Advances in Differential Equations and Control Processes
© 2022 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324322033
Volume 29, 2022, Pages 47-63 **P-ISSN: 0974-3243**

CLOSED FORM EXPRESSIONS OBTAINED FROM THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS OF THE PROBABILITY DENSITY FUNCTION OF THE BETA DISTRIBUTION

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Received: August 12, 2022; Revised: August 30, 2022; Accepted: September 5, 2022
2020 Mathematics Subject Classification: 34A05, 60E05.
Keywords and phrases: Beta distribution, closed form expression, ordinary differential equation, probability density function, shape parameter statistics.

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How to cite this article: Olasunmbo O. Agboola, Hilary I. Okagbue, Adedayo F. Adedotun and Paulinus O. Ugwoke, Closed form expressions obtained from the solution of ordinary differential equations of the probability density function of the Beta distribution, Advances in Differential Equations and Control Processes 29 (2022), 47-63.
http://dx.doi.org/10.17654/0974324322033

Published Online: October 7, 2022

Abstract

In this paper, some closed form expressions for selected parameters for the probability density function (PDF) of the beta distribution are obtained. The closed form expressions are recovered from the solution of the ordinary differential equations (ODEs), obtained from the differentiation of the PDF of the distribution. The paper shows that the shape of the distributions also determines the nature of the resulting ODE which has shown how distributions related to the beta distribution can be traced via the solutions of the ODEs. Numerical methods are unnecessary because the closed form expressions are the same with the values obtained from the standard statistical software.

Introduction

The beta distribution is one of the most important interval bounded probability distributions with wide applications in industries and various areas of research [1]. The probability density function (PDF), cumulative distribution function (CDF) and other functions of the distribution are complex and only the PDF has a semi-closed form expression. A research that *x*-rayed the extent of such complexity and intractability could be found in [2].

The complexity of the distribution means that different values of the PDF can be obtained for different values of the parameters that defined the distribution. The easiest way is to query the standard software like Microsoft EXCEL, R, SPSS, MINITAB and so on. The software outputs values but not closed form expressions which can help understand the structure of the distribution and hence, improve the applicability of the distribution. One way of obtaining the structure is to solve the ordinary differential equations (ODEs) obtained from the continuous differentiation and simplification of the ODE obtained from the PDF of the distribution. Similar works have been done by different authors who utilize the nature of ODEs whose output results in close form is used to propose some ODEs from the differentiation of probability functions of some probability distributions. Recently, such route was followed in the creation of ODEs of the probability functions of the power function distribution [3] and the authors used numerical methods to solve the resulting ODEs. Other authors have proposed the ODE of probability functions of the following probability distributions: Harris extended exponential [4], half-normal [5], Gumbel [6], exponentiated Pareto [7], exponential and truncated exponential [8], half-Cauchy and power Cauchy [9], exponentiated Fréchet [10], 3-parameter Weibull [11], Gompertz and Gamma Gompertz [12], Burr XII and Pareto [13], Cauchy, standard Cauchy and Log-Cauchy [14], exponentiated generalized exponential [15] and Fréchet [16].

The use of ODE in the PDF is now added to a growing literature of the use of ODE in probability functions. Hitherto, the use is mainly on the quantile function which is used to recover the quartiles form intractable CDF of probability distributions [17]. Examples are the use of ODE in the recovery of the quartiles of Gamma [18], Maxwell-Boltzmann [19], Erlang [20], Nakagami [21], Chi-square [22] and convoluted probability distributions [23]. The method has also been extended to some generalized family of distributions such as the generalized odd half-Cauchy family of distributions [24] and transmuted generalized gamma distribution [25].

We remark that this paper uses differentiation to obtain the ODE of the PDF of the beta distribution. The closed form expressions obtained for the PDF can be integrated to obtain their respective cumulative distribution function (CDF) which can be transformed to obtain their quantile functions. Consequently, the quantile functions can be used in simulation and modeling of real life phenomena.

Ordinary Differential Equations for the PDF of the Beta Distribution

The PDF of the beta distribution is given by

$$f(x)=\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \tag{1}$$

which is supported on $x\in(0,1)$; $a, b>0$ and $B(a, b)$ is the beta function.

Differentiating equation (1), one obtains

$$f'(x) = \frac{1}{B(a, b)}[x^{a-1}(b-1)(1-x)^{b-2}(-1) + (1-x)^{b-1}(a-1)x^{a-2}], \quad (2)$$

which becomes

$$f'(x) = \frac{1}{B(a, b)}\left[\frac{(a-1)x^{a-1}(1-x)^{b-1}}{x} - \frac{(b-1)x^{a-1}(1-x)^{b-1}}{1-x}\right] \quad (3)$$

upon simplification.

Substituting the PDF as presented in equation (1) to obtain

$$f'(x) = \frac{(a-1)f(x)}{x} - \frac{(b-1)f(x)}{1-x}. \quad (4)$$

This is termed the beta PDF first order ODE with the initial value condition given as $f(0) = 0$ or $f(1) = 0$.

It can be noted that no general solution of the ODE exists except for the particular values of the shape parameters *a* and *b*.

Five conditions are to be considered. Theoretically, the conditions are the different behaviours and orientations of the distribution.

Condition 1. When $a = b$.

This is when the distribution is symmetric. When $a = b$, equation (4) becomes

$$f'(x) = \frac{(a-1)f(x)}{x} - \frac{(a-1)f(x)}{1-x}. \quad (5)$$

Equation (5) can be simplified to get

$$f'(x) = \frac{(a-1)(1-2x)f(x)}{x(1-x)}. \quad (6)$$

When $a = b = 1$, equation (6) reduces to $f'(x) = 0$ which when solved, yields the standard uniform distribution.

When $a = b = 2$, equation (6) reduces to

$$f'(x) = \frac{(1-2x)f(x)}{x(1-x)}, \tag{7}$$

which can be put in the form

$$x(1-x)f'(x) - (1-2x)f(x) = 0. \tag{8}$$

Applying the variable separable method to solve equation (8) yields

$$f(x) = e^c(x^2 - x), \tag{9}$$

where c is the arbitrary constant of integration.

Solving equation (8) analytically with the initial condition yields

$$f(x) = -6x(x-1). \tag{10}$$

When $a = b = 3$, equation (6) reduces to

$$f'(x) = \frac{2(1-2x)f(x)}{x(1-x)}, \tag{11}$$

$$x(1-x)f'(x) - 2(1-2x)f(x) = 0. \tag{12}$$

Solving equation (12) by the variable separable method, we have

$$f(x) = e^c(x^4 - 2x^3 + x^2), \tag{13}$$

where c is the arbitrary constant of integration.

Solving with the initial value condition, we have

$$f(x) = 30x^2(x-1)^2. \tag{14}$$

When $a = b = 4$, equation (6) reduces to

$$f'(x) = \frac{3(1-2x)f(x)}{x(1-x)}, \tag{15}$$

$$x(1-x)f'(x) - 3(1-2x)f(x) = 0. \tag{16}$$

Solving equation (16) by the variable separable method and without any initial value condition, we have

$$f(x) = e^{c}(x^6 - 3x^5 + 3x^4 - x^3). \tag{17}$$

Solving equation (17) analytically with the initial value condition yields

$$f(x) = -140x^3(x-1)^3. \tag{18}$$

When $a = b = 5$, equation (6) reduces to

$$f'(x) = \frac{4(1-2x)f(x)}{x(1-x)}, \tag{19}$$

$$x(1-x)f'(x) - 4(1-2x)f(x) = 0. \tag{20}$$

By employing the variable separable method to equation (20) as usual, its solution is obtained as

$$f(x) = e^{c}(x^8 - 4x^7 + 6x^6 - 4x^5 + x^4). \tag{21}$$

Invoking the initial value condition analytically yields

$$f(x) = 630x^4(x-1)^4. \tag{22}$$

Corollary. *Similar results as follows are obtained*:

When $a = b = 6$,

$$f(x) = -2772x^5(x-1)^5. \tag{23}$$

When $a = b = 7$,

$$f(x) = 12012x^6(x-1)^6. \tag{24}$$

Generally, when $a = b = n$, *the PDF of the beta distribution has the following closed form expression given as*:

$$f(x) = \begin{cases} Ax^{n-1}(x-1)^{n-1}; & n = 3, 5, 7, 9, \ldots \\ -Bx^{n-1}(x-1)^{n-1}; & n = 2, 4, 6, 8, \ldots. \end{cases} \tag{25}$$

Condition 2. When $a = 1$ and $b > 1$.

This is when the distribution is positively skewed, strictly decreasing and reduces to reversed power function $[0, 1]$ distribution. When $a = 1$, equation (4) becomes

$$f'(x) = -\frac{(b-1) f(x)}{1-x}, \tag{26}$$

$$(1-x) f'(x) + (b-1) f(x) = 0. \tag{27}$$

When $a = 1$ and $b = 2$, equation (27) becomes

$$(1-x) f'(x) + f(x) = 0. \tag{28}$$

By using the variable separable method, the solution of equation (28) without any initial value condition gives

$$f(x) = e^{c}(1-x). \tag{29}$$

Invoking the initial value condition analytically yields

$$f(x) = 2(1-x). \tag{30}$$

When $a = 1$ and $b = 3$, equation (27) becomes

$$(1-x) f'(x) + 2 f(x) = 0. \tag{31}$$

In a similar manner, equation (31) can be solved by employing the variable separable method (without any initial value condition) to obtain

$$f(x) = e^{c}(1 - 2x + x^2). \tag{32}$$

Solving equation (31) together with the initial value condition analytically, we have

$$f(x) = 3(1-x)^2. \tag{33}$$

When $a = 1$ and $b = 4$, equation (27) becomes

$$(1-x) f'(x) + 3 f(x) = 0. \tag{34}$$

Solving equation (34) by the variable separable method without any initial value condition, we have

$$f(x) = e^c(1 - 3x + 3x^2 - x^3). \tag{35}$$

Solving equation (34) analytically with the initial value condition yields

$$f(x) = 4(1 - x)^3. \tag{36}$$

When $a = 1$ and $b = 5$, equation (27) becomes

$$(1 - x) f'(x) + 4 f(x) = 0. \tag{37}$$

By solving equation (37) by the variable separable method without imposing any initial value condition, we obtain

$$f(x) = e^c(1 - 4x + 6x^2 + 4x^3 - x^4). \tag{38}$$

Solving equation (37) with the initial value condition yields

$$f(x) = 5(1 - x)^4. \tag{39}$$

Corollary. *Similar results as follows are obtained*:

When $a = 1$ *and* $b = 6$,

$$f(x) = 6(1 - x)^5. \tag{40}$$

When $a = 1$ *and* $b = 7$,

$$f(x) = 7(1 - x)^6. \tag{41}$$

Generally, when $a = 1$ *and* $b = n$, *the PDF of the beta distribution has the following closed form expression given as*:

$$f(x) = n(1 - x)^{n-1}; \quad n = 2, 3, 4, 5, 6, 7, \ldots, \tag{42}$$

which is similar to the differentiation of $(x - 1)^n$.

Condition 3. When $b = 1$ and $a > 1$.

This is when the distribution is negatively skewed, strictly increasing and reduces to power function $[0, 1]$ distribution. When $b = 1,$ equation (4) becomes

$$f'(x) = \frac{(a-1)f(x)}{x}, \tag{43}$$

$$xf'(x) - (a-1)f(x) = 0. \tag{44}$$

When $b = 1$ and $a = 2,$ equation (44) becomes

$$xf'(x) - f(x) = 0. \tag{45}$$

Solving equation (45) by the variable separable method without any initial value condition, we have

$$f(x) = e^c x. \tag{46}$$

Solving equation (45) with the initial value condition yields

$$f(x) = 2x. \tag{47}$$

When $b = 1$ and $a = 3,$ equation (44) becomes

$$xf'(x) - 2f(x) = 0. \tag{48}$$

Solving equation (48) by the variable separable method without any initial value condition, we have

$$f(x) = e^c x^2. \tag{49}$$

Solving equation (48) with the initial value condition yields

$$f(x) = 3x^2. \tag{50}$$

When $b = 1$ and $a = 4,$ equation (44) becomes

$$xf'(x) - 3f(x) = 0. \tag{51}$$

Solving equation (51) by the variable separable method without any initial value condition, we have

$$f(x) = e^c x^3. \tag{52}$$

Solving equation (51) analytically with the initial value condition yields

$$f(x) = 4x^3. \tag{53}$$

When $b = 1$ and $a = 5$, equation (44) becomes

$$xf'(x) - 3f(x) = 0. \tag{54}$$

Solving equation (54) by the variable separable method without any initial value condition, we have

$$f(x) = e^c x^4. \tag{55}$$

Solving equation (54) with the initial value condition yields

$$f(x) = 5x^4. \tag{56}$$

Corollary. *Similar results as follows are obtained*:

When $b = 1$ *and* $a = 6$,

$$f(x) = 6x^5. \tag{57}$$

When $b = 1$ *and* $a = 7$,

$$f(x) = 7x^6. \tag{58}$$

Generally, *when* $b = 1$ *and* $a = n$, *the PDF of the beta distribution has the following closed form expression given as*:

$$f(x) = nx^{n-1}; \quad n = 2, 3, 4, 5, 6, 7, \dots. \tag{59}$$

Which is similar to the differentiation of x^n.

Condition 4. When $b > 1$, $a > 1$, $a \neq b$ and $b > a$.

This is when the distribution is positively skewed, U shaped and unimodal.

When $a = 2$ and $b = 3$, equation (4) becomes

$$f'(x) = \frac{f(x)}{x} - \frac{2f(x)}{1-x}, \tag{60}$$

$$x(1-x)f'(x) - (1-3x)f(x) = 0. \tag{61}$$

Solving equation (61) by the variable separable method without any initial value condition, we have

$$f(x) = e^c(-x^3 + 2x^2 - x). \tag{62}$$

Solving with the initial value condition, we have

$$f(x) = 12x(x-1)^2. \tag{63}$$

When $a = 2$ and $b = 4$, equation (4) becomes

$$f'(x) = \frac{f(x)}{x} - \frac{3f(x)}{1-x}, \tag{64}$$

$$x(1-x)f'(x) - (1-4x)f(x) = 0. \tag{65}$$

Solving equation (65) by the variable separable method without any initial value condition, we have

$$f(x) = e^c(x^4 - 3x^3 + 3x^2 - x). \tag{66}$$

Solving equation (65) with the initial value condition yields

$$f(x) = -20x(x-1)^3. \tag{67}$$

When $a = 2$ and $b = 5$, equation (4) becomes

$$f'(x) = \frac{f(x)}{x} - \frac{4f(x)}{1-x}, \tag{68}$$

$$x(1-x)f'(x) - (1-5x)f(x) = 0. \tag{69}$$

Solving equation (69) by the variable separable method without any initial value condition, we have

$$f(x) = e^c(-x^5 + 4x^4 - 6x^3 + 4x^2 - x). \tag{70}$$

Solving equation (69) with the initial value condition yields

$$f(x) = 30x(x-1)^4. \tag{71}$$

Corollary. *Similar results as follows are obtained*:

When $a = 2$ *and* $b = 6$,

$$f(x) = -42x(x-1)^5. \tag{72}$$

When $a = 2$ *and* $b = 7$,

$$f(x) = 56x(x-1)^6. \tag{73}$$

When $a = 2$ *and* $b = 8$,

$$f(x) = -72x(x-1)^7. \tag{74}$$

Generally, *when* $a = 2$ *and* $b = n$, *the PDF of the beta distribution has the following closed form expression given as*:

$$f(x) = \begin{cases} Cx(x-1)^{n-1}; & n = 3, 5, 7, 9, \ldots \\ -Dx(x-1)^{n-1}; & n = 4, 6, 8, 10, \ldots. \end{cases} \tag{75}$$

Condition 5. When $b > 1$, $a > 1$, $a \neq b$ and $a > b$.

This is when the distribution is negatively skewed, U shaped and unimodal.

When $a = 3$ and $b = 2$, equation (4) becomes

$$f'(x) = \frac{2f(x)}{x} - \frac{f(x)}{1-x}, \tag{76}$$

$$x(1-x)f'(x) - (2-3x)f(x) = 0. \tag{77}$$

Solving equation (77) by the variable separable method without any initial value condition, we have

$$f(x) = e^c(-x^3 + x^2). \tag{78}$$

Solving equation (77) analytically with the initial value condition yields

$$f(x) = -12(x-1)x^2. \tag{79}$$

When $a = 4$ and $b = 2$, equation (4) becomes

$$f'(x) = \frac{3f(x)}{x} - \frac{f(x)}{1-x}, \tag{80}$$

$$x(1-x)f'(x) - (3-4x)f(x) = 0. \tag{81}$$

Solving equation (81) by the variable separable method without any initial value condition, we have

$$f(x) = e^c(x^4 - x^3). \tag{82}$$

Solving equation (81) analytically with the initial value condition yields

$$f(x) = -20(x-1)x^3. \tag{83}$$

When $a = 5$ and $b = 2$, equation (4) becomes

$$f'(x) = \frac{4f(x)}{x} - \frac{f(x)}{1-x}, \tag{84}$$

$$x(1-x)f'(x) - (4-5x)f(x) = 0. \tag{85}$$

Solving equation (85) by the variable separable method without any initial value condition, we have

$$f(x) = e^c(-x^5 + x^4). \tag{86}$$

Solving equation (81) analytically with the initial value condition yields

$$f(x) = -30(x-1)x^4. \tag{87}$$

Corollary. *Similar results as follows are obtained*:

When $a = 6$ *and* $b = 2$,

$$f(x) = -42(x-1)x^5. \tag{88}$$

When $a = 7$ *and* $b = 2$,

$$f(x) = -56(x-1)x^6. \tag{89}$$

When $a = 8$ *and* $b = 2$,

$$f(x) = -72(x-1)x^7. \tag{90}$$

Generally, when $b = 2$ *and* $a = n$, *the PDF of the beta distribution has the closed form expression given as*:

$$f(x) = \begin{cases} E(x-1)x^{n-1}; & n = 3, 5, 7, 9, \dots \\ -F(x-1)x^{n-1}; & n = 4, 6, 8, 10, \dots, \end{cases} \quad (91)$$

where A, B, C, D, E *and* F *are the respective constants from the closed form expressions*.

Conclusion

Differentiation was successfully applied to obtain ordinary differential equations from the probability density function of the beta distribution. The work shows that no general solution is possible except for particular values of the two shape parameters that characterize the beta distribution. Four cases are considered which are also different shapes and orientation of the distribution. Closed form expressions were obtained from the solutions of the resulting ordinary differential equations. The paper considers only cases where the shape parameters are integers. The rationale is to find patterns within the probability density function and their relationship with other distributions related to the beta distribution. Moreover, numerical methods could be applied to validate the results. However, the various closed form expressions gave the same values as obtainable in the various standard statistical software.

Acknowledgements

The authors appreciate the kind comments from the reviewers. The research benefits from sponsorship from Covenant University, Ota, Nigeria.

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