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FINITE-APPROXIMATE CONTROLLABILITY OF NONLOCAL STOCHASTIC CONTROL SYSTEMS DRIVEN BY HYBRID NOISES

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Abstract

In this paper, a class of nonlocal stochastic control systems with Brownian motions and Poisson jumps is under consideration. In the setting of suitable function spaces and under certain assumptions, the finite-approximate controllability is discussed by means of variational

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method. After providing some properties of the variational functional, we use Schauder Fixed Point Theorem to obtain the existence of mild solutions. Finally, the finite-approximate controllability of the systems is concluded.

1. Introduction

Controllability concepts for various deterministic and stochastic semilinear evolution systems have been studied in many fields by means of various methods. There are many different concepts of controllability for evolution equations: approximate controllability, exact null controllability, finite-approximate controllability and so on. There are many papers on approximate controllability of semilinear evolution systems in infinite dimensional spaces, see [1-8] and the references therein. Some authors have studied exact controllability of differential control systems, see [9-13] and the references therein. Finite-approximate controllability of fractional semilinear evolution systems is studied in [14, 15].

Compared with the integer order calculus, fractional calculus, containing differentiation and integration of an arbitrary real order, has a history of more than three hundred years. Many real world phenomena can be better described by using fractional operators. In fact, there are many applications of fractional calculus in anomalous diffusion, random walk, nonlocal elasticity, and memory materials, see [16-22] and the references therein. The fractional calculus has been acknowledged as a promising mathematical tool to efficiently characterize the historical memory and global correlation of complex dynamic systems, phenomena or structures.

We are interested in the controllability of the following nonlocal stochastic control problem:

$$\begin{cases} dy + (-\Delta)^{\gamma} y dt = f(t, x, y)dt + \mathcal{B}v dt + \sigma(t)dB(t), \\ \quad + g(t, x, y)(dt)^{\alpha} + \int_U h(t, x, y; u)\tilde{\theta}(du, t), \; t \in (0, T], \; x \in \mathcal{O}, \\ y(0, x) = y_0(x), \; x \in \mathcal{O}, \\ y(t, x) = 0, \; x \in \mathbb{R}^N \setminus \mathcal{O}. \end{cases} \tag{1.1}$$

To better understand the meaning of problem (1.1), we explain the symbols and notions concerning problem (1.1) in the following:

- $\mathcal{O} \subset \mathbb{R}^N$ is a bounded domain.
- $\alpha \in \left(\frac{1}{2}, 1\right)$, $\gamma \in (0, 1)$, $T \in (0, \infty)$ and $N > 2\gamma$.
- $(-\Delta)^{\gamma}$ is the fractional Laplacian operator.
- The drift terms f and g are functions from $[0, T] \times \mathcal{O} \times \mathbb{R}$ to $\mathbb{R}$.
- $\sigma(t) \in \mathscr{L}_2^0(W, L^2(\mathcal{O}))$ for any $t \in [0, T]$ and B is a W-valued cylindrical Brownian motion, where W is a given real and separable Hilbert space.
- Let V be a Hilbert space. Assume that control $v \in L^2([0, T], V)$ and operator $\mathcal{B} \in \mathscr{L}(V, L^2(\mathcal{O}))$ which is the space of all bounded linear operators from V to $L^2(\mathcal{O})$.
- Let $(U, \mathscr{B}(U), e)$ be a σ-finite measurable space. h is a function from $[0, T] \times \mathcal{O} \times \mathbb{R} \times U$ to $\mathbb{R}$. $\tilde{\theta}$ denotes the compensated Poisson martingale measure.
- $(dt)^{\alpha}$ is a fractional differential in the sense of Jumarie [23].

Problem (1.1) strongly depends on the ranges of α, γ and N. There are two main contributions in this paper:

(1) Utilize a reasonable framework of mild solutions and overcome the complex calculations caused by not only fractional differential operators but also Brownian motions and Poisson jumps. Establish the existence and uniqueness of the mild solution to nonlocal stochastic control problems by Schauder Fixed Point Theorem.

(2) Establish sufficient conditions to the finite-approximate controllability of nonlocal stochastic control problems by means of a

variational method. Propose a variational functional and obtain some properties of the variational functional. At last conclude the finite-approximate controllability result.

The paper is organized as follows. In Section 2, we introduce basic concepts and results. In Section 3, we prove the existence and uniqueness of mild solution and establish the finite-approximate controllability of the nonlocal stochastic control problems.

2. Preliminaries

In this section, we introduce basic concepts and results.

2.1. Brownian motions

Let $(\Sigma, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\in[0,\infty)}$ which satisfies usual conditions. A one-dimensional fractional Brownian motion $\{\beta(t)\}_{t\in[0,T]}$ is a Gaussian process which has zero mean and its covariance is

$$\mathrm{Cov}(s, t) = \mathbb{E}[\beta(t), \beta(s)] = ts.$$

Brownian motions can be expressed by Wiener processes.

Let $\{w_i\}_{i=1}^{\infty}$ be an orthonormal basis of W and $\mathscr{L}(W)$ be the space of all bounded linear operators on W. Let $\mathcal{A} \in \mathscr{L}(W)$ be a symmetric, nonnegative operator and $\mathcal{A}w_i = \lambda_i w_i$, $i = 1, 2, \ldots,$ with $\mathrm{tr}\mathcal{A} = \sum_{i=1}^{\infty}\lambda_i < \infty.$ The infinite dimensional fractional Brownian motion $\{B_{\mathcal{A}}^H(t)\}_{t\in[0,T]}$ is defined by

$$B_{\mathcal{A}}(t) = \sum_{i=1}^{\infty}\lambda_i^{1/2} w_i \beta_i(t),$$

where $\{\beta_i\}_{i=1}^{\infty}$ is a sequence of independent one-dimensional fractional

Brownian motions. $B_{\mathcal{A}}$ has zero mean and covariance $\mathrm{Cov}\langle B_{\mathcal{A}}(t), \mu\rangle \langle B_{\mathcal{A}}(s), \nu\rangle = \mathrm{Cov}(t, s)\langle \mathcal{A}\mu, \nu\rangle$, for any $t, s \in [0, T]$ and $\mu, \nu \in W$. In this paper, we consider the cylindrical Brownian motion B, i.e., $\lambda_i = 1$, $i = 1, 2, \ldots$.

Let $\mathscr{L}_2^0(W, L^2(\mathcal{O}))$ be the space of Hilbert-Schmidt operators. An operator $\bar{g} \in \mathscr{L}_2^0(W, L^2(\mathcal{O}))$ satisfies $\bar{g} \in \mathscr{L}(W, L^2(\mathcal{O}))$ and

$$\| \bar{g} \|_{\mathscr{L}_2^0}^2 = \sum_{i=1}^{\infty} \| \bar{g} w_i \|_{L^2}^2 < \infty.$$

Endowed with the inner product $\langle \bar{g}, \tilde{g}\rangle_{\mathscr{L}_2^0} = \sum_{i=1}^{\infty} \langle \bar{g} w_i, \tilde{g} w_i\rangle_{L^2}$, $(\mathscr{L}_2^0(W, L^2(\mathcal{O})), \langle \cdot, \cdot\rangle_{\mathscr{L}_2^0})$ is a separable Hilbert space. We recall the following inequality:

Lemma 2.1 (See [18]). *Let* $g : [0, T] \to \mathscr{L}_2^0(W, L^2(\mathcal{O}))$ *and* $\int_0^T \| g(s) \|_{\mathscr{L}_2^0}^2 ds < \infty$. *Then stochastic integral* $\int_0^t g(s) dB(s)$ *is a well defined* $L^2(\mathcal{O})$*-valued random variable and satisfies*

$$\mathbb{E}\left\| \int_0^t g(s) dB(s) \right\|_{L^2}^2 = \int_0^t \| g(s) \|_{\mathscr{L}_2^0}^2 ds, \quad \forall t \in [0, T].$$

2.2. Poisson jumps

Let $(U, \mathscr{B}(U), e)$ be a σ-finite measurable space and $\{\pi_t\}_{t\in[0, \infty)}$ be a stationary Poisson point process which is defined on $(\Sigma, \mathscr{F}, \mathbb{P})$ and take values in U. The compensated Poisson martingale measure $\tilde{\theta}$ is defined as

$$\tilde{\theta}(du, t) = \theta(du, t) - e(du)t, \quad \forall t \in [0, \infty),$$

where θ is a counting measure which is generated by $\{\pi_t\}_{t\in[0,\infty)}$. $\tilde{\theta}$ has zero mean and variance $\mathbb{E}[\tilde{\theta}(du, t)]^2 = te(du)$.

Let $M_{\mathscr{F}}^{e,2}([0, T]\times U, L^2(\mathcal{O}))$ be the space of $\mathscr{F}\times\mathscr{B}(U)$ measurable processes with finite second moments and be equipped with the norm

$$\| h \|_{M_{\mathscr{F}}^{e,2}}^2 = \mathbb{E}\int_0^T \int_U \| h(s, y; u) \|_{L^2}^2 e(du)ds.$$

If $h \in M_{\mathscr{F}}^{e,2}([0, T]\times U, L^2(\mathcal{O}))$, then

$$\int_0^t \int_U h(s, x, y; u)\tilde{\theta}(du, ds), \quad \forall t \in [0, T]$$

is a $L^2(\Sigma; L^2(\mathcal{O}))$-valued random process which has zero mean. Furthermore, recall Theorem 6.1 in [24]:

Lemma 2.2. *If* $h \in M_{\mathscr{F}}^{e,2}([0, T]\times U, L^2(\mathcal{O}))$, *then the isomorphic formula*

$$\mathbb{E}\left\| \int_0^t \int_U h(s, y; u)\tilde{\theta}(du, ds) \right\|_{L^2}^2$$

$$= \mathbb{E}\int_0^t \int_U \| h(s, y; u) \|_{L^2}^2 e(du)ds, \quad \forall t \in [0, T]$$

holds. Moreover, the following inequality holds:

$$\sup_{t\in[0,T]} \mathbb{E}\left\| \int_0^t \int_U h(s, y; u)\tilde{\theta}(du, ds) \right\|_{L^2}^2$$

$$\leq 4\mathbb{E}\int_0^T \int_U \| h(t, y; u) \|_{L^2}^2 e(du)dt, \quad \forall T > 0.$$

2.3. Fractional differentials and fractional Laplacian operators

The relationship between the fractional differential and the classical differential of y is:

$$d^{\alpha} y = \Gamma(\alpha + 1) df, \quad 0 < \alpha \leq 1.$$

Here, "d^{α}" and "d" are referred to as the fractional and classical differentials, respectively. We have

Definition 2.3 (See [25]). Let $\alpha \in (0, 1]$, and y denote a continuous function. Then the *integral* of y with respect to $(dt)^{\alpha}$ is defined as

$$\int_0^t y(s)(ds)^{\alpha} = \alpha \int_0^t (t - s)^{\alpha-1} y(s) ds.$$

There are several approaches to define the fractional Laplacian operators. We introduce the definition by means of Fourier transform $\mathcal{F}$,

$$\mathcal{F}[(-\Delta)^{\gamma} y(x)] = |\zeta|^{2\gamma} \mathcal{F} y(x), \quad \gamma \in (0, 1).$$

Consider the following space-fractional diffusion equation:

$$\begin{cases} y_t + (-\Delta)^{\gamma} y = 0, & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ y(x, 0) = y_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{2.1}$$

By the theory of semigroups of bounded linear operators, the solution of (2.1) can be written as

$$y(t) = \mathcal{S}(t) y_0 = e^{-t(-\Delta)^{\gamma}} y_0,$$

while by means of Fourier transformation, we also have

$$y(x, t) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\gamma}} \mathcal{F}(y_0)) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\gamma}}) * y_0(x) = K_t * y_0,$$

where K_t is defined as

$$K_t(x) = \begin{cases} \dfrac{1}{(2\pi)^N} \displaystyle\int_{\mathbb{R}^N} e^{ix\cdot\xi} e^{-t|\xi|^{2\gamma}} d\xi, & \text{if } t \neq 0, \\ \delta(x), & \text{if } t = 0. \end{cases}$$

It is obvious that if $t \neq 0,$ then

$$K_t(x) = t^{-\frac{N}{2\gamma}} K\left(\frac{x}{t^{\frac{1}{2\gamma}}}\right),$$

where

$$K(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix\cdot\xi} e^{-|\xi|^{2\gamma}} d\xi.$$

Let the space

$$H^{\gamma}(\mathcal{O}) = \left\{ f \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2\gamma}) |\mathcal{F}f(\xi)|^2 d\xi < \infty \right.$$

$$\left. \text{and } u = 0, \text{ a.e. in } \mathbb{R}^N \backslash \mathcal{O} \right\}$$

be endowed with the norm

$$\| f \|_{H^{\gamma}(\mathcal{O})} = 2C(N, \gamma)^{-1} \int_{\mathbb{R}^N} |\xi|^{2\gamma} |\mathcal{F}f(\xi)|^2 d\xi,$$

where $C(N, \gamma)$ is a constant dependent on N and γ. Let $H_0^{\gamma}(\mathcal{O})$ denote the closure of $C_0^{\infty}(\mathcal{O})$ in $H^{\gamma}(\mathcal{O})$. We recall the embedding result in [26].

Lemma 2.4 (See [26]). *Let* $\gamma \in (0, 1)$, *the space* $H_0^{\gamma}(\mathcal{O})$ *is compactly embedded in* $L^2(\mathcal{O})$, *and*

$$\| f \|_2 \leq C \| f \|_{H_0^{\gamma}}.$$

Fractional Laplacian can be extended to $H_0^{\gamma}(\mathcal{O})$, i.e., $(-\Delta)^{\gamma} : H_0^{\gamma}(\mathcal{O}) \to L^2(\mathcal{O})$. Let $\{\mathcal{S}(t)\}_{t \geq 0}$ be the semigroup generated by $(-\Delta)^{\gamma}$. Then $\{\mathcal{S}(t)\}_{t \geq 0}$ is a C_0-contraction semigroup on $L^2(\mathcal{O})$. On the other hand, since the embedding from $H_0^{\gamma}(\mathcal{O})$ to $L^2(\mathcal{O})$ is compact, we get

Proposition 2.5. *$\mathcal{S}(t): L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is a compact operator for any $t \geq 0$.*

From [27], we have the following lemmas:

Lemma 2.6. *$K \in L^{\infty}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $\lim_{|x| \to \infty} K(x) = 0$.*

Lemma 2.7. *$|K(x)| \leq C(1 + |x|)^{-N-2s}$ and further $K \in L^p(\mathbb{R}^N)$ for any $p \in [1, \infty]$.*

Lemma 2.8. *$|\nabla K(x)| \leq C(1 + |x|)^{-N-1}$ and further $\nabla K \in (L^p(\mathbb{R}^N))^N$ for any $p \in [1, \infty]$.*

Let

$$\Upsilon^T(\cdot) = \int_0^T K_{T-s}\mathcal{B}\mathcal{B}^* K_{T-s}^*(\cdot)ds,$$

$$G^T = \int_0^T K_{T-s}\mathcal{B}\mathcal{B}^* K_{T-s}^* ds,$$

where $\mathcal{B}^*$ is the adjoint operator of $\mathcal{B}$. It is easy to see that $\Upsilon^T \in \mathscr{L}(L^2(\Sigma, L^2(\mathcal{O})))$ and $G^T \in L^2(\mathcal{O})$ can be called the controllability Gramian of the control problem (1.1). *By the boundedness of K and $\mathcal{B}$, we can assume that for any $t \in [0, T]$, $\| G^t \|_2 \leq M_0$ for some constant M_0.*

2.4. Function spaces

Let $L^2(\Sigma, L^2(\mathcal{O}))$ be the set of all $\mathscr{F}$ measurable random variables ζ such that

$$\| \zeta \|_{L^2}^2 = \mathbb{E}\{\| \zeta \|_2^2\} < \infty.$$

We use notation $L^2_{\mathscr{F}_t}([0, T]; H_0^{\gamma}(\mathcal{O}))$ to denote the space of all $\mathcal{F}_t$-adapted random processes such that

$$\| y \|^2_{L^2_{\mathscr{F}_t}} = \mathbb{E}\left\{\int_{-\tau}^{T} \| y(t, \cdot) \|^2_{H_0^\gamma} dt\right\} < \infty.$$

Let $C_{\mathscr{F}_t} \doteq C_{\mathscr{F}_t}([0, T]; L^2(\mathcal{O}))$ be the space of all continuous random processes from $[0, T]$ to $L^2(\mathcal{O})$, a.s., with essentially finite second moments and be endowed with the following norm:

$$\| y \|^2_{C_{\mathscr{F}_t}} = \sup_{t \in [0, T]} \mathbb{E}\{\| y(t, \cdot) \|^2_2\}.$$

Let $\mathscr{D}(T)$ be the set of all stochastic processes with the following properties:

(1) $y \in L^2_{\mathscr{F}_t}([0, T]; H_0^\gamma(\mathcal{O})) \cap C_{\mathscr{F}_t}([0, T]; L^2(\mathcal{O}))$;

(2) $y(0, x) = y_0(x)$, a.s., for all $x \in \mathcal{O}$;

(3) $y(t, x) = 0$, a.s., for all $(t, x) \in [0, T] \times (\mathbb{R}^N \backslash \mathcal{O})$.

Then the definition of mild solutions is given by

Definition 2.9. We call $y \in \mathscr{D}(T)$ a *mild solution* of problem (1.1) if for any $t \in [0, T]$, y satisfies the following formula:

$$y = K_t * y_0 + \int_0^t K_{t-s} * \mathcal{B} v ds + \int_0^t K_{t-s} * f ds$$

$$+ \int_0^t K_{t-s} * \sigma dB(s) + \alpha \int_0^t (t-s)^{\alpha-1} K_{t-s} * g ds$$

$$+ \int_0^t \int_U K_{t-s} * h \tilde{\theta}(du, ds), \text{ a.s.,}$$

or equivalently,

$$y(t, x, \omega) = \int_{\mathcal{O}} K_t(x-z)y_0(z)dz + \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z)\mathcal{B}v(s)(z)dzds$$

$$+ \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z)f(s, z, y(s, z, \omega))dzds$$

$$+ \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z)\sigma(s)dzdB(s)$$

$$+ \alpha \int_0^t (t-s)^{\alpha-1} \int_{\mathcal{O}} K_{t-s}(x-z)g(s, z, y(s, z, \omega))dzds$$

$$+ \int_0^t \int_U \int_{\mathcal{O}} K_{t-s}(x-z)h(s, z, y(s, z, \omega); u)dz\tilde{\theta}(du, ds), \text{ a.s.,}$$

if each integral is well defined.

Definition 2.10. Let $\mathcal{E}$ be a finite dimensional subspace of $L^2(\Sigma; L^2(\mathcal{O}))$ and denote by $\pi_{\mathcal{E}}$ the orthogonal projection from $L^2(\Sigma; L^2(\mathcal{O}))$ onto $\mathcal{E}$. The system of the control problem (1.1) is said to be *finite-approximately controllable* if for given any $y_0, y_T \in L^2(\Sigma; L^2(\mathcal{O}))$ and $\varepsilon > 0$, there exists a control $v_\varepsilon \in L^2([0, T], V)$ such that the solution $y_\varepsilon \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$ to the control problem (1.1) satisfies

$$\| y_\varepsilon(T) - y_T \|_{L^2} \leq \varepsilon,$$

$$\pi_{\mathcal{E}} y_\varepsilon(T) = \pi_{\mathcal{E}} y_T.$$

3. Finite-approximate Controllability

We propose the following hypotheses on nonlinear functions f, g, σ and h:

(H1) For any $t \in [0, T]$, $x \in \mathcal{O}$ and $y \in \mathbb{R}$, there exist a constant $\beta > 0$ and positive functions $\Lambda \in L[0, T]$ and $\Phi \in L(\mathcal{O})$ such that f, g and h

satisfy

$$|f(t, x, y)|^2 \leq \Lambda(t)\Phi(x),$$

$$|g(t, x, y)|^2 \leq \beta\Phi(x),$$

$$\int_U |h(t, x, y; u)|^2 e(du) \leq \Lambda(t)\Phi(x).$$

(H2) For any $t \in [0, T]$, $x \in \mathcal{O}$ and $y \in \mathbb{R}$, there exists a constant $L > 0$ such that *f*, *g* and *h* satisfy

$$\max\{|f(t, x, y_1) - f(t, x, y_2)|, |g(t, x, y_1) - g(t, x, y_2)|\}$$

$$\leq M_1 |y_1 - y_2|,$$

$$\int_U |h(t, x, y_1; u) - h(t, x, y_1; u)| e(du) \leq L|y_1 - y_2|.$$

(H3) $\sigma : [0, T] \to \mathscr{L}_2^0(W; L^2(\mathcal{O}))$ satisfies

$$\|\sigma(t)\|_{\mathscr{L}_2^0}^2 \leq M_2, \quad \forall t \in [0, T],$$

where the constant $M_2 > 0$.

(H4) $\kappa(\kappa I + G^T)^{-1} \in \mathscr{L}(L^2(\mathcal{O}))$ satisfies

$$\|\kappa(\kappa I + G^T)^{-1}\| \to 0 \text{ as } \kappa \to 0^+.$$

The finite-approximate controllability is based on the fact that the finite-approximate controllability can be viewed as the limit of a sequence of optimal control problems. More precisely, for $\varepsilon > 0$, we introduce the following functional:

$$I_\varepsilon(v, w) = \frac{1}{2}\mathbb{E}\left\{\int_{\mathcal{O}}\int_0^T\int_{\mathcal{O}} \|\mathcal{B}^* K_{T-t}^*(x - z)v(z, \omega)\| dz dt dx\right\}$$

$$+ \varepsilon\|(I - \pi_{\mathcal{E}})v(z, \omega)\|_{L^2} - \mathbb{E}\{\langle v, H(w)\rangle\},$$

where

$$H(z) = H_1(z) + H_2(z),$$

$$H_1(z) = y_T - \int_0^T \int_{\mathcal{O}} K_{T-t}(x-z) f(t, z, w(t, z, \omega)) dz dt$$

$$- \int_0^T \int_{\mathcal{O}} K_{T-t}(x-z) \sigma(t, z) dz dB(t)$$

$$- \alpha \int_0^T (T-t)^{\alpha-1} \int_{\mathcal{O}} K_{T-t}(x-z) g(t, z, w(t, z, \omega)) dz dt$$

$$- \int_0^T \int_U \int_{\mathcal{O}} K_{T-t}(x-z) h(t, z, w(t, z, \omega); u) dz \widetilde{\theta}(du, dt),$$

$$H_2(w) = -\int_{\mathcal{O}} K_T^*(x-z) y_0(z) dz.$$

Lemma 3.1. *The set* $W = \{H(w) : w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))\}$ *is relatively compact in* $L^2(\Sigma, L^2(\mathcal{O}))$.

Proof. For any $t \in [0, T]$, $x \in \mathcal{O}$ and $\omega \in \Sigma$, we have

$$\int_{\mathcal{O}} | f(t, x, w(t, x, \omega)) |^2 dx \leq \Lambda(t) \int_{\mathcal{O}} \Phi(x) dx \leq C(t),$$

where $C(t)$ is a constant dependent on t only. Let $\{w_n\}$ be a sequence in $C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$. By the compactness of $\mathcal{S}(t)$, there exists a subsequence of $\{w_n\}$, which is still denoted by $\{w_n\}$, such that

$$\xi(t, x, \omega) = \lim_{n \to \infty} \int_{\mathcal{O}} K_{T-t}(x-z) f(t, z, w_n(t, z, \omega)) dz \text{ a.e.}$$

for some $\xi \in L^2_{\mathscr{F}_t}([0, T]; L^2(\mathcal{O}))$. As

$$\left| \int_{\mathcal{O}} K_{T-t}(x-z) f(t, z, w_n(t, z, \omega)) dz \right|^2$$

$$\leq \Lambda(t) \int_{\mathbb{R}^N} K^2(x) dx \int_{\mathcal{O}} \Phi(x) dx \leq C\Lambda(t),$$

where C is a constant independent of t, x and ω, by Lebesgue Dominated Convergence Theorem, in $L^2_{\mathscr{F}_t}([0, T]; L^2(\mathcal{O}))$, we have

$$\lim_{n\to\infty} \int_{\mathcal{O}} K_{T-t}(x-z) f(t, z, w_n(t, z, \omega)) dz = \xi(t, x\ \omega),$$

i.e., in $L^2(\Sigma, L^2(\mathcal{O}))$,

$$\lim_{n\to\infty} \int_0^T \int_{\mathcal{O}} K_{T-t}(x-z) f(t, z, w(t, z, \omega)) dz dt = \int_0^T \xi(t, x\ \omega) dt.$$

In the same way, we can deal with the other terms and conclude that there exists a subsequence of $\{w_n\}$, which is still denoted by $\{w_n\}$, such that $\{H(w_n)\}$ is convergent in $L^2(\Sigma; L^2(\mathcal{O}))$. □

Lemma 3.2. *The operator* $H : C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O})) \to L^2(\Sigma, L^2(\mathcal{O}))$ *is continuous*.

Proof. Assume that $\{w_n\} \subset C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$ and $\{w_n\} \to w$ in $C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$. By Cauchy inequality, we get

$$\mathbb{E}\left\{\int_{\mathcal{O}} \int_0^T \int_{\mathcal{O}} K^2_{T-t}(x-z) | f(t, z, w_n(t, z, \omega)) - f(t, z, w(t, z, \omega))|^2 dzdtdx\right\}$$

$$\leq L \int_{\mathbb{R}^N} K^2(x) dx \mathbb{E}\left\{\int_{\mathcal{O}} \int_0^T | w_n(t, z, \omega) - w(t, z, \omega)|^2 dzdt\right\}$$

$$\to 0,$$

as $n \to \infty$.

In the same way, we can deal with the other terms and conclude that H is continuous from $C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$ to $L^2(\Sigma, L^2(\mathcal{O}))$. □

Lemma 3.3. *The functional* $I_\varepsilon : L^2(\Sigma, L^2(\mathcal{O})) \times C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O})) \to \mathbb{R}$ *has the following properties*:

(1) I_ε *is strictly convex and continuous with respect to* $v \in L^2(\Sigma; L^2(\mathcal{O}))$ *for any* $w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$.

(2) $\liminf\limits_{\| v \|_{L^2} \to \infty} \inf\limits_{w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))} \dfrac{I_\varepsilon(v, w)}{\| v \|_{L^2}} \geq \varepsilon.$

Proof. (1) It is obvious that I_ε is strictly convex with respect to $v \in L^2(\Sigma; L^2(\mathcal{O}))$.

For any $w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$, $\{v_n\} \subset L^2(\Sigma; L^2(\mathcal{O}))$ such that $\{v_n\} \to v$ in $L^2(\Sigma; L^2(\mathcal{O}))$, we have

$$\Bigg| \mathbb{E}\Bigg\{ \int_{\mathcal{O}} \int_0^T \int_{\mathcal{O}} \| B^* K_{T-t}^*(x - z) v_n(z, \omega) \|^2$$

$$- \| B^* K_{T-t}^*(x - z) v(z, \omega) \|^2 \, dz dt dz \Bigg\} \Bigg|$$

$$= \Bigg| \mathbb{E}\Bigg\{ \int_{\mathcal{O}} \int_0^T \int_{\mathcal{O}} \langle K_{T-t} BB^* K_{T-t}^* * v_n, v_n \rangle - \langle K_{T-t} BB^* K_{T-t}^* * v, v \rangle dz dt dx \Bigg\} \Bigg|$$

$$= \Bigg| \mathbb{E}\Bigg\{ \int_{\mathcal{O}} \int_0^T \int_{\mathcal{O}} \langle K_{T-t} BB^* K_{T-t}^* * v_n, v_n - v \rangle$$

$$+ \langle K_{T-t} BB^* K_{T-t}^* * (v_n - v), v \rangle dz dt dx \Bigg\} \Bigg|$$

$$\leq \| G^T \| \mathbb{E}\left\{\int_{\mathcal{O}} | v_n || v_n - v | + | v_n - v || v | dz\right\}$$

$$\leq \| G^T \| \mathbb{E}\left\{\int_{\mathcal{O}} | v_n |^2 + | v |^2 dz\right\}^{\frac{1}{2}} \mathbb{E}\left\{\int_{\mathcal{O}} | v_n - v |^2 dz\right\}^{\frac{1}{2}}$$

$$\to 0,$$

as $n \to \infty$. Therefore, I_ε is continuous with respect to $v \in L^2(\Sigma; L^2(\mathcal{O}))$.

(2) We prove by contradiction. If not, then here exist sequences $\{v_n\} \subset L^2(\Sigma; L^2(\mathcal{O}))$ and $\{w_n\} \subset C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$ such that $\| v \|_{L^2} \to \infty$, but

$$\liminf_{n\to\infty} \frac{I_\varepsilon(v_n, w_n)}{\| v_n \|_{L^2}} < \varepsilon.$$

By Lemma 3.1, the set

$$W = \{H(w) : w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))\}$$

is relatively compact in $L^2(\Sigma; L^2(\mathcal{O}))$, so there exists a subsequence, which is still denoted by $h(w_n)$, such that

$$H(w_n) \to h \text{ strongly in } L^2(\Sigma; L^2(\mathcal{O})),$$

for some $h \in L^2(\Sigma; L^2(\mathcal{O}))$.

Next, we normalize v_n by $\tilde{v}_n = \frac{v_n}{\| v_n \|_{L^2}}$. As $\| \tilde{v}_n \|_{L^2} = 1$, there exists a subsequence, which is still denoted by $\{\tilde{v}_n\}$ such that $\{\tilde{v}_n\}$ weakly converges in $L^2(\Sigma; L^2(\mathcal{O}))$ to some $\tilde{v}$ in $L^2(\Sigma; L^2(\mathcal{O}))$. The compactness of $\mathcal{S}(t)$ implies that

$$B^*K^*_{T-t} * \mathbb{E}\{\tilde{v}_n\} \to B^*K^*_{T-t} * \mathbb{E}\{\tilde{v}\} \text{ strongly in } L^2(\mathcal{O}).$$

According to the definition of $I_\varepsilon,$ we have

$$\frac{I_\varepsilon(v_n, w_n)}{\| v_n \|^2_{L^2}} = \frac{1}{2}\mathbb{E}\left\{\int_0^T \| B^*K^*_{T-t} * v_n \|^2 dt\right\}$$

$$+ \frac{1}{\| v_n \|_{L^2}}(\varepsilon\| (I - \pi_{\mathcal{E}})\tilde{v}_n \|_{L^2} - \mathbb{E}\{\langle \tilde{v}_n, H(w_n)\rangle\}). \tag{3.1}$$

In view of the fact $\| v_n \|_{L^2} \to \infty$ and equality (3.1) above, by Fatou Lemma, we have

$$\int_0^T \| B^*K^*_{T-t} * \mathbb{E}\{\tilde{v}\} \|^2 dt$$

$$\leq \liminf_{n\to\infty} \int_0^T \| B^*K^*_{T-t} * \mathbb{E}\{\tilde{v}_n\} \|^2 dt$$

$$\leq \liminf_{n\to\infty} \mathbb{E}\left\{\int_0^T \| B^*K^*_{T-t} * \tilde{v}_n \|^2 dt\right\}$$

$$= 0.$$

By assumption (H4), we have $\mathbb{E}\{\tilde{v}\} = 0$ according to [1] and further $\tilde{v} = 0.$ So we can deduce that

$$\tilde{v}_n \to 0 \text{ weakly in } L^2(\Sigma; L^2(\mathcal{O})).$$

Since $\mathcal{E}$ is a finite-dimensional space and $\pi_{\mathcal{E}}$ is compact, we get that $\pi_{\mathcal{E}}\tilde{v}_n \to 0$ in $L^2(\Sigma; L^2(\mathcal{O}))$ and so

$$\lim_{n\to\infty} \| (I - \pi_{\mathcal{E}})\tilde{v}_n \| = 1.$$

Then

$$\begin{aligned}\varepsilon &> \liminf_{n\to\infty} \frac{I_\varepsilon(v_n, w_n)}{\| v_n \|_{L^2}} \\ &\geq \liminf_{n\to\infty} (\varepsilon \| (I - \pi_{\mathcal{E}})\tilde{v}_n \| - \langle \tilde{v}_n, H(w_n) \rangle) \\ &= \varepsilon,\end{aligned}$$

which is a contradiction. Now we conclude the result. □

Lemma 3.4. *For any* $w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$, *the functional* $I_\varepsilon(\cdot, w)$ *admits a unique minimum* $\hat{\varphi}_\varepsilon \in L^2(\Sigma; L^2(\mathcal{O}))$.

Proof. Lemma 3.3(2) implies that the functional $I_\varepsilon(\cdot, w) : L^2(\Sigma; L^2(\mathcal{O})) \to \mathbb{R}$ is coercive, i.e., for any $w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$,

$$\liminf_{\| v \|_{L^2} \to \infty} \frac{I_\varepsilon(v, w)}{\| v \|_{L^2}} \geq \varepsilon.$$

Let $\{v_{\varepsilon, n}\}$ be a minimizing sequence of $I_\varepsilon(\cdot, w)$. Because of the coerciveness of I_ε with respect to v, the sequence $\{v_{\varepsilon, n}\}$ is bounded in $L^2(\Sigma; L^2(\mathcal{O}))$. Then there exists a subsequence, which is still denoted by $\{v_{\varepsilon, n}\}$, such that $\{v_{\varepsilon, n}\} \to \hat{v}_\varepsilon$ weakly in $L^2(\Sigma; L^2(\mathcal{O}))$. By Fatou Lemma, we have

$$I_\varepsilon(\hat{v}_\varepsilon, w) \leq \liminf_{n\to\infty} I_\varepsilon(v_{\varepsilon, n}, w) = \inf_{v \in L^2(\Sigma; L^2(\mathcal{O}))} I_\varepsilon(v, w).$$

Thus, $\hat{v}_\varepsilon$ is a minimum. By the strict convexity of $I_\varepsilon(\cdot, w)$, the minimum is unique. □

Hence, for any $w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$, the functional $I_\varepsilon(\cdot, w)$ admits a unique minimum $\hat{v}_\varepsilon$. Define an operator $\mathcal{T}_\varepsilon : C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O})) \to$

$L^2(\Sigma; L^2(\mathcal{O}))$ by $\mathcal{T}_\varepsilon : w \to \hat{v}_\varepsilon$. Then the operator $\mathcal{T}_\varepsilon$ has the following properties.

Lemma 3.5. *There exists* M_ε *such that* $\| \mathcal{T}_\varepsilon(w) \|_{L^2} \le M_\varepsilon$ *for any* $w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$.

Proof. By Lemma 3.3(2), there exists a constant $M_\varepsilon > 0$, such that

$$\inf_{w \in C_{\mathcal{F}_t}([0,T]; L^2(\mathcal{O}))} \frac{I_\varepsilon(v, w)}{\| v \|_{L^2}} \ge \frac{\varepsilon}{2}, \text{ if } \| v \|_{L^2} > M_\varepsilon.$$

On the other hand, by the definition of $\mathcal{T}_\varepsilon$, we have

$$I_\varepsilon(\mathcal{T}_\varepsilon(w), w) \le I_\varepsilon(0, w) = 0, \text{ for all } w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O})), \tag{3.2}$$

from which we conclude that

$$\| \mathcal{T}_\varepsilon(w) \|_{L^2} \le M_\varepsilon, \text{ for all } w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O})).$$

In fact, if there exists $w_0 \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$ such that $\| \mathcal{T}_\varepsilon(w_0) \|_{L^2} > M_\varepsilon$, then

$$\frac{I_\varepsilon(\mathcal{T}_\varepsilon(w_0); w_0)}{\| \mathcal{T}_\varepsilon(w_0) \|_{L^2}} \ge \inf_{w \in C_{\mathcal{F}_t}([0,T]; L^2(\mathcal{O}))} \frac{I_\varepsilon(v, w)}{\| v \|_{L^2}} \ge \frac{\varepsilon}{2},$$

which contradicts (3.2). □

Lemma 3.6. *For any* $\{w_n\} \subset C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$ *and* $w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$,

$$\{w_n\} \to w \ \textit{in} \ C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$$

implies

$$\lim_{n \to \infty} \| \mathcal{T}_\varepsilon(w_n) - \mathcal{T}_\varepsilon(w) \|_{L^2} = 0.$$

Proof. By Lemma 3.5, we get the boundedness of $\{\hat{v}_{\varepsilon,n}\} = \{\mathcal{T}_\varepsilon(w_n)\}$. Suppose that $\{\hat{v}_{\varepsilon,n}\}$ weakly converges to $\tilde{v}_\varepsilon$ as $n \to \infty$. Then, by the definitions of I_ε and $\mathcal{T}_\varepsilon$, we have

$$I_\varepsilon(\hat{v}_\varepsilon, w) \le I_\varepsilon(\tilde{v}_\varepsilon, w) \le \liminf_{n\to\infty} I_\varepsilon(\hat{v}_{\varepsilon,n}, w_n)$$

$$\le \lim_{n\to\infty} I_\varepsilon(\hat{v}_\varepsilon, w_n) = I_\varepsilon(\hat{v}_\varepsilon, w),$$

from which we know that

$$I_\varepsilon(\hat{v}_\varepsilon, w) = I_\varepsilon(\tilde{v}_\varepsilon, w)$$

and further $\tilde{v}_\varepsilon$ is also a minimum of $I_\varepsilon(\cdot, w)$. By the uniqueness of the minimum, we conclude that $\tilde{v}_\varepsilon = \hat{v}_\varepsilon$. So

$$\lim_{n\to\infty} I_\varepsilon(\hat{v}_{\varepsilon,n}, w_n) = I_\varepsilon(\hat{v}_\varepsilon, w),$$

$$\lim_{n\to\infty} \int_0^T \| B^* K_{T-t}^* * \hat{v}_{\varepsilon,n} \|^2 dt = \int_0^T \| B^* K_{T-t}^* * \hat{v}_\varepsilon \|^2 dt,$$

$$\lim_{n\to\infty} \langle \hat{v}_{\varepsilon,n}, H(w_n)\rangle = \langle \hat{v}_\varepsilon, H(w)\rangle,$$

$$\| (I - \pi_{\mathcal{E}})\tilde{v}_\varepsilon \|_{L^2} \le \liminf_{n\to\infty} \| (I - \pi_{\mathcal{E}})\tilde{v}_{\varepsilon,n} \|_{L^2}.$$

These relations show that

$$\lim_{n\to\infty} \| (I - \pi_{\mathcal{E}})\hat{v}_{\varepsilon,n} \|_{L^2} = \| (I - \pi_{\mathcal{E}})\hat{v}_\varepsilon \|_{L^2},$$

from which we conclude that $\{\hat{v}_{\varepsilon,n}\}$ strongly converges to $\hat{v}_\varepsilon$ in view of the fact that $L^2(\Sigma; L^2(\mathcal{O}))$ is a Hilbert space. □

Let $y_0, y_T \in L^2(\Sigma; L^2(\mathcal{O}))$ be fixed. Define an operator

$$\mathfrak{T}_\varepsilon : C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O})) \to C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$$

as

$$(\mathfrak{T}_\varepsilon y)(t, x, \omega) := \int_{\mathcal{O}} K_t(x-z)\, y_0(z)\,dz + \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z)\mathcal{B}v_\varepsilon(s, y)\,dzds$$

$$+ \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z) f(s, z, y(s, z, \omega))\,dzds$$

$$+ \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z)\sigma(s)\,dzdB(s)$$

$$+ \alpha \int_0^t (t-s)^{\alpha-1} \int_{\mathcal{O}} K_{t-s}(x-z)\, g(s, z, y(s, z, \omega))\,dzds$$

$$+ \int_0^t \int_U \int_{\mathcal{O}} K_{t-s}(x-z) h(s, z, y(s, z, \omega); u)\,dz\tilde{\theta}(du, ds),$$

where $v_\varepsilon(s, y) = \mathcal{B}^* K_{t-s}^* * \mathcal{T}_\varepsilon(y)$. It is immediate that a mild solution of the equation

$$dy + (-\Delta)^\gamma y dt = f(t, x, y)dt + \mathcal{B}v_\varepsilon(t, y)dt + \sigma(t)dB(t)$$

$$+ g(t, x, y)(dt)^\alpha + \int_U h(t, x, y; u)\tilde{\theta}(du) \tag{3.3}$$

is a solution of the operator equation $y = \mathfrak{T}_\varepsilon y$ which will be obtained by means of Schauder Fixed Point Theorem.

Lemma 3.7. *Under the assumptions* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$, *for any* $\varepsilon > 0$, *there exists* $r_\varepsilon > 0$ *such that* $\mathfrak{T}_\varepsilon(B(0, r_\varepsilon)) \subset B(0, r_\varepsilon)$, *where* $B(0, r_\varepsilon) = \{w \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O})) : \| w \|_{C_{\mathcal{F}_t}} \le r_\varepsilon\}$.

Proof. Assume that the conclusion is not true. Then there exists $\varepsilon > 0$ such that for any $r = r_\varepsilon > 0$, there exists a function $w_r \in B(0, r)$, but $\mathfrak{T}_\varepsilon(w_r) \notin B(0, r)$, i.e., there exists $t \in [0, T]$ such that $\mathbf{E}\left\{\int_{\mathcal{O}} (\mathfrak{T}_\varepsilon(w_r))^2\, dx\right\}$ $> r^2$. For such t, we can get

$$r^2 < \mathbf{E}\left\{\int_{\mathcal{O}} (\mathfrak{T}_\varepsilon(w_r))^2 dx\right\}$$

$$\leq 6\mathbf{E}\left\{\int_{\mathcal{O}} \left| \int_{\mathcal{O}} K_t(x-z) y_0(z) dz \right|^2 dx\right\}$$

$$+ 6\mathbf{E}\left\{\int_{\mathcal{O}} \left| \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z) \mathcal{B} v_\varepsilon(s, y) dz ds \right|^2 dx\right\}$$

$$+ 6\mathbf{E}\left\{\int_{\mathcal{O}} \left| \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z) f(s, z, y(s, z, \omega)) dz ds \right|^2 dx\right\}$$

$$+ 6\mathbf{E}\left\{\int_{\mathcal{O}} \left| \int_0^t \int_{\mathcal{O}} K_{t-s}(x-z) \sigma(s) dz dB(s) \right|^2 dx\right\}$$

$$+ 6\alpha\mathbf{E}\left\{\int_{\mathcal{O}} \left| \int_0^t (t-s)^{\alpha-1} \int_{\mathcal{O}} K_{t-s}(x-z) g(s, z, y(s, z, \omega)) dz ds \right|^2 dx\right\}$$

$$+ 6\mathbf{E}\left\{\int_{\mathcal{O}} \left| \int_0^t \int_U \int_{\mathcal{O}} K_{t-s}(x-z) h(s, z, y(s, z, \omega); u) dz \tilde{\theta}(du, ds) \right|^2 dx\right\}$$

$$=: 6\sum_{i=1}^{6} I_i.$$

Let us estimate I_i, $i = 1, \ldots, 6$, one by one. We have

$$I_1 \leq \int_{\mathbb{R}^N} K^2(x) dx \int_{\mathcal{O}} | y_0(z) |^2 dz,$$

$$I_2 \leq \| G^t \|_2^2 \| \mathcal{T}_\varepsilon(y) \|_{L^2}^2 \leq M_\varepsilon^2 M_0^2,$$

$$I_3 \leq \int_{\mathbb{R}^N} K^2(x) dx \int_0^T \Lambda(t) dt \int_{\mathcal{O}} \Phi(x) dx,$$

$$I_4 \leq M_2 M_3 \int_{\mathbb{R}^N} K^2(x) dx,$$

$$I_5 \leq \beta(1+T)\int_{\mathbb{R}^N} K^2(x)dx\int_{\mathcal{O}} \Phi(x)dx,$$

$$I_6 \leq \int_{\mathbb{R}^N} K^2(x)dx\int_0^T \Lambda(t)dt\int_{\mathcal{O}} \Phi(x)dx,$$

from which we conclude that

$$r \leq \widetilde{M}_\varepsilon, \tag{3.4}$$

where $\widetilde{M}_\varepsilon$ is a constant dependent on ε only. Dividing both the sides of (3.4) by r and taking limit as $r \to \infty$, we obtain that

$$1 \leq 0,$$

which is a contradiction. So $\mathfrak{T}(B(0, r_\varepsilon)) \subset B(0, r_\varepsilon)$ for some r_ε. □

Theorem 3.8. *Under the conditions* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$, *the system* (3.3) *has a mild solution in* $C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$.

Proof. Similar to Lemmas 3.1 and 3.2, $\mathfrak{T}_\varepsilon(B(0, r_\varepsilon))$ is relatively compact in $C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$ and $\mathfrak{T}_\varepsilon$ is continuous on $C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$. Hence $\mathfrak{T}_\varepsilon$ is a completely continuous operator on $C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$. From the Schauder Fixed Point Theorem, $\mathfrak{T}_\varepsilon$ has a fixed point in $C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$. In other words, for any $\varepsilon > 0$, there exists a solution $y_\varepsilon \in C_{\mathcal{F}_t}([0, T]; L^2(\mathcal{O}))$ to equation (3.3). □

Theorem 3.9. *Under the conditions* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$, *the system* (1.1) *is finite-approximately controllable on* $[0, T]$.

Proof. By Lemma 3.3(1), the strict convexity of I_ε implies that $I_\varepsilon(v, y_\varepsilon)$ has a unique critical point $\hat{v}_\varepsilon$ which minimizes $I_\varepsilon(v, y_\varepsilon)$, i.e.,

$$\hat{v}_\varepsilon \in L^2(\Sigma; L^2(\mathcal{O})) \quad \text{and} \quad I_\varepsilon(\hat{v}_\varepsilon, y_\varepsilon) = \min_{v \in L^2(\Sigma; L^2(\mathcal{O}))} I_\varepsilon(v, y_\varepsilon).$$

Then for any $v \in L^2(\Sigma; L^2(\mathcal{O}))$ and $\lambda \in \mathbb{R}$, we have

$$I_\varepsilon(\hat{v}_\varepsilon, y_\varepsilon) \le I_\varepsilon(\hat{v}_\varepsilon + \lambda v, y_\varepsilon).$$

Dividing the above inequality by $\lambda > 0$, we have

$$\begin{aligned} 0 &\le \frac{1}{\lambda}[I_\varepsilon(\hat{v}_\varepsilon + \lambda v, y_\varepsilon) - I_\varepsilon(\hat{v}_\varepsilon, y_\varepsilon)] \\ &= \mathbb{E}\left\{\int_0^T \langle B^* K_{T-t}^* * \hat{v}_\varepsilon, B^* K_{T-t}^* * v\rangle dt + \frac{\lambda}{2}\int_0^T \| B^* K_{T-t}^* * v \|^2 dt\right\} \\ &\quad + \varepsilon \frac{\| (I - \pi_{\mathcal{E}})(\hat{v}_\varepsilon + \lambda v) \|_{L^2} - \| (I - \pi_{\mathcal{E}})\hat{v}_\varepsilon \|_{L^2}}{\lambda} - \mathbb{E}\{\langle v, H(y_\varepsilon)\rangle\} \end{aligned}$$

and letting $\lambda \to 0^+$, we obtain

$$\begin{aligned} &\mathbb{E}\{\langle v, H(y_\varepsilon)\rangle\} \\ &\le \mathbb{E}\left\{\int_0^T \langle B^* K_{T-t}^* * \hat{v}_\varepsilon, B^* K_{T-t}^* * v\rangle dt\right\} \\ &\quad + \varepsilon \liminf_{\lambda \to 0^+} \frac{\| (I - \pi_{\mathcal{E}})(\hat{\varphi}_\varepsilon + \lambda v) \|_{L^2} - \| (I - \pi_{\mathcal{E}})\hat{v}_\varepsilon \|_{L^2}}{\lambda} \\ &\le \mathbb{E}\left\{\int_0^T \langle B^* K_{T-t}^* * \hat{v}_\varepsilon, B^* K_{T-t}^* * v\rangle dt\right\} + \varepsilon \| (I - \pi_{\mathcal{E}})v \|_{L^2}. \end{aligned}$$

Repeating the procedure for the case $\lambda < 0$, we finally get

$$\left| \mathbb{E}\left\{\int_0^T \langle B^* K_{T-t}^* * \hat{v}_\varepsilon, B^* K_{T-t}^* * v\rangle dt - \langle v, H(y_\varepsilon)\rangle\right\} \right| \le \varepsilon \| (I - \pi_{\mathcal{E}})v \|_{L^2}.$$

On the other hand, we have

$$\int_0^T \langle B^* K_{T-t}^* * \hat{v}_\varepsilon, B^* K_{T-t}^* * v\rangle dt = \int_0^T \langle K_{T-t}^* * Bu_\varepsilon(t, y_\varepsilon), v\rangle dt,$$

$$H(y_\varepsilon) = y_T - \int_{\mathcal{O}} K_T^*(x - z) y_0(z) dz$$

$$- \int_0^T \int_{\mathcal{O}} K_{T-t}(x-z) f(t, z, y_\varepsilon) dz dt$$

$$- \int_0^T \int_{\mathcal{O}} K_{T-t}(x-z) \sigma(t, z) dz dB(t)$$

$$- \alpha \int_0^T (T-t)^{\alpha-1} \int_{\mathcal{O}} K_{T-t}(x-z) g(t, z, y_\varepsilon) dz dt$$

$$- \int_0^T \int_U \int_{\mathcal{O}} K_{T-t}(x-z) h(t, z, y_\varepsilon; u) dz \widetilde{\theta}(du, dt).$$

From the arguments above, we conclude that

$$| \mathbb{E}\{\langle y_\varepsilon(T) - y_T, v \rangle\} | \leq \varepsilon \| (I - \pi_{\mathcal{E}}) v \|_{L^2}$$

holds for any $v \in L^2(\Sigma; L^2(\mathcal{O}))$, which implies that

$$\| y_\varepsilon(T) - y_T \|_{L^2} \leq \varepsilon,$$

$$\pi_{\mathcal{E}} y_\varepsilon(T) = \pi_{\mathcal{E}} y_T.$$

This concludes the finite-approximate controllability. □

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