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SOLUTIONS OF FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS VIA FIXED POINT THEOREMS AND PICARD APPROXIMATIONS

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Abstract

We investigate the following fractional hybrid differential equation:

$$\begin{cases} D^{\alpha}_{t_0+}[x(t) - f_1(t, x(t))] = f_2(t, x(t)) \quad \text{a.e} \quad t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.0}$$

where $D^{\alpha}_{t_0+}$ is the Riemann-Liouville differential operator order of $\alpha > 0$, $J = [t_0, t_0 + a]$, for some $t_0 \in \mathbb{R}$, $a > 0$, $f_1 \in C(J \times \mathbb{R}, \mathbb{R})$,

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$f_2 \in \mathcal{L}_p^\alpha(J \times \mathbb{R}, \mathbb{R})$, $p \geq 1$ and satisfies certain conditions. We investigate such equations in two cases: $\alpha \in (0, 1)$ and $\alpha \geq 1$. In the first case, we prove the existence and uniqueness of a solution of (1.0), which extends the main result of [1]. Moreover, we show that the Picard iteration associated to an operator $T : C(J \times \mathbb{R}) \to C(J \times \mathbb{R})$ converges to the unique solution of (1.0) for any initial guess $x \in C(J \times \mathbb{R})$. In particular, the rate of convergence is n^{-1}. In the second case, we investigate this equation in the space of k times differentiable functions. Naturally, the initial condition $x(t_0) = x_0$ is replaced by $x^{(k)}(t_0) = x_0$, $0 \leq k \leq n_{\alpha, p} - 1$ and the existence and uniqueness of a solution of (1.0) is established. Moreover, the convergence of the Picard iterations to the unique solution of (1.0) is shown. In particular, the rate of convergence is n^{-1}. Finally, we provide some examples to show the applicability of the abstract results. These examples cannot be solved by the methods demonstrated in [1].

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, and aerodynamics (see [2-5] and references therein). The main advantage of using fractional nonlinear differential equations is related to the fact that we can describe the dynamics of complex non-local systems with memory. In this line of taught, the equations involving various fractional orders are essential from both theoretical and applied views of points. In this line of taught, the equations involving various fractional orders are essential from both theoretical and applied views of points. Moreover, applications of fractional order differential equations in modeling viral disease transmission have been widely used and many significant results related to the fractional differential equations were obtained (see, for example [6-9] and references therein). However, still, there are many open problems in this direction.

There are various definitions of fractional derivative, but in this work, we investigate fractional order differential equations defined by Riemann-Liouville differential operator. The differential equations defined by Riemann-Liouville differential operators are very important in the modelling several real-life problems (see, for instance [10, 11]). Fractional order hybrid differential equations defined by Riemann-Liouville differential operators have been developing very fast in the last few years (see [12-14] and references therein). In the investigations of fractional order hybrid type differential equations, the hybrid type fixed point theorems are widely used. In this context, the hybrid type fixed point theorem refers to the Krasnoselskii-type fixed point theorem which is a combination of Banach's and Schauder's fixed point theorems. Next, we discuss some applications of Krasnoselskii-type fixed point theorems to the hybrid differential and hybrid fractional differential equations. In 2013, Dhage [12] and Dhage and Lakshmikantham [15] proved significant Krasnoselskii-type fixed point theorems and applied them to the following hybrid differential equation:

$$\begin{cases} \dfrac{d}{dt}\left[\dfrac{x(t)}{f_1(t, x(t))}\right] = f_2(t, x(t)), \quad t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $J = [t_0, t_0 + a]$, for some $t_0 \in \mathbb{R}$, $a > 0$, and the functions $f_1 : J \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ and $f_2 : J \times \mathbb{R} \to \mathbb{R}$ are continuous. Dhage and Jadhav [16] investigated the existence of solutions for hybrid differential equation

$$\begin{cases} \dfrac{d}{dt}[x(t) - f_1(t, x(t))] = f_2(t, x(t)), \quad \text{a.e} \quad t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.2}$$

by using a Krasnoselskii-type fixed point theorem, where $f_1 : J \times \mathbb{R} \to \mathbb{R} \backslash \{0\}$ is continuous and $f_2 : J \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Xu [17] applied a Krasnoselskii-type fixed point theorem to investigate the solutions of the fractional boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t)), \, t \in [0, 1], \, \alpha \in (1, 2), \\ x(1) = \mu \int_0^1 x(s)ds, \; x'(0) = x'(1) = 0, \end{cases} \tag{1.3}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In [1], Lu et al. investigated the following fractional hybrid differential equations:

$$\begin{cases} D^{\alpha}_{t_0+}[x(t) - f_1(t, x(t))] = f_2(t, x(t)), \quad t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.4}$$

where $\alpha \in (0, 1)$ and $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions and satisfy certain conditions. In order to show the existence of the solutions of equation (1.4), a Krasnoselskii-type fixed point theorem was used. For the uniqueness of the solution, there was required a stronger condition. Krasnoselskii-type fixed point theorems are considered powerful tools in showing the existence of solutions of nonlinear hybrid differential equations, however, these tools are not sufficient to show the uniqueness of the solutions. Therefore, in the investigations of equations (1.1)-(1.4), the uniqueness of the solutions is not studied mostly, or studied under some extra conditions. Unfortunately, there are some shortcomings found in the proof of the existence of solutions of equation (1.4) demonstrated by Lu et al. in [1] and for the uniqueness of the solution, there was required a stronger condition as we have mentioned above. Moreover, naturally, we would like to ask the following important question: *How to investigate equation* (1.4) *for all* $\alpha > 0$ *in a* "*wider*" *class of functions*? Motivated by the question above, in this work, we investigate equation (1.4) for all $\alpha > 0$ and $f_1 \in C(J \times \mathbb{R}, \mathbb{R}), \quad f_2 \in \mathcal{L}^{\alpha}_{p}(J \times \mathbb{R}, \mathbb{R}), \quad p \geq 1.$ Obviously, we have to consider two cases: $\alpha \in (0, 1)$ and $\alpha \geq 1$ separately. It is because the operator $D^{\alpha}_{t_0+}$ is well defined on the space of continuous functions when $\alpha \in (0, 1)$, however, it is not sufficient to consider this space when $\alpha \geq 1$, i.e., we need to consider $n_{\alpha, p} - 1$ time differentiable functions, where $n_{\alpha, p}$

is a positive integer depending on α and $p \geq 1$. In the case of $\alpha \in (0, 1)$, we prove the existence and uniqueness of a solution of (1.4) under some conditions of f_1 and f_2. We use a coupled fixed point method which is different than the methods of the works mentioned above. This method allows us to investigate the uniqueness of solution of equation (1.4). The shortcomings in [1] can be corrected by using Lemma 2.7 stated below. We discuss about these shortcomings in the forthcoming sections in detail. Our first theorem extends the main existence theorem of [1]. Moreover, we show that the Picard iteration associated to an operator $T : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ converges to the unique solution of (1.4) for any initial guess $x \in C(J, \mathbb{R})$. In particular case, we have shown that, the rate of convergence is n^{-1}. In the case of $\alpha \geq 1$, we investigate equation (1.4) in the space of $n_{\alpha, p} - 1$ time differentiable functions. In this case, naturally, the initial condition $x(t_0) = x_0$ is replaced by $x^{(k)}(t_0) = x_0$, $0 \leq k \leq n_{\alpha, p} - 1$ because our main equation is investigated in the space of $n_{\alpha, p} - 1$ time differentiable functions. Similar to first case, the existence and uniqueness of a solution of (1.4) is established. Moreover, the convergence of the Picard iterations to the unique solution of (1.4) is shown. In particular case, the rate of convergence is n^{-1}. Finally, at the end of the paper, we provide some examples to show the applicability of the abstract results. These examples cannot be solved by the methods demonstrated in [1].

2. Preliminaries

In this section, first we recall the definitions of the Riemann-Liouville fractional integral and derivative, and then we provide a composition relation between fractional differentiation and integration operators. Using the composition relation, we prove the equivalency of the fractional differential equation (1.4) to a fractional integral equation. Finally, we recall the definitions of nonlinear weak contractions.

2.1. Riemann-Liouville fractional integral and derivative

Definition 2.1 [18]**.** Let $f \in L_1(a, b).$ The *Riemann-Liouville fractional integral* $I_{a+}^{\alpha} f$ *of order* $\alpha > 0,$ is defined as

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - s)^{\alpha - 1} f(s) ds, \quad x > a.$$

Definition 2.2 [18]**.** The *Riemann-Liouville fractional derivative* $D_{a+}^{\alpha} f$ *of order* $\alpha \geq 0,$ is defined as

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dx^m} \int_a^x (x - s)^{m - \alpha - 1} f(s) ds, \quad x > a,$$

where $m = [\alpha] + 1.$

Next, we consider two cases: $\alpha \in (0, 1)$ and $\alpha \geq 1.$

2.1.1. Case $\alpha \in (0, 1)$

Our next discussion is related to the following lemma which was formulated in [1] as follows:

Lemma 2.3 [1]**.** *Let* $0 < \alpha < 1$ *and* $f \in L_1(a, b).$ *Then*

$(A1)$ *the equality* $(D_{a+}^{\alpha} I_{a+}^{\alpha} f)(x) = f(x)$ *holds*;

$(A2)$ *the equality*

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x) - \frac{(I_{a+}^{1-\alpha} f)(a)}{\Gamma(\alpha)} (x - a)^{\alpha - 1}$$

holds for almost everywhere on $[a, b].$

The proof of this lemma was not given in [1] but it was cited in [18] where more general facts were proven. While formulating Lemma 2.3, two shortcomings were done by the authors of [1]. The first one (the minor) is, the equality $(A1)$ holds for almost everywhere on $[a, b].$ The second one (the major) is not true in general because there can be found a function

$f \in L_1(a, b)$ such that the equality ($A2$) does not hold. Since this is not the main result, we will not provide an example but we can give a brief idea of finding such a function for the benefit of the reader. We find a function $\phi : [a, b] \to \mathbb{R}$ which is differentiable a.e. on $[a, b]$, $I_{a+}^{\alpha}\phi \in AC[a, b]$, $(I_{a+}^{\alpha}\phi)(a) = 0$, and $I_{a+}^{\alpha}D\phi$ is not well defined (i.e. $D\phi \notin L_1(a, b)$), where $AC[a, b]$ is the class of absolutely continuous functions on $[a, b]$ and $D = d/dx$ is the differential operator. It is not hard to find such a function. Then consider the following equation: $\phi(x) = (I_{a+}^{1-\alpha}f)(x)$. This equation is known as *Abel's equation*. This equation is solvable in $L_1(a, b)$ if and only if $I_{a+}^{\alpha}\phi \in AC[a, b]$ and $(I_{a+}^{\alpha}\phi)(a) = 0$ (see Theorem 2.1, [18]). So, by our construction, there is $f \in L_1(a, b)$ such that $\phi(x) = (I_{a+}^{1-\alpha}f)(x)$. By the definitions of D_{a+}^{α}, the function $(I_{a+}^{\alpha}D_{a+}^{\alpha}f)(x) = (I_{a+}^{\alpha}DI_{a+}^{1-\alpha}f)(x) = (I_{a+}^{\alpha}D\phi)(x)$ is not well defined. On the other hand, the left hand side of ($A2$) is well defined which leads to a contradiction. Next, we state the corrected version of Lemma 2.3 as follows:

Lemma 2.4. *Let* $\alpha \in (0, 1)$ *and* $f \in L_1(a, b)$. *Then*

($B1$) *the equality* $(D_{a+}^{\alpha}I_{a+}^{\alpha}f)(x) = f(x)$ *holds for almost everywhere on* $[a, b]$;

($B2$) *the equality*

$$(I_{a+}^{\alpha}D_{a+}^{\alpha}f)(x) = f(x) - \frac{(I_{a+}^{1-\alpha}f)(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}$$

holds for almost everywhere on $[a, b]$, *provided that* $I_{a+}^{1-\alpha}f \in AC[a, b]$.

This lemma will be used below. For the proof of this lemma, see Theorem 2.4 in [18]. To simplify our experiments, we use the notation $\Phi_f(t, x) = x - f(t, x)$. Our next discussion is related to the following lemma:

Lemma 2.5 [1]. *Assume that* $f \in C(J \times \mathbb{R}, \mathbb{R})$ *and the function* $x \mapsto \Phi_f(t, x)$ *is increasing in* $\mathbb{R}$ *for all* $t \in J$. *Then, for any* $y \in C(J, \mathbb{R})$ *and* $\alpha \in (0, 1)$, *the function* $x \in C(J, \mathbb{R})$ *is a solution of initial value problem*

$$\begin{cases} D^{\alpha}_{t_0+}[\Phi_f(t, x(t))] = y(t), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

if and only if $x(t)$ *satisfies the hybrid fractional integral equation*

$$\Phi_f(t, x(t)) = \Phi_f(t_0, x_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} y(s) ds, \quad t \in J. \tag{2.2}$$

Unfortunately, there are some shortcomings found in the proof of Lemma 2.5. That is, after applying $D^{\alpha}_{t_0+}$ on both the sides of (2.2), the constant $\Phi_f(t_0, x_0)$ does not disappear, but another function emerges due to the definition of $D^{\alpha}_{t_0+}$. Indeed, let us consider the following simple example. If we take $f(t, x) \equiv 1$, $y(t) \equiv 1$ for $t \in J = [0, 1]$, then the function $x(t) = 2 + (I^{\alpha}_{0+}1)(t)$ is a solution to (2.2) but not to (2.1).

Our next aim is to rectify these shortcomings. First, we prove the following:

Lemma 2.6. *Let* $\alpha > 0$. *If* $g \in C(J, \mathbb{R})$, *then* $I^{\alpha}_{t_0+}g \in C(J, \mathbb{R})$ *and* $(I^{\alpha}_{t_0+}g)(t_0) = 0$.

Proof. For any $t, z \in J$ verifying $t > z$, we have

$$(I^{\alpha}_{t_0+}g)(t) - (I^{\alpha}_{t_0+}g)(z) = \frac{1}{\Gamma(\alpha)}\left[\int_{t_0}^{z} ((t - s)^{\alpha-1} - (z - s)^{\alpha-1}) g(s) ds\right]$$

$$+ \frac{1}{\Gamma(\alpha)}\left[\int_{z}^{t} ((t - s)^{\alpha-1}) g(s) ds\right]$$

$$= -\frac{(t-z)^{\alpha}}{\alpha\Gamma(\alpha)} g(\hat{z}) + \frac{(t-z)^{\alpha}}{\alpha\Gamma(\alpha)} g(\tilde{z})$$

$$+ \left[\frac{(t-t_0)^{\alpha}}{\alpha\Gamma(\alpha)} - \frac{(z-t_0)^{\alpha}}{\alpha\Gamma(\alpha)}\right] g(\hat{z}),$$

where $\hat{z} \in (t_0, z)$ and $\tilde{z} \in (z, t)$ which satisfy the mean value theorem. This implies that $I^{\alpha}_{t_0+}g$ is continuous. Furthermore, from the obvious inequalities

$$\frac{\min_{t\in J} g(t)}{\alpha\Gamma(\alpha)}(t-t_0)^{\alpha} \leq (I^{\alpha}_{t_0+}g)(t) \leq \frac{\max_{t\in J} g(t)}{\alpha\Gamma(\alpha)}(t-t_0)^{\alpha},$$

we get $(I^{\alpha}_{t_0+}g)(t_0) = 0.$ □

Now we state the corrected version of Lemma 2.5. This lemma will be also used below.

Lemma 2.7. *Let* $\alpha \in (0, 1)$. *Assume that* $f \in C(J \times \mathbb{R}, \mathbb{R})$ *and the function* $y \in L_1(J, \mathbb{R})$ *satisfies* $I^{\alpha}_{t_0+}y \in C(J, \mathbb{R})$ *and* $(I^{\alpha}_{t_0+}y)(t_0) = 0.$ *Then the function* $x(t) \in C(J, \mathbb{R})$ *is a solution of the fractional differential equation*

$$\begin{cases} D^{\alpha}_{t_0+}[\Phi_f(t, x(t))] = y(t) \quad \text{a.e} \quad t \in J, \\ x(t_0) = x_0, \end{cases} \tag{2.3}$$

if and only if $x(t)$ *satisfies the fractional integral equation*

$$\Phi_f(t, x(t)) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}y(s)ds \tag{2.4}$$

and $f(t_0, x(t_0)) = x_0.$

Proof. Let $x(t)$ be a solution of equation (2.3) with initial condition $x(t_0) = x_0$. By the definition of $D^{\alpha}_{t_0+}$, we have

$$D_{t_0+}^{\alpha}[\Phi_f(t, x(t))] = DI_{t_0+}^{1-\alpha}[\Phi_f(t, x(t))] = y(t).$$

It implies that the function $I_{t_0+}^{1-\alpha}[\Phi_f(t, x(t))]$ is absolutely continuous since $y(t) \in L_1(J, \mathbb{R})$. Using Lemma 2.4, we get

$$I_{t_0+}^{\alpha} D_{t_0+}^{\alpha}[\Phi_f(t, x(t))] = \Phi_f(t, x(t)) - \frac{I_{t_0+}^{1-\alpha}[\Phi_f(t, x(t))]|_{t=t_0}}{\Gamma(\alpha)}(t - t_0)^{\alpha-1}. \quad (2.5)$$

By Lemma 2.6, we have

$$I_{t_0+}^{1-\alpha}[\Phi_f(t, x(t))]\Big|_{t=t_0} = 0$$

since $\Phi_f(t, x(t))$ is continuous. From this and equality (2.5),

$$I_{t_0+}^{\alpha} D_{t_0+}^{\alpha}[\Phi_f(t, x(t))] = \Phi_f(t, x(t)).$$

Taking into account this and applying $I_{t_0+}^{\alpha}$ on both the sides of (2.3), we get

$$\Phi_f(t, x(t)) = (I_{t_0+}^{\alpha} y)(t),$$

i.e.,

$$\Phi_f(t, x(t)) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} y(s) ds. \quad (2.6)$$

Since $x(t_0) = x_0$ and $(I_{t_0+}^{\alpha} y)(t_0) = 0,$ substituting $t = t_0$ in (2.6) yields

$$\Phi_f(t_0, x(t_0)) = x_0 - f(t_0, x(t_0)) = 0.$$

On the contrary, assume that $x(t)$ satisfies (2.4) and $x_0 = f(t_0, x(t_0))$. Then applying $D_{t_0+}^{\alpha}$ on both the sides of (2.4), we get

$$D_{t_0+}^{\alpha}[\Phi_f(t, x(t))] = y(t) \quad \text{a.e} \quad t \in J$$

due to Lemma 2.4. Again, substituting $t = t_0$ in equation (2.4), we have

$$\Phi_f(t_0, x(t_0)) = x(t_0) - x_0 = 0$$

because of $(I^{\alpha}_{t_0+}y)(t_0) = 0.$ The proof is completed. □

Note that for $\alpha \geq 1,$ the conclusion of Lemma 2.7 no longer holds. More precisely, the continuity of *f* is not sufficient. In this case, in order to get an analogical result, it is necessary to put some restrictions on *f*.

2.1.2. Case $\alpha \geq 1$

In this case, we deal with the following space of functions. Define

$$\mathcal{I}^{\alpha}_{p}(J, \mathbb{R}) := \mathcal{I}^{\alpha}(L_p(J, \mathbb{R}))$$

$$= \{f : J \to \mathbb{R},\ f = I^{\alpha}_{t_0+}\varphi,\ \varphi \in L_p(J, \mathbb{R}),\ p \geq 1\}.$$

We can easily see that $\mathcal{I}^{\alpha}_{p}(J, \mathbb{R})$ is a linear space and $I^{\alpha}_{t_0+}\varphi = 0,$ $\varphi \in L_p(J, \mathbb{R})$ only in the case $\varphi = 0.$ Thus we may introduce the norm in $\mathcal{I}^{\alpha}_{p}(J, \mathbb{R})$ by relation

$$\| f \|_{\alpha, p} := \| \varphi \|_p, \tag{2.7}$$

where $\| \cdot \|_p$ is the L_p-norm. A trivial verification shows that the space $\mathcal{I}^{\alpha}_{p}(J, \mathbb{R})$ with norm (2.7) is a Banach space. Our next goal is to extend Lemma 2.7 for the case of $\alpha \geq 1.$ In achieving this goal, the following theorem plays a key role.

Theorem 2.8 [18]**.** *Let* $\alpha > 0.$ *Then*

$(C1)$ *the equality* $(D^{\alpha}_{a+}I^{\alpha}_{a+}f)(x) = f(x)$ *holds for almost everywhere on* $[a, b]$ *for any summable function f*;

(*C*2) *the equality*

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x)$$

holds for almost everywhere on $[a, b]$, *provided that* $f \in \mathcal{I}_1^{\alpha}([a, b], \mathbb{R})$.

Here and below, for given $\alpha > 1/p > 0$, we denote $n_{\alpha, p} = [\alpha - 1/p] + 1$, where $[\cdot]$ is the integer part of a number. Now we state an extension of Lemma 2.7 for $\alpha \geq 1$.

Lemma 2.9. *Let* $\alpha \geq 1$, $p \geq 1$ *and* $p > 1/\alpha$. *Let* $y \in L_p(J, \mathbb{R})$ *be a function satisfying*

$$\left. \frac{d^k (I_{t_0+}^{\alpha} y)(t)}{dt^k} \right|_{t=t_0} = 0, \quad 0 \leq k \leq n_{\alpha, p} - 1. \tag{2.8}$$

Assume that $f(t, x(t)) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$ *for any* $x(t) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$. *Then the function* $x(t) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$ *is a solution of the fractional differential equation*

$$\begin{cases} D_{t_0+}^{\alpha}[\Phi_f(t, x(t))] = y(t) \quad \text{a.e} \quad t \in J, \\ x^{(k)}(t_0) = x_0, \quad 0 \leq k \leq n_{\alpha, p} - 1 \end{cases} \tag{2.9}$$

if and only if $x(t)$ *satisfies fractional integral equation*

$$\Phi_f(t, x(t)) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} y(s) ds \tag{2.10}$$

and $\left. \frac{d^k f(t, x(t))}{dt^k} \right|_{t=t_0} = x_0, \ 0 \leq k \leq n_{\alpha, p} - 1.$

Proof. Let $x(t) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$ be a solution of equation (2.9) satisfying $x^{(k)}(t_0) = x_0, \ 0 \leq k \leq n_{\alpha, p} - 1$. Then by the assumption of the lemma, we have $f(t, x(t)) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$. We can get $x(t) - f(t, x(t)) = \Phi_f(t, x(t)) \in$

$\mathcal{I}_p^{\alpha}(J, \mathbb{R})$ since $\mathcal{I}_p^{\alpha}(J, \mathbb{R})$ is a linear space. Thus, applying $I_{t_0}^{\alpha}$ on both the sides of (2.9), we get

$$\Phi_f(t, x(t)) = (I_{t_0+}^{\alpha} y)(t)$$

due to Theorem 2.8. Replacing $\Phi_f(t, x(t))$ by $x(t) - f(t, x(t))$ in the last equation and differentiating the resulting equation k times with respect to t at t_0, we get $\left.\frac{d^k f(t, x(t))}{dt^k}\right|_{t=t_0} = x_0$, $0 \leq k \leq n_{\alpha, p} - 1$. On the contrary, assume that $x(t) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$ satisfies (2.10) and $\left.\frac{d^k f(t, x(t))}{dt^k}\right|_{t=t_0} = x_0$, $0 \leq k \leq n_{\alpha, p} - 1$. Then applying $D_{t_0+}^{\alpha}$ on both the sides of (2.10), we get

$$D_{t_0+}^{\alpha}[\Phi_f(t, x(t))] = y(t) \quad \text{a.e} \quad t \in J$$

due to Theorem 2.8. Again, by (2.8) and $\left.\frac{d^k f(t, x(t))}{dt^k}\right|_{t=t_0} = x_0$, $0 \leq k \leq n_{\alpha, p} - 1$, we get $x^{(k)}(t_0) = x_0$, $0 \leq k \leq n_{\alpha, p} - 1$ which completes the proof. □

2.2. Dominating and altering distance functions

Let $\mathbb{R}_0^+$ denote the set of all non-negative real numbers. The following definitions will be used in the subsequent part of this paper.

Definition 2.10. A function $\Lambda : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a *dominating function* or, in short, $\mathcal{D}$*-function* if it is an upper semi-continuous, non-decreasing, $\Lambda(r) < r$ for $r > 0$, and $\Lambda(0) = 0$.

Statement 2.11. If $\Lambda : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $\Upsilon : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are $\mathcal{D}$-functions, then for any $c_1, c_2 > 0$ such that $c_1 + c_2 < 1$, the function $c_1\Lambda + c_2\Upsilon$ is also a $\mathcal{D}$-function.

Proof. The proof of this statement is straightforward. □

Definition 2.12 [19]**.** A continuous and non-decreasing function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be an *altering distance function* if

$$\psi(t) = 0 \Leftrightarrow t = 0.$$

3. Formulation of Main Results

In this section, we formulate our main results. Their proofs will be given in the forthcoming sections. In order to formulate main results, we consider the following two cases: $\alpha \in (0, 1)$ and $\alpha \geq 1$. Let $\alpha \in (0, 1)$, $p \geq 1$, and $J = [t_0, t_0 + a]$, for some $t_0 \in \mathbb{R}$, $a > 0$. Suppose that the function $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(a_1) $f(t, \xi(t)) \in L_p(J, \mathbb{R})$ for any $\xi(t) \in C(J, \mathbb{R})$;

(a_2) $I_{t_0+}^{\alpha}[f(t, \xi(t))]|_{t=t_0} = 0$ for any $\xi(t) \in C(J, \mathbb{R})$.

We denote by $\mathcal{L}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$ the set of all functions $f : J \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions (a_1)-(a_2). It is obvious that $C(J \times \mathbb{R}, \mathbb{R}) \subset \mathcal{L}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$. Consider the following fractional hybrid differential equation:

$$\begin{cases} D_{t_0+}^{\alpha}[\Phi_{f_1}(t, x(t))] = f_2(t, x(t)) \quad \text{a.e} \quad t \in J, \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where $\alpha \in (0, 1)$, $p \geq 1$, $f_1 \in C(J \times \mathbb{R}, \mathbb{R})$, and $f_2 \in \mathcal{L}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$. To formulate our first main result, we need the following hypotheses.

3.1. Hypotheses for fractional hybrid differential equation (3.1)

We assume:

$(A1)$ $f_1(t_0, x) = x_0$ for all $x \in \mathbb{R}$.

$(A2)$ $f_1(t, \cdot)$ is non-decreasing, $f_2(t, \cdot)$ is non-increasing for all $t \in J$.

$(A3)$ There exist $\mathcal{D}$-functions $\Lambda_1, \Lambda_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and constants $c_1, c_2 \geq 0$ satisfying

$$c_1 + c_2 \frac{|J|^\alpha}{\Gamma(\alpha + 1)} \leq 1$$

such that the functions $f_i(t, \cdot)$, $i = 1, 2$ satisfy the following weak contraction conditions:

$$|f_i(t, x) - f_i(t, y)| \leq c_i \Lambda_i(|x - y|), \quad i = 1, 2,$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

$(A4)$ There exist $\beta_1, \beta_2 \in C(J, \mathbb{R})$ such that

$$f_1(t, \beta_1(t)) + I_{t_0+}^\alpha [f_2(t, \beta_2(t))] \geq \beta_1(t)$$

and

$$f_1(t, \beta_2(t)) + I_{t_0+}^\alpha [f_2(t, \beta_1(t))] \leq \beta_2(t).$$

We can now formulate our first main result.

Theorem A. *Let* $\alpha \in (0, 1)$ *and* $p \geq 1$. *Assume that the functions* $f_1 \in C(J \times \mathbb{R}, \mathbb{R})$ *and* $f_2 \in \mathcal{L}_p^\alpha(J \times \mathbb{R}, \mathbb{R})$ *satisfy the hypotheses* $(A1)$-$(A4)$. *Then the fractional hybrid differential equation* (3.1) *has a unique solution* $x^*(t) \in C(J \times \mathbb{R})$. *Furthermore, the sequence* $(x_n(t))_{n \in \mathbb{N}_0} \in C(J \times \mathbb{R})$ *constructed as*

$$x_n(t) = f_1(t, x_{n-1}(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t - s)^{\alpha - 1} f_2(s, x_{n-1}(s)) ds, \quad n \geq 1,$$

converges to $x^*(t)$ *in the* C^0*-norm for any initial guess* $x_0(t) = x(t) \in C(J \times \mathbb{R})$, *that is*,

$$\lim_{n\to\infty} \| x_n - x^* \|_{C^0} = 0.$$

In particular, *if* $\mathcal{D}$*-functions* $\Lambda_1, \Lambda_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *in hypothesis* $(A3)$ *satisfy* $\Lambda_1 = \Lambda_2 = \Lambda_\kappa^4$ *for some* $\kappa > 0$, *then there exists a constant* $C := C(\kappa, x_0, x^*) > 0$ *such that*

$$\| x_n - x^* \|_{C^0} \leq \frac{C}{n}$$

for all $n \geq 1$.

Next, we consider the case $\alpha \geq 1$. In this case, to investigate equation (3.1), the classes of functions $C(J \times \mathbb{R}, \mathbb{R})$ and $\mathcal{L}_p^\alpha(J \times \mathbb{R}, \mathbb{R})$ are not sufficient because the operators $D_{t_0+}^\alpha$ and $I_{t_0+}^\alpha$ are defined on high degree of smoothness functions. Let $\alpha \geq 1$, $p \geq 1$ and $p > 1/\alpha$. Suppose that the function $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(b_1) $f(t, \xi(t)) \in L_p(J \times \mathbb{R})$ for any $\xi(t) \in \mathcal{I}_p^\alpha(J \times \mathbb{R})$.

(b_2) We have

$$\left.\frac{d^k (I_{t_0+}^\alpha [f(t, \xi(t))])}{dt^k}\right|_{t=t_0} = 0, \quad 0 \leq k \leq n_{\alpha, p} - 1$$

for any $\xi(t) \in \mathcal{I}_p^\alpha(J \times \mathbb{R})$.

We denote by $\hat{\mathcal{L}}_p^\alpha(J \times \mathbb{R}, \mathbb{R})$ the set of all functions $f : J \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions (b_1)-(b_2). We also denote by $\mathcal{C}_p^\alpha(J \times \mathbb{R}, \mathbb{R})$ the set of all functions $f : J \times \mathbb{R} \to \mathbb{R}$ satisfying the following condition:

$f(t, \xi(t)) \in \mathcal{I}_p^{\alpha}(J \times \mathbb{R})$ for any $\xi(t) \in C(J \times \mathbb{R})$. Consider the following fractional hybrid differential equation:

$$\begin{cases} D_{t_0+}^{\alpha}[\Phi_{f_1}(t, x(t))] = f_2(t, x(t)) \quad \text{a.e} \quad t \in J, \\ x^{(k)}(t_0) = x_0, \quad 0 \le k \le n_{\alpha, p} - 1, \end{cases} \tag{3.2}$$

where $\alpha \ge 1$, $p \ge 1$, $p > 1/\alpha$, $f_1 \in \mathcal{C}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$, and $f_2 \in \hat{\mathcal{L}}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$. To formulate our second main result, we need the following hypotheses.

3.2. Hypotheses for fractional hybrid differential equation (3.2)

We assume:

($B1$) For any $x(t) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$, we have

$$\left. \frac{d^k f_1(t, x(t))}{dt^k} \right|_{t=t_0} = x_0$$

for $0 \le k \le n_{\alpha, p} - 1$.

($B2$) The functions f_1 and f_2 satisfy ($A2$).

($B3$) There exist $\mathcal{D}$-functions $\Lambda_1, \Lambda_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and constants $c_1, c_2 \ge 0$ satisfying

$$c_1 + c_2 \frac{|J|^{\alpha}}{\Gamma(\alpha + 1)} \le 1$$

such that for any $x(t), y(t) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$, we have

$$\| f_i(\cdot, x(\cdot)) - f_i(\cdot, y(\cdot)) \|_{\alpha, p} \le c_i \Lambda_i(\| x - y \|_{\alpha, p}), \quad i = 1, 2.$$

($B4$) There exist $\beta_1, \beta_2 \in C(J, \mathbb{R})$ such that

$$f_1(t, \beta_1(t)) + I_{t_0+}^{\alpha}[f_2(t, \beta_2(t))] \ge \beta_1(t)$$

and

$$f_1(t, \beta_2(t)) + I_{t_0+}^{\alpha}[f_2(t, \beta_1(t))] \leq \beta_2(t).$$

The following is our second main result.

Theorem B. *Let* $\alpha \geq 1$, $p \geq 1$ *and* $p > 1/\alpha$. *Assume that the functions* $f_1 \in \mathcal{C}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$ *and* $f_2 \in \hat{\mathcal{L}}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$ *satisfy the hypotheses* $(B1)$-$(B4)$. *Then the fractional hybrid differential equation* (3.2) *has a unique solution* $x_*(t) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$. *Furthermore, the sequence* $(x_n(t))_{n \in \mathbb{N}_0} \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$ *constructed as*

$$x_n(t) = f_1(t, x_{n-1}(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} f_2(s, x_{n-1}(s)) ds, \quad n \geq 1,$$

converges to $x_*(t)$ *in the* $\| \cdot \|_{\alpha, p}$*-norm for any initial guess* $x_0(t) = x(t) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$, *that is,*

$$\lim_{n \to \infty} \| x_n - x_* \|_{\alpha, p} = 0.$$

In particular, if $\mathcal{D}$*-functions* $\Lambda_1, \Lambda_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *in hypothesis* $(B3)$ *satisfy* $\Lambda_1 = \Lambda_2 = \Lambda_\kappa^4$ *for some* $\kappa > 0$, *then there exists a constant* $C := C(\kappa, x_0, x_*, \alpha, p) > 0$ *such that*

$$\| x_n - x_* \|_{\alpha, p} \leq \frac{C}{n}$$

for all $n \geq 1$.

4. Basic Notions of Fixed Point Theory and a Coupled Fixed Point Theorem

We begin by recalling the notions of fixed point theory. The definitions and theorems in this section are borrowed from [19].

Definition 4.1 [19]**.** Let X be a nonempty set and $A : X \times X \to X$ be a mapping. Then an element $(x, y) \in X \times X$ is called a *coupled fixed point* of A if

$$x = A(x, y) \quad \text{and} \quad y = A(y, x).$$

Definition 4.2 [19]**.** Let $(X, \preceq)$ be a partially ordered set and let $A : X \times X \to X$ be a mapping. Then we say that the mapping A has the *mixed monotone property* if A is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument. That is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \Rightarrow A(x_1, y) \preceq A(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Rightarrow A(x, y_1) \succeq A(x, y_2).$$

Definition 4.3 [19]**.** Let $(X, \preceq)$ be a partially ordered set. We define $\preceq_2$ to be a *partially order* in $X \times X$ as follows. For any $(x, y), (u, v) \in X \times X$, we say $(x, y) \preceq_2 (u, v)$ if $x \preceq u$ and $y \succeq v$.

The following theorem plays a key role in the proof of Theorem A. We denote by $(X, d, \preceq)$ a complete partially ordered metric space.

Theorem 4.4 [19]**.** *Assume that the mapping* $A : X \times X \to X$ *satisfies the following conditions*:

(1) *there exist an altering distance function* ψ, *an upper semi-continuous function* $\theta : [0, \infty) \to [0, \infty)$, *and a lower semi-continuous function* $\varphi : [0, \infty) \to [0, \infty)$ *such that for all* $(x, y), (u, v) \in X \times X$ *with* $(x, y) \preceq_2 (u, v)$,

$$\psi(d(A(x, y), A(u, v)))$$
$$\leq \theta(\max\{d(x, u), d(y, v)\}) - \varphi(\max\{d(x, u), d(y, v)\}),$$

where $\theta(0) = \varphi(0) = 0$ *and* $\psi(t) - \theta(t) + \varphi(t) > 0$ *for all* $t > 0$;

(2) *there exist* $x_0, y_0 \in X$ *such that* $x_0 \preceq A(x_0, y_0)$ *and* $y_0 \succeq A(y_0, x_0)$;

(3) *A has the mixed monotone property*;

(4) *A is continuous*.

Then A has a coupled fixed point $(x^*, y^*) \in X \times X$. *Moreover, if for any* $(x, y), (u, v) \in X \times X$, *there exists* $(w, z) \in X \times X$ *such that* $(x, y) \preceq_2 (w, z)$ *and* $(u, v) \preceq_2 (w, z)$, *then* (x^*, x^*) *is the unique coupled fixed point of A*.

Remark 4.5. Note that a weaker contraction condition (so-called (ψ, θ, φ)-weak contraction condition) has been successfully applied in multidimensional fixed point theorems and their applications to the system of matrices equations and nonlinear integral equations (see, for instance [19-26]). Our previous studies encourage us to believe that the techniques of multidimensional fixed point theorem under (ψ, θ, φ)-weak contraction conditions can be successfully applied in the investigations of fractional hybrid differential equations.

5. Proof of Main Theorems

5.1. Proof of Theorem A

Proof. *Existence*. In this subsection, we denote by X the class of continuous functions $f : J \to \mathbb{R}$ and $\|\cdot\|_{C^0}$ the uniform norm in X, that is, $X = C(J, \mathbb{R})$ and $\|x\|_{C^0} = \max_{t \in J} |x(t)|$. Obviously, $(X, \|\cdot\|_{C^0})$ is a Banach space. In this space, the partial order $\preceq$ is defined as follows: for the given $x(t), y(t) \in X$, we say

$$x(t) \preceq y(t) \text{ iff } x(t) \le y(t).$$

By the assumptions of the theorem, $f_1 \in C(J \times \mathbb{R}, \mathbb{R})$, $f_2 \in \mathcal{L}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$, $\alpha \in (0, 1)$ and $p \geq 1$. Moreover, the functions f_1 and f_2 satisfy hypotheses $(A1)$-$(A4)$. From Lemma 2.7, it implies that $x(t)$ is a solution of equation (3.1) if and only if it satisfies the hybrid fractional integral equation

$$x(t) = f_1(t, x(t)) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} f_2(s, y(s))ds \tag{5.1}$$

and $f_1(t_0, x(t_0)) = x_0$. Consider the operators $F_1 : X \to X$, $F_2 : X \to X$, and $A : X \times X \to X$ defined as follows:

$$F_1 x(t) = f_1(t, x(t)),\ F_2 y(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} f_2(s, y(s))ds, \quad t \in J,$$

and $A(x, y) = F_1 x + F_2 y$. We show that the operator A satisfies all hypotheses of Theorem 4.4. First, we show that the operator A satisfies the condition (1) of Theorem 4.4 with $\psi(t) = t$, $\varphi(t) = 0$ and

$$\theta(t) = c_1\Lambda_1(t) + c_2\frac{|J|^{\alpha}}{\Gamma(\alpha+1)}\Lambda_2(t).$$

Note that the function θ is a $\mathcal{D}$-function due to Statement 2.11. Let $(x, y), (u, v) \in X \times X$ with $(x, y) \preceq_2 (u, v)$. By hypothesis $(A3)$, we have

$$\begin{aligned}
&|A(x(t), y(t)) - A(u(t), v(t))| \\
&\leq |f_1(t, x(t)) - f_1(t, u(t))| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|f_2(s, y(s)) - f_2(s, v(s))|ds \\
&\leq c_1\Lambda_1(\|x-u\|_{C^0}) + c_2\frac{|J|^{\alpha}}{\Gamma(\alpha+1)}\Lambda_2(\|y-v\|_{C^0}) \\
&\leq c_1\Lambda_1(\max\{\|x-u\|_{C^0}, \|y-v\|_{C^0}\}) \\
&\quad + c_2\frac{|J|^{\alpha}}{\Gamma(\alpha+1)}\Lambda_2(\max\{\|x-u\|_{C^0}, \|y-v\|_{C^0}\}).
\end{aligned} \tag{5.2}$$

Taking maximum over t from the left hand side, we obtain

$$\psi(\| A(x,\ y) - A(u,\ v) \|_{C^0})$$

$$\leq \theta(\max\{\| x - u \|_{C^0},\ \| y - v \|_{C^0}\}) - \varphi(\max\{\| x - u \|_{C^0},\ \| y - v \|_{C^0}\}).$$

Since θ is a $\mathcal{D}$-function, we have

$$\psi(t) - \theta(t) + \varphi(t) = t - c_1\Lambda_1(t) - c_2 \frac{| J |^{\alpha}}{\Gamma(\alpha + 1)} \Lambda_2(t) > 0 \tag{5.3}$$

for all $t > 0$. Hence the operator A satisfies the condition (1) of Theorem 4.4. Next, we show that the operator A satisfies the condition (2) of Theorem 4.4. Let $x_0(t) \equiv \beta_1(t)$ and $y_0(t) \equiv \beta_2(t)$. By hypothesis (A4), we have

$$A(x_0(t),\ y_0(t))$$

$$= F_1(x_0(t)) + F_2(y_0(t))$$

$$= f_1(t,\ \beta_1(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} f_2(s,\ \beta_2(s)) ds \geq \beta_1(t) = x_0(t)$$

and

$$A(y_0(t),\ x_0(t))$$

$$= F_1(y_0(t)) + F_2(x_0(t))$$

$$= f_1(t,\ \beta_2(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha - 1} f_2(s,\ \beta_1(s)) ds \leq \beta_2(t) = y_0(t).$$

Hence A satisfies the condition (2). Hypothesis (A2) implies that $A(\cdot,\ y)$ is non-decreasing and $A(x,\ \cdot)$ is non-increasing, so A has the mixed monotone property which satisfies the condition (3) of Theorem 4.4. Finally, A is continuous due to inequality (5.2). Thus, it satisfies the condition (4) of Theorem 4.4. We have shown that the operator A satisfies the conditions (1)-(4) of Theorem 4.4. Therefore, A has a coupled fixed point, that is, there exists an element $(x^*,\ y^*) \in X \times X$ such that

$$x^* = A(x^*, y^*) \quad \text{and} \quad y^* = A(y^*, x^*).$$

Uniqueness. Next, we show that the operator A has the unique coupled fixed point. For any $(x(t), y(t)), (u(t), v(t)) \in X \times X$, we set

$$\omega(t) = \max\{x(t), y(t), u(t), v(t)\}$$

and

$$z(t) = \min\{x(t), y(t), u(t), v(t)\}.$$

It is obvious that

$$x(t), u(t) \leq \omega(t) \quad \text{and} \quad y(t), v(t) \geq z(t),$$

that is,

$$(x(t), y(t)) \preceq_2 (w(t), z(t)) \quad \text{and} \quad (u(t), v(t)) \preceq_2 (w(t), z(t)).$$

Hence, there exists a unique $x^* \in X$ such that

$$x^* = A(x^*, x^*) \tag{5.4}$$

due to Theorem 4.4. This implies that there exists a unique $x^*(t) \in X$ such that

$$x^*(t) = f_1(t, x^*(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f_2(s, x^*(s))ds.$$

By hypothesis ($A1$), we have $f_1(t_0, x^*(t_0)) = x_0$. As a result, equation (3.1) has a unique solution $x^*(t)$ in $C(J, \mathbb{R})$ and the solution satisfies $x^*(t_0) = x_0$. This proves the existence and uniqueness of the solution of equation (3.1).

From the relation (5.4), it follows that the operator A has a unique coupled fixed point (x^*, x^*) satisfying

$$x^* = A(x^*, x^*).$$

From this, it implies that the coupled fixed point of A lies on the main diagonal of $X \times X$. This allows us to investigate the Picard iterations of A on the main diagonal of $X \times X$. Let us denote by $T : X \to X$ the restriction of A to the main diagonal of $X \times X$, that is,

$$Tx = A(x, x).$$

For any initial guess $x_0 = x$, we construct the sequence $(x_n)_{n \in \mathbb{N}_0}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, as follows:

$$\begin{aligned} x_n &= Tx_{n-1} = A(x_{n-1}, x_{n-1}) \\ &= f_1(t, x_{n-1}(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f_2(s, x_{n-1}(s)) ds, \quad n \geq 1. \end{aligned}$$

Recall that the sequence $(x_n)_{\mathbb{N}_0}$ is called nth *Picard iteration* of T. Next, we prove the second assertion of the theorem. Utilizing hypotheses ($A3$), ($A4$) and equation (5.4), we get

$$\begin{aligned} &| x_n(t) - x^*(t) | \\ &= | A(x_{n-1}(t), x_{n-1}(t)) - A(x^*(t), x^*(t)) | \\ &\leq | F_1(x_{n-1}(t)) - F_1(x^*(t)) | + | F_2(x_{n-1}(t)) - F_2(x^*(t)) | \\ &= | f_1(t, x_{n-1}(t)) - f_1(t, x^*(t)) | \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} | f_2(s, x_{n-1}(s)) - f_2(s, x^*(s)) | ds \\ &\leq c_1 \Lambda_1(\| x_{n-1} - x^* \|_{C^0}) + c_2 \frac{| J |^{\alpha}}{\Gamma(\alpha+1)} \Lambda_2(\| x_{n-1} - x^* \|_{C^0}) \\ &\leq \| x_{n-1} - x^* \|_{C^0}. \end{aligned} \tag{5.5}$$

Taking maximum over t from the left hand side, we obtain

$$\| x_n - x^* \|_{C^0} \leq \| x_{n-1} - x^* \|_{C^0}.$$

Thus, the sequence $(\| x_n - x^* \|_{C^0})_{n \in \mathbb{N}_0}$ is a non-increasing sequence and bounded below. Hence, there exists $r \geq 0$ such that

$$\lim_{n \to \infty} \| x_n - x^* \|_{C^0} = r.$$

It remains to show $r = 0$. On the contrary, suppose that $r > 0$. From equation (5.5), it follows that

$$\| x_n - x^* \|_{C^0} \leq c_1 \Lambda_1(\| x_{n-1} - x^* \|_{C^0}) + c_2 \frac{|J|^\alpha}{\Gamma(\alpha+1)} \Lambda_2(\| x_{n-1} - x^* \|_{C^0}). \tag{5.6}$$

Since the functions Λ_1 and Λ_2 are upper semi-continuous and the sequences $\Lambda_1(\| x_{n-1} - x^* \|_{C^0})$ and $\Lambda_2(\| x_{n-1} - x^* \|_{C^0})$ are bounded by taking the limit $n \to \infty$ from (5.6), we get

$$\begin{aligned} r &= \lim_{n \to \infty} \| x_n - x^* \|_{C^0} \\ &\leq \limsup_{n \to \infty} \left[c_1 \Lambda_1(\| x_{n-1} - x^* \|_{C^0}) + c_2 \frac{|J|^\alpha}{\Gamma(\alpha+1)} \Lambda_2(\| x_{n-1} - x^* \|_{C^0}) \right] \\ &\leq c_1 \limsup_{n \to \infty} [\Lambda_1(\| x_{n-1} - x^* \|_{C^0})] \\ &\quad + c_2 \frac{|J|^\alpha}{\Gamma(\alpha+1)} \limsup_{n \to \infty} [\Lambda_2(\| x_{n-1} - x^* \|_{C^0})] \\ &\leq c_1 \Lambda_1(r) + c_2 \frac{|J|^\alpha}{\Gamma(\alpha+1)} \Lambda_2(r). \end{aligned} \tag{5.7}$$

This is contrary to (5.3). Hence

$$\lim_{n \to \infty} \| x_n - x^* \|_{C^0} = 0.$$

Next, we prove the third assertion of the theorem. By assumption, there exists a constant $\kappa > 0$ such that

$$\Lambda_1(t) = \Lambda_2(t) = \frac{t}{1+\kappa t}.$$

It follows from (5.6) that

$$\| x_n - x^* \|_{C^0} \leq \frac{\| x_{n-1} - x^* \|_{C^0}}{1 + \kappa \| x_{n-1} - x^* \|_{C^0}}. \tag{5.8}$$

Without loss of generality, we can assume $x_n \neq x^*$ for all $n \in \mathbb{N}_0$. Inequality (5.8) implies that

$$\exp\left(-\frac{1}{\| x_n - x^* \|_{C^0}}\right) \leq \lambda \exp\left(-\frac{1}{\| x_{n-1} - x^* \|_{C^0}}\right),$$

where $\lambda = e^{-\kappa}$. Iterating the last inequality, we get

$$\exp\left(-\frac{1}{\| x_n - x^* \|_{C^0}}\right) \leq \lambda^n \exp\left(-\frac{1}{\| x_0 - x^* \|_{C^0}}\right).$$

Solving for $\| x_n - x^* \|_{C^0}$, we get

$$\| x_n - x^* \|_{C^0} \leq \frac{C}{n}.$$

This completes the proof of Theorem A. □

Remark 5.1. In recent paper [26], the main result has been proved by using Lemma 2.5 which was incorrect. That shortcoming can be corrected by using Lemma 2.7 and the same manner as in the proof of the existence part of Theorem A.

5.2. Proof of Theorem B

Proof. In this subsection, our notation is slightly different from the above section. Here as space X, we take $\mathcal{I}_p^{\alpha}(J, \mathbb{R})$ and as a norm, we take $\| \cdot \|_{\alpha, p}$. A trivial verification shows that the space $(X, \| \cdot \|_{\alpha, p})$ is a Banach space. The partial order $\preceq$ remains the same as in the previous section. We prove this theorem, similar to those of the preceding section. By assumptions of Theorem B, the functions $f_1 \in \mathcal{C}_p^\alpha(J \times \mathbb{R}, \mathbb{R})$ and $f_2 \in \mathcal{L}_p^\alpha(J \times \mathbb{R}, \mathbb{R})$ satisfy hypotheses ($B1$)-($B4$). From Lemma 2.9, it follows that $x(t) \in \mathcal{I}_p^\alpha$ is a solution of equation (3.1) if and only if it satisfies the hybrid integral equation

$$x(t) = f_1(t, x(t)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f_2(s, y(s)) ds \tag{5.9}$$

and

$$\left. \frac{d^k f_1(t, x(t))}{dt^k} \right|_{t=t_0} = x_0,$$

for all $0 \leq k \leq n_{\alpha, p} - 1$. The same as the previous section, consider the operators $F_1 : X \to X$, $F_2 : X \to X$ and $A : X \times X \to X$ defined as follows:

$$F_1 x(t) = f_1(t, x(t)),$$

$$F_2 y(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f_2(s, y(s)) ds, \quad t \in J$$

and $A(x, y) = F_1 x + F_2 y.$ We can easily check that the operator A satisfies all hypotheses of Theorem 4.4, with the exception of (2), in the space $(X, \|\cdot\|_{\alpha, p})$. The reason for not satisfying hypothesis (2) of Theorem 4.4 is the functions $\beta_1, \beta_2 \in C(J, \mathbb{R})$ in hypothesis ($B4$), might not be in $\mathcal{I}_p^\alpha(J, \mathbb{R})$. Therefore, to prove the first assertion of the theorem, it only remains to find functions $\hat{\beta}_1, \hat{\beta}_2 \in \mathcal{I}_p^\alpha(J, \mathbb{R})$ such that ($B4$) holds. In order to find such $\hat{\beta}_1$ and $\hat{\beta}_2$, we use hypotheses ($B2$) and ($B4$). Taking $\hat{\beta}_1 := A(\beta_1, \beta_2)$, $\hat{\beta}_2 := A(\beta_2, \beta_1)$ for $\beta_1, \beta_2 \in C(J, \mathbb{R})$ in hypothesis ($B4$), yields

$\hat{\beta}_1, \hat{\beta}_2 \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$ since $f_1(t, x(t)) \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$ and $f_2 \in \mathcal{L}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$. Moreover, $(\beta_1, \beta_2) \preceq_2 (\hat{\beta}_1, \hat{\beta}_2)$ due to ($B4$). On the other hand, from the hypothesis ($B2$), it follows that the operator $\mathcal{A} : X \times X \to X \times X$ defined as

$$\mathcal{A}(x, y) = (A(x, y), A(y, x))$$

is non-decreasing with respect to $\preceq_2$. This together with last inequality gives

$$\mathcal{A}(\beta_1, \beta_2) \preceq_2 \mathcal{A}(\hat{\beta}_1, \hat{\beta}_2).$$

It means that

$$\hat{\beta}_1 \leq A(\hat{\beta}_1, \hat{\beta}_2) \quad \text{and} \quad \hat{\beta}_2 \geq A(\hat{\beta}_2, \hat{\beta}_1).$$

We conclude that there exist $\hat{\beta}_1, \hat{\beta}_2 \in \mathcal{I}_p^{\alpha}(J, \mathbb{R})$ such that

$$f_1(t, \hat{\beta}_1(t)) + I_{t_0+}^{\alpha}[f_2(t, \hat{\beta}_2(t))] \geq \hat{\beta}_1(t)$$

and

$$f_1(t, \hat{\beta}_2(t)) + I_{t_0+}^{\alpha}[f_2(t, \hat{\beta}_1(t))] \leq \hat{\beta}_2(t).$$

Hence the operator A satisfies all hypotheses of Theorem 4.4. So A has a coupled fixed point, that is, there exists a point $(x_*, y_*) \in X \times X$ such that

$$x_* = A(x_*, y_*) \quad \text{and} \quad y_* = A(y_*, x_*).$$

Analysis similar to that in the proof of Theorem A, in part Uniqueness, shows that $x_* = y_*$. Consequently, x_* satisfies equation (5.9) and belongs to $\mathcal{I}_p^{\alpha}(J, \mathbb{R})$ since $f_1 \in \mathcal{C}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$ and $f_2 \in \mathcal{L}_p^{\alpha}(J \times \mathbb{R}, \mathbb{R})$. By ($B1$), we have

$$\left. \frac{d^k f_1(t, x_*(t))}{dt^k} \right|_{t=t_0} = x_0$$

for all $0 \le k \le n_{\alpha, p} - 1.$ This proves the theorem. The proof of the second and third assertions of this theorem follows exactly the same way that of Theorem A. □

6. Illustrative Example and Comparison of Results

In this section, we first provide an example to show the applicability of Theorem A. Moreover, we compare our results with the main result of [1]. As we mentioned in Introduction, an existence theorem for the fractional hybrid differential equation (1.4) was proved by Lu et al. in [1] under different hypotheses. In their work, the contraction condition for f_1 was chosen as follows:

($A3a$) There exist constants $M \ge L > 0$ such that

$$| f_1(t, x) - f_1(t, y) | \le \frac{L| x - y |}{M + | x - y |}$$

for all $t \in J$ and $x, y \in \mathbb{R}.$

The conditions ($A3$) posed on the map f_1 and ($A3a$) do not imply from each other. Next, we provide an example which satisfies ($A3$) but does not satisfy ($A3a$) for any $M \ge L > 0.$

Example 6.1. Let $J = [0, 1].$ Consider the following fractional hybrid differential equation:

$$\begin{cases} D^{\frac{1}{2}}[x(t) - te^{t-1} \tanh(x(t))] = t^3, \\ x(0) = 0, \end{cases} \tag{6.1}$$

where $t \in J.$ We show that equation (6.1) has a unique solution in $C(J, \mathbb{R}).$ Denote

$$f_1(t, x) = te^{t-1} \tanh(x) \quad \text{and} \quad f_2(t, x) = t^3.$$

Clearly, $f_2 \in \mathcal{L}_1^{0.5}(J \times \mathbb{R}, \mathbb{R})$, $f_1 \in C(J \times \mathbb{R}, \mathbb{R})$ and $f_1(0, x) = 0$ for all $x \in \mathbb{R}$. It is not difficult to see that $f_1(t, \cdot)$ is increasing and $f_2(t, \cdot)$ is non-increasing. We can easily show that f_1 satisfies the hypothesis $(A3)$ with constant $c_1 = 1$, that is,

$$| f_1(t, x) - f_1(t, y) | \leq \tanh(| x - y |)$$

for all $t \in J$. Now we show that f_1 does not satisfy the hypothesis $(A3a)$. For this, it is enough to prove the following statement:

Statement 6.2. For any $M > 0$, there exist $t_0 \in J$ and $\tau_0 > 0$ such that

$$| f_1(t_0, \tau) | \geq \frac{M\tau}{M + \tau}$$

for all $0 < \tau < \tau_0$.

Proof. For given $M > 0$ and $|\tau| < \pi/2$, consider the function

$$\Phi_M(\tau) = (M + \tau) f_1(1, \tau) - M\tau = (M + \tau)\tanh(\tau) - M\tau$$

$$= (M + \tau)\sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n} - 1)B_{2n}}{(2n)!}\tau^{2n-1} - M\tau$$

$$= (M + \tau)\left(\tau - \frac{\tau^3}{3} + O(\tau^5)\right) - M\tau$$

$$= \tau^2\left(1 - \frac{M\tau}{3} - \frac{\tau^2}{3} + O(\tau^3)\right),$$

where B_n is the Bernoulli numbers. It can be easily shown that for any $M > 0$, there exists $\tau_0 > 0$ such that

$$1 - \frac{M\tau}{3} - \frac{\tau^2}{3} + O(\tau^3) \geq 0$$

for all $0 < \tau < \tau_0$. It implies that

$$f_1(1, \tau) \geq \frac{M\tau}{M + \tau}$$

for all $0 < \tau < \tau_0$ which completes the proof of the statement. □

We now turn to check the hypothesis ($A4$). The hypothesis ($A4$) is satisfied with $\beta_1(t) \equiv 0$ and $\beta_2(t) \equiv 1 + 2/\sqrt{\pi}$. Indeed, it can be easily shown that $f_1(t, \beta_1(t)) = f_1(t, 0) = 0$ and

$$f_1(t, \beta_1(t)) + I_{0+}^{0.5}[f_2(t, \beta_2(t))] = \frac{1}{\Gamma(0.5)}\int_0^t \frac{s^3 ds}{\sqrt{t-s}} \geq 0 = \beta_1(t)$$

for all $t \in J$. On the other hand, we have $f_1(t, \beta_2(t)) \leq 1$ and

$$f_1(t, \beta_2(t)) + I_{0+}^{0.5}[f_2(t, \beta_1(t))] \leq 1 + \frac{1}{\Gamma(0.5)}\int_0^t \frac{s^3 ds}{\sqrt{t-s}} \leq 1 + \frac{2}{\sqrt{\pi}} = \beta_2(t)$$

for all $t \in J$. Thus, hypothesis ($A4$) is satisfied. It follows from Theorem A that the hybrid differential equation (6.1) has a unique solution. However, this equation cannot be solved by the methods demonstrated in [1].

We continue comparing the main result of [1] with our result. In [1], for f_2, the following condition was required:

($A3b$) There exists a continuous function $h \in C(J, \mathbb{R})$ such that

$$|f_2(t, x)| \leq h(t)$$

for all $t \in J$ and $x \in \mathbb{R}$.

Generally, the conditions ($A3$) posed on the map f_2 and ($A3b$) do not imply from each other. Our next goal is to construct an example satisfying hypotheses ($A1$)-($A4$) but not ($A3b$).

Example 6.3. Again we take $J = [0, 1]$. Denote by X the set of all continuous and non-negative functions, and consider in it the following fractional hybrid differential equation:

$$\begin{cases} D^{\frac{1}{2}}[x(t) - 0.5\sin(t)\arctan(x(t))] = k(t) - \dfrac{\sqrt{\pi}}{4}\tanh(x(t)), \\ x(0) = 0, \end{cases} \tag{6.2}$$

where $t \in J$ and $k(t) = t^{-0.4}$ if $t \in (0, 1]$ and $k(t) = 0$ if $t = 0$. Denote $f_1(t, x) = 0.5\sin(t)\arctan(x)$ and $f_2(t, x) = k(t) - \frac{\sqrt{\pi}}{4}\tanh(x)$. It is a simple matter to see $f_2(t, \xi(t)) \in L_1(J \times \mathbb{R}, \mathbb{R})$ for any $\xi(t) \in C(J, \mathbb{R})$. It implies that $f_2 \in \mathcal{L}_1^{0.5}(J \times \mathbb{R}, \mathbb{R})$. Let $\xi(t) \in C(J, \mathbb{R})$. An easy computation shows that

$$\begin{aligned} I_{0+}^{0.5}[f_2(t, \xi(t))] &= I_{0+}^{0.5}[t^{-0.4}] - I_{0+}^{0.5}[\tanh(\xi(t))] \\ &= \frac{\Gamma(0.6)}{\Gamma(1.1)} t^{0.1} - I_{0+}^{0.5}[\tanh(\xi(t))]. \end{aligned}$$

Hence $I_{0+}^{0.5}[f_2(t, \xi(t))] \in C(J, \mathbb{R})$ and $I_{0+}^{0.5}[f_2(t, \xi(t))]\Big|_{t=0} = 0$ since $\tanh(\xi(t)) \in C(J, \mathbb{R})$. Thus $f_2 \in \mathcal{L}_1^{0.5}(J \times \mathbb{R}, \mathbb{R})$. We show that f_1 and f_2 satisfy hypotheses $(A1)$-$(A4)$. It is clear that $f_1(t, \cdot)$ is increasing, $f_2(t, \cdot)$ is decreasing and $f_1(0, x) = 0$ for all $x \in \mathbb{R}$. We claim that f_1 and f_2 satisfy the hypothesis $(A3)$ with $\mathcal{D}$-functions $\hat{\Lambda}^2$ and $\hat{\Lambda}^1$ and constants $c_1 = 0.5$ and $c_2 = \sqrt{\pi}/4$, respectively. Indeed, for any $x, y \in \mathbb{R}_0^+$, we can easily get

$$\begin{aligned} | f_1(t, x) - f_1(t, y) | &\le 0.5| \arctan(x) - \arctan(y) | \\ &\le 0.5\arctan\left|\frac{x - y}{1 + xy}\right| \le 0.5\arctan| x - y | \\ &= 0.5\hat{\Lambda}^2(| x - y |) \end{aligned}$$

and for f_2, it is clear from above example. Similarly, as in the above examples, we can show that inequalities

$$f_1(t, \beta_1(t)) + I^{\alpha}_{t_0+}[f_2(t, \beta_2(t))] \geq \beta_1(t)$$

and

$$f_1(t, \beta_2(t)) + I^{\alpha}_{t_0+}[f_2(t, \beta_1(t))] \leq \beta_2(t)$$

hold for $\beta_1(t) \equiv 0$ and $\beta_2(t) \equiv 1 + \Gamma(0.6)/\Gamma(1.1)$. Hence the hypotheses ($A1$)-($A4$) are satisfied. Therefore, equation (6.2) has a unique solution in $C(J, \mathbb{R})$. However, this equation cannot be solved by methods demonstrated in [1] because f_2 does not satisfy ($A3b$). Conversely, suppose that there is $h \in C(J, \mathbb{R})$ such that

$$| f_2(t, x) | \leq h(t)$$

for all $t \in J$ and $x \in \mathbb{R}$. It is known that h is bounded since it is continuous. This is impossible because $k(t)$ is unbounded.

7. Conclusion

In this paper, we have investigated fractional hybrid differential equation (1.0). We have considered two cases: $\alpha \in (0, 1)$ and $\alpha \geq 1$. In the first case, we prove the existence and uniqueness of a solution of (1.0). This theorem extends the main result of [1]. Moreover, we show that the Picard iteration associated to an operator $T : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ converges to the unique solution of (1.0) for any initial guess $x \in C(J, \mathbb{R})$. In particular, the rate of convergence is n^{-1}. In the second case, we have considered this equation in the space of k times differentiable functions. Naturally, the initial condition $x(t_0) = x_0$ is replaced by $x^{(k)}(t_0) = x_0$, $0 \leq k \leq n_{\alpha, p} - 1$. We have shown the existence and uniqueness of a solution of (1.0). The proof of the main theorems is based on a coupled fixed point method which is different from the methods of the previous works mentioned above. This method allowed us not only to investigate the existence of the solution but it is allowed us to investigate the uniqueness of solution of equation (1.4) also. We believe that this method will be applied to investigate the existence and uniqueness of the solutions of other nonlinear integral and differential equations. Moreover, it is shown the convergence of the Picard iterations to the unique solution of (1.0) and in particular, the rate of convergence is n^{-1}. Finally, we have provided two examples to show the applicability of the abstract results. These examples cannot be solved by the methods demonstrated in [1].

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