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2D-WAVELETS BASED EFFICIENT SCHEME FOR SOLVING SOME PDEs

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Abstract

We propose two-dimensional Hermite wavelet method for solving some applications of partial differential equations. Kronecker tensor product has been utilized to resolve and control huge matrices operations and calculations. Proposed method is based on the approximation of largest mixed derivatives of the given partial differential equation into a series of two-dimensional Hermite wavelet basis functions. To validate the efficiency and accuracy of the proposed technique, some numerical examples are presented.

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Introduction

Wavelets theory is one of the most recent and upcoming fields which has become very important for mathematical research as it is being used as a powerful tool for solving various types of differential equations. Hermite wavelet method (HWM) has been successfully implemented to get accurate solutions of linear and nonlinear boundary value problems of fifth and sixth order in [1]. By extension of the same method, other linear and nonlinear diversified physical problems of complex nature can also be solved. In [2], an approximate solution of a system of linear differential equations has been obtained by using the Haar wavelet method, whereas in [3], Haar wavelet bases and their spline derivatives have been utilized to find the numerical approximation of differential operators and thereafter applied to sketch solution of a linear diffusive equation. In [4], the wavelet solution of an evolution (parabolic-hyperbolic) problem has been defined using the Haar wavelet and then approximate solution, at a given fixed scale, results from the superimposition of a small set of fundamental wavelets, which also gives a physical interpretation to wavelets. In [5], the operational matrix of integration derived from Haar wavelets has been presented for solving lumped and distributed parameter systems and a new method using Haar wavelet based operational matrix has been developed for optimizing a dynamic system in [6]. A numerical method established with the help of Hermite wavelets has been favorably compared with optimal homotopy asymptotic method (OHAM) for solving the coupled system of nonlinear fractional differential equations like Jaulent-Miodek equations in [7]. In [8], a numerical approach based on Hermite wavelets has been presented to obtain the solution for Bratu's problem, whereas in [9], a method has been proposed to solve PDEs with the help of two-dimensional Haar wavelets.

In [10], Legendre wavelets have been utilized to obtain the solutions of partial differential equations (PDEs), whereas in [11], a numerical approach based on Hermite wavelets has been presented for solving the two-dimensional hyperbolic telegraph equation and hyperbolic partial differential equations such as wave and sinh-Gordon equations. In [12], numerical solution of nonlinear singular initial value problems with the help of Hermite wavelet operational matrix of integration has been presented and in [13], second-order nonlinear singular boundary value problems have been solved with the help of operational matrix of integration using Hermite wavelets and the results have been favorably compared with the exact solutions. In [14], Hermite wavelets based collocation method has been proved to be more efficacious than the Haar wavelet collocation method for solving nonlinear differential equations of Bernoulli's type and in [15], a numerical technique based on Hermite wavelets has been presented for solving oscillatory electrical circuit equations. In [16], Haar wavelets have been used for the solution of fourth order nonlinear Kuramoto-Sivashinsky equation, whereas in [17], nonlinear Volterra integral equations of the first kind have been solved by the Haar wavelet method by converting them into linear Volterra integral equations of the second kind. In [18], an effective numerical method has been proposed for finding the solution of generalized Burger's type equations by converting them into nonlinear ordinary differential equations and then solving the algebraic system of linear equations thus obtained by using Haar wavelet based collocation method. The accuracy of physicists Hermite wavelet method (PHWM) has been demonstrated to obtain the solutions of singular differential equations (SDEs) in [19], whereas in [20], Legendre wavelets have been efficiently applied for the solution of initial value problems of Bratu-type, which is widely applicable in fuel ignition of the combustion theory and heat transfer.

Kronecker tensor product of two matrices

The Kronecker tensor product of two matrices A and B of the same order $m \times n$ is defined as:

$$A \otimes B = \begin{bmatrix} k_{11}B & k_{12}B & \cdots & k_{1n}B \\ k_{21}B & k_{22}B & \cdots & k_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ k_{m1}B & k_{m2}B & \cdots & k_{mn}B \end{bmatrix}.$$

Between 1858 and 1868, Johann Georg Zehfuss carried out the documented work on Kronecker products. In MATLAB, the command for finding the Kronecker product of two matrices is $\text{kron}(A, B)$.

Basic Concepts of Hermite Wavelets

Wavelets form a family of mathematical functions $\psi_{c,d}$ obtained from change of scale (dilation) and change of position (translation) of a single function ψ called the *mother wavelet*. If the dilation parameter '*c*' and translation parameter '*d*' are taken to vary continuously, then the family of continuous wavelets can be denoted as:

$$\psi_{c,d}(t) = \frac{1}{\sqrt{c}}\psi\left(\frac{t-d}{c}\right), \quad c > 0, \quad d \in R. \tag{1}$$

Discretize the values of parameters *c* and *d* as:

$$c = c_0^{-k}, \quad d = nd_0c_0^{-k}, \quad c_0 > 1, \quad d_0 > 0.$$

The family of discrete wavelets is obtained as:

$$\psi_{k,n}(t) = |\, c\,|^{-1/2}\psi(c_0^k t - nd_0), \quad \forall c, d \in R, \quad c \neq 0.$$

Here $\psi_{k,n}$ form a wavelet basis of $L^2(R)$.

In particular, select $c_0 = 2$ and $d_0 = 1$, then $\psi_{k,n}(t)$ represents an orthonormal basis. Basis of Hermite wavelet is given as follows:

$$\psi_{n,m}(t) = \begin{cases} \dfrac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} h_m(2^k t - 2n + 1), & \dfrac{n-1}{2^{k-1}} \leq t < \dfrac{n}{2^{k-1}}, \\ 0, & \text{elsewhere,} \end{cases} \tag{2}$$

where $m = 0, 1, \ldots, M-1$, and $h_m(t)$ is the Hermite polynomial of degree *m*. Hermite polynomials, $h_n(t)$, are the solutions of Hermite's differential equation given by

$$x'' - 2tx' + 2nx = 0, \quad n = 0, 1, 2, 3, \ldots.$$

These polynomials are defined in the interval $(-\infty, \infty)$ and are derived from the Rodrigue's formula

$$h_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}(e^{-t^2}).$$

By taking $k = 1$ and $M = 6$ $(m = 0, 1, 2, 3, 4, 5)$, the six basis functions on $[0, 1)$ are given by

$$\psi_{1,0}(t) = \frac{2}{\sqrt{\pi}},$$

$$\psi_{1,1}(t) = \frac{2}{\sqrt{\pi}}(4t - 2),$$

$$\psi_{1,2}(t) = \frac{2}{\sqrt{\pi}}(16t^2 - 16t + 2),$$

$$\psi_{1,3}(t) = \frac{2}{\sqrt{\pi}}(64t^3 - 96t^2 + 36t - 2),$$

$$\psi_{1,4}(t) = \frac{2}{\sqrt{\pi}}(256t^4 - 512t^3 + 320t^2 - 64t + 2),$$

$$\psi_{1,5}(t) = \frac{2}{\sqrt{\pi}}(1024t^5 - 2560t^4 + 2240t^3 - 800t^2 + 100t - 2).$$

Function Approximation

Let $u(x, t)$ be any integrable function. We suppose that it can be expanded in terms of infinite series of two-dimensional Hermite wavelet basis functions as:

$$u(x, t) = \sum_{p_1=1}^{\infty} \sum_{q_1=0}^{\infty} \sum_{p_2=1}^{\infty} \sum_{q_2=0}^{\infty} C_{p_1, q_1, p_2, q_2} \psi_{p_1, q_1}(x).\psi_{p_2, q_2}(t),$$

where C_{p_1, q_1, p_2, q_2} are the constants of this infinite series known as *wavelet coefficients*. For numerical approximation, we truncate the above infinite series up to a finite number of terms as follows:

$$u(x,\ t) = \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} \psi_{p_1, q_1}(x).\psi_{p_2, q_2}(t), \tag{3}$$

where C and ψ are matrices of order $2^{k-1}M \times 1$ defined as:

$$C^T(x) = [C_{1,0}, \ldots, C_{1,M-1}, \ldots, C_{2^{k-1},0}, \ldots, C_{2^{k-1},M-1}]$$

and

$$\psi = [\psi_{1,0}, \ldots, \psi_{1,M-1}, \ldots, \psi_{2^{k-1},0}, \ldots, \psi_{2^{k-1},M-1}]^T,$$

where T means the transpose of a matrix.

Hermite Wavelet Method for Solving Diffusion Equation

Consider the diffusion equation

$$\frac{\partial v}{\partial t} = C\frac{\partial^2 v}{\partial x^2}, \quad \{x,\ t\} \in [0,\ 1] \tag{4}$$

with initial conditions $v(x,\ 0) = f_1(x)$, $v(0,\ t) = f_2(t)$ and boundary conditions $v(x,\ 1) = f_3(x)$, $v(1,\ t) = f_4(t)$. Assume that $k_1 = k_2 = k$ and $M_1 = M_2 = M$. The wavelet solution is sought in the form

$$\frac{\partial^3 v}{\partial t \partial x^2} = \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} \psi_{p_1, q_1}(x).\psi_{p_2, q_2}(t). \tag{5}$$

Integrating (5) with respect to t, from 0 to t, we obtain

$$\frac{\partial^2 v}{\partial x^2}(x,\ t) = \frac{\partial^2 v}{\partial x^2}(x,\ 0) + \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} \psi_{p_1, q_1}(x).P_{p_2, q_2}(t), \tag{6}$$

where

$$P_{p_2,q_2}(t) = \int_0^t \psi_{p_2,q_2}(t)\,dt.$$

Integrating (5), twice with respect to x, from 0 to x, we obtain

$$\frac{\partial^2 v}{\partial t \partial x}(x, t) = \frac{\partial^2 v}{\partial t \partial x}(0, t) + \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1,q_1,p_2,q_2} P_{p_1,q_1}(x).\psi_{p_2,q_2}(t) \quad (7)$$

and

$$\frac{\partial v}{\partial t}(x, t) = \frac{\partial v}{\partial t}(0, t) + x.\frac{\partial^2 v}{\partial t \partial x}(0, t) + \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1,q_1,p_2,q_2} Q_{p_1,q_1}(x).\psi_{p_2,q_2}(t), \quad (8)$$

where

$$Q_{p_1,q_1}(t) = \int_0^t \int_0^t \psi_{p_1,q_1}(t)\,dt\,dt.$$

Putting $x = 1$ in (8), we obtain

$$\frac{\partial^2 v}{\partial t \partial x}(0, t) = \frac{\partial v}{\partial t}(1, t) - \frac{\partial v}{\partial t}(0, t) - \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1,q_1,p_2,q_2} Q_{p_1,q_1}(1).\psi_{p_2,q_2}(t). \quad (9)$$

From (8) and (9), we obtain

$$\frac{\partial v}{\partial t}(x, t) = \frac{\partial v}{\partial t}(0, t) + x.\left\{\frac{\partial v}{\partial t}(1, t) - \frac{\partial v}{\partial t}(0, t)\right\}$$

$$+\sum_{p_1=1}^{2^{k-1}}\sum_{q_1=0}^{M-1}\sum_{p_2=1}^{2^{k-1}}\sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2}$$

$$.\{Q_{p_1, q_1}(x) - x.Q_{p_1, q_1}(1)\}.\psi_{p_2, q_2}(t). \tag{10}$$

Integrating (10) with respect to t, from 0 to t, we obtain

$$v(x, t) = v(x, 0) + v(0, t) - v(0, 0) + x.\{v(1, t) - v(1, 0) - v(0, t) + v(0, 0)\}$$

$$+\sum_{p_1=1}^{2^{k-1}}\sum_{q_1=0}^{M-1}\sum_{p_2=1}^{2^{k-1}}\sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2}$$

$$.\{Q_{p_1, q_1}(x) - x.Q_{p_1, q_1}(1)\}.P_{p_2, q_2}(t). \tag{11}$$

Substituting the values from (6) and (10) into (4), we obtain

$$\sum_{p_1=1}^{2^{k-1}}\sum_{q_1=0}^{M-1}\sum_{p_2=1}^{2^{k-1}}\sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2}.[\{Q_{p_1, q_1}(x) - x.Q_{p_1, q_1}(1)\}$$

$$.\psi_{p_2, q_2}(t) - \psi_{p_1, q_1}(x).P_{p_2, q_2}(t)]$$

$$= \frac{\partial^2 v}{\partial x^2}(x, 0) - \frac{\partial v}{\partial t}(0, t) - x.\left\{\frac{\partial v}{\partial t}(1, t) - \frac{\partial v}{\partial t}(0, t)\right\}. \tag{12}$$

Using collocation points x_l and t_l, we obtain

$$\sum_{p_1=1}^{2^{k-1}}\sum_{q_1=0}^{M-1}\sum_{p_2=1}^{2^{k-1}}\sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2}.[S_{p_1, q_1, p_2, q_2}(x_l, t_l) - T_{p_1, q_1, p_2, q_2}(x_l, t_l)]$$

$$= F(x_l, t_l), \tag{13}$$

where

$$S_{p_1, q_1, p_2, q_2}(x_l, t_l) = \{Q_{p_1, q_1}(x_l) - x_l.Q_{p_1, q_1}(1)\}.\psi_{p_2, q_2}(t_l)$$

$$\simeq \{W_{p_1, q_1}(x_l)\}.\{\psi_{p_2, q_2}(t_l)\},$$

$$T_{p_1, q_1, p_2, q_2}(x_l, t_l) = \{\psi_{p_1, q_1}(x_l)\}.\{P_{p_2, q_2}(t_l)\}$$

and

$$F(x_l, t_l) = F_1(x_l).F_2(t_l).$$

In matrix form, it can be written as

$$[C].[S - T] = [F],$$

where

$$S = W \otimes \Psi, \quad T = \Psi \otimes P$$

and

$$F = F_1 \otimes F_2.$$

The values of matrices W, Ψ and P are defined as:

$$W = \begin{bmatrix} W_{1,0}(x_l) & \cdots & W_{1,0}(x_{M-1}) & \cdots & W_{1,0}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_{1,M-1}(x_l) & \cdots & W_{1,M-1}(x_{M-1}) & \cdots & W_{1,M-1}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ W_{2^{k-1},M-1}(x_l) & \cdots & W_{2^{k-1},M-1}(x_{M-1}) & \cdots & W_{2^{k-1},M-1}(x_{(2^{k-1}).(M-1)}) \end{bmatrix},$$

$$\Psi = \begin{bmatrix} \psi_{1,0}(x_l) & \cdots & \psi_{1,0}(x_{M-1}) & \cdots & \psi_{1,0}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_{1,M-1}(x_l) & \cdots & \psi_{1,M-1}(x_{M-1}) & \cdots & \psi_{1,M-1}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_{2^{k-1},M-1}(x_l) & \cdots & \psi_{2^{k-1},M-1}(x_{M-1}) & \cdots & \psi_{2^{k-1},M-1}(x_{(2^{k-1}).(M-1)}) \end{bmatrix}$$

and

$$P = \begin{bmatrix} P_{1,0}(x_l) & \cdots & P_{1,0}(x_{M-1}) & \cdots & P_{1,0}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ P_{1,M-1}(x_l) & \cdots & P_{1,M-1}(x_{M-1}) & \cdots & P_{1,M-1}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ P_{2^{k-1},M-1}(x_l) & \cdots & P_{2^{k-1},M-1}(x_{M-1}) & \cdots & P_{2^{k-1},M-1}(x_{(2^{k-1}).(M-1)}) \end{bmatrix}.$$

From (13), we obtain wavelet coefficients. The wavelet solution is obtained by substituting the values of wavelets coefficients into (11).

Hermite Wavelet Method for Solving Poisson Equations

Consider the Poisson equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = f(x, y), \quad \{x, y\} \in [0, 1] \tag{14}$$

with initial conditions $v(x, 0) = g_1(x)$, $v(0, y) = g_2(y)$ and boundary conditions $v(x, 1) = g_3(x)$, $v(1, y) = g_4(y)$. Assume that $k_1 = k_2 = k$ and $M_1 = M_2 = M$. Then the wavelet solution is sought in the form

$$\frac{\partial^4 v}{\partial x^2 \partial y^2} = \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} \psi_{p_1, q_1}(x).\psi_{p_2, q_2}(y). \tag{15}$$

Integrating (15) twice, with respect to *x*, from 0 to *x*, we obtain

$$\frac{\partial^3 v(x, y)}{\partial x \partial y^2} = \frac{\partial^3 v(0, y)}{\partial x \partial y^2}$$

$$+ \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} P_{p_1, q_1}(x).\psi_{p_2, q_2}(y) \tag{16}$$

and

$$\frac{\partial^2 v(x, y)}{\partial y^2} = \frac{\partial^2 v(0, y)}{\partial y^2} + x.\frac{\partial^3 v(0, y)}{\partial x \partial y^2}$$

$$+ \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} Q_{p_1, q_1}(x).\psi_{p_2, q_2}(y). \tag{17}$$

Putting $x = 1$ in (17), we obtain

$$\frac{\partial^3 v(0,\ y)}{\partial x \partial y^2} = \frac{\partial^2 v(1,\ y)}{\partial y^2} - \frac{\partial^2 v(0,\ y)}{\partial y^2}$$

$$- \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} Q_{p_1, q_1}(1).\psi_{p_2, q_2}(y). \quad (18)$$

From (17) and (18), we obtain

$$\frac{\partial^2 v(x,\ y)}{\partial y^2} = \frac{\partial^2 v(0,\ y)}{\partial y^2} + x.\left\{\frac{\partial^2 v(1,\ y)}{\partial y^2} - \frac{\partial^2 v(0,\ y)}{\partial y^2}\right\}$$

$$+ \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2}$$

$$.\{Q_{p_2, q_2}(x) - x.Q_{p_2, q_2}(1)\}.\psi_{p_1, q_1}(y). \quad (19)$$

Again, integrating (15) twice, with respect to *y*, from 0 to *y*, we obtain

$$\frac{\partial^3 v(x,\ y)}{\partial x^2 \partial y} = \frac{\partial^3 v(x,\ 0)}{\partial x^2 \partial y}$$

$$+ \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} \psi_{p_1, q_1}(x).P_{p_2, q_2}(y) \quad (20)$$

and

$$\frac{\partial^2 v(x,\ y)}{\partial x^2} = \frac{\partial^2 v(x,\ 0)}{\partial x^2} + y.\frac{\partial^3 v(x,\ 0)}{\partial x^2 \partial y}$$

$$+ \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} \psi_{p_1, q_1}(x).Q_{p_2, q_2}(y). \quad (21)$$

Putting $y = 1$ in (21), we obtain

$$\frac{\partial^3 v(x,0)}{\partial x^2 \partial y} = \frac{\partial^2 v(x,1)}{\partial x^2} - \frac{\partial^2 v(x,0)}{\partial x^2} - \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1,q_1,p_2,q_2} \psi_{p_1,q_1}(x).Q_{p_2,q_2}(1). \quad (22)$$

From (21) and (22), we obtain

$$\frac{\partial^2 v(x,y)}{\partial x^2} = \frac{\partial^2 v(x,0)}{\partial x^2} + y.\left\{\frac{\partial^2 v(x,1)}{\partial x^2} - \frac{\partial^2 v(x,0)}{\partial x^2}\right\} + \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1,q_1,p_2,q_2} \psi_{p_1,q_1}(x) .\{Q_{p_2,q_2}(y) - y.Q_{p_2,q_2}(1)\}. \quad (23)$$

Integrating (23) twice, with respect to x, from 0 to x, we obtain

$$\frac{\partial v(x,y)}{\partial x} = \frac{\partial v(0,y)}{\partial x} + \frac{\partial v(x,0)}{\partial x} - \frac{\partial v(0,0)}{\partial x} + y.\left\{\frac{\partial v(x,1)}{\partial x} - \frac{\partial v(0,1)}{\partial x} - \frac{\partial v(x,0)}{\partial x} + \frac{\partial v(0,0)}{\partial x}\right\} + \sum_{p_1=1}^{2^{k-1}} \sum_{q_1=0}^{M-1} \sum_{p_2=1}^{2^{k-1}} \sum_{q_2=0}^{M-1} C_{p_1,q_1,p_2,q_2} P_{p_1,q_1}(x) .\{Q_{p_2,q_2}(y) - y.Q_{p_2,q_2}(1)\} \quad (24)$$

and

$$v(x,y) = v(0,y) + v(x,0) - v(0,0) + x.\left\{\frac{\partial v(0,y)}{\partial x} - \frac{\partial v(0,0)}{\partial x} - y.\frac{\partial v(0,1)}{\partial x} + y.\frac{\partial v(0,0)}{\partial x}\right\} + y.\{v(x,1) - v(0,1) - v(x,0) + v(0,0)\}$$

$$+\sum_{p_1=1}^{2^{k-1}}\sum_{q_1=0}^{M-1}\sum_{p_2=1}^{2^{k-1}}\sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} Q_{p_1, q_1}(x)$$

$$.\{Q_{p_2, q_2}(y) - y.Q_{p_2, q_2}(1)\}. \tag{25}$$

Putting $x = 1$ in (25), we obtain

$$\left\{\frac{\partial v(0,\ y)}{\partial x} - \frac{\partial v(0,\ 0)}{\partial x} - y.\frac{\partial v(0,\ 1)}{\partial x} + y.\frac{\partial v(0,\ 0)}{\partial x}\right\}$$

$$= v(1,\ y) - v(0,\ y) - v(1,\ 0) + v(0,\ 0)$$

$$- y.\{v(x,\ 1) - v(0,\ 1) - v(x,\ 0) + v(0,\ 0)\}$$

$$-\sum_{p_1=1}^{2^{k-1}}\sum_{q_1=0}^{M-1}\sum_{p_2=1}^{2^{k-1}}\sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2} Q_{p_1, q_1}(1).\{Q_{p_2, q_2}(y) - y.Q_{p_2, q_2}(1)\}. \tag{26}$$

From (25) and (26), we obtain

$$v(x,\ y) = v(0,\ y) + v(x,\ 0) - v(0,\ 0)$$

$$+ x.\{v(1,\ y) - v(0,\ y) - v(1,\ 0) + v(0,\ 0)\}$$

$$- x.\{y.\{v(x,\ 1) - v(0,\ 1) - v(x,\ 0) + v(0,\ 0)\}\}$$

$$+ y.\{v(x,\ 1) - v(0,\ 1) - v(x,\ 0) + v(0,\ 0)\}$$

$$+\sum_{p_1=1}^{2^{k-1}}\sum_{q_1=0}^{M-1}\sum_{p_2=1}^{2^{k-1}}\sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2}\{Q_{p_2, q_2}(y) - y.Q_{p_2, q_2}(1)\}$$

$$.\{Q_{p_2, q_2}(y) - y.Q_{p_2, q_2}(1)\}. \tag{27}$$

Substituting the values from (19) and (23) into (14), we obtain

$$\sum_{p_1=1}^{2^{k-1}}\sum_{q_1=0}^{M-1}\sum_{p_2=1}^{2^{k-1}}\sum_{q_2=0}^{M-1} C_{p_1, q_1, p_2, q_2}.[R_{p_1, q_1, p_2, q_2}(x_l,\ y_l) + U_{p_1, q_1, p_2, q_2}(x_l,\ y_l)]$$

$$= G(x_l,\ y_l), \tag{28}$$

where

$$R_{p_1, q_1, p_2, q_2}(x_l,\ y_l) = \psi_{p_1, q_1}(x_l).\{Q_{p_2, q_2}(y_l) - y_l .Q_{p_2, q_2}(1)\}$$

$$\simeq \{\psi_{p_1, q_1}(x_l)\}.\{A_{p_2, q_2}(y_l)\},$$

$$U_{p_1, q_1, p_2, q_2}(x_l,\ y_l) = \{Q_{p_2, q_2}(x_l) - x_l .Q_{p_2, q_2}(1)\}.\psi_{p_1, q_1}(y_l)$$

$$\simeq \{B_{p_2, q_2}(x_l)\}.\{\psi_{p_1, q_1}(y_l)\}$$

and

$$G(x_l,\ y_l) = f(x_l,\ y_l) - \frac{\partial^2 v(x, 0)}{\partial x^2} - y_l .\left\{\frac{\partial^2 v(x, 1)}{\partial x^2} - \frac{\partial^2 v(x, 0)}{\partial x^2}\right\}$$

$$- \frac{\partial^2 v(0,\ y_l)}{\partial y^2} - x.\left\{\frac{\partial^2 v(1,\ y_l)}{\partial y^2} - \frac{\partial^2 v(0,\ y_l)}{\partial y^2}\right\} \simeq G_1(x_l).G_2(y_l).$$

In matrix form,

$$[C].[R + U] = [G],$$

where

$$R = \Psi \otimes A, \quad U = B \otimes \Psi$$

and

$$G = G_1 \otimes G_2.$$

The values of matrices A, Ψ and B are defined as:

$$A = \begin{bmatrix} A_{1,0}(x_l) & \cdots & A_{1,0}(x_{M-1}) & \cdots & A_{1,0}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{1, M-1}(x_l) & \cdots & A_{1, M-1}(x_{M-1}) & \cdots & A_{1, M-1}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{2^{k-1}, M-1}(x_l) & \cdots & A_{2^{k-1}, M-1}(x_{M-1}) & \cdots & A_{2^{k-1}, M-1}(x_{(2^{k-1}).(M-1)}) \end{bmatrix},$$

$$\Psi = \begin{bmatrix} \psi_{1,0}(x_l) & \cdots & \psi_{1,0}(x_{M-1}) & \cdots & \psi_{1,0}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_{1,M-1}(x_l) & \cdots & \psi_{1,M-1}(x_{M-1}) & \cdots & \psi_{1,M-1}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi_{2^{k-1},M-1}(x_l) & \cdots & \psi_{2^{k-1},M-1}(x_{M-1}) & \cdots & \psi_{2^{k-1},M-1}(x_{(2^{k-1}).(M-1)}) \end{bmatrix}$$

and

$$B = \begin{bmatrix} B_{1,0}(x_l) & \cdots & B_{1,0}(x_{M-1}) & \cdots & B_{1,0}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{1,M-1}(x_l) & \cdots & B_{1,M-1}(x_{M-1}) & \cdots & B_{1,M-1}(x_{(2^{k-1}).(M-1)}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ B_{2^{k-1},M-1}(x_l) & \cdots & B_{2^{k-1},M-1}(x_{M-1}) & \cdots & B_{2^{k-1},M-1}(x_{(2^{k-1}).(M-1)}) \end{bmatrix}.$$

From (28), we obtain wavelets coefficients. The wavelet solution is obtained by substituting the values of wavelets coefficients into (27).

Numerical Experiments

In this section, we perform some numerical examples to prove the efficacy of the proposed numerical scheme based on Hermite wavelets. The accuracy of the numerical results so obtained is verified from the following relation:

$$\text{Absolute Error} = | y_{\text{Exact}} - y_{\text{Approximate}} |.$$

To establish the efficiency of the presented numerical scheme with Hermite wavelet basis functions, a comparison study is also presented in this research by using Haar wavelets.

Example 1. Consider the diffusion equation

$$\frac{\partial v}{\partial t} = \frac{1}{\pi^2} \frac{\partial^2 v}{\partial x^2}, \quad \{x, t\} \in [0, 1]$$

with initial conditions $v(x, 0) = \sin \pi x$, $v(0, t) = 0$ and boundary conditions $v(x, 1) = e^{-1}.\sin \pi x$, $v(1, t) = 0$. The exact solution is $u(x, t) = e^{-t}.\sin \pi x$. The maximum absolute errors are given as:

Table 1. Maximum absolute errors of Example 1

$\{k, M\}$	Maximum absolute errors
{1, 2}	8.2341e-003
{1, 3}	5.1135e-004
{1, 4}	2.8480e-004
{1, 5}	7.5619e-006
{1, 6}	5.0426e-006
{1, 7}	1.1721e-007

Table 1 shows the maximum values of absolute errors for different values of k and M. Figure 1 shows the physical behavior of solutions of Example 1.

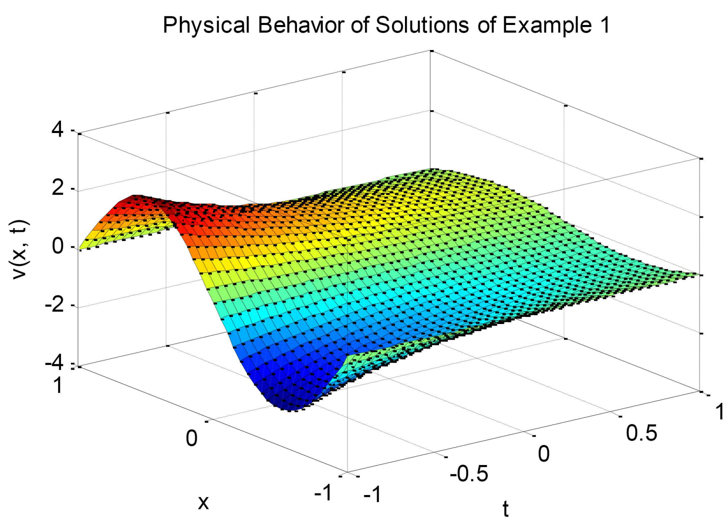

Figure 1

Example 2. Consider the Poisson equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -2\pi^2 \sin \pi x \sin \pi y, \quad \{x, y\} \in [0, 1]$$

with initial conditions $v(x, 0) = 0$, $v(0, t) = 0$ and boundary conditions $v(x, 1) = 0$, $v(1, t) = 0$. The exact solution is $u(x, t) = \sin \pi x . \sin \pi y$. Table 2 shows the maximum values of absolute errors for different values of k and M. Figure 2 shows the physical behavior of solutions of Example 2.

The maximum absolute errors are given as:

Table 2. Maximum absolute errors for Example 2

$\{k, M\}$	Maximum absolute errors
{1, 2}	3.7362e-002
{1, 3}	2.9600e-003
{1, 4}	1.1705e-003
{1, 5}	2.0826e-005
{1, 6}	1.3503e-005
{1, 7}	1.7728e-007

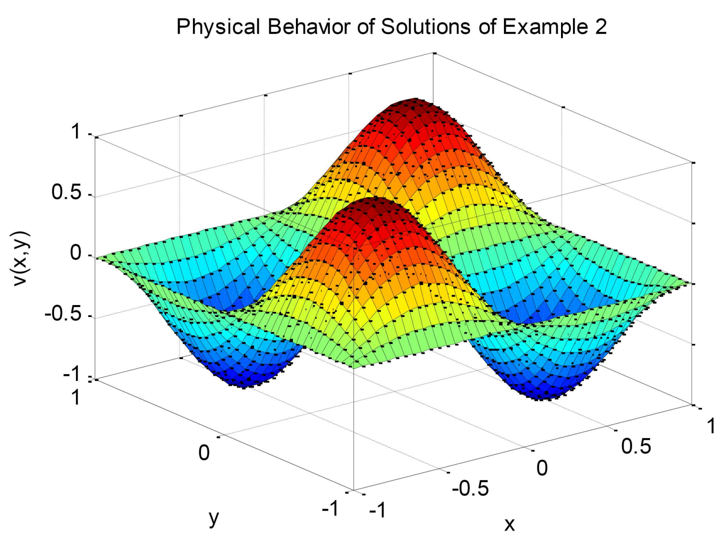

Figure 2

Conclusion

The computational and graphical representation of numerical results of the above illustrative examples exhibits that two-dimensional Hermite wavelets are a powerful numerical tool for solving partial differential equations such as diffusion equation and Poisson equation. The presented work can prove to be very beneficial for the progress of further research in the field of numerical analysis and differential equations.

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