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ON THE ENERGY EQUALITY FOR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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Abstract

In this paper, we first introduce the concept of absolutely continuous functions of order s $(0 < s \leq 1)$. Next, we prove the energy equality for weak solutions of the Navier-Stokes equations (NSE) in bounded

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three dimensional domains if and only if u is an absolutely continuous solution of order $1/2$. Finally, we present a sufficient condition for the energy equality of weak solutions to NSE. Here, we prove that if $u \in L^2(0, T; H^s) \cap L^4(0, T; L^{\frac{12}{2s+1}}) \left(1 \leq s < \frac{5}{2}\right)$, then the energy equality holds.

1. Introduction

We consider the three dimensional initial boundary value problem for NSE

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} - \Delta u_i + \frac{\partial p}{\partial x_i} = 0 \text{ in } \Omega^T := (0, T) \times \Omega, \; i = \overline{1, 3}, \tag{1.1}$$

$$\operatorname{div}(u) = \sum_{i=0}^{3} \frac{\partial u_i}{\partial x_i} = 0 \text{ in } \Omega^T, \tag{1.2}$$

$$u(0, x) = u_0(x) \text{ in } \Omega, \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, and u_0 is a given vector-function satisfying the condition $\operatorname{div}(u_0) = 0$.

We recall the definitions of the spaces $C_{0,\sigma}^{\infty}(\Omega)$, $W_{0,\sigma}^{1,2}(\Omega)$, $L_{\sigma}^2(\Omega)$:

$C_{0,\sigma}^{\infty}(\Omega) = \{u \in C_0^{\infty}(\Omega), \operatorname{div}(u) = 0\}$,

$W_{0,\sigma}^{1,2}(\Omega) =$ the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in the topology $H_0^1(\Omega)$,

$L_{\sigma}^2(\Omega) =$ the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in the topology $L^2(\Omega)$.

The space $L_{\sigma}^2(\Omega)$ is equipped with the usual scalar product $(\cdot, \cdot)$ and the norm

$$\| u \|_2 = \| u \|_{2,\Omega} = \| u \|_{L^2} := \left(\int_{\Omega} | u(x) |^2 dx \right)^{\frac{1}{2}}.$$

The space $W^{1,2}_{0,\sigma}(\Omega)$ is a Hilbert space with scalar product

$$((u, v)) = \sum_{i=1}^{3} (D_i u, D_i v).$$

The norm in $W^{1,2}_{0,\sigma}(\Omega)$ is defined by

$$\| u \|_{W^{1,2}_{0,\sigma}(\Omega)} = \| u \|_{1,2} := \left(\sum_{|\alpha| \le 1} | D^{\alpha} u(x) |_2^2 dx \right)^{\frac{1}{2}}.$$

The space $W^{1,2}_{0,\sigma}(\Omega)$ is contained and dense in $L^2_\sigma(\Omega)$, and the injection is continuous. Let $W^{-1,2}_{0,\sigma}(\Omega)$ denote the dual space of $W^{1,2}_{0,\sigma}(\Omega)$. By the Riesz representation theorem, we have

$$W^{1,2}_{0,\sigma}(\Omega) \subset L^2_\sigma(\Omega) \subset W^{-1,2}_{0,\sigma}(\Omega).$$

For each u in $W^{1,2}_{0,\sigma}(\Omega)$, there exists a unique element of $W^{-1,2}_{0,\sigma}(\Omega)$ which we denote by Au such that

$$\langle Au, v \rangle = ((u, v)), \quad \forall v \in W^{1,2}_{0,\sigma}(\Omega).$$

We denote by H^s (or sometimes $H^s(\Omega)$) the domain of definition of $A^{\frac{s}{2}}$. For the definition of $A^{\frac{s}{2}}$, we refer the readers to [9].

We define a trilinear continuous form by setting

$$\mathbf{b}(u, v, w) = \sum_{i,j=1}^{3} \int_\Omega u_i (D_i v_j) w_j.$$

For u, v in $W^{1,2}_{0,\sigma}(\Omega)$, we denote by $B(u, v)$ the element of $W^{-1,2}_{0,\sigma}(\Omega)$ defined by

$$\langle B(u, v), w\rangle = \mathbf{b}(u, v, w), \text{ for all } w \in W_{0,\sigma}^{1,2}(\Omega)$$

and we set

$$Bu = B(u, u) \in W_{0,\sigma}^{-1,2}(\Omega), \text{ for all } u \in W_{0,\sigma}^{1,2}(\Omega).$$

By projecting on space $W_{0,\sigma}^{-1,2}(\Omega)$, equation (1.1) can write in the form

$$\frac{du}{dt} + Au + Bu = 0. \tag{1.4}$$

Definition 1.1. A vector field

$$u \in L^{\infty}(0, T; L_{\sigma}^{2}(\Omega)) \cap L_{loc}^{2}(0, T; W_{0}^{1,2}(\Omega))$$

is called a *weak solution of NSE* if the relation

$$-(u, w_t)_{\Omega,T} + (\nabla u, \nabla w)_{\Omega,T} - (uu, \nabla w)_{\Omega,T} = (u_0, w(0))_{\Omega}$$

is satisfied for all test functions $w \in C_0^{\infty}(0, T; C_{0,\sigma}^{\infty}(\Omega))$.

In this definition, $(\cdot, \cdot)_{\Omega}$ means the usual pairing of functions on Ω, $(\cdot, \cdot)_{\Omega,T}$ means the corresponding pairing on $[0, T) \times \Omega$. Finally, $uu = (u_i u_j)_{i,j=1}^{3}$ for $u = (u_1, u_2, u_3)$ and we have $u \cdot \nabla u = (u \cdot \nabla)u = \operatorname{div}(uu)$ when $\operatorname{div}(u) = 0$.

Leray [6] and Hopf [4] showed the global existence of weak solutions to NSE satisfying the energy inequality

$$\frac{1}{2}\| u(t)\|_2^2 + \int_0^t \| \nabla u(\tau)\|_2^2 d\tau \leq \frac{1}{2}\| u_0\|_2^2.$$

However, the question whether or not every weak solution satisfies the energy inequality remains still an open problem. Solutions satisfying the above energy inequality are called *Leray-Hopf weak solutions*.

It is well-known that the classical solutions of NSE satisfy the energy equality

$$\frac{1}{2}\| u(t) \|_2^2 + \int_0^t \| \nabla u(\tau) \|_2^2 d\tau = \frac{1}{2}\| u_0 \|_2^2.$$

Serrin [7] showed that if a weak solution u belongs to $L^s(0, T; L^q(\Omega))$ for some $q > 3,\ s > 2$ with

$$\frac{3}{q} + \frac{2}{s} \leq 1,$$

then u satisfies the energy equality.

Later, Shinbrot [8] derived the same conclusion if the weak solution u belongs to $L^s(0, T; L^q(\Omega))$ with

$$\frac{2}{q} + \frac{2}{s} \leq 1 \quad (q \geq 4).$$

Sohr [9] proved the energy equality for a weak solution u if uu belongs to $L^4_{loc}(0, T; L^4(\Omega))$. Here, the Serrin index of $L^4_{loc}(0, T; L^4(\Omega))$ is $\frac{5}{4}$.

In 2008, Cheskidov et al. [1] proved the energy equality in a function class not covered by the class considered by Sohr. If $\Omega = \mathbb{R}^3$, then they obtained the energy equality for weak solutions belonging to $L^3(0, T; B^{\frac{1}{3}}_{3,\infty}(\mathbb{R}^3))$. For a general domain Ω, the energy equality for weak solutions belonging to $L^3(0, T; B^{\frac{1}{3}}_{3,\infty}(\Omega))$ is still not known. Later on in [2], the authors showed that if Ω is a bounded domain with C^2-boundary and if a weak solution u belongs to $L^3(0, T; H^{\frac{5}{6}}(\Omega))$, then u satisfies the energy equality. Notice that the Serrin index of $L^3(0, T; H^{\frac{5}{6}}(\Omega))$ is $\frac{4}{3}$. For general unbounded domains, Farwig and Taniuchi [3] proved the energy equality for weak solutions in $L^3(0, T; \tilde{D}^{\frac{1}{2}}_{\frac{18}{7}}(\Omega))$, where $\tilde{D}^{\frac{1}{2}}_{\frac{18}{7}}(\Omega)$ is a space obtained by interpolation between the Sobolev space $H^{\frac{5}{6}}(\Omega)$ and the Besov space $B^{\frac{1}{3}}_{3,\infty}(\Omega)$. Observe that the Serrin index of $L^3(0, T; \tilde{D}^{\frac{1}{2}}_{\frac{18}{7}}(\Omega))$ is also $\frac{4}{3}$.

In this paper, we first introduce the concept of absolutely continuous functions of order s $(0 < s \leq 1)$. Next, we prove the energy equality for weak solutions of the Navier-Stokes equations (NSE) in bounded three dimensional domains if and only if u is an absolutely continuous solution of order $1/2$. Finally, we present a sufficient condition for the energy equality of weak solutions to NSE. Here, we prove that if $u \in L^2(0, T; H^s(\Omega)) \cap L^4(0, T; L^{\frac{12}{2s+1}}(\Omega))$ $\left(1 \leq s < \frac{5}{2}\right)$, then the energy equality holds. Note that the Serrin index of $L^2(0, T; H^s(\Omega)) \cap L^4(0, T; L^{\frac{12}{2s+1}}(\Omega))$ $\left(1 \leq s < \frac{5}{2}\right)$ is also $4/3$ but our space is not contained in $L^3(0, T; H^{\frac{5}{6}}(\Omega))$ nor in $L^3(0, T; \tilde{D}^{\frac{1}{2}}_{\frac{18}{7}}(\Omega))$.

2. Energy Equality for Absolutely Continuous Solutions of Order $1/2$

Definition 2.1. A function f is called *absolutely continuous* of order s $(0 < s \leq 1)$ on $[a, b]$ if for every $\varepsilon > 0$, there exists $\delta > 0$ so that for all finite separate intervals $[a_i, b_i]$, $i = 1, 2, ..., n$, $[a_i, b_i] \subset [a, b]$ satisfying

$$\sum_{i=1}^{n} | b_i - a_i |^s < \delta,$$

we have

$$\sum_{i=1}^{n} | f(b_i) - f(a_i) | < \varepsilon.$$

Remark. (i) If $s = 1$, then the above definition is the definition of the absolute continuity.

(ii) For $0 < s_1 \leq s_2 \leq 1$, if the function f is absolutely continuous of order s_2 on $[a, b]$, then f is absolutely continuous of order s_1 on $[a, b]$.

Theorem 2.2. *Suppose that u is a weak solution of the Navier-Stokes equations. Then there exists a closed set* $K \subset [0, T]$ *whose* $1/2$*-dimensional Hausdorff measure vanishes, and such that u is (at least) continuous from* $[0, T] \backslash K$ *into* H^1.

This theorem is Theorem 5.1 in [10].

Theorem 2.3. *Suppose that u is a weak solution of the Navier-Stokes equations and* $u \in C(a, b; H^1(\Omega))$, $[a, b] \subset [0, T]$. *Then the energy equality holds*:

$$\frac{1}{2}(\| u(a) \|_2^2 - \| u(b) \|_2^2) = \int_a^b \| \nabla u(\tau) \|_2^2 d\tau.$$

This theorem is a direct consequence of Theorem 1.4.1, V, see [9].

Theorem 2.4. *Suppose that u is a weak solution of the Navier-Stokes equations. A necessary and sufficient condition for the weak solution u satisfying the energy equality*

$$\frac{1}{2}(\| u(0) \|_2^2 - \| u(T) \|_2^2) = \int_0^T \| \nabla u(t) \|_2^2 dt$$

is that the function $\| u(t) \|_2^2$ *is absolutely continuous of order* $1/2$ *on* $[0, T]$.

Proof. The sufficient condition. We have that the function $\|\nabla u\|_2^2$ is integrable on $[0, T]$. For $\varepsilon > 0$, because of the absolute continuity of the Lebesgue integral, there exists $\delta_1 > 0$ so that

$$\int_M \|\nabla u(t)\|_2^2 dt < \varepsilon$$

with any measurable set $M \subset [0, T]$, $\mu(M) < \delta_1$, where μ denotes the Lebesgue measure of a set in $\mathbb{R}^1$. Since $\|u(t)\|_2^2$ is absolutely continuous of order $1/2$ on $[0, T]$, there exists $\delta_2 > 0$ so that

$$\sum_{i=1}^{n} |\|u(a_i)\|_2^2 - \|u(b_i)\|_2^2| < \varepsilon$$

with any finite separate intervals $[a_i, b_i]$, $i = 1, 2, ..., n$, $[a_i, b_i] \subset [0, T]$ satisfying

$$\sum_{i=1}^{n} |b_i - a_i|^{\frac{1}{2}} < \delta_2.$$

From Theorem 2.2, there exists an open set $\Omega \subset [0, T]$ so that $K = [0, T]\backslash\Omega$, $\mu_{\frac{1}{2}}(K) = 0$ and $u \in C(\Omega, H^1)$. Here, $\mu_s(M)$ denotes the s-dimensional Hausdorff measure of a set in $\mathbb{R}^1$, $s \in \mathbb{R}^+$.

By the definition of the Hausdorff measure, there exist open intervals G_i, $i = 1, 2, ..., G_i \subset [0, T]$ such that

$$K \subset \bigcup_{i=1}^{\infty} G_i$$

satisfying

$$\sum_{i=1}^{\infty} |G_i|^{\frac{1}{2}} < \delta := \min(\delta_1, \delta_2).$$

Here, $|H|$ is the diameter of any set $H, |H| = \sup\{|x - y| : x, y \in H\}$.

Because K is compact, there exists $n \in \mathbb{N}$ so that

$$K \subset \bigcup_{i=1}^{n} G_i, \quad \sum_{i=1}^{n} |G_i|^{\frac{1}{2}} \leq \sum_{i=1}^{\infty} |G_i|^{\frac{1}{2}} \leq \delta.$$

It is easy to see that, there exist sets $H_i = (a_i, b_i)$, $1 \leq i \leq m$, such that

$$\bigcup_{i=1}^{n} G_i = \bigcup_{i=1}^{m} H_i, \quad H_i \cap H_j = \varnothing$$

and

$$H_i = \bigcup_{k=1}^{k_i} G_{i_k},$$

with $1 \leq i \leq m$, $1 \leq i_{k_1} < i_{k_2} < \cdots < i_{k_i} \leq n$. We have

$$|H_i|^{\frac{1}{2}} = \left|\bigcup_{k=1}^{k_i} G_{i_k}\right|^{\frac{1}{2}} \leq \sum_{k=1}^{k_i} |G_{i_k}|^{\frac{1}{2}},$$

$1 \leq i \leq m$. Therefore,

$$\sum_{i=1}^{m} |H_i|^{\frac{1}{2}} \leq \sum_{i=1}^{m} \sum_{k=1}^{k_i} |G_{i_k}|^{\frac{1}{2}} = \sum_{i=1}^{n} |G_i|^{\frac{1}{2}} < \delta.$$

Thus,

$$\sum_{i=1}^{m} |b_i - a_i|^{\frac{1}{2}} < \delta.$$

Let

$$M = [0, T] \backslash \bigcup_{i=1}^{m} (a_i, b_i).$$

Then

$$M = \bigcup_{i=1}^{m+1} [c_i,\, d_i],$$

where $[c_i,\, d_i] \cap [c_k,\, d_k] = \varnothing,\ i,\ k = 1,\, 2,\, ...,\, m+1,\ i \neq k.$ We have

$$\| u(0) \|_2^2 - \| u(T) \|_2^2$$

$$= \sum_{i=1}^{m+1} (\| u(c_i) \|_2^2 - \| u(d_i) \|_2^2) + \sum_{i=1}^{m} (\| u(a_i) \|_2^2 - \| u(b_i) \|_2^2).$$

Since $u \in C([c_i,\, d_i];\, H^1),$ we have by Theorem 2.3,

$$\frac{1}{2} (\| u(c_i) \|_2^2 - \| u(d_i) \|_2^2) = \int_{c_i}^{d_i} \| \nabla u(t) \|_2^2 dt.$$

From

$$\sum_{i=1}^{m} | b_i - a_i |^{\frac{1}{2}} < \delta \leq \delta_2,$$

it follows that

$$\sum_{i=1}^{m} (\| u(a_i) \|_2^2 - \| u(b_i) \|_2^2) < \varepsilon.$$

Since $\delta < 1,$

$$\sum_{i=1}^{m} | b_i - a_i | \leq \sum_{i=1}^{m} | b_i - a_i |^{\frac{1}{2}} < \delta \leq \delta_1,$$

we deduce that

$$\sum_{i=1}^{m} \int_{a_i}^{b_i} \| \nabla u(\tau) \|_2^2 d\tau < \varepsilon.$$

Therefore,

$$\left| \frac{1}{2}(\| u(0)\|_{L^2}^2 - \| u(T)\|_{L^2}^2) - \int_0^T \| \nabla u(t)\|_2^2 dt \right|$$

$$= \left| \frac{1}{2}\sum_{i=1}^{m}(\| u(a_i)\|_2^2 - \| u(b_i)\|_2^2) - \sum_{i=1}^{m}\int_{a_i}^{b_i} \| \nabla u(\tau)\|_2^2 d\tau \right| \leq \frac{3\varepsilon}{2}$$

for any $\varepsilon > 0$. Thus,

$$\frac{1}{2}(\| u(0)\|_2^2 - \| u(T)\|_2^2) = \int_0^T \| \nabla u(t)\|_2^2 dt.$$

The necessary condition. If u satisfies the energy equality, then the function $\| u(t)\|_{L^2}^2$ is absolutely continuous of order $s = 1$. Therefore, $\| u(t)\|_{L^2}^2$ is absolutely continuous of order $s = \frac{1}{2}$. □

3. A Sufficient Condition for the Energy Equality

We first recall two lemmas that we need later.

Lemma 3.1. *Let X be a given Banach space with dual* X' *and let u and g be two functions belonging to* $L^1(a, b; X)$. *Then the following three conditions are equivalent*:

(i) *u is equal to a primitive function of g*:

$$u(t) = \xi + \int_0^t g(s)ds,\ \xi \in X,\ t \in [a, b].$$

(ii) *For each test function* $\phi \in D(a, b)$,

$$\int_a^b u(t)\phi'(t)dt = -\int_a^b g(t)\phi(t)dt \quad \left(\phi' = \frac{d\phi}{dt}\right).$$

(iii) *For each* $\eta \in X'$,

$$\frac{d}{dt}\langle u, \eta\rangle = \langle g, \eta\rangle,$$

in the scalar distribution sense, *on* (a, b).

In particular, *if* (i)-(iii) *are satisfied*, *then u is equal to a continuous function from* $[a, b]$ *into X*.

This lemma is Lemma 1.1 in [11].

Lemma 3.2. *Let* V, H, V' *be three Hilbert spaces*, *each space continuously included in the following one*, V' *being the dual of* V, $H \equiv H'$. *If a function u belongs to* $L^2(0, T; V)$ *and its derivative* u' *belongs to* $L^2(0, T; V')$, *then u is almost everywhere equal to a function continuous from* $[0, T]$ *into H and we have the following equality*, *which holds in the scalar distribution sense on* $(0, T)$:

$$\frac{d}{dt}|u|^2 = 2\langle u', u\rangle.$$

This lemma is a particular case of a general theorem of interpolation of Lions-Magenes [5].

Theorem 3.3. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain*, $1 \leq s < \frac{5}{2}$ *and u be a weak solution of the Navier-Stokes equations*. *Suppose additionally that*

$$u \in L^2(0, T; H^s) \cap L^4(0, T; L^{\frac{12}{2s+1}}).$$

Then, *after a redefinition on a null set of* $[0, T]$, $u \in C(0, T; L^2)$, *and the energy equality holds*

$$\frac{1}{2}\| u(t)\|_2^2 + \int_0^t \|\nabla u(\tau)\|_2^2 d\tau = \frac{1}{2}\| u_0\|_2^2$$

for all $t \in [0, T)$.

Proof. If v is an element of $C^{\infty}_{0,\sigma}(\Omega)$, then one have by definition of weak solutions

$$\frac{d}{dt}(u,\, v) + ((u,\, v)) + b(u,\, u,\, v) = 0.$$

In other words, we can rewrite the above equation in another form

$$u' = -Au - Bu. \tag{3.1}$$

Let

$$q = \frac{6}{5 - 2s}.$$

By the Sobolev embedding inequality, we have

$$\| v \|_{W^{1,q}} \leq c\| v \|_{W^{s,2}} \leq c\| v \|_{H^s}.$$

Now, using the Hölder inequality with $\frac{2}{p} + \frac{1}{q} = 1,$ we obtain inequality

$$| b(u,\, u,\, v) | \leq c\| u \|^2_{L^p} \| \nabla v \|_{L^q} \leq c\| u \|^2_{L^p} \| v \|_{W^{1,q}}.$$

Thus, we have

$$| b(u,\, u,\, v) | \leq c\| u \|^2_{L^{\frac{12}{2s+1}}} \| v \|_{H^s}.$$

Hence,

$$\| Bu(t) \|^2_{H^{-s}} \leq c\| u(t) \|^4_{L^{\frac{12}{2s+1}}}.$$

Therefore, by hypotheses of the theorem, Bu belongs to $L^2(0,\, T;\, H^{-s})$. From $u \in L^2(0,\, T;\, H^s)$ with $s > 1$, it follows that $Au \in L^2(0,\, T;\, H^{-s})$. Hence, since both Au and Bu belong to $L^2(0,\, T;\, H^{-s})$, the function $-Au - Bu$ belongs to $L^2(0,\, T;\, H^{-s})$. It then follows that

$$u' = -Au - Bu \in L^2(0, T; H^{-s}).$$

Applying Lemma 3.2 with $H = L^2$, $V = H^s$, $V' = H^{-s}$, we obtain $u \in C(0, T; L^2)$ and

$$\frac{d}{dt}\| u \|_2^2 = 2\langle u', u \rangle. \tag{3.2}$$

We notice that $\langle Bu(t), u(t) \rangle \in L^1(0, T)$ since $Bu \in L^2(0, T; H^{-s})$, $u \in L^2(0, T; H^s)$. By pairing (3.1) with u, integrating in τ from 0 to t, we get

$$\frac{1}{2}\| u(t) \|_2^2 + \int_0^t \| \nabla u(\tau) \|_2^2 d\tau - \int_0^t \langle Bu(\tau), u(\tau) \rangle d\tau = \frac{1}{2}\| u_0 \|_2^2$$

for all $t \in [0, T)$. However, $\langle Bu(\tau), u(\tau) \rangle = 0$ almost everywhere on $[0, T)$, then $\int_0^t \langle Bu(\tau), u(\tau) \rangle d\tau = 0$ and we arrive at the conclusion that

$$\frac{1}{2}\| u(t) \|_2^2 + \int_0^t \| \nabla u(\tau) \|_2^2 d\tau = \frac{1}{2}\| u_0 \|_2^2$$

for all $t \in [0, T)$. The proof of the theorem is complete. □

Remark. If $s = \frac{7}{6}$, then the hypotheses in Theorem 3.3 mean

$$u \in L^2(0, T; H^{\frac{7}{6}}) \cap L^4(0, T; L^{\frac{18}{5}}).$$

Here, Serrin's index of the spaces $L^2(0, T; H^{\frac{7}{6}}) \hookrightarrow L^2(0, T; L^9)$ and $L^4(0, T; L^{\frac{18}{5}})$ is $\frac{4}{3}$. The Serrin index of our spaces is the same as the Serrin index of the space considered in [2]. However, our space is different from theirs and the method of the proof is also different.

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