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SEMIADDITIVITY OF THE ENTROPY RAYLEIGH-RITZ OPERATOR IN THE PROBLEM OF REALIZATION OF AN INVARIANT POLYLINEAR REGULATOR OF A NON-STATIONARY HYPERBOLIC SYSTEM

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Abstract

Such qualitative issues which bound up with existence of a solution for the inverse problem of systems analysis as realization solvability (sufficient conditions) of the operator functions of the polylinear regulator for a non-stationary hyperbolic system, which contains given (finite/countable/continual) nonlinear bundles of infinite-dimensional controlled dynamic processes in the capacity of admissible solutions in a separable Hilbert space, are investigated.

0. Introduction

The theory of differential realization represents (and will probably represent in the future) a rather active domain of mathematical investigations. In the present context, this paper implies continuation of investigations conducted in the domain indicated [1-3]. Meanwhile, this paper has been constructed in some sense as a philosophical sketch. The point is that its principal objective presumes investigation of the existential proving of existence of special operator functions[1] of the invariant polylinear regulator for a non-stationary hyperbolic system; although, some of the results may be extended also to stationary cases [4-7]. Invariance of the regulator presupposes that the scrutinized (modeled) hyperbolic system shall contain a fixed finite family of nonlinear dynamic bundles-processes in the class of feasible solutions; furthermore, each bundle is unbounded with respect to the power (finite/countable/continual) and induced by its (individual) regulator.

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[1]This circumstance reduces the problem of structural identification of the system's nonlinear regulator (understood as *a posteriori* realization of its general polylinear structure) to a more tangible solvability problem of realization of the adaptation (adjusted *a posteriori*) operator functions presuming multiplicative representation of *i*-linear additive terms of this construction; another (more radical) approach is bound up with identification of operators from *i*-multiple tensor products of Hilbert spaces, in particular, subspaces of the Fock space [8].

1. Terminology and Problem Statement

From now on, $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z_i, \|\cdot\|_Z)$, $i = 1, ..., n$, are real separable Hilbert spaces (the property of pre-Hilbertability [8] is determined by norms $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$), $U := Y \times Z_1 \times \cdots \times Z_n$ is a Hilbert space product with the norm

$$\|(y, z_1, ..., z_n)\|_U := \left(\| y \|_Y^2 + \sum_{i=1,...,n} \| z_i \|_Z^2\right)^{1/2},$$

$L(Y, X)$ is a Banach space with the operator norm $\|\cdot\|_{L(Y, X)}$ of all linear continuous operators acting from Y into X (similarly, $(L(X, X), \|\cdot\|_{L(X, X)})$ and $(L(Z_i, X), \|\cdot\|_{L(Z, X)})$), X^i is the ith Cartesian degree of space X, $\mathcal{L}(X^i, Z_i)$ is the space of all continuous i-linear (polylinear) mappings[2] from X into Z_i.

Let $T := [t_0, t_1]$ be a segment of a numerical line R with the Lebesgue measure μ; and $\wp_\mu$ be σ-algebra of all μ-measurable subsets from T. When $(\mathcal{B}, \|\cdot\|)$ denotes some Banach space below, $L_p(T, \mu, \mathcal{B})$, $p \in [1, \infty)$ denotes a Banach factor space of classes of μ-equivalence for all the Bochner integrable [8] mappings $f : T \to \mathcal{B}$ with norm

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[2]Space $\mathcal{L}(X^k, Z)$ is linear (as the spaces of functions with the values laying in the vector space Z: addition and multiplication by a scalar is conducted pointwisely), furthermore, $\mathbb{B} \in \mathcal{L}(X^k, Z)$ is understood as a relation $z = \mathbb{B}(x_1, ..., x_k)$ between the ordered systems $(x_1, ..., x_k)$ of elements from X and the elements from Z, which is linear with respect to each x_i, other elements being fixed, and for some $c < 0$, it satisfies $\| \mathbb{B}(x_1, ..., x_k)\|_Z \le c\| x_1 \|_X \cdots \| x_k \|_X$; this is equivalent: for each i and for any $x_1, ..., x_{i-1}, x_{i+1}, ..., x_k \in X$, it is true that $x_i \mapsto \mathbb{B}(x_1, ..., x_i, ..., x_k) \in L(X, Z)$.

$\left(\int_T \| f(\tau)\|^p \mu(d\tau)\right)^{1/p} < \infty$; $L_\infty(T, \mu, \mathcal{B})$ denotes a space of all (equivalent class) μ-measurable and μ-substantially bounded functions from T into $\mathcal{B}$. Furthermore, henceforth, $AC^1(T, X)$ is a set of all functions $\varphi : T \to X$, whose first derivative is absolutely continuous on T (with respect to measure μ), moreover, for the purpose of simplicity, let us introduce the denotation

$$\Pi := AC^1(T,\ X) \times L_2(T,\ \mu,\ Y) \times L_2(T,\ \mu,\ Z_1) \times \cdots \times L_2(T,\ \mu,\ Z_n).$$

Denote by

$$H_2 := L_2(T,\ \mu,\ Y) \times L_2(T,\ \mu,\ Z_1) \times \cdots \times L_2(T,\ \mu,\ Z_n)$$

a space-product with the topology induced by the norm

$$\|(w_0, \ldots, w_n)\|_H := \left(\int_T \|(w_0(\tau), \ldots, w_n(\tau))\|_U^2 \mu(d\tau)\right)^{1/2}, \ (w_0, \ldots, w_n) \in H_2;$$

obviously, H_2 is a Hilbert space (due to [8] the construction of norm $\|\cdot\|_H$).

Next, consider the Banach space-product

$$\mathbf{L}_2 := L_2(T,\ \mu,\ L(Y,\ X)) \times L_2(T,\ \mu,\ L(Z_1,\ X)) \times \cdots \times L_2(T,\ \mu,\ L(Z_n,\ X))$$

for μ-equivalence classes of ordered systems of operator functions with the norm

$$\|(B_0, \ldots, B_n)\|_{\mathbf{L}} := \left(\int_T \left(\|B_0(\tau)\|_{L(Y,X)}^2 + \sum_{i=1,\ldots,n} \|B_i(\tau)\|_{L(Z,X)}^2\right)\mu(d\tau)\right)^{1/2}.$$

Let there be given operator functions $A_0, A_1 \in L_1(T,\ \mu,\ L(X,\ X))$, $A_2 \in L_\infty(T,\ \mu,\ L(X,\ X))$; μ-almost everywhere on T operator $A_0(t)$, $t \in T$, is self-conjugated and strongly positive definite; furthermore, fixed are the natural number n, i-linear mappings $\mathbb{B}_i \in \mathcal{L}(X^i,\ Z_i)$, $i = 1, \ldots, n$ and

$$N_1 \subset \{(x,\ u,\ \mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x)) \in \Pi\}, \text{ Card } N_1 \le \exp \aleph_0,$$

$$N_2 \subset \{(x, u, \mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x)) \in \Pi\}, \text{ Card } N_2 \le \exp \aleph_0, \qquad (1)$$

represent two fixed variants of behavior of the scrutinized dynamic system with trajectories x, programmed control u and positional feedbacks (forms) $\mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x)$, furthermore, $N_1 \cap N_2 = \varnothing$; henceforth, Card N_j is power of the set (bundle) N_j, $\aleph_0$-aleph-zero. It is clear that

$$\mathbb{B}_i(x, \ldots, x) \in L_\infty(T, \mu, Z_i), \quad i = 1, \ldots, n,$$

$$(x, u, \mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x)) \in N_j, \quad j = 1, 2.$$

Let us also agree to differentiate in the denotations between the vector function $(x, u, \mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x)) \in \Pi$ as a class of equivalence $(\text{mod}\,\mu)$ and a definite representative of this class, i.e., an "individual" vector function $t \mapsto (x(t), u(t), \mathbb{B}_1(x(t)), \ldots, \mathbb{B}_n(x(t), \ldots, x(t)))$.

Furthermore, we assume that in reality, the controlled dynamic bundles N_1, N_2 are solutions of *one* hyperbolic system with *different* polylinear regulators:

$$\exists (B_{01}, \ldots, B_{n1}) \in \mathbf{L}_2 : \forall (x, u, \mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x)) \in N_1,$$

$$A_2 d^2x/dt^2 + A_1 dx/dt + A_0 x = B_{01}u + \sum_{i=1,\ldots,n} B_{i1}\mathbb{B}_i(x, \ldots, x),$$

$$\exists (B_{02}, \ldots, B_{n2}) \in \mathbf{L}_2 : \forall (x, u, \mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x)) \in N_2,$$

$$A_2 d^2x/dt^2 + A_1 dx/dt + A_0 x = B_{02}u + \sum_{i=1,\ldots,n} B_{i2}\mathbb{B}_i(x, \ldots, x),$$

$$(B_{01}, \ldots, B_{n1}) \ne (B_{02}, \ldots, B_{n2}); \qquad (2)$$

from now on, on account of Lemma 1 [1], as regards the analytical construction of the x-solution, we follow the provisions of Section 121 in [9].

Consider the following problem: Determine (in terms of trajectories of the joint bundle $N_1 \cup N_2$) the analytical conditions of existence of the ordered system of operator functions $(B_0^+, ..., B_n^+) \in \mathbf{L}_2$, for which differential realization of the dynamic bundle $N_+ := N_1 \cup N_2$ of the form

$$A_2 d^2x/dt^2 + A_1 dx/dt + A_0 x = B_0^+ u + \sum\nolimits_{i=1,...,n} B_i^+ \mathbb{B}_i(x, ..., x),$$

$$(x, u, \mathbb{B}_1(x), ..., \mathbb{B}_n(x, ..., x)) \in N_+; \tag{3}$$

is implementable. Meanwhile, statement of the converse problem (3) gives a number of theoretical schemes[3], which correctly explain the physics reality. Furthermore, this approach is a way to development of a new kind of mathematical intuition. Such an intuition is based on the "differential - *a posteriori*" modeling of complex hyperbolic systems [10].

Remark 1. Note that there are no structural obstacles[4] for extending the results obtained and described below onto the qualitative theory of realization of the invariant regulator of hyperbolic system (3), which includes into its content polylinear operators, i.e., programmed positional links from $\mathcal{L}(X^i \times Y, Z_i)$, and includes (in the capacity of additional variables) *k*-times $(k \leq i)$ derivative dx/dt and once programmed control *u*; obviously, for the given problem statement, $\mathbb{B}(x, ..., dx/dt, ..., u) \in L_2(T, \mu, Z_i)$ for any mapping $\mathbb{B} \in \mathcal{L}(X^i \times Y, Z_i)$. Furthermore, if the problem of solvability of realization of polylinear forms from $\mathcal{L}(X^i \times Y, Z_i)$, $i = 1, ..., n$, is stated for the differential realization (3), then

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[3]In [2], another (geometric) consideration of the converse problem (3) is briefly discussed, and a variant of its solution (Theorem 3 [2]) is given. In Section 4, this consideration obtained further development (it is embodied in the formulation of Theorem 2 and its Corollary 3) under definite constraints upon the power of dynamic bundles N_1, N_2.

[4]This may not be stated with regard to the structure of the regulator with the programmed positional links from $\mathcal{L}(X^i \times Y^j, Z_{ij})$, $j \geq 2$, because in the given case, i.e., when the domain of definition of operator $\mathbb{B} \in \mathcal{L}(X^i \times Y^j, Z_{ij})$, $j \geq 2$ includes variable *u* (control) *j*-times, condition $\mathbb{B}(x, ..., dx/dt, ..., u) \in L_2(T, \mu, Z_{ij})$ may fail to be satisfied (see the comment in Note 2 above).

construction of the tensor product[5] of Hilbert spaces [8] may be the foundation of the mathematical apparatus, because its structure reduces the investigation of polylinear mappings needed to the consideration of linear mappings by introducing a new operation bound up with the category of linear spaces.

2. Constructions of the Concomitant Mathematical Formalism

Let us denote by $L(T, \mu, R)$ the space classes of μ-equivalence of all real functions μ-measurable on T, and let $\leq_L$ be quasi-ordering in $L(T, \mu, R)$ such that $\phi_1 \leq_L \phi_2$; when $\phi_1(t) \leq \phi_2(t) - \mu$ -almost everywhere on T. Let us denote the least upper bound for a subset $W \subset L(T, \mu, R)$ as $\sup_L W$, when this bound exists for a subset W in the structure of partial ordering $\leq_L$.

Definition 1 [2, 3]**.** Consider operator $\Psi : \Pi \to L(T, \mu, R)$, which is constructed according to the rule

$$\Psi(q, w_0, ..., w_n)(t)$$
$$:= \begin{cases} \| A_2(t) d^2 q(t)/dt^2 + A_1(t) dq(t)/dt + A_0(t) q(t) \|_X \\ \cdot \left(\| w_0(t) \|_Y^2 + \sum_{i=1,\dots,n} \| w_i(t) \|_Z^2 \right)^{-1/2}, \text{ when } (w_0(t), ..., w_n(t)) \neq 0 \in U; \\ 0 \in R, \text{ when } (w_0(t), ..., w_n(t)) = 0 \in U; \end{cases} \quad (4)$$

according to the terminology [10, 12, 13], operator (4) is called the *entropy Rayleigh-Ritz operator*.

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[5]Some elements of this tensor-theoretic extension can be found in [6], as well as in the algorithm of structural parametric identification (2.1)-(2.2) [11]; on its basis, we have constructed Example 1 [11] of *a posteriori* reconstruction of differential equations for nonlinear dynamics of spatial-rotational motion described by both the Euler equation with polylinear terms with respect to the coordinates of angular rates and the regulator of their damping.

In the construction of operator Ψ, the inclusion $d^2q/dt^2 \in L_1(T, \mu, X)$ is correct (due to Lemma 1 [1]).

Let $N \subset \{(x, u, \mathbb{B}_1(x), ..., \mathbb{B}_n(x, ..., x)) \in \Pi\}$, Card $N \leq \exp \aleph_0$ and Q be some (i.e., any) consuming set in Span N; as far as geometry of the consuming set is concerned, we follow [8], i.e., $\bigcup\{\alpha Q\}_{\alpha>0} = \text{Span}\, N$. Having fixed the terminology (motivations can be found in Theorem 2 [2]), let us speak that the bundle of controlled dynamic processes N is *regular* for the 3-tuple of operator functions (A_0, A_1, A_2) of the hyperbolic system (3) if and only if the following hold:

$$\{t \in T : \| A_2(t)d^2q(t)/dt^2 + A_1(t)dq(t)/dt + A_0(t)q(t) \|_X = 0\}$$

$$\supset \left\{t \in T : \| w_0(t)\|_Y + \sum\nolimits_{i=1,\ldots,n} \| w_i(t)\|_Z = 0\right\} (\text{mod}\, \mu), \;\; (q, w_0, ..., w_n) \in Q.$$

Remark 2. (i) If in the process of analysis of the dynamic bundle N, it appears obvious that

$$\bigcup\nolimits_{i=1,\ldots,n} \text{supp}\| \mathbb{B}_i(x, ..., x) \|_Z = \text{supp}\| x \|_X (\text{mod}\, \mu),$$

$$(x, u, \mathbb{B}_1(x), ..., \mathbb{B}_n(x, ..., x)) \in N,$$

then bundle N is regular for any 3-tuple of operator functions

$$(A_0, A_1, A_2) \in L_1(T, \mu, L(X, X)) \times L_1(T, \mu, L(X, X)) \times L_\infty(T, \mu, L(X, X));$$

(ii) due to Theorem 2 [2] and representation (2), N_1, N_2 from (1) are regular dynamic bundles.

Next, we have

Lemma 1 (Modification of Lemma 1 [14])**.** *If* $\ker \mathbb{B}_1 = 0$, *then the dynamic bundle N is regular for any operator functions*

$$A_0, A_1 \in L_1(T, \mu, L(X, X)), \quad A_2 \in L_\infty(T, \mu, L(X, X)).$$

Proof. Obviously, provision $\ker \mathbb{B}_1 = 0$ incurs that

$$\bigcup_{i=1,\dots,n} \operatorname{supp}\| \mathbb{B}_i(x, \dots, x)\|_Z = \operatorname{supp}\| x \|_X \,(\operatorname{mod} \mu),$$

$$(x, u, \mathbb{B}_1(x), \dots, \mathbb{B}_n(x, \dots, x)) \in N.$$

So, it is sufficient to show that for any functions $f \in AC(T, X)$ and $g \in L(T, \mu, R)$, equality $df(t)/dt = 0$ is satisfied μ-almost everywhere in $T_{fg} := \{t \in T : \| f(t)\|_X + | g(t)| = 0\}$; hence, in the structure of the proof, the 2-tuple $(f, df/dt)$ plays a dual role, i.e., the role of the 2-tuple $(x, dx/dt)$ and the 2-tuple $(dx/dt, d^2x/dt^2)$.

Let $T_f := \{t \in T : f(t) = 0\}$. Since $T_f \supset T_{fg}$, in the case when $\mu(T_f) = 0$, the proposition $\{t \in T : \| df(t)/dt \|_X = 0\} \supset T_{fg} (\operatorname{mod} \mu)$ is transparent. So, consider the variant when $\mu(T_f) \neq 0$.

Introduce the denotation $T_0 := \{t \in T_f : \exists \delta > 0, \mu((t - \delta, t + \delta) \cap T_f) = 0\}$. Let us show that $\mu(T_0) = 0$. To this end, let us put constant $\delta_t^* > 0$ in correspondence to each $t \in T_0$, so that $\mu((t - \delta_t^*, t + \delta_t^*) \cap T_f) = 0$. Let us find rational numbers δ_t', δ_t'' such that $\delta_t' \in (t - \delta_t^*, t)$, $\delta_t'' \in (t, t + \delta_t^*)$ and let $I_t := (\delta_t', \delta_t'')$. Hence the family of intervals $\{I_t\}_{t\in T_0}$ covers the set T_0, and since each interval I_t is open with rational ends, the family $\{I_t\}_{t\in T_0}$ contains a countable subfamily $\{I_{ti}\}_{i=1,2,\dots,}$ which also represents the coverage of set T_0. Next, since for any index $i = 1, 2, \dots,$ it is true that $I_{ti} \subset (t_i - \delta_{ti}^*, t_i + \delta_{ti}^*)$, $\mu(I_{ti} \cap T_\delta) = 0$, and so, the following "chain" of μ-relations:

$$\mu(T_0) = \mu\Big(T_0 \cap \Big(\bigcup_{i=1,2,\dots} I_{ti}\Big)\Big) = \mu\Big(\bigcup_{i=1,2,\dots} T_0 \cap I_{ti}\Big)$$
$$\leq \sum_{i=1,2,\dots} \mu(T_0 \cap I_{ti}) = 0,$$

whence $\mu(T_0) = 0.$ Now let us conduct the following concluding part of the proof.

Let $t \in T_f \backslash T_0,$ hence for any $\delta > 0,$ it is true that $\mu((t - \delta, t + \delta) \cap T_f) > 0,$ and since $f \in AC(T, X),$ it is possible to find a set $T^* \subset T$ such that $\mu(T^*) = 0$ and $\forall t \in T_f \backslash T^*,$ there exists $df(t)/dt.$ Let us show that $df(t)/dt = 0$ for $t \in T_f \backslash (T_0 \cup T^*).$ Indeed, for any natural $k,$ we have $\mu((t - 1/k, t + 1/k) \cap T_f) > 0$ and, consequently, one can find a moment $t_k \neq t, |t_k - t| < 1/k, t_k \in T_f.$ But then, the following limit transition may be executed in the structure of strong topology:

$$df(t)/dt = \lim\{(f(t - \Delta t) - f(t))/\Delta t : \Delta t \to 0\}$$

$$= \lim\{(f(t_k) - f(t))/(t_k - t) = 0 \in X : k \to \infty\} = 0 \in X. \quad \square$$

Corollary 1. *If* $(x, u, \mathbb{B}_1(x), ..., \mathbb{B}_n(x, ..., x)) \in \Pi$ *and* $\ker \mathbb{B}_1 = 0,$ *then* $\wp_\nu \subset \wp_{\nu\_},$ *where* ν *and* $\nu\_$ *are the respective Lebesgue-complemented behavioral measures*

$$\nu(S) := \int_S \Big(\| u(\tau) \|_Y^2 + \sum\nolimits_{i=1,\dots,n} \| \mathbb{B}_i(x(\tau), ..., x(\tau)) \|_Z^2 \Big) \mu(d\tau), \quad S \in \wp_\mu,$$

$$\nu\_(S) := \int_S \| A_2(\tau)\, d^2x(\tau)/d\tau^2 + A_1(\tau) dx(\tau)/d\tau + A_0(\tau) x(\tau) \|_X \, \mu(d\tau),$$

$$S \in \wp_\mu.$$

Furthermore, if $\operatorname{Im} \mathbb{B}_1 = Z_1,$ *then*

$$\| A_2 d^2x/dt^2 + A_1 dx/dt + A_0 x \|_X$$

$$\cdot \Big(\| u \|_Y^2 + \sum\nolimits_{i=1,\dots,n} \| \mathbb{B}_i(x, ..., x) \|_Z^2 \Big)^{-1/2} \in L_2(T, \mu, R)$$

$$\Leftrightarrow \| A_2 d^2 x/dt^2 + A_1 dx/dt + A_0 x \|_X$$

$$\cdot \left(\| u \|_Y^2 + \| x \|_X^2 + \sum_{i=2,\ldots,n} \| \mathbb{B}_i(x, \ldots, x) \|_Z^2 \right)^{-1/2} \in L_2(T, \mu, R),$$

for any operator functions A_0, $A_1 \in L_1(T, \mu, L(X, X))$, $A_2 \in L_\infty(T, \mu, L(X, X))$.

It obviously follows from the functional construction (4) that the Rayleigh-Ritz operator satisfies the following simple (but important) relations:

$$0 \leq_L \Psi(\phi), \text{ where } 0 \in L(T, \mu, R), \phi \in \Pi, \; \Psi_{(\pm r\phi)} = \Psi(\phi), 0 \neq r \in R. \quad (5)$$

Before proceeding further, we need to introduce some additional terms.

Definition 2 [12, 13]**.** The Rayleigh-Ritz operator is called *semiadditive* with weight $\alpha \in R$ on set $E \subset \Pi$, when for any tuple $(\phi_1, \phi_2) \in E \times E$, it is true that

$$\Psi(\phi_1 + \phi_2) \leq_L \alpha\Psi(\phi_1) + \alpha\Psi(\phi_2).$$

Lemma 2. *Semiadditivity* (*with the fixed weight*) *of the Rayleigh-Ritz operator is the property of finite character for a subset of* Π.

Proof. Suppose that the Rayleigh-Ritz operator Ψ is semiadditive on a set $E \subset \Pi$ with weight α. Then the given operator is semiadditive with this weight on a finite subset of E. On the other hand, if Ψ is semiadditive with the same weight on any finite subset of set E, then for any tuple of vector functions $(\phi_1, \phi_2) \in E \times E$, the relationship $\Psi(\phi_1 + \phi_2) \leq_L \alpha\Psi(\phi_1) + \alpha\Psi(\phi_2)$ is satisfied because operator Ψ is semiadditive with weight α on subset $\{\phi_1, \phi_2\} \subset E$. □

The interrelation between Lemma 2 and the Teichmüller-Tukey lemma[6] explicit the important geometric characteristic of semiadditivity of the

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[6]We have to remind the reader that the Teichmüller-Tukey lemma is an alternative form of the axiom of choice [15].

Rayleigh-Ritz operator in Π. There exist maximum sets, on which operator (4) is semiadditive with some weight $\alpha > 0$; furthermore, the given sets cannot be *linear* when $\alpha \in (0, 1)$. To make sure, it is sufficient to consider the effect of Ψ on the tuple $(\phi, 0) \in E \times E$, $\phi \neq 0$, with the exception of the trivial variant when $E = \{0\} \subset \Pi$. In this connection, below, in Lemma 3 (and later by default), we assume that the weight of semiadditivity of operator Ψ is a fixed constant $\alpha \in [1, \infty)$.

Lemma 3. *Let* $\alpha \in [1, \infty)$, *hence in* Π, *there exists a* (*nonunique*) *maximum* (*with respect to the set-theoretic inclusion*) *linear set E*, *on which the Rayleigh-Ritz operator is semiadditive with weight* α.

Proof. Let $(q_1, w_{01}, ..., w_{n1})$ be a nonzero element in Π. Hence due to (5), operator Ψ is semiadditive with weight α on the linear hull E_1 over $(q_1, w_{01}, ..., w_{n1})$. Next, let $(q_2, w_{02}, ..., w_{n2}) \in \Pi$, $(q_2, w_{02}, ..., w_{n2}) \notin E_1$ and Ψ be semiadditive on $E_1 \bigcup \{(q_2, w_{02}, ..., w_{n2})\}$ with weight α; if such an element does not exist, then E_1 is a desired maximum set. Let us choose an arbitrary element

$$\beta_1(q_1, w_{01}, ..., w_{n1}) + \beta_2(q_2, w_{02}, ..., w_{n2}), \quad \beta_1, \beta_2 \in R, \quad \beta_2 \neq 0,$$

in the set $E_1 + E_2$, where E_2 is a linear hull over $(q_2, w_{02}, ..., w_{n2})$.

Under such a problem statement, the relations

$$\Psi(\beta_1(q_1, w_{01}, ..., w_{n1}) + \beta_2(q_2, w_{02}, ..., w_{n2}))$$

$$= \Psi(\beta_1\beta_2^{-1}(q_1, w_{01}, ..., w_{n1}) + (q_2, w_{02}, ..., w_{n2}))$$

$$\leq_L \alpha\Psi(\beta_1\beta_2^{-1}(q_1, w_{01}, ..., w_{n1}) + \alpha\Psi(q_2, w_{02}, ..., w_{n2}))$$

$$= \alpha\Psi(\beta_1(q_1, w_{01}, ..., w_{n1})) + \alpha\Psi(\beta_2(q_2, w_{02}, ..., w_{n2}))$$

are satisfied in accordance with formulas (5), whence it follows that operator Ψ is semiadditive on the linear manifold $E_1 + E_2$ with weight α. By similar speculations, it is possible to show that E_1 may be replaced in the upper computations with any nonzero linear subset from Π, on which Ψ is semiadditive with the weight equal to α.

Other explications will be bound up with the chains, hence let *P* be a family of all ordered tuples $(E^{\#}, \alpha^{\#})$, where $E^{\#}$ is a nonzero linear set in Π and $\alpha^{\#} \in [1, \infty)$, furthermore, the Rayleigh-Ritz operator is semiadditive on $E^{\#}$ with weight $\alpha^{\#}$. Introduce the operation of partial ordering $\ll$ in *P*, while assuming that

$$(E^{\#}, \alpha^{\#}) \ll (E^{\#\#}, \alpha^{\#\#}) \Leftrightarrow E^{\#} \subset E^{\#\#}, \quad \alpha^{\#} = \alpha^{\#\#}.$$

According to the Hausdorff theorem (the Hausdorff maximum principle [15]), in family *P*, there exists a maximum chain Ω (a maximum linear ordered set), which contains the chain $(E_1, \alpha) \ll (E_1 + E_2, \alpha)$. Let $\mathbb{E}$ be a set of all linear sets E_γ in Π such that $(E_\gamma, \alpha) \in \Omega$. Hence $\mathbb{E}$ is linear ordered with respect to the set-theoretic inclusion, and, consequently, disjunction $E := \bigcup\{E_\gamma : E_\gamma \in \mathbb{E}\}$ (trivially) forms a linear manifold in Π. Next, if $(\phi_1, \phi_2) \in E \times E$, then obviously $(\phi_1, \phi_2) \in E_\gamma \times E_\gamma$ for some set $E_\gamma \in \mathbb{E}$, whence we draw the conclusion on the weight semiadditivity for the tuple (ϕ_1, ϕ_2) of operator Ψ:

$$\Psi(\phi_1, \phi_2) \leq_L \alpha\Psi(\phi_1) + \alpha\Psi(\phi_2),$$

and, consequently, $(E, \alpha) \in \Omega$. Furthermore, if manifold *E* would fail to be maximum in Π, on which our operator Ψ is semiadditive with weight α, then the linear extension construction indicated above would have allowed one to obtain an element (E^*, α) in family *P*, for which E^* strongly contains *E*, but this would contradict to the criterion of maximum for chain Ω in family *P*. $\square$

3. Existence of an Invariant Polylinear Regulator in the Constructions of the Rayleigh-Ritz Operator

This section presumes investigation of the problem of existence of operator functions of the invariant polylinear regulator of system (3). We shall avoid the formulation of the best positive result in this direction. This result will be given in the form of a compact theorem (while roving a simple sufficient condition). All the parts of proving this condition have in essence been already prepared by us. It is necessary only to integrate these parts.

Theorem 1. *Let* $N_1, N_2 \subset \Pi$ *be the bundles of dynamic processes* (1) *and* (2). *Hence problem* (3) *is solvable when the Rayleigh-Ritz operator is semiadditive with some weight on* $Span\, N_1 + Span\, N_2$.

Remark 3. The following issue remains open: whether Theorem 1 is equivalent to Theorem 3 [2], which is the solution of the problem of invariant extension of the differential realization in terms of angular metric of Hilbert space subspaces; in this case, Theorem 1 gives evidence of validity of the weight construction of semiadditivity of operator (4) in the process of discussion of the issue of extension for the bundles of dynamic processes assuming differential realization (3).

Proof of Theorem 1. Since linear hulls $\mathrm{Span}\, N_1$ and $\mathrm{Span}\, N_2$ are absorbing sets, one, due to Theorem 2 [2], can find the two functions $\phi_1, \phi_2 \in L_2(T, \mu, R)$, for which the following two functional inequalities are satisfied[7]

$$\sup_L \Psi[\mathrm{Span}\, N_1] \le_L \phi_1, \quad \sup_L \Psi[\mathrm{Span}\, N_2] \le_L \phi_2.$$

---

[7]There is (see Theorem 17 [16, p. 68]) a countable set $Q^* \subset \mathrm{Span}\, N_i$ such that if $\sup_L \Psi(\mathrm{Span}\, N_i) \in L(T, \mu, R)$, then $\phi := \sup_L \Psi(\mathrm{Span}\, N_i)$ is due to the sup-construction: $t \mapsto \phi(t) = \sup\{\Psi(q, w_0, \ldots, w_n)(t) \in R : (q, w_0, \ldots, w_n) \in Q^*\}$.

Let us choose this manifold in the manifold $\text{Span}\, N_1 + \text{Span}\, N_2$ in the capacity of its absorbing set. Hence due to semiadditivity Ψ (with weight α) on $\text{Span}\, N_1 + \text{Span}\, N_2$, we obtain

$$\sup_L \Psi[\text{Span}\, N_1 + \text{Span}\, N_2]$$

$$\leq_L \alpha \sup_L \Psi[\text{Span}\, N_1] + \alpha \sup_L \Psi[\text{Span}\, N_2] \leq_L \alpha(\phi_1, \phi_2),$$

whence, proceeding from Theorem 2 [2], it follows (on account of $\text{Span}\, N_1 \bigcup N_2 = \text{Span}\, N_1 + \text{Span}\, N_2$ and item (ii) of Remark 2 that the set of processes $N_1 \bigcup N_2$ possesses differential realization (3). □

Corollary 2. *Let sets* $N_1, N_2, \ldots, N_k \subset \Pi$ *possess realizations* (2). *Hence* $\bigcup_{i=1,\ldots,k} N_i$ *is a family of solutions of system* (3) *for some ordered system* $(B_0^+, \ldots, B_n^+) \in \mathbf{L}_2$, *when* Ψ *is semiadditive with the weight on the linear manifold of a sum of linear hulls of these sets*:

$$\text{Span}\, N_1 + \cdots + \text{Span}\, N_k.$$

Corollary 2 allows one to construct the algebra of a set of dynamic processes with a unit $\bigcup_{i=1,\ldots,k} N_i$, all the elements of which (as a set of all subsets of the unit) possess realization (2) with a fixed model (3). Furthermore, the issue of "individual" characteristic indicator of differential realization for each separate bundle N_i $(i = 1, \ldots, k)$ can be quite easily (constructively) solved on the family of one-element $N_i = \{(x, u, \mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x))_i\}$:

$$\Psi((x_i, u_i, \mathbb{B}_1(x), \ldots, \mathbb{B}_n(x, \ldots, x))_i) \in L_2(T, \mu, R), \quad i = 1, \ldots, k;$$

what is the analytical fact of Theorem 1 [2]. If these relations (or some of them) fail to be satisfied, it is possible to state a problem of synthesis

$\mathbb{B}_i \in \mathcal{L}(X^i, Z_i)$, $i = 1, ..., n$, which provide for the indicated conditions[8], furthermore, this problem may be methodologically interpreted as structural identification[9] of nonlinear components of equation (3) (in this aspect, see also the provisions discussed in [11, 17-19]).

4. Related Issues for the Variant Card $N_1 \leq \aleph_0$, Card $N_2 = 1$

Now consider what analytical results in the solution of the problem of existence of the realization of operator functions $(B_0^+, ..., B_n^+) \in \mathbf{L}_2$ of hyperbolic system (3) introduce the following conditions:

$$N_1 \subset \{(x, u, \mathbb{B}_1(x), ..., \mathbb{B}_n(x, ..., x)) \in \Pi\}, \quad 1 \leq \text{Card } N_1 \leq \aleph_0$$

is not more than a countable bundle of system (2) solutions with the ordered system of operator functions $(B_{01}, ..., B_{n1}) \in \mathbf{L}_2$,

$$N_2 := (x^*, u^*, \mathbb{B}_1(x^*), ..., \mathbb{B}_n(x^*, ..., x^*)) \in \Pi$$

is a solution of system (2) with the set of operator functions $(B_{02}, ..., B_{n2})$ $\in \mathbf{L}_2$, furthermore, $(B_{02}, ..., B_{n2}) \neq (B_{01}, ..., B_{n1})$ and $(x^*, u^*, \mathbb{B}_1(x^*), ..., \mathbb{B}_n(x^*, ..., x^*)) \notin \text{Span } N_1$; therefore, the "integrated" dynamic bundle $N_+ := N_1 \cup N_2$ is either finite or countable.

Obviously, expansion of vector $(u^*(t), \mathbb{B}_1(x^*(t)), ..., \mathbb{B}_n(x^*(t), ..., x^*(t)))$ in Hilbert space U into a projection in

$$\text{Span}\{(u(t), \mathbb{B}_1(x(t)), ..., \mathbb{B}_n(x(t), ..., x(t))) : (x, u, \mathbb{B}_1(x), ..., \mathbb{B}_n(x, ..., x)) \in N_1\},$$

---

[8]It is possible also to search for the functions $\varphi_{jp} : X \to X$ such that $\Psi((x, u, \mathbb{B}_1(\varphi_{11}(x)), ..., \mathbb{B}_n(\varphi_{n1}(x), ..., \varphi_{nn}(x)))_i) \in L_2(T, \mu, R)$, $i = 1, ..., k$.

[9]Explanation: if the assertoric relation $\langle\langle = \rangle\rangle$ is sought for in some R^n by *parametric* identification, then by *structural* identification $\mathbb{B}_i \in \mathcal{L}(X^i, Z_i)$, $i = 1, ..., n$ explicated is the apodictic relation $\langle\langle \in \rangle\rangle$ in $L_2(T, \mu, R)$.

which is designated with $(u^*(t),\ \mathbb{B}_1(x^*(t)),\ ...,\ \mathbb{B}_n(x^*(t),\ ...,\ x^*(t)))$, and a complement

$$(u^*(t),\ \mathbb{B}_1(x^*(t)),\ ...,\ \mathbb{B}_n(x^*(t),\ ...,\ x^*(t)))_\perp$$

$$:= (u^*(t),\ \mathbb{B}_1(x^*(t)),\ ...,\ \mathbb{B}_n(x^*(t),\ ...,\ x^*(t)))$$

$$- (u^*(t),\ \mathbb{B}_1(x^*(t)),\ ...,\ \mathbb{B}_n(x^*(t),\ ...,\ x^*(t)))_-$$

is possible at any point $t \in T$.

For the given problem statement, it can readily be ascertained that vector functions

$$t \mapsto (u^*(t),\ \mathbb{B}_1(x^*(t)),\ ...,\ \mathbb{B}_n(x^*(t),\ ...,\ x^*(t)))_- : T \to U,$$

$$t \mapsto (u^*(t),\ \mathbb{B}_1(x^*(t)),\ ...,\ \mathbb{B}_n(x^*(t),\ ...,\ x^*(t)))_\perp : T \to U$$

are μ-measurable[10]. Next, denote by $E_1(N_1),\ E_2(N_2) \subset H_2$ the subspaces from the formulation of Theorem 3 [2], by $E_\perp(N_2)$ - the closure in H_2 of the linear hull $\mathrm{Span}\{\chi \cdot (u^*,\ \mathbb{B}_1(x^*),\ ...,\ \mathbb{B}_n(x^*,\ ...,\ x^*))_\perp : \chi \in F\}$, $F \subset L(T,\ \mu,\ R)$ is a family of equivalence classes $(\mathrm{mod}\ \mu)$ of all the characteristic functions induced by elements of σ-algebra $\wp_\mu$.

Lemma 4. *Subspaces* E_1 *and* $E_\perp$ *are orthogonal in the Hilbert space* H_2.

Let us agree from now on that for two closed subspaces from spaces H_2, such that their intersection is $\{0\} \subset H_2$, and the vector sum is closed in H_2, the sign of their vector composition is denoted by $\oplus$, in particular, Theorem 14.C [9] and Lemma 4 make the notation $E_1 \oplus E_\perp$ correct.

---

[10]Due to separability of space U, weak and strong measurabilities coincide (Theorem IV.22 [8]).

Let us answer the question: Under what analytical conditions imposed on the sets of controlled dynamic processes N_1 and N_2, and "extended" family of processes, N_+ possesses differential realization (2)? One of the ways of geometric solving this problem presumes constructing the characteristic indicator (see Lemma 5 below), which defines the quality

$$E_1 + E_2 = E_1 \oplus E_\perp, \tag{6}$$

because existence of the partial form of equality (6), i.e., the form

$$E_1 \oplus E_2 = E_1 \oplus E_\perp, \tag{7}$$

gives a positive answer to the question about the differential realization of an extended bundle N_+ in the context of the approach to the geometric solution of the problem of existence of a general polylinear regulator for dynamic bundles N_1, N_2, which is based on Theorem 14.C [9] and Theorem 3 [2]. Below, one characteristic property of equality (7) is ascertained by Theorem 2.

Further speculations necessitate involvement of additional constructions. So, assume that

$$\mathcal{T} := \{t \in T : (u^*(t), \mathbb{B}_1(x^*(t)), \ldots, \mathbb{B}_n(x^*(t), \ldots, x^*(t)))_\perp = 0\},$$

and let $\nu_\perp^*, \nu^*$ be the Lebesgue complements (on the respective extensions of σ-algebras) of the measures

$$\int_S \| (u^*(\tau), \mathbb{B}_1(x^*(\tau)), \ldots, \mathbb{B}_n(x^*(\tau), \ldots, x^*(\tau)))_\perp \|_U^2 \mu(d\tau), \quad S \in \wp_\mu,$$

$$\int_S \| (u^*(\tau), \mathbb{B}_1(x^*(\tau)), \ldots, \mathbb{B}_n(x^*(\tau), \ldots, x^*(\tau))) \|_U^2 \mu(d\tau), \quad S \in \wp_\mu.$$

Lemma 5. *Equality* $E_1 + E_2 = E_1 \oplus E_\perp$ *holds if and only if*

$$L_2(T, \nu_\perp^*, R) = \chi_\perp \cdot L_2(T, \nu^*, R),$$

where $\chi_\perp$ *is a characteristic function of set* $T \backslash \mathcal{T}$.

Consider now a variant of the characteristic conditions for equality (7) with a sketch of the proof.

Theorem 2. *When equality* $\mathcal{F} = \varnothing \,(\mathrm{mod}\,\mu)$ *holds, the following proposition is valid*:

$$E_1 \oplus E_2 = E_1 \oplus E_\perp \Leftrightarrow L_2(T, \nu_\perp^*, R) = L_2(T, \nu^*, R).$$

Proof. Proposition $E_1 + E_2 = E_1 \oplus E_\perp \Leftrightarrow L_2(T, \nu_\perp^*, R) = L_2(T, \nu^*, R)$ is a direct statement of Lemma 5. On the other hand, confirmation of the fact that $E_1 \cap E_2 = \{0\} \subset H_2$ follows from

$$\{t \in T : (u^*(t), \mathbb{B}_1(x^*(t)), \ldots, \mathbb{B}_n(x^*(t), \ldots, x^*(t)))_\perp = 0\} = \varnothing \,(\mathrm{mod}\,\mu)$$

and from Corollary 3 of Theorem III.5 (the Hahn-Banach theorem) [8]. □

Lemma 5 and Theorem 2 in the context of Theorem 14.C [9] and Theorem 3 [2] validate the following corollary:

Corollary 3. (i) *The following three properties are equivalent*:

$$L_2(T, \nu_\perp^*, R) \subset \chi_\perp \cdot L_2(T, \nu^*, R)$$

$$\Leftrightarrow L_2(T, \nu_\perp^*, R) = \chi_\perp \cdot L_2(T, \nu^*, R)$$

$$\Leftrightarrow E_1 + E_2 = E_1 \oplus E_\perp.$$

(ii) *If* $\mathcal{F} = \varnothing \,(\mathrm{mod}\,\mu)$, *then existence of any property from* (i) *transforms the dynamic bundle* N_+ *into a set of nonlinear controlled processes with the differential realization* (3).

The hypothesis (in essence, conversion of Theorem 3 [2] and part (ii) of Corollary 3):

"*if* $\mathcal{F} = \varnothing \,(\mathrm{mod}\,\mu)$ *and bundle* N_+ *possesses realization* (3), *then* $L_2(T, \nu_\perp^*, R) \subset L_2(T, \nu^*, R)$" does not find confirmation in the general case, what may be illustrated by the following simple example.

Example. Let $X = Y = Z = R$, $T = [-1, 1]$, $n = 2$. Assume that parameters (coefficients) of the differential system and the dynamic bundles modelled have the following representation:

$$A_0 = 2,\ A_1 = 4,\ A_2 = 2,\ \mathbb{B}_1 = 1,\ \mathbb{B}_2 = (1, 1),$$

$$N_1 = \{t \mapsto (e^{-2t}, 0, e^{-2t}, e^{-4t}) : t \in T\},$$

$$N_2 = \{t \mapsto (t^2 + 1, 4t + 2, t^2 + 1, (t^2 + 1)^2) : t \in T\}.$$

Obviously, bundles N_1, N_2 have differential realizations (2) with regulators $(B_{01}, B_{11}, B_{21}) = (1, 2, 0)$, $(B_{02}, B_{12}, B_{22}) = (2, 2, 0)$. Furthermore, the integrated dynamic bundle $N_+ := N_1 \cup N_2$ possesses realization (3) with the regulator, for which $B_0^+ = B_1^+ = 2$, $B_2^+ = 0$. Moreover, the relations

$$(u^*(t), \mathbb{B}_1(x^*(t)), \mathbb{B}_2(x^*(t), x^*(t))) = (4t + 2, t^2 + 1, (t^2 + 1)^2),$$

$$(u^*(t), \mathbb{B}_1(x^*(t)), \mathbb{B}_2(x^*(t), x^*(t)))_\perp = (4t + 2, 0, 0),$$

hold, and this leads to the expected fact that $\mathcal{T} = \varnothing \,(\mathrm{mod}\, \mu)$ (although $\mathcal{T} = \{-2^{-1}\} \neq \varnothing$) and to the transparent statement that

$$L_2(T, \nu^*, R) = L_2(T, \mu, R),$$

$$L_2(T, \nu_\perp^*, R), \quad \nu_\perp^* = \int (4\tau + 2)^2 \mu(d\tau).$$

Therefore, for the given problem statement, we have

$$1/(4t + 2) \in L_2(T, \nu_\perp^*, R), \quad 1/(4t + 2) \notin L_2(T, \mu, R),$$

whence $L_2(T, \nu_\perp^*, R) \not\subset L_2(T, \nu^*, R)$. Furthermore, due to Corollary 3, a conclusion may be drawn that for N_1, N_2, it is true that $E_1 + E_2 \neq E_1 \oplus E_2$. So, this example shows that, in the general case, Theorem 3 [2] does not have a converse.

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References

[1] V. A. Rusanov, L. V. Antonova and A. V. Daneev, Inverse problem of nonlinear systems analysis: a behavioral approach, Advances in Differential Equations and Control Processes 10(2) (2012), 69-88.

[2] A. V. Lakeev, Yu. É. Linke and V. A. Rusanov, To the structure identification of a nonlinear regulator for a nonstationary hyperbolic system, Doklady Mathematics 93(3) (2016), 339-343.

[3] A. V. Lakeev, Yu. É. Linke and V. A. Rusanov, Realization of a polylinear controller as a second-order differential system in a Hilbert space, Differ. Equ. 53(8) (2017), 1070-1081.

[4] A. V. Daneev, V. A. Rusanov and M. V. Rusanov, From Kalman-Mesarovic realization to a normal-hyperbolic linear model, Cybernet. Systems Anal. 41(6) (2005), 909-923.

[5] Y. Chen, A new one-parameter inhomogeneous differential realization of the spl(2, 1) super-algebra, Internat. J. Theoret. Phys.s 51(12) (2012), 3763-3768.

[6] V. A. Rusanov, A. V. Daneev, A. V. Lakeev and Yu. É. Linke, On the differential realization theory of non-linear dynamic processes in Hilbert space, Far East J. Math. Sci. (FJMS) 97(4) (2015), 495-532.

[7] V. A. Rusanov, A. V. Daneev, A. V. Lakeyev and V. N. Sizykh, Higher-order differential realization of polylinear-controlled dynamic processes in a Hilbert space, Advances in Differential Equations and Control Processes 19(3) (2018), 263-274.

[8] M. Reed and B. Simon, Methods of Modern Mathematical Physics. 1. Functional Analysis, Academic Press, New York, 1972.

[9] J. L. Massera and J. J. Schaffer, Linear Differential Equations and Function Spaces, Academic Press, New York, 1966.

[10] V. A. Rusanov, A. V. Banshchikov, R. A. Daneev and A. V. Lakeyev, Maximum entropy principle in the differential second-order realization of a nonstationary bilinear system, Advances in Differential Equations and Control Processes 20(2) (2019), 223-248.

[11] V. A. Rusanov and D. Yu. Sharpinskii, The theory of the structural identification of nonlinear multidimensional systems, J. Appl. Math. Mech. 74 (2010), 84-94.

[12] V. A. Rusanov, A. V. Daneev and Yu. É. Linke, To the geometrical theory of the differential realization of dynamic processes in a Hilbert space, Cybernet. Systems Anal. 53(4) (2017), 554-564.

[13] V. A. Rusanov, A. V. Daneev, Yu. É. Linke and P. A. Plesnyov, Existence of a bilinear delay differential realization of nonlinear neurodynamic process in the constructions of entropy Rayleigh-Ritz operator, Advances in Dynamical Systems and Applications 15(2) (2020), 199-215.

[14] V. A. Rusanov, A. V. Lakeev and Yu. É. Linke, Solvability of differential realization of minimum dynamic order for a family of nonlinear input-output processes in Hilbert space, Differ. Equ. 51(4) (2015), 533-547.

[15] J. L. Kelley, General Topology, New York, 1957.

[16] L. V. Kantorovich and G. P. Akilov, Functional Analysis, Nauka, Moscow, 1977.

[17] V. A. Rusanov, R. A. Daneev, A. V. Lakeyev and Yu. É. Linke, Differential realization of second-order bilinear system: a functional geometric approach, Advances in Differential Equations and Control Processes 19(3) (2018), 303-321.

[18] A. V. Daneev, A. V. Lakeyev and V. A. Rusanov, Existence of a bilinear differential realization in the constructions of tensor product of Hilbert spaces, WSEAS Trans. Math. 19 (2020), 99-107.

[19] V. A. Rusanov, A. V. Daneev, A. V. Lakeyev and Yu. É. Linke, On the theory of differential realization: criteria of continuity of the nonlinear Rayleigh-Ritz operator, International Journal of Functional Analysis, Operator Theory and Applications 12(1) (2020), 1-22.