Advances in Differential Equations and Control Processes
© 2022 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324322030
Volume 29, 2022, Pages 1-11 **P-ISSN: 0974-3243**

SOLUTION OF SOME LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Anarkul Urdaletova[1], Syrgak Kydyraliev[2] and Elena Burova[2,*]

[1]Department of Mathematics
Kyrgyz-Turkish Manas University
Bishkek, Kyrgyz Republic
e-mail: anarkul.urdaletova@manas.edu.kg

[2]Department of Applied Mathematics and Informatics
American University of Central Asia
Bishkek, Kyrgyz Republic
e-mail: kiydyraliev_s@auca.kg
burova_e@auca.kg

Received: June 9, 2022; Accepted: July 7, 2022

2020 Mathematics Subject Classification: 34A30, 34K05, 34M03.

Keywords and phrases: ordinary differential equations, linear systems, variable coefficients, integrable combinations, explicit form of the solution.

*Corresponding author

How to cite this article: Anarkul Urdaletova, Syrgak Kydyraliev and Elena Burova, Solution of some linear systems of ordinary differential equations with variable coefficients, Advances in Differential Equations and Control Processes 29 (2022), 1-11.

http://dx.doi.org/10.17654/0974324322030

Published Online: August 23, 2022

Abstract

The problem of integrability of ordinary differential equations to find their exact solutions is a celebrated problem in the theory of differential equations which attracted attention of several workers in the area. This is due to the fact that: (a) differential equations are the most widely used continuous models of dynamic systems in physics, medicine, economics, biology and other sciences that study the surrounding reality, for which the explicit trajectory of the dynamic system's behavior is important as the explicit solution contains in itself the maximum information about the behavior of the system; (b) an explicit solution of the equation is necessary to confirm the mathematical and physical intuition, to compare the solutions obtained by various approximate methods and to compare these methods. It is also worth noting that in the presence of various methods for obtaining an explicit form of solving differential equations, the advantage is given to simpler algorithms.

This paper presents a method for finding an explicit form of the solution of one class of systems of linear ordinary differential equations of the first order with variable coefficients. Examples are given for illustration. This method includes elements of the well-known classical methods of the theory of integration of ordinary differential equations: the Leonard Euler method, based on the roots of the characteristic equation, and the Jean Leron D'Alembert method of integrable combinations.

1. Introduction

In the field of mathematical sciences, the 18th century bequeathed to the 19th century a great problem that has not been completely solved to this day is the integration of differential equations. To construct a physical theory of a dynamical system for scientists of the 19th century meant first of all to find differential equations describing the motion of all parts of the system under study, be it the planets of the solar system, or tiny particles of gas invisible to the eye. Laplace believed that the entire Universe from a mathematical point of view is just a huge set of differential equations. A mind capable of grasping and solving these equations at once could predict the future of the world. Therefore, the ability to solve them, to integrate, as mathematicians say, was an urgent need of the time [1].

Of course, since then, mathematical science has gone far ahead. It has long been proven that not all ordinary differential equations, even of the first order, such as the Riccati equations [2], are integrable in quadratures. This circumstance contributed to the rapid development of methods for the approximate solutions of differential equations. At the same time, despite the great progress in this direction, which is essentially associated with computerization, the problem of determining the types of differential equations and their systems that can be integrated in quadratures remains relevant.

The method presented in this work is an extension of the method of the authors used to find a solution to systems of non-homogeneous equations with constant coefficients [3, 4] for a certain class of systems of differential equations with variable coefficients.

2. Solution of the Second Order Systems of the Linear Ordinary Differential Equations with Variable Coefficient at the Derivatives

Consider a linear system of ordinary differential equations of the form:

$$\begin{cases} q(x)\,y' = ay + bz + f(x), \\ q(x)\,z' = cy + dz + g(x), \end{cases} \tag{1}$$

where the coefficients a, b, c, d are constants and q, f and g are given functions. Unknown functions are denoted by $y(x)$ and $z(x)$.

Note that for $q(x)$ identically equal to one, the system (1) is a standard system with constant coefficients, which can be solved both by classical methods [5, 6] and by the method described in [3, 4].

In accordance with the classical theory, despite the fact that there is a variable coefficient, let say that the characteristic equation of system (1) is the equation:

$$\Delta(\lambda) = 0, \tag{2}$$

where the function $\Delta(\lambda)$ has the form:

$$\Delta(\lambda) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - c \cdot b.$$

The characteristic numbers of system (1) are the roots of equation (2).

We show how, using the characteristic numbers, it is possible to find integrable combinations and, by using them, construct a solution to the specified system of differential equations.

Theorem 2.1. *If p and q are the roots of the characteristic equation* (2) *of the system* (1), *then the solution of system* (1) *is found from the system of algebraic equations* $y - kz = F(x; C_1)$, $y - mz = G(x; C_2)$, *where the coefficients k and m are the proportionality coefficients of the rows of the determinant* $\Delta(\lambda)$, *and the functions* $F(x; C_1)$ *and* $G(x; C_2)$ *are functions determined by the free terms of equations of system* (1) *and its coefficients*; C_1, C_2 *are arbitrary constants*.

Proof. Let p and q be the characteristic numbers of the system (1). Then, in order to solve the system, at the first step, we subtract the function py from the left and right sides of the first equation of system (1) and pz from both sides of the second equation. This will lead us to the system:

$$\begin{cases} q(x)\,y' - py = (a-p)\,y + bz + f(x), \\ q(x)\,z' - pz = cy + (d-p)\,z + g(x). \end{cases} \tag{3}$$

Since p is the characteristic number, the determinant $\begin{vmatrix} a-p & b \\ c & d-p \end{vmatrix}$ is equal to zero, which means that the rows of this determinant are proportional. Let the proportionality factor be k. Multiply the second equation by k. Then the system will take the form:

$$\begin{cases} q(x)y' - py = (a-p)y + bz + f(x), \\ k[q(x)z' - pz] = kcy + k(d-p)z + kg(x). \end{cases}$$

Now, we subtract the second equation from the first and write it in the form:

$$q(x)[y - kz]' - p[y - kz] = f(x) - kg(x). \tag{4}$$

Equation (4) is a linear ordinary differential equation of the first order with respect to the unknown function $y - kz$. Integrating it, we obtain an expression for $y - kz$ in the form:

$$y - kz = F(x; C_1), \tag{5}$$

where F is some definite function.

We repeat the procedure, taking q instead of p, and get

$$y - mz = G(x; C_2), \tag{6}$$

where G is some definite function.

Consider equalities (5) and (6) as an algebraic system with respect to the unknowns y and z. The solution to this algebraic system will be the general solution of the system of differential equations (1). □

In the next section, we illustrate Theorem 2.1 with examples.

3. The Case of Different Characteristic Numbers

Example 3.1. Consider the system of equations:

$$\begin{cases} x^2 y' = 5y + z + 8e^{-7/x}, \\ x^2 z' = 4y + 5z + 21. \end{cases} \tag{7}$$

The characteristic numbers of system (7) are the roots of the equation:

$$\begin{vmatrix} 5-\lambda & 1 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)(5-\lambda) - 4 \cdot 1 = 0 \Rightarrow \lambda_1 = 3;\ \lambda_2 = 7.$$

(1) Using the root $\lambda = 3$, we rewrite system (7) in the form:

$$\begin{cases} x^2 y' - 3y = 2y + z + 8e^{-7/x}, \\ x^2 z' - 3z = 4y + 2z + 21. \end{cases}$$

We subtract the second equation from the doubled first equation and get

$$x^2(2y' - z') - 3(2y - z) = 16e^{-7/x} - 21.$$

The solution of this equation is the function:

$$2y - z = 4e^{-7/x} + 7 + Ae^{-3/x}.$$

(2) We repeat the procedure, taking the second characteristic number $\lambda = 7$:

$$\begin{cases} x^2y' - 7y = -2y + z + 8e^{-7/x}, \\ x^2z' - 7z = 4y - 2z + 21. \end{cases}$$

Add to the doubled first equation the second equation:

$$x^2(2y' + z') - 7(2y + z) = 16e^{-7/x} + 21.$$

Now, we integrate the last equation and get

$$2y + z = \frac{-16}{x}e^{-7/x} - 3 + Be^{-7/x}.$$

It remains to find y and z. To do this, we need to solve an algebraic system of equations of the form:

$$\begin{cases} 2y - z = 4e^{-7/x} + 7 + Ae^{-3/x}, \\ 2y + z = \dfrac{-16}{x}e^{-7/x} - 3 + Be^{-7/x}. \end{cases}$$

Adding these equations, we get

$$4y = \left(4 - \frac{16}{x}\right)e^{-7/x} + 4 + Ae^{-3/x} - Be^{-7/x}.$$

At the same time, the difference between these equations is a function:

$$-2z = \left(4 + \frac{16}{x}\right)e^{-7/x} + 10 + Ae^{-3/x} - Be^{-7/x}.$$

So, it turns out that the solution of the system (7) is a pair of functions:

$$\begin{cases} y = \left(1 - \dfrac{4}{x}\right)e^{-7/x} + 1 + 0.25Ae^{-3/x} + 0.25Be^{-7/x}, \\ z = \left(-2 - \dfrac{8}{x}\right)e^{-7/x} - 5 - 0.5Ae^{-3/x} + 0.5Be^{-7/x}, \end{cases}$$

where A and B are arbitrary constants.

In order to obtain a solution of the system (7), we used two linear combinations $2y - z$ and $2y + z$, each of which was determined by the corresponding characteristic number. But what if the characteristic equation has only one multiple root? Let us consider this case in the following section.

4. The Case of One Multiple Characteristic Number

Example 4.1. Consider the system of equations:

$$\begin{cases} tgx \cdot y' = 4y - z + 4\sin^5 x, \\ tgx \cdot z' = y + 2z. \end{cases} \tag{8}$$

Characteristic equation of the system:

$$\begin{vmatrix} 4 - \lambda & -1 \\ 1 & 2 - \lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = 0$$

has only one multiple root equal to 3.

Using this root, we can rewrite system (8) in the form:

$$\begin{cases} tgx \cdot y' - 3y = y - z + 4\sin^5 x, \\ tgx \cdot z' - 3z = y - z. \end{cases} \tag{9}$$

Let us subtract the second equation of the system (9) from the first equation. This will give us the following equation:

$$tgx(y' - z') - 3(y - z) = 4\sin^5 x.$$

We have obtained a linear ordinary differential equation of the first order for the unknown function $y - z$. We integrate it and get

$$y - z = (\sin^2 x - \cos^2 x + C)\sin^3 x. \tag{10}$$

In order to complete the process of solving the system, we substitute the value of the function $y - z$ from (10) into the right side of any of the equations of system (9). For example, into the first equation:

$$tgx \cdot y' - 3y = (\sin^2 x - \cos^2 x + C)\sin^3 x + 4\sin^5 x.$$

Now, we write the last equation in a form convenient for integration:

$$tgx\left(\frac{y}{\sin^3 x}\right) = 5\sin^2 x - \cos^2 x + C;$$

$$\left(\frac{y}{\sin^3 x}\right) = 5\sin x \cdot \cos x - \frac{\cos^3 x}{\sin x} + C\frac{\cos x}{\sin x}.$$

We integrate this equation and get the value of the function $y(x)$:

$$\frac{y}{\sin^3 x} = 2.5\sin^2 x - [ln|\sin x| - 0.5\sin^2 x] + Cln|\sin x| + B;$$

$$y = \{3\sin^2 x + (C-1)ln|\sin x| + B\}\sin^3 x.$$

In order to determine the value of z, it is enough to perform an elementary algebraic operation:

$$z = y - (y - z) = \{3\sin^2 x + (C-1)ln|\sin x| + B\}\sin^3 x$$

$$-(\sin^2 x - \cos^2 x + C)\sin^3 x$$

$$= (\sin^2 x + 1 - C + (C-1)ln|\sin x| + B)\sin^3 x.$$

Thus, it turns out that the following is the solution to system (8):

$$y(x) = \{3\sin^2 x + (C-1)ln|\sin x| + B\}\sin^3 x,$$

$$z(x) = \{\sin^2 x + 1 - C + (C-1)ln|\sin x| + B\}\sin^3 x,$$

where B and C are arbitrary constants.

We managed to obtain the solution of the system (8) due to the fact that the linear combination of solutions $y - z$ turned out to be on the right side of the transformed system (8) in the system (9). Perhaps this is a lucky break. Repeating, practically word for word, the proof of the corresponding statement for the systems with constant coefficients [3, 4], it can be shown that in the case of multiple roots of the characteristic equation, the linear combination of solutions of the system, determined by the characteristic number, will always be on the right side of the transformed system.

5. The Case of Complex Characteristic Numbers

Example 5.1. Consider the system of equations:

$$\begin{cases} xy' = 5y + z + 2x^6, \\ xz' = -y + 5z. \end{cases} \tag{11}$$

The characteristic numbers of system (11) are the roots of the equation:

$$\begin{vmatrix} 5-\lambda & 1 \\ -1 & 5-\lambda \end{vmatrix} = (5-\lambda)(5-\lambda) - (-1)\cdot 1 = 0 \Rightarrow \lambda_1 = 5+i; \;\; \lambda_2 = 5-i.$$

(1) Using the root $\lambda = 5 + i$, we rewrite system (11) in the form:

$$\begin{cases} xy' - (5+i)y = -iy + z + 2x^6, \\ xz' - (5+i)z = -y - iz. \end{cases}$$

Subtracting from the 1st equation the 2nd, multiplied by i, we get

$$x(y' - iz') - (5+i)(y - iz) = 2x^6.$$

The solution to the resulting equation is the function:

$$y - iz = (1+i)x^6 + Ax^{5+i}.$$

(2) We repeat the procedure, taking the second characteristic number $\lambda = 5 - i$:

$$\begin{cases} xy' - (5-i)\,y = iy + z + 2x^6, \\ xz' - (5-i)\,z = -y + iz. \end{cases}$$

Add to the 1st equation the 2nd, multiplied by i:

$$x(y' + iz') - (5-i)(y + iz) = 2x^6,$$

and integrating, we get

$$y + iz = (1-i)\,x^6 + Bx^{5-i}.$$

It remains to find y and z. To do this, we need to solve the system:

$$\begin{cases} y - iz = (1+i)\,x^6 + Ax^{5+i}, \\ y + iz = (1-i)\,x^6 + Bx^{5-i}. \end{cases}$$

Adding the equations, we get $2y = 2x^6 + Ax^{5+i} + Bx^{5-i}$.

The difference between the equations of the system is a function

$$-2iz = 2ix^6 + Ax^{5+i} - Bx^{5-i}.$$

So, it turns out that the following pair constitutes the solution to the system (11):

$$\begin{cases} y = x^6 + 0.5x^5\,[Ax^i + Bx^{-i}], \\ z = -x^6 + 0.5x^5 i\,[Ax^i - Bx^{-i}]. \end{cases}$$

In order to find the solution, we select the real and imaginary parts of the functions $y(x)$ and $z(x)$:

$$\begin{cases} y = x^6 + 0.5x^5\,[(A+B)\cos(\ln(x)) + i(A-B)\sin(\ln(x))], \\ z = -x^6 + 0.5x^5\,[-(A+B)\sin(\ln(x)) + i(A-B)\cos(\ln(x))]. \end{cases}$$

Since the original system is specified in real functions, we write the answer in the form:

$$\begin{cases} y(x) = x^6 + x^5 \left(C_1 \cos(\ln(x)) + C_2 \sin(\ln(x)) \right), \\ z(x) = -x^6 + x^5 \left(-C_1 \sin(\ln(x)) + C_2 \cos(\ln(x)) \right). \end{cases}$$

6. Conclusion

The results of this work demonstrate a powerful synergy effect arising from the combination of the approaches of Euler and D'Alembert to solve systems of linear ordinary differential equations. In this case, it is demonstrated how the combined approach allows one, without special theoretical difficulties, to pass from systems with constant coefficients to a certain class of systems of linear ordinary differential equations of the first order with variable coefficients. Note that the presence of an explicit form of solutions to systems of this kind allows expanding the possibilities of qualitative analysis of the behavior of their solutions. In particular, the examples given in the article allow us to simulate a situation in which the terms with derivatives degenerate when approaching the left boundary of the segment $[0; a]$.

References

[1] A. A. Tyapkin and A. S. Shibanov, Poincaré, Moscow, Mol. Guard, 1982.

[2] A. I. Egorov, Riccati Equations, Moscow, Fizmatlit, 2001.

[3] S. K. Kydyraliev and A. B Urdaletova, A new method for solving systems of linear differential and difference equations, Vestnik KAZNU 63(4) (2009), 21-29.

[4] S. K. Kydyraliev and A. B. Urdaletova, Direct integration of systems of linear differential and difference equations, Filomat 33(5) (2019), 1453-1461.

[5] M. M. Guterman and Z. H. Nitecki, Differential Equations: A First Course, Philadelphia, Saunders College Publishing, 1992.

[6] W. E. Boyce and R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems, New York, Wiley, 2012.