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CHOICE OF A BASIS TO SOLVE THE LANE-EMDEN EQUATION

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Abstract

If the set of basis functions is chosen by overlooking physics of a problem, then the results can be misleading. It is shown that for the Lane-Emden equation, a set of functions with semi-infinite domain sometimes fails to produce results of desired accuracy. A qualitative analysis of the problem shows that the solution is bounded when m is an odd integer but is unbounded when m is even. Solution of the Lane-Emden equation with rational Legendre functions, as basis, is poorer in accuracy when $m = 2$ as compared with the one when $m = 3$ with the same basis. Since the physically important region is contained in a finite interval, a set of scaled Legendre polynomials, as basis, produces results which are much more accurate on the interval of interest.

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1. Introduction

The spectral method is an important tool for the approximation of a function by employing members of an orthogonal set of polynomials. Let $\{\varphi_i(x)\}_0^\infty$ be a set of polynomials orthogonal on an interval *I* with respect to a weight function $w(x)$. Let an arbitrary function *f* be approximated on *I* as

$$f(x) \simeq \sum_{i=0}^{n} a_i^{(n)} \varphi_i(x). \tag{1}$$

Then the square of the error is minimized by choosing the coefficients a_i, $i = 0, 1, ..., n$ by solving the following system of equations:

$$f(x_k) = \sum_{i=0}^{n} a_i^{(n)} \varphi_i(x_k), \quad k = 1, 2, ..., n+1, \tag{2}$$

where x_k, $k = 1, ..., n+1$, are $n+1$ zeros of the polynomial $\varphi_{n+1}(x)$. The reason behind the success of the above scheme is that for any set of orthogonal polynomials $\{\varphi_i(x)\}_0^\infty$, corresponding to the *integral* orthogonality relation

$$\int_I w(x) \varphi_i(x) \varphi_j(x) = 0, \text{ if } i \neq j,$$

there is a *discrete* orthogonality relation,

$$\sum_{k=1}^{n+1} W_k \varphi_i(x_k) \varphi_j(x_k) = 0, \text{ if } i \neq j,$$

where x_k, $k = 1, ..., n+1$, are defined as above.

A nice introduction to the theory of the spectral methods may be found in the lecture notes of Gheorghiu [1] or Boyd [2]. This method has application in the areas of ordinary as well as partial differential equations. See, for example [1, 4].

In this paper, we wish to highlight the importance of a proper choice of basis functions to solve an equation which models a physical problem. For this purpose, we choose the Lane-Emden equation, because its solutions depend critically on a parameter m. This equation describes temperature variation of a spherical gas cloud [13]. In its standard form, the problem reduces to the following initial value problem:

$$y'' + \frac{2}{x} y' + y^m = 0, \quad x > 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{3}$$

Approximate solutions of the problem have been found by several authors, see for example [14, 15] and references therein. Parand et al. have solved this problem by spectral methods first by using the set of rational Legendre functions [16], and then by employing Hermite functions [17]. Primary reason for the preference of the above mentioned sets of functions appears to be the semi-infinite domain of these functions. As a result, collocation points are distributed on $[0, \infty)$ and the approximate result is supposed to represent solution of the Lane-Emden equation on $[0, \infty)$. However, when $m = 2$ or 4, solution of the problem (1) is unbounded while the opposite is true if m happens to be an odd integer. Thus the approximate solution is not likely to be very accurate in case of even m, especially for large x. Main interest lies in the region between the origin and the first zero of the solution, if it exists hence it is advisable to restrict the domain to an interval which contains $[0, z_1]$, where z_1 denotes the first zero of the solution.

Associated with the coefficients, $a_i^{(n)}, i = 0, 1, ..., n,$ which define an approximate expression at level n, we define a coefficient vector A_n in the following manner:

$$A_n = \{a_0^{(n)}, a_1^{(n)}, ..., a_n^{(n)}, 0, 0, ...\}. \tag{4}$$

Thus the first $n+1$ elements of A_n are the coefficients in the expression (1) which defines an approximation for a function f on a given interval. All subsequent elements upto infinity are defined to be zero, i.e.,

$$a_i^{(n)} = 0, \text{ if } i > n. \tag{5}$$

For a good choice of a basis, the coefficient, $a_i^{(n)}$, at the nth level should not differ much from the coefficient $a_i^{(n+1)}$ at the subsequent level for $i = 0, 1, ..., n.$ To formalize this concept, we define a *proximity index*

$$p_n = \| A_{n+1} - A_n \|, \quad n = 1, ..., \infty. \tag{6}$$

It was recently observed that, in certain problems, the accuracy of an approximate expression, at the nth level, obtained by a pseudo-spectral method does not improve monotonically with n, rather it follows a periodic pattern. Also a dip in the values of the proximity index is accompanied by a spike in the accuracy of the corresponding approximate expression. It should be of interest to see whether a similar phenomenon exists in the present case.

2. Qualitative Analysis of the Lane-Emden Equation

The Lane-Emden equation describes temperature variation of a spherical gas cloud [13]. In its standard form, the problem reduces to the following initial value problem:

$$y'' + \frac{2}{x} y' + y^m = 0, \quad x > 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{7}$$

The equation has been treated in great detail by Bellman [18, Chapter 7]. Below we outline a few simple results. Theorem 1, which is the main result of this section, is not contained in Bellman. We write the equation in its self-adjoint form:

$$Ly = -x^2 y^m, \quad y(0) = 1, \quad y'(0) = 0, \tag{8}$$

where $Ly = (x^2 y')'$. The Cauchy function for $Ly = 0$ is $y(x, s) = \frac{x - s}{xs}$ [19, Chapter 5], therefore the Lane-Emden problem can be expressed in the integral form:

$$y(x) = 1 - \frac{1}{x}\int_0^x s(x - s)\, y^m(s)\, ds. \tag{9}$$

Also, the following results are easily obtained directly from the equations:

$$y'(x) = -\frac{1}{x^2}\int_0^x s^2 y^m(s)\, ds, \tag{10}$$

$$(y'(x))^2 + \int_0^x \frac{4}{s}(y'(s))^2 + \frac{2}{m+1}\, y^{m+1}(x) = \frac{2}{m+1}. \tag{11}$$

From (9), it is clear that when m is even, slope of the curve representing the solution monotonically decreases. If this curve crosses the x-axis, it will do so with a negative slope which continues to decrease. Thus the curve will never asymptotically approach a horizontal line, hence the solution must approach minus infinity as x becomes infinitely large. Also, the solution can vanish at most once in this case.

On the other hand, if m is an odd integer, the point where the curve representing the solution intersects the x-axis is an inflection point of the curve. Both the solutions as well as their derivatives are bounded. Equation (10) shows that

$$|\, y(x)\,| \le 1, \quad |\, y'(x)\,| \le \sqrt{\frac{2}{m+1}}. \tag{12}$$

Let $y_n(x)$ and $y_m(x)$, respectively, denote solutions of the Lane-Emden problem when the index equals n and m, then we have the following:

Theorem 1. *Let* $n > m$ *and* $[0, a]$ *be an interval on which both* $y_n(x)$ *and* $y_m(x)$ *are nonnegative. Then* $y_n(x) \ge y_m(x)$ *for* $0 \le x \le a$.

Proof. Using (9), we obtain

$$\begin{aligned} y_n(x) - y_m(x) &= -\frac{1}{x}\int_0^x s(x-s)([y_n(s)]^n - [y_m(s)]^m)ds \\ &\geq -\frac{1}{x}\int_0^x s(x-s)([y_n(s)]^n - [y_m(s)]^n)ds \\ &= -\frac{1}{x}\int_0^x s(x-s)(y_n(s) - y_m(s))g(y_n, y_m)ds, \end{aligned}$$

where $g(y_n(s), y_m(s))$ is a function which is nonnegative for $0 \leq s \leq a$. For example if $n = 3$, then $g(y_n, y_m) = y_n^2 + y_n y_m + y_m^2$. The second line follows from the first because on $[0, a]$, $0 \leq y_m(x) \leq 1$. Now assume $y_n(x) - y_m(x) < 0$ on $(0, \alpha]$ for some $\alpha \leq a$. Letting $x = \alpha$ in the above equation, we get

$$0 \geq y_n(\alpha) - y_m(\alpha) \geq -\frac{1}{\alpha}\int_0^\alpha s(\alpha - s)(y_n(s) - y_m(s))g(y_n, y_m)ds, \quad (13)$$

which is a contradiction, since the right hand side is clearly positive. Hence no such α exists and the proof is complete.

From the above theory, we can draw the following conclusions concerning the solution $y(x)$ of the Lane-Emden problem.

(1) Let $z_1^{(m)}$, $m = 1, 2, \ldots,$ denote the first zero, if it exists, of $y_m(x)$. If $n > m$, then $z_1^{(n)} > z_1^{(m)}$.

(2) The exact solution $y_5(x) = \left(1 + \frac{1}{3}x^2\right)^{-\frac{1}{2}}$ is known. It is bounded, positive on $(0, \infty)$ and asymptotically approaches zero. All solutions $y_m(x)$, $m \geq 5$ have these properties.

(3) The solution is unbounded only for $m = 2$ or $m = 4$. It is bounded for all other m.

3. Approximate Solution by Pseudo-spectral Methods

Let $\{\varphi_n(x)\}_0^\infty$ be a set of basis functions and let

$$u_n(x) = \sum_{i=0}^{n} a_i^{(n)}\varphi_i(x) \tag{14}$$

be an approximate solution of the Lane-Emden problem. Fix m and define the *residual* $R_n(x)$ by

$$R_n(x) = xu_n''(x) + u_n'(x) + x(u_n(x))^m. \tag{15}$$

In order to find the $n+1$ coefficients, we need a system of $n+1$ equations. Two linear equations are found by applying the initial conditions to the approximate solution. This gives

$$\sum\nolimits_{i=0}^{n} a_i^{(n)}\varphi_i(0) = 1, \tag{16}$$

$$\sum\nolimits_{i=0}^{n} a_i^{(n)}\varphi_i'(0) = 0. \tag{17}$$

Rest of the $n-1$ equations are obtained by setting the residual to zero at $n-1$ of the $n+1$ nodes x_i, $i = 1, 2, ..., n+1.$ These nodes may be the zeros of $\varphi_{n+1}(x)$ or $\varphi_n(x) + \varphi_{n+1}(x).$ In the latter case, they are called *Gauss-Radau nodes*. In keeping with Parand et al. [16], we use the Gauss-Radau nodes. Let

$$R_n(x_i) = 0, \quad i = 2, 3, ..., n. \tag{18}$$

Then the system (16)-(18) is usually solved by Newton's method. The process starts with a small value of n and the values obtained at each step serve as the initial values for the next step.

Rational Legendre functions as basis

Let $\{P_n(x)\}_{n=0}^\infty$ be a set of Legendre polynomials. It is simply orthogonal on $[-1, 1]$. Let $L > 0$ and define

$$R_n(x) = P_n\left(\frac{x-L}{x+L}\right). \tag{19}$$

Then the set $\{R_n(x)\}_{n=0}^{\infty}$ is called the *set of rational Legendre functions*. This set is orthogonal on $[0, \infty)$ with respect to the weight function $w(x) = \frac{2L}{(x+L)^2}$. We choose $L = 3$. This means half of the collocation points are located in $[0, 3)$ while the rest lie in $(3, \infty)$. We use this set as a basis to find a sequence of approximate solutions $\{u_n(x)\}_{n=4}^{25}$ for the Lane-Emden problem with $m = 2$ and display the results in Table 1.

Table 1. Convergence for the rational Legendre functions, $m = 2$

n	Proximity index	$\max_{0\le t\le 5} \lvert u(t) - u_n(t) \rvert$	$\int_0^5 [u(t) - u_n(t)]^2 dt$	Location of the first zero
4	0.233327	0.0292798	0.00139989	4.57333
5	1.24966	0.0101751	0.000123557	4.35131
6	0.297183	0.0365693	0.000542606	4.47947
7	1.01127	0.0259539	0.000290643	4.38915
8	0.350996	0.0376806	0.000527263	4.51454
9	0.161265	0.0333052	0.000349242	4.45086
10	1.74167	0.0200174	0.000114241	4.40586
11	2.05692	0.0114619	0.000178643	4.2892
12	2.78926	0.0365666	0.000753846	4.41111
13	3.44182	0.0250355	0.000836198	4.55891
14	7.55011	0.0189436	0.000442732	4.33291
15	8.51485	0.0392917	0.00229154	4.06477
16	1.90194	0.0258961	0.0005041	4.53317
17	0.950711	0.0173157	0.000131082	4.42356
18	3.40476	0.0193844	0.000239182	4.46971
19	6.29386	0.0177251	0.000331708	4.43223
20	0.469752	0.0153862	0.00010614	4.35814
21	3.24371	0.014954	0.00011152	4.41453
22	3.26882	0.0174042	0.000108038	4.3971
23	7.35233	0.0207061	0.000244302	4.46549
24	19.449	0.0135399	0.000189781	4.44872
25	3.15838	0.0288659	0.000964772	4.2463

It is clear that:

(1) The truncation error does not decrease as the number of terms, n, in the approximate solution increases. For $n = 10,$ maximum deviation of the approximate solution from the MATHEMATICA generated numerical solution is 0.0200 which is less than 0.0207 for $n = 23.$

(2) There appears to be no correlation between the proximity index and other entries of the table, which are measures of the accuracy of the approximate solution.

The poor level of convergence was expected in the light of the qualitative analysis of Section 2, since the solution $y_2(x)$ was shown to be unbounded on $[0, \infty)$ which leads to a large interpolation error.

Rational Legendre functions as basis, $m = 3$

We use the same set of basis functions, with $L = 3,$ and repeat the above calculations for the Lane-Emden problem with $m = 3.$ Results are presented in Table 2. There is remarkable gain in accuracy for any n as compared with the corresponding result in Table 1. We note that

(1) The interval over which the maximum deviation and the integral of the square of discretization error are being considered is double in length as compared with the one in Table 1. However, the maximum deviation for $n = 8$ over $[0, 10]$ is only 0.002, an order of magnitude less than the corresponding result for $m = 2$ over $[0, 5].$ The same holds for other entries of the table.

(2) The truncation error does not decrease monotonically with $n.$ However, no periodic pattern is discernible.

(3) The best result occurs for $n = 24,$ where the proximity index is at its lowest and rest of the bench marks appear to be at their best.

Superiority of the results for the Lane-Emden problem with $m = 3$ over those for $m = 2$ is obviously due to the fact that the solution in the first case is bounded which is not the case for $m = 2.$

Table 2. Convergence for the rational Legendre variables, $m = 3$

n	Proximity index	$\max_{0 \le t \le 10} \lvert u(t) - u_n(t) \rvert$	$\int_0^{10} [u(t) - u_n(t)]^2 dt$	Location of the first zero
4	0.569181	0.085488	0.00798008	6.87111
5	0.32664	0.0294279	0.00133518	7.13709
6	0.125567	0.0417983	0.00647965	6.27012
7	0.111105	0.0146701	0.000390386	6.8724
8	0.200473	0.00200499	0.0000142829	6.93704
9	0.101023	0.00662752	0.000123083	6.80788
10	0.112639	0.00981965	0.000160349	6.96266
11	0.0688939	0.0056047	0.0000837939	7.02797
12	0.0870926	0.00136887	5.63536×10^{-6}	6.89557
13	0.191389	0.00307848	0.0000128457	6.92568
14	0.0301019	0.000686243	1.00176×10^{-6}	6.91026
15	0.0611039	0.000267723	1.74298×10^{-7}	6.89587
16	0.0508278	0.0010208	1.35758×10^{-6}	6.90875
17	0.0869031	0.0012116	3.17855×10^{-6}	6.91163
18	0.194333	0.000791822	1.37807×10^{-6}	6.88983
19	0.133893	0.00130068	2.93767×10^{-6}	6.91754
20	0.0442609	0.00202164	9.13353×10^{-6}	6.90385
21	0.0614723	0.00220189	0.000011131	6.86311
22	0.078738	0.00193925	6.44782×10^{-6}	6.87266
23	0.0957474	0.000892308	1.76617×10^{-6}	6.90101
24	0.0122488	0.000129399	5.04586×10^{-8}	6.89963
25	0.0235659	0.000327454	1.59284×10^{-7}	6.89426

Scaled Legendre polynomials as basis, $m = 2$

Lane-Emden problem originally arose in the physics of stellar structure. The first zero z_1, say, of the solution of Lane-Emden problem is proportional to the radius of the star, therefore the region beyond z_1 is of little physical interest. For the same reason a Lane-Emden problem with $m > 5$ is devoid of physical interest since the solution does not vanish at all.

We have seen that for $m = 2,$ the approximate solution, found by employing rational Legendre functions as a basis, is poor in accuracy on $[0, 5]$ which indeed happens to be the interval on which physics would like it be as accurate as possible. In order to achieve this objective, we shall use, as basis, the set of scaled Legendre polynomials $\{SP_i(x)\}_{i=0}^{\infty},$ where

$$SP_i(x) = P_i\left(\frac{2x}{p} - 1\right), \tag{20}$$

for $p > 0.$ This set of polynomials is simply orthogonal on $[0, p].$ We choose $p = 6$ and find a set of approximate solutions for the Lane-Emden problem with $m = 2.$ Results are presented in Table 3. The accuracy is superior by several orders of magnitude. It increases monotonically until $n = 17.$ Beyond that, entries in the third and fourth columns appear to oscillate slightly about the values at $n = 17.$ However, for $n \geq 18,$ numerical integration using Gaussian quadrature, with increasing number of nodes, does not produce a convergent sequence. This indicates that we have reached the *roundoff plateau*.

Table 3. Convergence for the translated Legendre polynomials, $m = 2$

n	Proximity index	$\max_{0 \leq t \leq 6} \lvert u(t) - u_n(t) \rvert$	$\int_0^6 [u(t) - u_n(t)]^2 dt$	Location of the first zero
4	0.0403619	0.0495227	0.00148828	4.594513642
5	0.0185134	0.0184143	0.000180824	4.302005300
6	0.00914559	0.0138726	0.0000319557	4.367199411
7	0.00181412	0.00370478	1.4129×10^{-6}	4.353509186
8	0.000423812	0.000182853	4.59788×10^{-8}	4.353155536
9	0.000296566	0.000472107	2.12375×10^{-8}	4.352435228
10	0.0000709972	0.00015906	1.43119×10^{-9}	4.352862975
11	0.0000100162	6.16252×10^{-6}	2.18322×10^{-11}	4.352871343
12	9.21333×10^{-6}	0.0000146217	1.51152×10^{-11}	4.352884036
13	2.55969×10^{-6}	5.82808×10^{-6}	1.38332×10^{-12}	4.352876116
14	2.44975×10^{-7}	1.09733×10^{-7}	1.68913×10^{-14}	4.352874440

15	2.70333×10^{-7}	3.69867×10^{-7}	1.19933×10^{-14}	4.352874407
16	8.62672×10^{-8}	2.19345×10^{-7}	3.88206×10^{-15}	4.352874524
17	7.61286×10^{-9}	4.92069×10^{-8}	2.38322×10^{-15}	4.352874607
18	7.5252×10^{-9}	4.71518×10^{-8}	2.39128×10^{-15}	4.352874599
19	2.7574×10^{-9}	5.27375×10^{-8}	2.40315×10^{-15}	4.352874599
20	2.90678×10^{-10}	4.60559×10^{-8}	2.39624×10^{-15}	4.352874595
21	1.99324×10^{-10}	4.64095×10^{-8}	2.39679×10^{-15}	4.352874596
22	8.44095×10^{-11}	4.68503×10^{-8}	2.39714×10^{-15}	4.352874596
23	1.11283×10^{-11}	4.66329×10^{-8}	2.39696×10^{-15}	4.352874596
24	5.02218×10^{-12}	4.66632×10^{-8}	2.39698×10^{-15}	4.352874596
25	2.49079×10^{-12}	4.66739×10^{-8}	2.39699×10^{-15}	4.352874596

4. Conclusions

We highlight the importance of a proper choice of basis functions to solve an equation which models a physical problem. For this purpose, Lane-Emden equation is chosen as its solutions depend critically on a parameter. This equation describes temperature variation of a spherical gas cloud [13].

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