Advances in Differential Equations and Control Processes
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324322021
Volume 28, 2022, Pages 1-28 **P-ISSN: 0974-3243**

SOME RESULTS ON NONLINEAR MIXED FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Received: January 18, 2022; Accepted: April 25, 2022

2020 Mathematics Subject Classification: 26A33, 34A08, 34A12, 47H10.

Keywords and phrases: fractional mixed integrodifferential equation, existence and uniqueness of solution, fixed point theorem, integral inequality.

How to cite this article: H. L. Tidke, V. V. Kharat and G. N. More, Some results on nonlinear mixed fractional integrodifferential equations with nonlocal conditions, Advances in Differential Equations and Control Processes 28 (2022), 1-28.
http://dx.doi.org/10.17654/0974324322021

Published Online: May 28, 2022

Abstract

In this paper, we study the existence, uniqueness and other properties of solutions of fractional Volterra Fredholm integrodifferential equation involving Caputo fractional derivative of special class $n-1 < \alpha \leq n, \quad n > 1$. The result of existence and uniqueness is obtained with help of well known Banach contraction principle and the integral inequality which provides explicit bound on the unknown function. The obtained some results are illustrated through example.

1. Introduction

In the present paper, we study existence, uniqueness and other properties of solutions of the following nonlinear Caputo fractional mixed integrodifferential equations with constant coefficient $\lambda \in (0, 1)$ of the form:

$$ {}^{c}D^{\alpha} y(t) = \lambda y(t) + f\left(t,\ y(t), \int_0^t k(s,\ y(s))ds, \int_0^b h(s,\ y(s))ds\right), \quad (1.1)$$

for $t \in [0, b] = I,\ n-1 < \alpha \leq n,\ n > 1,\ \lambda \in (0, 1);$ with nonlocal conditions:

$$y^{(j)}(0) = c_j + g_j(y),\ (j = 0, 1, 2, ..., n-1), \quad (1.2)$$

where $f : I \times X \times X \times X \to X$ and $k,\ h : I \times X \to X$ and

$$g_j : C([0, b],\ X) \to X\ (j = 0, 1, 2, ..., n-1)$$

are continuous functions and $c_j\ (j = 0, 1, 2, ..., n-1)$ are given points in X.

For the most of differential or integrodifferential equations of fractional order, we know that every solution is presented in terms of equivalent integral equation with singular kernel and few inequalities are there to study other properties of special version of such equations. Further, in case of singular kernel, there several research papers in the literature using the fact that $(t-s)^{\alpha-1} \leq b^{\alpha},\ s \leq t \in [0, b]$ with $0 < \alpha < 1$. This is not the correct, in fact for $\alpha = \frac{1}{2}$ and the interval $[0, 1]$ with $t = \frac{1}{2},\ s = \frac{1}{3},$ one can observe that

$$(t - s)^{\alpha - 1} = \left(\frac{1}{2} - \frac{1}{3}\right)^{\frac{1}{2} - 1} = \left(\frac{3 - 2}{6}\right)^{-\frac{1}{2}} = \left(\frac{1}{6}\right)^{-\frac{1}{2}} = \sqrt{6} \nleq b^{\alpha} = 1^{\frac{1}{2}} = 1.$$

By keeping these in mind, authors considered a class of special equations where singularities are removed and we are free to use general integral inequalities to discuss the various properties of solutions. This study may be the new motivation towards the class of more general type.

Recently, several researchers have been studied the results such as existence, uniqueness and other properties of solutions for the nonlinear fractional equations involving various types of fractional derivatives by different techniques, see [2-8, 10, 13, 14, 17-19] and the detailed literature for fractional calculus can be found in [1, 9, 11, 12, 16, 20].

The paper is organized as follows. In Section 2, we present the preliminaries and hypotheses. Section 3 deals with the existence and uniqueness of the solution employing contraction principle. Section 4 is devoted to the existence of at most one solution and estimates on solutions via inequality. In Section 5, we discuss results on continuous dependence of solutions on initial data, functions involved therein and parameters. In the final Section 6, we present the suitable example to demonstrate the results.

2. Preliminaries

Before proceeding to the statement of our main results, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

Suppose $L^1(I)$ denotes the space of Lebesgue-integrable functions $y : I \to \mathbf{X}$ with the norm

$$\| y \|_{L^1} = \int_0^b | y(t) | dt.$$

Definition 2.1 [16]**.** The *Riemann-Liouville fractional integral* of a function $h \in L^1(I, \mathbf{R}_+)$ of order $\alpha \in \mathbf{R}_+$ is defined by

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds,$$

where Γ is the Euler gamma function.

Definition 2.2 [9]**.** The *Caputo fractional derivative* of order $\alpha > 0$ of a function $h \in L^1(I, \mathbf{R}_+)$ is defined as

$${}^cD^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3 [9]**.** *Let* $\alpha > 0$ *and* $n = [\alpha] + 1$. *Then*

$$I^{\alpha}({}^cD^{\alpha}f(t)) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}t^k,$$

where $f^{(k)}(t)$ *is the usual derivative of* $f(t)$ *of order k.*

Lemma 2.4 [16]**.** *For* $\alpha > 0$, *the fractional differential equation*

$${}^cD^{\alpha}h(t) = 0,$$

has a solution $h(t) = c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}$, *where* c_i, $i = 0, 1, 2, \ldots, n-1$ *are constants and* $n = [\alpha] + 1$.

Let *X* be a Banach space with norm $\|\cdot\|$ and $I = [0, b]$ denotes an interval of the real line $\mathbb{R}$. We define $B = C^r(I, X)$ (where $r = n$ for $\alpha \in \mathbb{N}$ and $r = n-1$ for $\alpha \notin \mathbb{N}$) is a Banach space of all continuous functions from *I* into *X*, endowed with the norm

$$\|x\|_B = \sup\{\|x(t)\| : x \in B\}, \quad t \in I.$$

From the above lemma, it is easy to observe that if $y \in B$, then $y(t)$ satisfies the following integral equation which is equivalent to (1.1)-(1.2):

$$y(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!} t^j + g_0(y) + \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} g_j(y) ds$$
$$+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s,\ y(s),\ \int_0^s k(\tau,\ y(\tau)) d\tau,\ \int_0^b h(\tau,\ y(\tau)) d\tau\right) ds. \tag{2.1}$$

We require the following lemma known as Pachpatte's inequality in our further discussion.

Lemma 2.5 [15]. *Let* $u(t),\ p(t),\ q(t),\ r(t) \in C([a, b], \mathbb{R}_+)$ *and* $c \geq 0$ *be a real constant and for* $t \in [a, b]$,

$$u(t) \leq c + \int_a^t p(s)\left[u(s) + \int_a^s q(\sigma)u(\sigma)d\sigma + \int_a^b r(\sigma)u(\sigma)d\sigma\right] ds.$$

If

$$d = \int_a^b r(\sigma) \exp\left(\int_a^\sigma [p(\tau) + q(\tau)] d\tau\right) d\sigma < 1,$$

then

$$u(t) \leq \frac{c}{1-d} \exp\left(\int_a^t [p(s) + q(s)] ds\right), \text{ for } t \in [a, b].$$

We list the following hypotheses for our convenience.

(H_1) $f : I \times X \times X \times X \to X$ is a continuous and there exists a function $p_1 \in C(I, \mathbb{R}_+)$ such that

$$\| f(t,\ x(t),\ y(t),\ z(t)) - f(t,\ \bar{x}(t),\ \bar{y}(t),\ \bar{z}(t)) \|$$
$$\leq p_1(t)[\| x(t) - \bar{x}(t) \| + \| y(t) - \bar{y}(t) \| + \| z(t) - \bar{z}(t) \|].$$

(H_2) $k, h : I \times X \to X$ are continuous functions and there exist functions $p_2, p_3 \in C(I, \mathbb{R}_+)$ such that

$$\| k(t, x) - k(t, y) \| \leq p_2(t) \| x(t) - y(t) \|,$$

and

$$\| h(t, x) - h(t, y) \| \leq p_3(t) \| x(t) - y(t) \|.$$

(H_3) Each $g_j : C(I, X) \to X, (j = 0, 1, ..., n-1)$ are continuous functions and there exist constants $L_j, (j = 0, 1, ..., n-1)$ such that

$$\| g_j(y) - g_j(z) \| \leq L_j \| y(t) - z(t) \|.$$

(H_4) Assume that $N = \sup_{t \in I} \left\| f\left(t, 0, \int_0^t k(s, 0)ds, \int_0^b h(s, 0)ds\right) \right\|$ and

$$M = \sup_{t \in I} \sum_{j=0}^{n-1} \frac{\|c_j\|}{j!} t^j \text{ and } G_j = \| g_j(0) \|, 0 \leq j \leq n-1.$$

3. Existence and Uniqueness

The following theorem deals with existence and uniqueness of solution of the problem (1.1)-(1.2).

Theorem 3.1. *Assume that hypotheses* (H_1)-(H_4) *hold. If*

$$\beta = L + \left[\frac{\lambda + P_1(1 + (P_2 + P_3)b)}{\Gamma(\alpha + 1)} \right] b^\alpha < 1,$$

where $P_i = \sup_{t \in I} \{p_i(t)\}$, $(i = 1, 2, 3)$, *and* $G = \sup_{t \in I} \sum_{j=0}^{n-1} \frac{G_j t^j}{j!}$ *and* $L = \sup_{t \in I} \sum_{j=0}^{n-1} \frac{L_j t^j}{j!}$, *then the nonlocal problem* (1.1)-(1.2) *has a unique solution at* $y \in B$ *on I.*

Proof. We use the Banach contraction principle to prove existence and uniqueness of solution to the problem (1.1)-(1.2). Let

$$E_r = \{y \in B : \| y \|_B \le r\},$$

where

$$r \ge \left[1 - L + \left(\frac{\lambda + P_1(1 + (P_2 + P_3)b)}{\Gamma(\alpha + 1)}\right)b^{\alpha}\right]^{-1}\left[M + G + \frac{Nb^{\alpha}}{\Gamma(\alpha + 1)}\right]$$

be closed and bounded set. Define an operator on the Banach space B by

$$(Ty)(t) = \sum_{j=0}^{n-1} \frac{c_j}{j!} t^j + g_0(y) + \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} g_j(y) ds$$

$$+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \int_0^s k(\tau, y(\tau)) d\tau, \int_0^b h(\tau, y(\tau)) d\tau\right) ds. \tag{3.1}$$

Firstly, we show that the operator T maps E_r into itself.

By using hypotheses, we have

$$\| (Ty)(t) \|$$

$$\le \sum_{j=0}^{n-1} \frac{\| c_j \|}{j!} t^j + \| g_0(y) \| + \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \| g_j(y) \| ds$$

$$+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| y(s) \| ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\left(s, y(s), \int_0^s k(\tau, y(\tau)) d\tau, \int_0^b h(\tau, y(\tau)) d\tau\right) \right\| ds$$

$$\leq M + \| g_0(y) - g_0(0) \| + \| g_0(0) \| + \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \| g_j(0) \| ds$$

$$+ \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \| g_j(y) - g_j(0) \| ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} r ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\left(s, 0, \int_0^s k(\tau, 0) d\tau, \int_0^b h(\tau, 0) d\tau\right) \right\| ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| f\left(s, y(s), \int_0^s k(\tau, y(\tau)) d\tau, \int_0^b h(\tau, y(\tau)) d\tau\right) \right.$$

$$\left. - f\left(s, 0, \int_0^s k(\tau, 0) d\tau, \int_0^b h(\tau, 0) d\tau\right) \right\| ds$$

$$\leq M + G_0 + \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} G_j ds + L_0 \| y(t) \|$$

$$+ \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} L_j \| y(s) \| ds$$

$$+ \frac{N b^{\alpha}}{\Gamma(\alpha+1)} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} r ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p_1(s)]$$

$$\cdot \left[\| y(s) \| + \int_0^s p_2(\tau) \| y(\tau) \| d\tau + \int_0^b p_3(\tau) \| y(\tau) \| d\tau \right] ds$$

$$\leq M + \left(G_0 + \sum_{j=1}^{n-1} G_j \frac{t^j}{j!} \right) + \left(L_0 + \sum_{j=1}^{n-1} L_j \frac{t^j}{j!} \right) \| y(s) \|_B$$

$$+ \frac{N b^{\alpha}}{\Gamma(\alpha+1)} + \frac{\lambda b^{\alpha}}{\Gamma(\alpha+1)} r$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} P_1[r+P_2br+P_3br]ds$$

$$\leq M+\sum_{j=0}^{n-1} G_j \frac{t^j}{j!}+\sum_{j=0}^{n-1} L_j \frac{t^j}{j!} r+\frac{Nb^\alpha}{\Gamma(\alpha+1)}+\frac{\lambda b^\alpha}{\Gamma(\alpha+1)} r$$

$$+\frac{P_1[1+(P_2+P_3)b]b^\alpha r}{\Gamma(\alpha+1)}$$

$$\leq M+G+Lr+\frac{Nb^\alpha}{\Gamma(\alpha+1)}+\left(\frac{\lambda+P_1[1+(P_2+P_3)b]}{\Gamma(\alpha+1)}\right)b^\alpha r$$

$$\leq \left(M+G+\frac{Nb^\alpha}{\Gamma(\alpha+1)}\right)+\left[L+\left(\frac{\lambda+P_1[1+(P_2+P_3)]}{\Gamma(\alpha+1)}\right)b^\alpha\right]r$$

$$\leq \left(1-\left[L+\left(\frac{\lambda+P_1[1+(P_2+P_3)]}{\Gamma(\alpha+1)}\right)b^\alpha\right]\right)r$$

$$+\left[L+\left(\frac{\lambda+P_1[1+(P_2+P_3)]}{\Gamma(\alpha+1)}\right)b^\alpha\right]r$$

$$= r.$$

Thus,

$$\|(Ty)\|_B \leq r. \tag{3.2}$$

The equation (3.2) shows that the operator T maps E_r into itself.

Now, for every $x,\ y\in E_r$ and for $t\in I$, we obtain

$$\|(Tx)(t)-(Ty)(t)\|$$

$$\leq \|g_0(x)-g_0(y)\|+\sum_{j=1}^{n-1}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}\|g_j(x)-g_j(y)\|ds$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|x(s)-y(s)\|ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\| f\left(s,\ x(s),\ \int_0^s k(\tau,\ x(\tau))d\tau,\ \int_0^b h(\tau,\ x(\tau))d\tau\right)\right.$$

$$\left. - f\left(s,\ y(s),\ \int_0^s k(\tau,\ y(\tau))d\tau,\ \int_0^b h(\tau,\ y(\tau))d\tau\right)\right\| ds$$

$$\leq L_0 \| x(t) - y(t) \| + \sum_{j=1}^{n-1} L_j \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \| x(s) - y(s) \| ds$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \| x(s) - y(s) \| ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p_1(s)$$

$$\times\left[\| x(s) - y(s) \| + \int_0^s p_2(\tau) \| x(\tau) - y(\tau) \| d\tau \right.$$

$$\left. + \int_0^b p_3(\tau) \| x(\tau) - y(\tau) \| d\tau \right] ds$$

$$\leq L_0 \| x - y \|_B + \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} L_j \| x - y \|_B + \frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} \| x - y \|_B$$

$$+ \frac{1}{\Gamma(\alpha)} P_1 [1 + P_2 b + P_3 b] \frac{t^{\alpha}}{\alpha} \| x - y \|_B$$

$$\leq L_0 \| x - y \|_B + \sum_{j=1}^{n-1} \frac{t^j}{j!} L_j \| x - y \|_B$$

$$+ \left[\frac{\lambda b^{\alpha}}{\Gamma(\alpha+1)} + \frac{P_1 [1 + P_2 b + P_3 b]}{\Gamma(\alpha+1)} b^{\alpha} \right] \| x - y \|_B$$

$$\leq \sum_{j=0}^{n-1} \frac{t^j}{j!} L_j \| x - y \|_B + \left(\frac{\lambda + P_1 [1 + (P_2 + P_3) b]}{\Gamma(\alpha+1)} \right) b^{\alpha} \| x - y \|_B$$

$$\leq \left[L + \left(\frac{\lambda + P_1 [1 + (P_2 + P_3) b]}{\Gamma(\alpha+1)} \right) b^{\alpha} \right] \| x - y \|_B.$$

Hence, we have

$$\| (Tx) - (Ty) \|_B \leq \beta \| x - y \|_B,$$

where $0 < \beta < 1$. This proves that the operator T is a contraction on the complete metric space B. Therefore, by Banach fixed point theorem, the operator T has a unique fixed point in the space B and this is the required unique solution of the nonlocal problem (1.1)-(1.2) on I. □

4. Estimates on Solutions

The following theorem deals with the uniqueness of solutions to the nonlocal problem (1.1)-(1.2) without the existence part.

Theorem 4.1. *Suppose that the hypotheses* (H_1)-(H_3) *hold and*

$$d = \int_0^b p_3(s) \exp\left(\int_0^s \left[\frac{A(\tau)}{1 - L_0} + p_2(\tau) \right] d\tau \right) ds < 1,$$

where

$$A(s) = \left[\left(\frac{\lambda + p_1(s)}{\Gamma(\alpha)} \right) (b - s)^{\alpha - 1} + \sum_{j=1}^{n-1} L_j \frac{(b - s)^{j-1}}{(j - 1)!} \right].$$

Then the nonlocal problem (1.1)-(1.2) *has at most one solution on* I.

Proof. Let $y(t)$ and $z(t)$ be two solutions of the problem (1.1)-(1.2) and $u(t) = \| y(t) - z(t) \|$, $t \in I$. Now by using hypotheses, we have

$$\begin{aligned} & u(t) \\ & = \| y(t) - z(t) \| \\ & \leq L_0 \| y(t) - z(t) \| + \sum_{j=1}^{n-1} \int_0^t \frac{(t - s)^{j-1}}{(j - 1)!} L_j \| y(s) - z(s) \| ds \\ & \quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \| y(s) - z(s) \| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} p_1(s) \end{aligned}$$

$$\times\left[\| y(s)-z(s)\| + \int_0^s p_2(\tau)\| y(\tau)-z(\tau)\| d\tau\right.$$

$$\left. + \int_0^b p_3(\tau)\| y(\tau)-z(\tau)\| d\tau\right] ds$$

$$\leq L_0 u(t) + \sum_{j=1}^{n-1} L_j \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} u(s)ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} u(s)ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p_1(s)\left[u(s) + \int_0^s p_2(\tau)u(\tau)d\tau + \int_0^b p_3(\tau)u(\tau)d\tau\right] ds$$

$$\leq L_0 u(t) + \int_0^t \sum_{j=1}^{n-1} L_j \frac{(t-s)^{j-1}}{(j-1)!} u(s)ds$$

$$+ \int_0^t \left(\frac{\lambda + p_1(s)}{\Gamma(\alpha)}\right)(t-s)^{\alpha-1}\left[u(s) + \int_0^s p_2(\tau)u(\tau)d\tau + \int_0^b p_3(\tau)u(\tau)d\tau\right] ds$$

$$(1-L_0)u(t)$$

$$\leq \int_0^t A(s)\left[u(s) + \int_0^s p_2(\tau)u(\tau)d\tau + \int_0^b p_3(\tau)u(\tau)d\tau\right] ds.$$

Thus,

$$u(t) \leq \int_0^t \frac{A(s)}{(1-L_0)}\left[u(s) + \int_0^s p_2(\tau)u(\tau)d\tau + \int_0^b p_3(\tau)u(\tau)d\tau\right] ds. \quad (4.1)$$

By applying Pachpatte's inequality to the inequality (4.1) with $u(t) = \| y(t)-z(t)\|$, $p(s) = \frac{A(s)}{1-L_0}$, $q(s) = p_2(s)$, $r(s) = p_3(s)$ and $c = 0$, we obtain

$$\begin{aligned} u(t) &\leq \frac{0}{1-d}\exp\left(\int_0^t \left[\frac{A(s)}{1-L_0} + p_2(s)\right] ds\right) \\ &\leq 0 \\ &\Rightarrow u(t) = 0. \end{aligned}$$

Therefore $y(t) = z(t)$, which proves that there exists at most one solution. □

The following theorem deals with the estimates on the solutions of the nonlocal problem (1.1) - (1.2).

Theorem 4.2. *Suppose that the hypotheses* (H_1)-(H_4) *hold and*

$$d = \int_0^b p_3(s)\exp\left(\int_0^s \left[\frac{A(\tau)}{1-L_0} + p_2(\tau)\right]d\tau\right)ds < 1.$$

If $y(t),\ t \in I$ *is any solution of the problem* (1.1)-(1.2), *then*

$$\|y(t)\|$$

$$\leq \frac{\left[M + G + \frac{Nb^{\alpha}}{\Gamma(\alpha+1)}\right]\frac{1}{1-L_0}}{1-d}\exp\left(\int_0^t\left[\frac{A(s)}{1-L_0} + p_2(s)\right]ds\right)ds,\ \text{for } t \in I,$$

where $A(s)$ *is defined as in Theorem* 4.1.

Proof. By using the fact that the solution $y(t)$ of the problem (1.1)-(1.2) satisfies the equivalent equation (1.1) and the hypotheses, we have

$$\|y(t)\|$$

$$\leq \sum_{j=0}^{n-1}\frac{\|c_j\|}{j!}t^j + L_0\|y(t)\| + G_0 + \sum_{j=1}^{n-1}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}G_j ds$$

$$+ \sum_{j=1}^{n-1}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}L_j\|y(s)\|ds$$

$$+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y(s)\|ds + \frac{Nb^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p_1(s)$$

$$\cdot\left[\|y(s)\| + \int_0^s p_2(\tau)\|y(\tau)\|d\tau + \int_0^b p_3(\tau)\|y(\tau)\|d\tau\right]ds$$

$$\leq M + G_0 + \sum_{j=1}^{n-1} G_j \frac{t^j}{j!} + L_0\| y(s)\| + \frac{Nb^{\alpha}}{\Gamma(\alpha+1)}$$

$$+ \int_0^t \left[\sum_{j=1}^{n-1} L_j \frac{(b-s)^{j-1}}{(j-1)!} + \frac{\lambda + p_1(s)}{\Gamma(\alpha)} (b-s)^{\alpha-1} \right]$$

$$\times \left[\| y(s)\| + \int_0^s p_2(\tau)\| y(\tau)\| d\tau + \int_0^b p_3(\tau)\| y(\tau)\| d\tau \right] ds$$

$$\leq \left(M + G + \frac{Nb^{\alpha}}{\Gamma(\alpha+1)} \right) + L_0\| y(s)\|$$

$$+ \int_0^t A(s) \left[\| y(s)\| + \int_0^s p_2(\tau)\| y(\tau)\| d\tau + \int_0^b p_3(\tau)\| y(\tau)\| d\tau \right] ds,$$

which can be written as

$$\| y(t)\| \leq \frac{1}{1-L_0} \left[M + G + \frac{Nb^{\alpha}}{\Gamma(\alpha+1)} \right]$$

$$+ \int_0^t \frac{A(s)}{1-L_0} \left[\| y(s)\| + \int_0^s p_2(\tau)\| y(\tau)\| d\tau + \int_0^b p_3(\tau)\| y(\tau)\| d\tau \right] ds. \tag{4.2}$$

Hence, by an application of Lemma 2.5 to (4.2) with

$$u(t) = \| y(t)\|, \ p(s) = \frac{A(s)}{1-L_0}, \ q(s) = p_2(s), \ r(s) = p_3(s),$$

$$c = \frac{1}{1-L_0} \left[M + G + \frac{Nb^{\alpha}}{\Gamma(\alpha+1)} \right],$$

we obtain

$$\| y(t) \|$$

$$\leq \frac{\left[M + G + \frac{Nb^{\alpha}}{\Gamma(\alpha+1)} \right] \frac{1}{1 - L_0}}{1 - d} \exp\left(\int_0^t \left[\frac{A(s)}{1 - L_0} + p_2(s) \right] ds \right), \text{ for } t \in I. \quad (4.3)$$

□

5. Continuous Dependence

In this section, we shall deal with continuous dependence of solution of the problem (1.1)-(1.2) on the initial data, functions induced therein and also on parameters.

5.1. Dependence on initial data

We first discuss dependence of solution on given initial data.

Theorem 5.1. *Suppose that the hypotheses* (H_1)-(H_3) *and let*

$$d = \int_0^b p_3(s) \exp\left(\int_0^s \left[\frac{A(\tau)}{1 - L_0} + p_2(\tau) \right] d\tau \right) ds < 1,$$

where $A(s)$ *is defined as in Theorem* 4.1. *If* $y(t)$ *and* $z(t)$ *are solutions of* (1.1) *with initial data*

$$y^{(j)}(0) = c_j + g_j(y), \ (j = 0, 1, 2, ..., n-1) \quad (5.1)$$

and

$$z^{(j)}(0) = d_j + h_j(z), \ (j = 0, 1, 2, ..., n-1), \quad (5.2)$$

$$\| g_j(y) - h_j(y) \| \leq \delta_j, \ \textit{where} \ (j = 0, 1, 2, ..., n-1), \quad (5.3)$$

respectively, *then*

$$\| y(t) - z(t) \| \leq \frac{\left(\frac{\overline{M} + \overline{G}}{1 - L_0} \right)}{1 - d} \exp\left(\int_0^t \left[\frac{A(s)}{1 - L_0} + p_2(s) \right] ds \right), \ \textit{for} \ t \in I,$$

where

$$\overline{M} = \sup_{t\in I} \sum_{j=0}^{n-1} \frac{\| c_j - d_j \|}{j!}, \quad \overline{G} = \sup_{t\in I} \sum_{j=0}^{n-1} \frac{\delta_j t^j}{j!}.$$

Proof. By using the fact that $y(t)$ and $z(t)$ are solutions of (1.1) and $u(t) = \| y(t) - z(t) \|$, $t \in I$. Then by the hypotheses, we have

$$u(t)$$

$$= \| y(t) - z(t) \| \le \sum_{j=0}^{n-1} \frac{\|c_j - d_j\|}{j!} t^j + \| g_0(y) - h_0(z) \|$$

$$+ \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \| g_j(y) - h_j(z) \| ds$$

$$+ \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| y(s) - z(s) \| ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_1(s)$$

$$\times \Big[\| y(s) - z(s) \| + \int_0^s p_2(\tau) \| y(\tau) - z(\tau) \| d\tau$$

$$+ \int_0^b p_3(\tau) \| y(\tau) - z(\tau) \| d\tau \Big] ds$$

$$\le \overline{M} + \| g_0(y) - g_0(z) \| + \| g_0(z) - h_0(z) \|$$

$$+ \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \| g_j(y) - g_j(z) \| ds$$

$$+ \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \| g_j(z) - h_j(z) \| ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_1(s) \Big[u(s) + \int_0^s p_2(\tau) u(\tau) d\tau + \int_0^b p_3(\tau) u(\tau) d\tau \Big] ds$$

$$\leq \overline{M} + \delta_0 + L_0\| y(t) - z(t) \| + \sum_{j=1}^{n-1} \delta_j \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} ds$$

$$+ \sum_{j=1}^{n-1} L_j \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \| y(s) - z(s) \| ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_1(s) \left[u(s) + \int_0^s p_2(\tau) u(\tau) d\tau + \int_0^b p_3(\tau) u(\tau) d\tau \right] ds$$

$$\leq \overline{M} + \left(\delta_0 + \sum_{j=1}^{n-1} \delta_j \frac{t^j}{j!} \right) + L_0 u(t)$$

$$+ \int_0^t \left[\left(\frac{\lambda + p_1(s)}{\Gamma(\alpha)} \right) (b-s)^{\alpha-1} + \sum_{j=1}^{n-1} L_j \frac{(b-s)^{j-1}}{(j-1)!} \right]$$

$$\times \left[u(s) + \int_0^s p_2(\tau) u(\tau) d\tau + \int_0^b p_3(\tau) u(\tau) d\tau \right] ds$$

$$\leq \overline{M} + \left(\sum_{j=0}^{n-1} \delta_j \frac{t^j}{j!} \right) + L_0 u(t)$$

$$+ \int_0^t A(s) \left[u(s) + \int_0^s p_2(\tau) u(\tau) d\tau + \int_0^b p_3(\tau) u(\tau) d\tau \right] ds \tag{5.4}$$

$$\Rightarrow u(t) \leq \left(\frac{\overline{M} + \overline{G}}{1 - L_0} \right)$$

$$+ \int_0^t \frac{A(s)}{1 - L_0} \left[u(s) + \int_0^s p_2(\tau) u(\tau) d\tau + \int_0^b p_3(\tau) u(\tau) d\tau \right] ds. \tag{5.5}$$

Now, an application of Lemma 2.5 to (5.5) with $c = \frac{\overline{M} + \overline{G}}{1 - L_0}$, $p(s) = \frac{A(s)}{1 - L_0}$, $q(s) = p_2(s)$, $r(s) = p_3(s)$, $u(t) = \|y(t) - z(t)\|$, we obtain

$$\|y(t) - z(t)\| \le \frac{\left(\frac{\overline{M} + \overline{G}}{1 - L_0}\right)}{1 - d} \exp\left(\int_0^t \left[\frac{A(s)}{1 - L_0} + p_2(s)\right] ds\right), \; t \in I;$$

which shows the dependency of solutions of equation (1.1) on given initial data. □

5.2. Dependence on functions

Consider the problem (1.1)-(1.2) and the corresponding problem

$$^cD^{\alpha} z(t) = \lambda z(t) + \bar{f}\left(t, z(t), \int_0^t k(s, z(s)) ds, \int_0^b h(s, z(s)) ds\right) \tag{5.6}$$

for $t \in I = [0, b]$, $b > 0$, $n - 1 < \alpha \le n$, $n > 1$, $\lambda \in (0, 1)$ with nonlocal conditions:

$$z^{(j)}(0) = d_j + h_j(z), \; (j = 0, 1, 2, \ldots, n - 1), \tag{5.7}$$

where $\bar{f}$ is defined as f.

The following theorem deals with the continuous dependence of solutions of the problem (1.1)-(1.2) on the functions involved therein.

Theorem 5.2. *Suppose that the hypotheses* (H_1)-(H_3) *hold and functions* g_j, h_j *satisfying the conditions* (4.1). *Let*

$$d = \int_0^b p_3(s) \exp\left(\int_0^s \left[\frac{A(\tau)}{1 - L_0} + p_2(\tau)\right] d\tau\right) ds < 1.$$

Furthermore, suppose that

$$\left\| f\left(t, y(t), \int_0^t k(s, y(s)) ds, \int_0^b h(s, y(s)) ds\right) - \bar{f}\left(t, y(t), \int_0^t k(s, y(s)) ds, \int_0^b h(s, y(s)) ds\right) \right\| \le \varepsilon,$$

where $\varepsilon > 0$ *is an arbitrary small constant and* $z(t)$ *is a solution of the* *problem* (5.6)-(5.7). *Then the solution* $y(t),\ t \in I$ *of the problem* (1.1)-(1.2) *depends continuously on the functions involved in the right side of equation* (1.1).

Proof. Let $y(t)$ and $z(t)$ be solutions of the problem (1.1)-(1.2) and (5.6)-(5.7), respectively, and let $u(t) = \| y(t) - z(t) \|,\ t \in I.$

Now, by hypotheses, we have

$$
\begin{aligned}
& u(t) \\
&= \| y(t) - z(t) \| \\
&\le \sum_{j=0}^{n-1} \frac{\| c_j - d_j \|}{j!} t^j + \delta_0 + L_0 \| y(t) - z(t) \| + \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} \delta_j ds \\
&\quad + \sum_{j=1}^{n-1} \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} L_j \| y(s) - z(s) \| ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| y(s) - z(s) \| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_1(s) \Big[\| y(s) - z(s) \| d\tau + \int_0^s p_2(\tau) \| y(\tau) - z(\tau) \| d\tau \\
&\quad + \int_0^b p_3(\tau) \| y(\tau) - z(\tau) \| d\tau \Big] ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varepsilon ds \\
&\le \overline{M} + \left(\delta_0 + \sum_{j=1}^{n-1} \frac{\delta_j t^j}{j!} \right) \\
&\quad + L_0 u(t) + \sum_{j=1}^{n-1} \int_0^t L_j \frac{(t-s)^{j-1}}{(j-1)!} u(s) ds + \frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds
\end{aligned}
$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p_1(s)\left[u(s)+\int_0^s p_2(\tau)u(\tau)d\tau+\int_0^b p_3(\tau)u(\tau)d\tau\right]ds$$

$$\leq \overline{M}+\overline{G}+\frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)}+L_0 u(t)$$

$$+\int_0^t A(s)\left[u(s)+\int_0^s p_2(\tau)u(\tau)d\tau+\int_0^b p_3(\tau)u(\tau)d\tau\right]ds$$

$$\Rightarrow u(t)\leq \frac{1}{1-L_0}\left[\overline{M}+\overline{G}+\frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)}\right]$$

$$+\int_0^t \frac{A(s)}{1-L_0}\left[u(s)+\int_0^s p_2(\tau)u(\tau)d\tau+\int_0^b p_3(\tau)u(\tau)d\tau\right]ds. \quad (5.8)$$

Therefore, on application of Lemma 2.5 to (5.8), with

$$u(t)=\| y(t)-z(t)\|,\ p(t)=\frac{A(s)}{1-L_0},$$

$$q(s)=p_2(s),\ r(s)=p_3(s),\ c=\frac{1}{1-L_0}\left[\overline{M}+\overline{G}+\frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)}\right],$$

we get

$$\| y(t)-z(t)\|\leq \frac{\left[\overline{M}+\overline{G}+\frac{\varepsilon b^{\alpha}}{\Gamma(\alpha+1)}\right]}{(1-L_0)(1-d)}\exp\left(\int_0^t\left[\frac{A(s)}{1-L_0}+p_2(s)\right]ds\right), \quad (5.9)$$

$t\in I$. From (5.9), it follows that the solution of the problem (1.1)-(1.2) depends continuously on the functions involved in the right side of the problem (1.1). □

Remark 5.3. The result given in Theorem 5.2 rotates the solutions of problems (1.1)-(1.2) and (5.6)-(5.7) in the sense that if f is close to $\overline{F}$, $c_j \to d_j$, and $g_j \to h_j$, $(j=0, 1, ..., n-1)$, then the solutions of the problem (1.1)-(1.2) and the problem (5.6)-(5.7) are also close to each other.

5.3. Dependence on parameters

We next consider the following problem

$$ {}^{c}D^{\alpha}y(t) = \lambda y(t) + F\left(t,\ y(t),\ \int_0^t k(s,\ y(s))ds,\ \int_0^b h(s,\ y(s))ds,\ \mu_1\right), \quad (5.10)$$

for $t \in I = [0, b]$, $b > 0$, $n - 1 < \alpha \leq n$, $n > 1$, $\lambda \in (0, 1)$ with nonlocal conditions:

$$ y^{(j)}(0) = c_j + g_j(y),\ (j = 0, 1, 2, ..., n-1) \quad (5.11)$$

and

$$ {}^{c}D^{\alpha}z(t) = \lambda z(t) + F\left(t,\ z(t),\ \int_0^t k(s,\ z(s))ds,\ \int_0^b h(s,\ z(s))ds,\ \mu_2\right), \quad (5.12)$$

for $t \in I = [0, b]$, $b > 0$, $n - 1 < \alpha \leq n$, $n > 1$, $\lambda \in (0, 1)$ with nonlocal conditions:

$$ z^{(j)}(0) = d_j + h_j(z),\ (j = 0, 1, 2, ..., n-1), \quad (5.13)$$

where $F \in C(I \times X \times X \times X \times \mathbb{R},\ X)$, $h, k \in C(I,\ X)$ and constants μ_1 and μ_2 are real parameters.

The following theorem shows that the dependency of solutions of the problems (5.10)-(5.11) and (5.12)-(5.13) on parameters.

Theorem 5.4. *Assume that* (H_2)-(H_3) *hold and the functions* g_j, h_j *satisfy the condition* (5.3). *Also*, *the function F satisfying the conditions*

$$\| F(t,\ x(t),\ y(t),\ z(t), \mu_1) - F(t,\ \bar{x}(t),\ \bar{y}(t),\ \bar{z}(t), \mu_1) \|$$

$$\leq p_4(t)[\| x(t) - \bar{x}(t) \| + \| y(t) - \bar{y}(t) \| + \| z(t) - \bar{z}(t) \|], \quad (5.14)$$

and

$$\| F(t,\ x(t),\ y(t),\ z(t), \mu_1) - F(t,\ x(t),\ y(t),\ z(t), \mu_2) \| \leq p_5(t) | \mu_1 - \mu_2 |, \quad (5.15)$$

where $p_4,\ p_5 \in C(I, \mathbb{R}_+)$. *Let*

$$d = \int_0^b p_3(t)\exp\left(\int_0^t \left[\frac{B(s)}{1-L_0} + p_2(s)\right]ds\right)dt < 1,$$

where

$$B(s) = \left[\frac{\lambda + p_4(s)}{\Gamma(\alpha)}(b-s)^{\alpha-1} + \sum_{j=1}^{n-1}\frac{L_j(b-s)^{j-1}}{(j-1)!}\right].$$

If $y(t)$ *and* $z(t)$ *be the solutions of the problem* (5.10)-(5.11) *and* (5.12)-(5.13). *Then*

$$\| y(t) - z(t) \|$$

$$\leq \frac{\frac{1}{1-L_0}\left[\overline{M} + \overline{G} + \frac{|\mu_1 - \mu_2|\overline{P}}{\Gamma(\alpha+1)}b^{\alpha}\right]}{1-d}\exp\left(\int_0^t\left\{\frac{B(s)}{1-L_0} + p_2(s)\right\}ds\right),\ t \in I,$$

where $\overline{P} = \sup_{t \in I}\{p_5(t)\}$.

Proof. Let $u(t) = \| y(t) - z(t) \|,\ \ t \in I$. From the hypotheses, it follows that

$$u(t)$$

$$\leq \sum_{j=0}^{n-1}\frac{\| c_j - d_j \| t^j}{j!} + \delta_0 + \sum_{j=1}^{n-1}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}\delta_j ds + L_0\| y(t) - z(t) \|$$

$$+ \sum_{j=1}^{n-1}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}L_j \| y(s) - z(s) \| ds$$

$$+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\| y(s) - z(s) \| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\| F\left(s,\ y(s),\ \int_0^s k(\tau,\ y(\tau))d\tau,\ \int_0^b h(\tau,\ y(\tau))d\tau,\ \mu_1\right)\right.$$

$$\left. -F\left(s,\ z(s),\ \int_0^s k(\tau,\ z(\tau))d\tau,\ \int_0^b h(\tau,\ z(\tau))d\tau,\ \mu_2\right)\right\| ds$$

$$\leq \overline{M}+\sum_{j=0}^{n-1}\frac{\delta_j t^j}{j!}+L_0 u(t)$$

$$+\sum_{j=1}^{n-1}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}L_j u(s)ds+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\| F\left(s,\ y(s),\ \int_0^s k(\tau,\ y(\tau))d\tau,\ \int_0^b h(\tau,\ y(\tau))d\tau,\ \mu_1\right)\right.$$

$$\left. -F\left(s,\ z(s),\ \int_0^s k(\tau,\ z(\tau))d\tau,\ \int_0^b h(\tau,\ z(\tau))d\tau,\ \mu_1\right)\right\| ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left\| F\left(s,\ z(s),\ \int_0^s k(\tau,\ z(\tau))d\tau,\ \int_0^b h(\tau,\ z(\tau))d\tau,\ \mu_1\right)\right.$$

$$\left. -F\left(s,\ z(s),\ \int_0^s k(\tau,\ z(\tau))d\tau,\ \int_0^b h(\tau,\ z(\tau))d\tau,\ \mu_2\right)\right\| ds$$

$$\leq \overline{M}+\overline{G}+L_0 u(t)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p_5(s)|\mu_1-\mu_2|ds$$

$$+\sum_{j=1}^{n-1}\int_0^t \frac{(t-s)^{j-1}}{(j-1)!}L_j u(s)ds$$

$$+\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p_4(s)$$

$$\times\left[\| y(s)-z(s)\|+\int_0^s p_2(\tau)\| y(\tau)-z(\tau)\| d\tau\right.$$

$$+ \int_0^b p_3(\tau) \| y(\tau) - z(\tau) \| d\tau \Big] ds$$

$$\leq \overline{M} + \overline{G} + L_0 u(t) + \frac{|\mu_1 - \mu_2| \overline{P}}{\Gamma(\alpha+1)} b^{\alpha}$$

$$+ \int_0^t B(s) \left[u(s) + \int_0^s p_2(\tau) u(\tau) d\tau + \int_0^b p_3(\tau) u(\tau) d\tau \right] ds. \quad (5.16)$$

Thus

$$u(t) \leq \frac{1}{1 - L_0} \left[\overline{M} + \overline{G} + \frac{|\mu_1 - \mu_2| \overline{P}}{\Gamma(\alpha+1)} b^{\alpha} \right]$$

$$+ \int_0^t \frac{B(s)}{1 - L_0} \left[u(s) + \int_0^s p_2(\tau) u(\tau) d\tau + \int_0^b p_3(\tau) u(\tau) d\tau \right] ds. \quad (5.17)$$

Now, an application of Lemma 2.5 to (5.17), with

$$u(t) = \| y(t) - z(t) \|, \; p(s) = \frac{B(s)}{1 - L_0},$$

$$q(s) = p_2(s), \; r(s) = p_3(s), \; c = \frac{1}{1 - L_0} \left[\overline{M} + \overline{G} + \frac{|\mu_1 - \mu_2| \overline{P}}{\Gamma(\alpha+1)} b^{\alpha} \right],$$

we obtain

$$\| y(t) - z(t) \|$$

$$\leq \frac{\frac{1}{1 - L_0} \left[\overline{M} + \overline{G} + \frac{|\mu_1 - \mu_2| \overline{P}}{\Gamma(\alpha+1)} b^{\alpha} \right]}{1 - d} \exp\left(\int_0^t \left\{ \frac{B(s)}{1 - L_0} + p_2(s) \right\} ds \right], \quad (5.18)$$

$t \in I$, which shows the dependence of solutions of the problems (5.10)-(5.11) and (5.12)-(5.13) on parameters μ_1 and μ_2. □

6. Example

In the last section, we can illustrate our results through the following example by taking the fractional order α, $1 < \alpha \leq 2$.

Example 6.1. Consider the following fractional integrodifferential equation

$$ {}^{c}D^{3/2}y(t) = \frac{1}{10}\,y(t) + \frac{e^{-t}}{(8+e^{t})}\left[\frac{|\,y(t)\,|}{1+|\,y(t)\,|}\right] $$

$$ + \frac{1}{9}\int_0^t \frac{e^{-s}}{(2+s)^2}\,y(s)ds + \frac{1}{9}\int_0^1 \frac{e^{-s}}{(3+s)^2}\,y(s)ds, \tag{6.1} $$

for $t \in I = [0, 1], 1 < \alpha \leq 2, \lambda \in (0, 1)$ with conditions:

$$ y(0) = c_1 + \frac{1}{9}\sin y, \quad y'(0) = c_2 + \frac{1}{10}\sin y. \tag{6.2} $$

Problem (6.1)-(6.2) is of the form (1.1)-(1.2) with $\alpha = \frac{3}{2}, \lambda = \frac{1}{10}$,

$$ f\left(t,\ y(t),\ \int_0^t k(s,\ y(s))ds,\ \int_0^1 h(s,\ y(s))ds\right) $$

$$ = \frac{e^{-t}}{(8+e^{t})}\left[\frac{|\,y(t)\,|}{1+|\,y(t)\,|}\right] + \frac{1}{9}\int_0^t \frac{e^{-s}}{(2+s)^2}\,y(s)ds + \frac{1}{9}\int_0^1 \frac{e^{-s}}{(3+s)^2}\,y(s)ds. $$

Clearly, for each $y, z, u, \bar{y}, \bar{z}, \bar{u} \in X$ and $t \in [0, 1]$,

$$ \|\, f(t,\ y,\ z,\ u) - f(t,\ \bar{y},\ \bar{z},\ \bar{u})\,\| \leq \frac{1}{9}[\|\, y-\bar{y}\,\| + \|\, z-\bar{z}\,\| + \|\, u-\bar{u}\,\|]. $$

Also, we have

$$ \|\, k(t,\ y) - k(t,\ \bar{y})\,\| \leq \frac{1}{9}\|\, y-\bar{y}\,\|, $$

$$ \|\, h(t,\ y) - h(t,\ \bar{y})\,\| \leq \frac{1}{9}\|\, y-\bar{y}\,\|, $$

$$ \|\, g_1(y) - g_1(\bar{y})\,\| \leq \frac{1}{9}\|\, y-\bar{y}\,\|, $$

$$ \|\, g_2(y) - g_2(\bar{y})\,\| \leq \frac{1}{10}\|\, y-\bar{y}\,\|. $$

Hence all hypotheses (H_1)-(H_4) are satisfied with $\lambda = \frac{1}{10}$, $L_1 = \frac{1}{9}$, $L_2 = \frac{1}{10}$, $P_1 = \frac{1}{9}$, $P_2 = \frac{1}{9}$, $P_3 = \frac{1}{9}$. Therefore, we have

$$L = \sup_{t \in [0, 1]} \{L_1 + L_2 t\} \leq L_1 + L_2 = \frac{1}{9} + \frac{1}{10} = \frac{19}{90}.$$

Now, we estimate the value

$$\beta + L + \left[\frac{\lambda + P_1(1 + (P_2 + P_3)b)}{\Gamma(\alpha + 1)}\right] b^{\alpha}$$

$$= \frac{19}{90} + \left[\frac{\frac{1}{10} + \frac{1}{9}\left(1 + \left[\frac{1}{9} + \frac{1}{9}\right]1\right)}{\Gamma\left(\frac{5}{2}\right)}\right]$$

$$= 0.2111 + 0.1774$$

$$= 0.3886$$

$$< 1.$$

It follows from Theorem 3.1 that the problem (6.1)-(6.2) has a unique solution on $[0, 1]$.

Acknowledgement

Author G. N. More would like to acknowledge Y. C. I. S., Satara for partial financial support through the research corpus and RUSA funds.

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