1 \geq 0$ and $\lambda_2 \geq 0$ imply $\lambda_1 x + \lambda_2 y \in K$;

(b) $x \in K$ and $-x \in K$ imply that $x = \theta$, where θ is the zero element of K.

The following is the well-known Krasnosel'skii fixed point theorem in a cone.

Lemma 2.1. *Let X be a Banach space, and $K \subset X$ be a cone in X. Assume that Ω_1, Ω_2 are open bounded subsets of X with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$\Phi : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\| \Phi x \| \leq \| x \|, \forall x \in K \cap \partial\Omega_1$ *and* $\| \Phi x \| \geq \| x \|, \forall x \in K \cap \partial\Omega_2$, *or*

(ii) $\| \Phi x \| \geq \| x \|, \forall x \in K \cap \partial\Omega_1$ *and* $\| \Phi x \| \leq \| x \|, \forall x \in K \cap \partial\Omega_2$.

Then Φ has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

We define the set of ω-periodic continuous functions by

$$X = \{x(t) : x(t + \omega) = x(t) \in R,\ t \in R\}. \tag{2.1}$$

It is obvious that X is a Banach space with the norm

$$\| x \| = \sup_{0 \leq t \leq \omega} | x(t) |$$

for any $x \in X$. For an element $x \in X$ satisfying $x(t) > 0$ for $t \in R$, let

$$\hat{x}(t) = x(t) \text{ for } t \in [-\bar{\tau}, \infty)$$

and

$$\hat{\phi}(t) = x(t) \text{ for } t \in [-\bar{\tau}, 0].$$

We can prove that a positive ω-periodic function $\hat{x}(t)$ is the solution of (1.3) with initial function $\phi = \hat{\phi}$ if and only if the original function $x \in X$ satisfies

$$x(t) > 0 \text{ and } x(t) = \int_t^{t+\omega} F(t, s; x) \sum_{i=1}^{m} b_i(s) g_i(s, x(s - \tau_i(s)))ds, \quad (2.2)$$

where

$$F(t, s; x) = \frac{e^{\int_t^s \left(a(r) + \frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)} \right) dr}}{e^{\int_0^\omega \left(a(r) + \frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)} \right) dr} - 1} \text{ for } s \in [t, t + \omega]. \quad (2.3)$$

The inequality $l \le \frac{\sum_{j=1}^n H_j(t, u)}{a(t)u} + 1 \le L$ can be deduced easily from (1.4). Then we can estimate that

$$\delta^l = e^{l \int_0^\omega a(r) dr} \le e^{\int_0^\omega \left(a(r) + \frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)} \right) dr} \le e^{L \int_0^\omega a(r) dr} = \delta^L. \quad (2.4)$$

Hence, we have

$$F(t, t; x) = \frac{e^{\int_t^t \left(a(r) + \frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)} \right) dr}}{e^{\int_0^\omega \left(a(r) + \frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)} \right) dr} - 1}$$

$$= \frac{1}{e^{\int_0^\omega \left(a(r) + \frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)} \right) dr} - 1} \ge \frac{1}{\delta^L - 1}$$

and

$$F(t, t+\omega; x) = \frac{e^{\int_t^{t+\omega}\left(a(r)+\frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)}\right)dr}}{e^{\int_0^{\omega}\left(a(r)+\frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)}\right)dr} - 1}$$

$$= \frac{e^{\int_0^{\omega}\left(a(r)+\frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)}\right)dr}}{e^{\int_0^{\omega}\left(a(r)+\frac{\sum_{j=1}^n H_j(r, x(r))}{x(r)}\right)dr} - 1} \leq \frac{\delta^l}{\delta^l - 1}.$$

Therefore, it follows from (1.5) that

$$\alpha = \frac{1}{\delta^L - 1} \leq F(t, t; x) \leq F(t, s; x) \leq F(t, t+\omega; x) \leq \frac{\delta^l}{\delta^l - 1} = \beta. \quad (2.5)$$

Moreover, the functional F satisfies the periodic relationship

$$F(t+\omega, s+\omega; x) = F(t, s; x). \quad (2.6)$$

Let $K = \{x \in X : x(t) \geq \rho\| x \|\}$ be a cone in X. Then we define an operator Φ by

$$(\Phi x)(t) = \int_t^{t+\omega} F(t, s; x)\sum_{i=1}^m b_i(s)g_i(s, x(s-\tau_i(s)))ds \quad \text{for } x \in K. \quad (2.7)$$

Noting the solution representation (2.2), we see that the solution $x(; \hat{\phi})$ of (1.3) is a fixed point of Φ.

Lemma 2.2. *The operator* $\Phi : K \to K$ *is completely continuous.*

Proof. First, we show that Φ maps K into K. For any $x \in K \subset X$, it follows from (2.6) that

$$(\Phi x)(t+\omega)$$

$$= \int_{t+\omega}^{t+2\omega} F(t+\omega, s; x)\sum_{i=1}^{m} b_i(s) g_i(s, x(s-\tau_i(s)))ds$$

$$= \int_{t}^{t+\omega} F(t+\omega, s+\omega; x)\sum_{i=1}^{m} b_i(s+\omega) g_i(s+\omega, x(s+\omega-\tau_i(s+\omega)))ds$$

$$= \int_{t}^{t+\omega} F(t, s; x)\sum_{i=1}^{m} b_i(s) g_i(s, x(s-\tau_i(s)))ds = (\Phi x)(t).$$

Hence, $\Phi x \in X$.

From (2.5), we obtain

$$(\Phi x)(t) \leq \beta \int_{t}^{t+\omega} \sum_{i=1}^{m} b_i(s) g_i(s, x(s-\tau_i(s)))ds$$

and

$$(\Phi x)(t) \geq \alpha \int_{t}^{t+\omega} \sum_{i=1}^{m} b_i(s) g_i(s, x(s-\tau_i(s)))ds.$$

Hence, we have

$$\| \Phi \| = \sup_{0 \leq t \leq \omega} |(\Phi x)(t)| \leq \frac{\beta}{\alpha}(\Phi x)(t),$$

which implies that

$$(\Phi x)(t) \geq \rho \| \Phi \|.$$

Thus, we see that $\Phi x \in K$. By a basic calculation, we can verify that Φ is completely continuous. We omit its proof.

3. Proof of Theorem 1.1

We will prove Theorem 1.1 by means of Lemma 2.1. Since the operator Φ defined by (2.7) has the property shown in Lemma 2.2, it is sufficient to check the assumptions (i) and (ii) to apply Lemma 2.1 to our result.

Let $\overline{b_i} = \sup_{0 \le t \le \omega} b_i(t)$ for $i = 1, 2, ..., m$. Then we choose a constant ε_0 satisfying $0 < \varepsilon_0 < 1/\beta\omega\sum_{i=1}^{m} \overline{b_i}$, in which $\beta = \delta^i/(\delta^L - 1)$ is given in Section 1. Let $\underline{c} = \min_{1 \le i \le m} \inf_{0 \le t \le \omega} c_i(t)$. Then we define $h(u) = u^\gamma/e^{\underline{c}u}$ for $u \ge 0$. The function h is strictly increasing on $(0, u_h)$ and strictly decreasing on (u_h, ∞), where $u_h = \gamma/\underline{c}$. Moreover,

$$\lim_{u \to 0} \frac{h(u)}{u} = \lim_{u \to \infty} h(u) = 0.$$

Hence, we can find numbers v_1 and v_2 with $0 < v_1 < u_g < v_2$ such that

$$h(u) < \varepsilon_0 u < \varepsilon_0 v_1 \text{ for } 0 \le u \le v_1, \tag{3.1}$$

and

$$h(u) < \varepsilon_0 v_2 \text{ for } u \ge v_2, \tag{3.2}$$

where $u_g = \gamma/\bar{c}$ is given in Section 1. Let $v_3 = v_2/\rho > v_2$. Recall that $g(u) = u^\gamma/e^{\bar{c}u}$. From the condition (1.6), there exists a constant $\delta_0 \in (0, v_2 - u_g)$ such that

$$\alpha\omega\sum_{i=1}^{m} \underline{b}_i g(\rho u_g) > u_g + \delta_0. \tag{3.3}$$

We define subsets Ω_1, Ω_2, Ω_3 and Ω_4 of X as follows:

$$\Omega_1 = \{x \in X : \|x\| < v_1\}; \quad \Omega_2 = \{x \in X : \|x\| < u_g\};$$

$$\Omega_3 = \{x \in X : \|x\| < u_g + \delta_0\}; \quad \Omega_4 = \{x \in X : \|x\| < v_3\}.$$

Then $\Omega_i (1 \le i \le 4)$ are open and bounded subsets of X, and satisfy that

$$\Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_3 \subset \overline{\Omega}_3 \subset \Omega_4 \subset \overline{\Omega}_4.$$

Certainly, the zero element θ of K belongs to Ω_1.

Part (1). Let x be any element of $K \cap \partial\Omega_1 \subset X$. Then x is a ω-periodic function and satisfies $\| x \| = v_1$. Hence, we have

$$0 \le x(s - \tau_i(s)) \le v_1 \text{ for } s \in R \text{ and } i = 1, 2, ..., m. \tag{3.4}$$

For each $1 \le i \le m$, we see that $g_i(t, u) = u^\gamma e^{-c_i(t)u} \le u^\gamma / e^{\underline{c}u} = h(u)$ for $t \in R$ and $u \ge 0$. From (2.5), (3.1) and (3.4), we obtain

$$(\Phi x)(t) < \beta \int_t^{t+\omega} \left(\sum_{i=1}^m \bar{b}_i h(x(s - \tau_i(s))) \right) ds \le \beta\omega\varepsilon_0 \sum_{i=1}^m \bar{b}_i v_1 < v_1 \text{ for } t \in R.$$

Therefore, $\| \Phi x \| < v_1 = \| x \|$ for $x \in K \cap \partial\Omega_1$.

Part (2). Let x be any element of $K \cap \partial\Omega_2 \subset X$. Then x is an ω-periodic function and satisfies $x(t) \ge \rho \| x \| = \rho u_g$ for $t \in R$. Hence, it follows that

$$\rho u_g \le x(s - \tau_i(s)) \le u_g \text{ for } s \in R \text{ and } i = 1, 2, ..., m. \tag{3.5}$$

From the unimodal property of g, we see that

$$\min_{\rho u_g \le u \le u_g} g(u) = g(\rho u_g). \tag{3.6}$$

For each $1 \le i \le m$, we see that $g_i(t, u) = u^\gamma e^{-c_i(t)u} \ge u^\gamma / e^{\bar{c}u} = g(u)$ for $t \in R$ and $u \ge 0$. By (1.6), (2.5), (3.5) and (3.6), we have

$$(\Phi x)(t) > \alpha \int_t^{t+\omega} \left(\sum_{i=1}^m \underline{b}_i g(x(s - \tau_i(s))) \right) ds \ge \alpha\omega \sum_{i=1}^m \underline{b}_i g(\rho u_g) > u_g$$

for $t \in R$. This leads to $\| \Phi x \| > u_g = \| x \|$ for $x \in K \cap \partial\Omega_2$.

Hence, we checked that the assumption (i) of Lemma 2.1 is satisfied. Lemma 2.1 shows that the operator Φ defined by (2.7) has a fixed point x_1 in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point x_1 satisfies that $x_1(t) \ge \rho \| x_1 \|$ for $t \in R$

and $0 < v_1 \leq \| x_1 \| \leq u_g$. Namely, we see that x_1 is a positive ω-periodic function. Let $\hat{x}_1(t) = x_1(t)$ for $t \in [-\bar{\tau}, \infty)$ and $\hat{\phi}_1(t) = x_1(t)$ for $t \in [-\bar{\tau}, 0]$. Then $\hat{x}_1$ is a solution of (1.3) with the initial function $\phi = \hat{\phi}_1$.

Part (3). Let x be any element of $K \cap \partial\Omega_3 \subset X$. Then x is an ω-periodic function and satisfies $x(t) \geq \rho\| x \| = \rho(u_g + \delta_0)$ for $t \in R$. Hence, we have

$$\rho(u_g + \delta_0) \leq x(s - \tau_i(s)) \leq u_g + \delta_0 \text{ for } s \in R \text{ and } i = 1, 2, ..., m.$$

Since δ_0 is a sufficiently small positive constant, we have the relationship that

$$\min_{\rho(u_g + \delta_0) \leq u \leq u_g + \delta_0} g(u) = g(\rho(u_g + \delta_0)) > g(\rho u_g).$$

From (2.5) and the above two inequalities, we obtain

$$(\Phi x)(t) > \alpha \int_t^{t+\omega} \left(\sum_{i=1}^m \underline{b}_i g(x(s - \tau_i(s))) \right) ds \geq \alpha\omega \sum_{i=1}^m \underline{b}_i g(\rho(u_g + \delta_0))$$

$$> \alpha\omega \sum_{i=1}^m \underline{b}_i g(\rho u_g)$$

for $t \in R$. Then (3.3) derives that $\| \Phi x \| > u_g + \delta_0 = \| x \|$ for $x \in K \cap \partial\Omega_3$.

Part (4). Let x be any element of $K \cap \partial\Omega_4 \subset X$. Then x is an ω-periodic function and satisfies $\| x \| = v_3$ and $\rho v_3 \leq x(s - \tau_i(s)) \leq v_3$ for $s \in R$ and $i = 1, 2, ..., m$. Note that $u_g + \delta_0 < v_2 = \rho v_3$. From (2.5) and (3.2), we have

$$(\Phi x)(t) \leq \beta \int_t^{t+\omega} \left(\sum_{i=1}^m \bar{b}_i h(x(s - \tau_i(s))) \right) ds \leq \beta\varepsilon_0\omega \sum_{i=1}^m \bar{b}_i v_2 < v_2 < v_3$$

for $t \in R$, which implies that $\| \Phi x \| < v_3 = \| x \|$ for $x \in K \cap \partial\Omega_4$.

Hence, we checked that the assumption (ii) of Lemma 2.1 is satisfied. Lemma 2.1 shows that the operator Φ defined by (2.7) has a fixed point x_2 in $K \cap (\overline{\Omega}_4 \backslash \Omega_3)$. This fixed point x_2 satisfies that $x_2(t) \geq \rho \| x_2 \|$ for $t \in R$ and $0 < u_g + \delta_0 \leq \| x_2 \| \leq v_3$. Hence, x_2 is a positive ω-periodic function. Let $\hat{x}_2(t) = x_2(t)$ for $t \in [-\bar{\tau}, \infty)$ and $\hat{\phi}_2(t) = x_2(t)$ for $t \in [-\bar{\tau}, 0]$. Then $\hat{x}_2$ is a solution of (1.3) with the initial function $\phi = \hat{\phi}_2$.

4. Example and Numerical Simulation

We will give an example to illustrate Theorem 1.1 in this section.

Example 4.1. Consider the equation

$$x'(t) = -a(t)x(t) + \sum_{i=1}^{2} b_i(t)x^2(t - \tau_i(t))e^{-c_i(t)x(t-\tau_i(t))} - \sum_{j=1}^{2} H_j(t, x(t)). \tag{4.1}$$

Let

$$a(t) = \frac{1}{4} + \frac{1}{6}\sin\frac{\pi}{2}t, \tag{4.2}$$

$$b_1(t) = 22 + 2\sin\frac{\pi}{2}t, \quad b_2(t) = 6 + \cos\frac{\pi}{2}t, \tag{4.3}$$

$$c_1(t) = \frac{3}{4} + \frac{1}{4}\cos\frac{\pi}{2}, \quad c_2(t) = \frac{2}{3} + \frac{1}{3}\sin\frac{\pi}{2}t, \tag{4.4}$$

$$H_1(t, u) = \left(\frac{1}{40} + \frac{1}{480}\sin\frac{\pi}{2}t\right)u, \quad H_2(t, u) = \left(\frac{1}{48} + \frac{1}{480}\cos\frac{\pi}{2}t\right)u. \tag{4.5}$$

The delays are defined by $\tau_1(t) = 5$ and $\tau_2(t) = 1$. Then equation (4.1) has at least two positive 4-periodic solutions.

It is clear that the period $\omega = 4$. From (4.2), it turns out that

$$\delta = \exp\left\{\int_0^4 \left(\frac{1}{4} + \frac{1}{6}\sin\frac{\pi}{2}r\right)dr\right\} = e.$$

In view of (4.5), we have

$$\frac{1}{10} = \frac{\frac{1}{40} - \frac{1}{480} + \frac{1}{48} - \frac{1}{480}}{\frac{1}{4} + \frac{1}{6}} \leq \frac{\sum_{j=1}^{2} H_j(t, u)}{ua(t)}$$

$$\leq \frac{\frac{1}{40} + \frac{1}{480} + \frac{1}{48} + \frac{1}{480}}{\frac{1}{4} - \frac{1}{6}} = \frac{3}{5},$$

and hence, $l = 1.1$ and $L = 1.6$. Then we see that

$$\alpha = \frac{1}{e^{1.6} - 1} = 0.25297\cdots \text{ and } \rho = \frac{e^{1.1} - 1}{e^{1.1}(e^{1.6} - 1)} = 0.16876\cdots.$$

It follows from (4.4) that $\bar{c} = \max\{1, 1\} = 1.$ Note that $\gamma = 2.$ Then the function $g(u) = u^{\gamma}/e^{\bar{c}u} = u^2/e^u$ and $u_g = 2.$ Because of the unimodal property of g, we can estimate that

$$g(\rho u_g) > g(0.33752) = \frac{0.33752^2}{e^{0.33752}} = 0.08128\cdots.$$

Moreover, (4.3) implies that $\underline{b}_1 = 20$ and $\underline{b}_2 = 5.$ Therefore, we obtain

$$\alpha\omega\sum_{i=1}^{2} \underline{b}_i g(\rho u_g) > 0.25297 \times 4 \times (20 + 5) \times 0.08128$$

$$= 2.05614\cdots > 2 = u_g.$$

Thus, condition (1.6) is satisfied. Theorem 1.1 shows that equation (4.1) has at least two positive 4-periodic solutions.

In Figure 1, the two solutions of (4.1) are drawn. Numerical simulations show that the two solutions approach a periodic solution as t increases. In other words, equation (4.1) has a positive 4-periodic solution that is asymptotically stable.

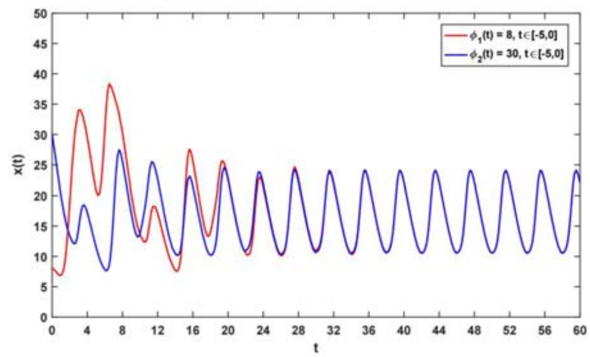

Figure 2. Numerical simulations of solutions of (4.1) with initial function $\phi_1(t) = 8$ and $\phi_2(t) = 30$.

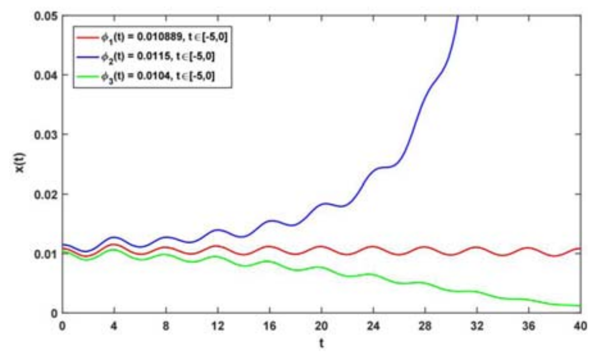

Figure 3. Equation (4.1) has a positive 4-periodic solution with the initial function $\phi_1(t) = 0.010889$ for $t \in [-5, 0]$. This positive 4-periodic solution is unstable.

Theorem 1.1 guarantees the existence of at least two positive periodic solutions of (4.1). Hence, equation (4.1) has another positive 4-periodic solution. This periodic solution is generally considered to be an unstable solution. In this example, we can fortunately simulate an unstable positive 4-periodic solution (see Figure 2). The approximate value of the initial function of the unstable periodic solution is $\phi_1(t) = 0.010889$ for $t \in [-5, 0]$.

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