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ANALYTIC EVALUATION OF PIEZOMETRIC HEAD FOR A CREEPING FLOW PAST A FULLY CONSTRAINED OBSTACLE

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Abstract

The paper presents a mathematical formulation of an incompressible two-dimensional groundwater creeping flow past a fully constrained impermeable obstacle. The physical boundary of this obstacle is modeled as a smooth surface having negligible roughness. Referring to the impact of boundary roughness, it is known that from Hydrodynamics point of view, a solid surface is called "smooth" when the average depth of the surface irregularities is less than the thickness of the laminar sublayer over the surface. In this framework, a theoretical evaluation of the piezometric head is exhibited and

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concurrently the position of velocity distribution local extrema is determined.

1. Introduction

In the current literature as well as in the usual engineering practices [1-5], the treatment of physical problems regarding creeping flows, is mainly carried out by means of computational fluid dynamics (CFD) approaches or trivial graphical methods, leading to the calculation of many distinct values of piezometric head. We may also remark that for creeping flows the piezometric head coincides with the hydraulic one since Reynolds number is very low [5]. In general, irregular boundaries are mostly modeled as spatially homogeneous random processes. Nonetheless, the spatial variations are many times very small for computational grids [6]. On the other hand, the mathematical approaches of such problems via analytic methods, which indeed are more rigorous when compared with graphical or CFD ones, result in a closed - form representation of piezometric head. In [7] a creeping flow past a fluid sphere with a solid core was investigated, whereas for a valuable analytic study on a nonstationary creeping flow past a solid sphere, we may refer to [8]. The most popular analytic methods to confront such problems are separation of variables and conformal mapping. Here, we shall use the second one.

2. Analysis

To perform a realistic engineering problem, let us consider on the basis of the author's diploma thesis [9], that the fully constrained obstacle is a pile sheet founded in a layer of alluvium sand, in which digging took place until depth 5m and this pile sheet buttresses up the resulting slopes. Before digging, a pumping parametrically with a system of well - points was carried out hence the water level instrument sustained a depression 4.5m, [see, Figure 1]. The flow takes place only in the alluvium sand layer. Darcy's law is expressed in differential representation as follows:

$$\vec{V} = -K \cdot \operatorname{grad} H \Rightarrow \begin{cases} V_x = -K \cdot \dfrac{\partial H}{\partial x} \\ \wedge \\ V_y = -K \cdot \dfrac{\partial H}{\partial y}, \end{cases} \tag{1a,b}$$

where V_x, V_y are velocity components and H denotes the hydraulic head which is supposed to be equal to the piezometric one and the constant K denotes the hydraulic conductivity of the porous medium. Also,

$$\operatorname{rot} \vec{V} = -K \cdot \operatorname{rotgrad} H = 0. \tag{2}$$

Thus, the flow field is irrotational and therefore

$$|\,\Phi\,| = K \cdot |\,H\,|. \tag{3}$$

The continuity equation reads

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \cdot \vec{V}) = 0 \Leftrightarrow \frac{\partial \rho}{\partial t} + \rho \cdot \operatorname{div} \vec{V} + (\operatorname{grad} \rho) \cdot \vec{V} = 0. \tag{4}$$

Since the flow is steady, it follows $\operatorname{div}\vec{V} = 0 \Rightarrow \operatorname{divgrad} \Phi = 0 \Rightarrow$

$$\nabla^2 \Phi = 0$$

$$\wedge$$

$$\nabla^2 H = 0. \tag{5a,b}$$

Concurrently, the stream function Ψ is defined as

$$\Psi : R^2 \to R : \frac{\partial \Psi}{\partial y} = V_x \wedge -\frac{\partial \Psi}{\partial x} = V_y \Rightarrow \frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial^2 \Phi}{\partial y \cdot \partial x}; \frac{\partial^2 \Psi}{\partial x^2} = -\frac{\partial^2 \Phi}{\partial x \cdot \partial y} \tag{6}$$

and therefore

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0. \tag{7}$$

Obviously, both stream and potential functions satisfy Cauchy-Riemann (C-R) conditions:

$$\frac{\partial \Psi}{\partial y} = \frac{\partial \Phi}{\partial x} \wedge -\frac{\partial \Psi}{\partial x} = \frac{\partial \Phi}{\partial y}. \quad \text{(8a,b)}$$

We may estimate the piezometric head provided that its values at the boundaries are known beforehand. Since the flow is steady (which implies that initial conditions do not exist), the following relationships hold:

$$\frac{\partial \vec{V}}{\partial t} = 0$$

$$\wedge$$

$$-\frac{\partial \rho}{\partial t} = 0. \quad \text{(9a,b)}$$

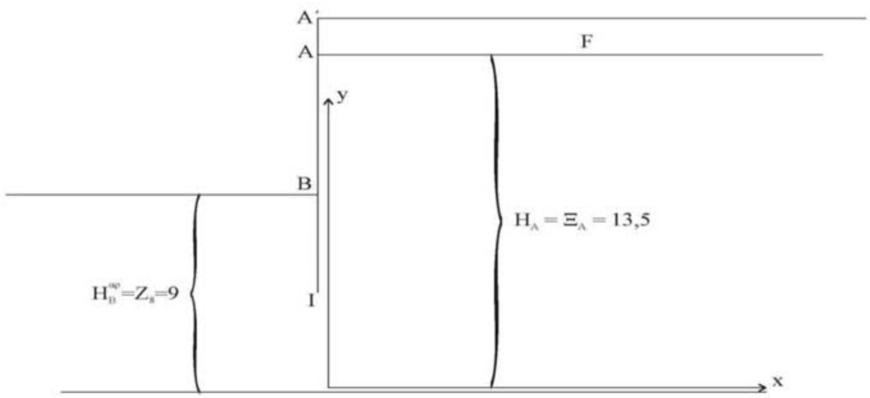

Figure 1. Geometric interpretation of physical problem.

In the meanwhile, Bernoulli's equation states that

$$H = z + \frac{p}{\gamma}. \quad \text{(10)}$$

Specifically for the surface AA' the following relationship holds:

$$H = 13.5 \left(\text{for } y = 13.5 \wedge x > 0\right) \text{ or } \Phi = -13.5 \cdot \mathrm{K}. \quad \text{(11a)}$$

Besides, for the surface $\mathrm{B}^{\text{left}}\mathrm{B}'$, it is valid that

$$H = 9 \, \Phi(x, y) = -9 \, \mathrm{K} \left(\text{for } y = 9 \wedge x < 0\right). \quad \text{(11b)}$$

The above conditions are of Dirichlet type. However, for $(x, y) \in \{0\} \times (3, 9]$, the potential function does not verify the mapping: $R \times R \to R$, since it gives two different values $\forall (x, y) \{0\} \times (3, 9]$. Thus, it is not defined over this subset of the domain. The same conclusions hold for the stream function. The points from $B(0, 9)$ until $I(0, 3)$, have different values of piezometric head left and right, thus the flow takes place in direction: $A \to B^{\text{right}} \to I \to B^{\text{left}}$. Thus, these positions have different potential values. The function Φ has domain of definition as the set $D(\Phi) = R \times R - \{0\} \times (3, 9]$ and range: $\Re(\Phi) = [0, 13.5 \cdot K]$. Also, the domain of definition and the range of function H are $D(H) = D(\Phi)$ and $\Re(H) = [0, 13.5]$, respectively. Potential function is linked with stream function by C-R conditions, thus neither $\Phi(x, y)$ nor $\Psi(x, y)$ can be defined over the set: $\{0\} \times (3, 9]$.

2.1. Boundary conditions for the surface

(a) No-slip condition: the tangential component of velocity vector equals the speed of the surface which here is zero. Hence,

$$V_x(x, y) = 0 \;\forall (x, y) \in \{0\} \times (3, 9]. \tag{12}$$

(b) Kinematic condition: since the surface is impermeable, it implies that

$$\vec{\imath} \cdot V_x(x, y) = 0. \tag{13}$$

In addition, the equation of streamlines reads as

$$V_x \cdot dy = V_y \cdot dx \Rightarrow \frac{dx}{dy} = \frac{V_x(x, y)}{V_y(x, y)} \Rightarrow x' = \frac{V_x(x, y)}{V_y(x, y)}. \tag{14}$$

The quotient: $\frac{V_x(x, y)}{V_y(x, y)}$ over the set $\{0\} \times (3, 9]$ takes the form: $\left(\frac{0}{0}\right)$.

Thus, the derivative $x' = \frac{dx}{dy}$ equals an arbitrary value $\lambda_0 \in R$. Hence,

$$x = \lambda_0 y + \mu_0, \; \mu_0 \in R. \tag{15}$$

Equation of streamlines also concerns the sheet pile $\forall(x, y) \in \{0\} \times (3, 9]$. Thus, $\forall y \in (3, 9] \Rightarrow x = 0$ and therefore

$$\lambda_0 = \mu_0 = 0. \tag{16}$$

2.2. Boundary condition for ground layer

(a) No-slip condition

The velocity component which is tangential to the solid surface should vanish. Thus

$$V_x(x, y) = 0 \Rightarrow \begin{cases} \dfrac{\partial \Phi}{\partial x} = 0 \\ \wedge \\ \dfrac{\partial \Psi}{\partial x}. \end{cases} \tag{17}$$

(b) Kinematic condition

The velocity component which is perpendicular to the surface should vanish. Thus

$$V_y(x, y) = 0 \Rightarrow \begin{cases} \Phi = C_1 \\ \wedge \\ \Psi = C_2, \end{cases} \tag{18}$$

where $C_1, C_2 \in R$. At $y = 0$ and $x \in R$, we may deduce that

$$\frac{-\Phi(x, 0)}{K} = 0 \Rightarrow \Phi(x, 0) = 0. \tag{19}$$

Hence, the conduct line with ground layer constitutes an equipotential line with equation:

$$\Phi = 0. \tag{20}$$

Also, this line constitutes a streamline with equation:

$$\Psi = C_2. \tag{21}$$

For facility reasons, let us create an artificial boundary surrounding the flow field [see, Figure 2]. Assume that

$$\begin{cases} -13.5 \le x \le 13.5 \\ -0 \le y \le 13.5. \end{cases} \tag{22}$$

Since $(x, y) \notin \{0\} \times (3, 9]$, it follows:

$$(x, y) \in [-13.5, 13.5] \times [0, 13.5] - \{0\} \times (3, 9] \equiv D(H). \tag{23}$$

The boundary of $D(H)$, $\partial D(H)$ consists of the following subsets:

$$B_1 = \{(x, y) : x \in [-13.5, 13.5] \wedge y = 0\} = \{-13.5, 13.5\} \times \{0\}.$$

Along segment B_1 it is valid that $\operatorname{grad} H = 0$ and $H = 0$

$$B_2 = \{(x, y) : x = -13.5 \wedge y \in [0, 9]\} = \{-13.5\} \times [0, 9].$$

On segment B_2 the hydrostatic conditions hold. Hence, $H = ct$

$$B_3 = \{(x, y) : x \in [-13.5, 0) \wedge y = 9\} = [-13.5, 0) \times \{9\}.$$

On segment B_3 it is valid that: $H = 9$

$$B_4 = \{(x, y) : x = 0 \wedge y \in (9, 13.5]\} = \{0\} \times (9, 13.5].$$

On segment B_4 it is valid that: $\operatorname{grad} H = 0 \Rightarrow H = ct$

$$B_5 = \{(x, y) : x \in [0, 13.5] \wedge y = 13.5\} = [0, 13.5] \times \{13.5\}.$$

On segment B_5 it is valid that: $H = 13.5$

$$B_6 = \{(x, y) : x - 13.5 \wedge y \in [0, 13.5]\} = \{13.5\} \times [0, 13.5].$$

On segment B_6 hydrostatic conditions hold, hence: $H = ct$.

Obviously,

$$\partial D(H) = U_{i=1}^{6} B_i. \tag{24}$$

3. Evaluation of Piezometric Head

Along the artificial boundaries B_2 and B_6, hydrostatic conditions hold. Thus

$$\begin{cases} B_2 : P_2 = (9 - y) \cdot \gamma_W \\ B_6 : P_6 = (13.5 - y) \cdot \gamma_W . \end{cases} \tag{25}$$

By taking into account Bernoulli's equation, we obtain

$$\begin{cases} H_6 = y + P_6/\gamma_W = 13.5 \\ H_2 = y + P_2/\gamma_W = 9. \end{cases} \tag{26a,b}$$

Hence, on B_2 it is valid that $H = 9$ whilst on B_6 the value of hydraulic head is 13.5. Thus, the study of this flow field reduces to a boundary value problem, with homogeneous boundary conditions of Dirichlet and Neumann type [10].

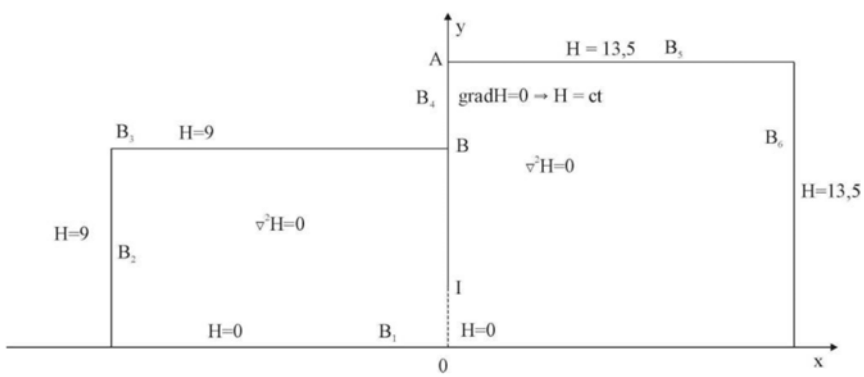

Figure 2. Geometric interpretation of boundary value problem.

Along the boundary B_4, the following relationships hold:

$$V_x = 0 \wedge V_y = 0. \tag{27A}$$

Since it is valid that $\vec{V} = K \cdot \operatorname{grad} H$, we conclude that $\operatorname{grad} H = 0 \Rightarrow H = ct$. According to Bernoulli's equation it implies that

$$\frac{P}{\gamma_W} + y = ct = c \Rightarrow P = (c - y) \cdot \gamma_W . \tag{27B}$$

However, since $c - y = (13.5\ y) \cdot \xi,\ \xi \in R^*,$ we may deduce that

$$P = (13.5y) \cdot \xi \cdot \gamma_W. \tag{27C}$$

Thus, on B_4 the pressure varies linearly with depth. This is unrealistic, since the fluid layer is not a motionless surface. Thus, the final form of $D(H)$ is

$$D(H) = ([-13.5, 13.5] \times [0, 13.5] - (\{0\} \times (3, 9]) \cup \{0\} \times (9, 13.5])$$

$$\Rightarrow D(H) = [-13.5, 13.5] \times [0, 13.5] - \{0\} \times (3, 13.5].$$

Hence, we arrived at a linear boundary value problem which is referred to as S_0. S_0 consists of a second order differential operator and five boundary operators. Hence $S_0 : L_0[H(x, y)] = \nabla^2 H(x, y) = 0$ and $\sum_i([x], [y]) = \alpha_i,$ where $i = 1, 2, 3, 4, 5$ and $\alpha_i = 0, 9, 9, 13.5$ and 13.5, respectively. Next, consider a holomorphic function $F : C \to C : F(z) = \Phi + i \cdot \Psi\ \forall z \in C.$ Thus

$$ReF = \Phi \wedge ImF = \Psi. \tag{28}$$

We may define as complex flow potential, the holomorphic function $W : C \to C : W = \Phi + i \cdot \Psi\ \forall z \in C.$ Thus

$$W = F(z). \tag{29}$$

So the initial problem reduces to the definition of *F*, which according to Riemann's, theorem of mapping [11, 12] is unique, one to one and also $D(F) \equiv \Re(F).$ Equation (29) is written as

$$\Phi + i \cdot \Psi = F(x + i \cdot y) \Leftrightarrow (\Phi, \Psi) = F(x, y). \tag{30}$$

Here, we may observe that a one to one mapping from plane defined by axes *Ox*, *Oy* to the plane defined by axes Φ, Ψ takes place [see, Figure 3]. The latter is referred to as a complex potential plane and its design is based on the boundary conditions [13, 14]. We are seeking a function *F*, such that to relate the points in complex potential plane W, with the Eulerian coordinates of the flow. The boundaries are the sets:

$$\{z \in C : Imz = 0\} \tag{31}$$

and

$$P_L = U_{K=1}^{4}[z_{K-1}, z_K], \tag{32}$$

where

$$z_0 : Rez_0 \to -\infty \wedge Imz = 0$$

$$z_1 : Rez_1 \to 0^- \wedge Imz_1 = 9$$

$$z_2 : Rez_2 \to 0^+ \wedge Imz_2 = 3$$

$$z_3 : Rez_3 \to 0^+ \wedge Imz_3 = 13.5$$

$$z_4 : Rez_4 \to +\infty \wedge Imz_4 = 13.5. \tag{33}$$

Actually, the set P_L is an orthogonal polygonal line.

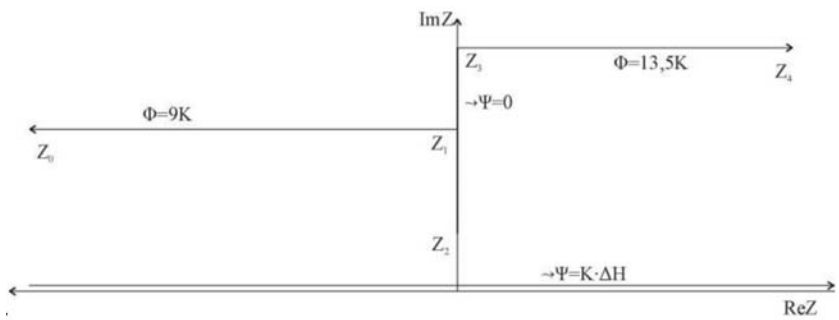

Figure 3. Geometric interpretation of the two associated planes.

Here the boundaries are the sets: $\{z \in C : Rez < 0 \wedge Imz = 9\}$, $\{z \in C : Rez > 0 \wedge Imz = 13.5\}$ and $\{z \in C : Rez > 0 \wedge 3 \leq Imz \leq 13.5\}$. The latter is the segment which connects the points z_2, z_4 and is written as

$$[z_2, z_4] = \{t \cdot z_2 + (1 + t) \cdot z_4 : t \in [0, 1]\}. \tag{34}$$

Along $[z_2, z_4]$ the following statement holds: $\Psi = 0$. To define the boundary of the complex potential field we assume that $\Psi = 0\ \forall z \in [z_2, z_4]$: $z = x + i \cdot y$. Thus, a polygon is drawn the sides of which are parallel to axes $O\Phi$ and $O\Psi$ [see, Figure 4].

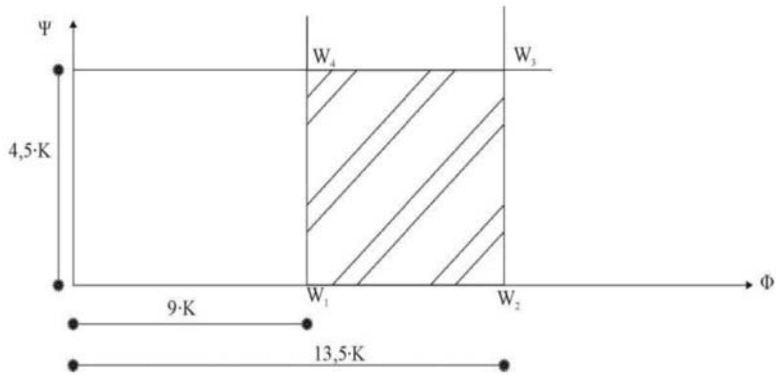

Figure 4. Complex potential field.

By means of mapping F the flow field can be imaged to the plane of complex potential, referred to as W-plane. The elements of the set: $\{z \in \mathbb{C} : Imz \geq 0\}$, will be imaged injectively to interior space or to the boundary of the above polygon. The functions which are able to image the set $\{z \in \mathbb{C} : Imz \geq 0\}$, to interior of a polygon are in a particular form. In this context, a mapping is introduced from the points of real axis to the points of the polygon boundary, i.e., the polygonal line: $U_{\iota=1}^{n}[W_i, W_{i+1}]$. The function $F : F(z) = W$, can be obtained from Schwarz-Christoffel formula [15, 16]:

$$F(z) = A \cdot \int_{Z_0}^{Z} (z - x_1)^{\frac{\Phi_1}{\pi} - 1} \cdot (z - x_2)^{\frac{\Phi_2}{\pi} - 1} \cdots (z - x_n)^{\frac{\Phi_n}{\pi} - 1} dz + B, \quad (35)$$

where $x_1, x_2, ..., x_n \in \{z \in \mathbb{C} : ImZ = 0\}$ and are imaged to the tops $W_1, W_2, ..., W_n$ of polygon, meaning $W_i = F(x_i)$.

Also the angles $\varphi_1, \varphi_2, ..., \varphi_n$ are the interior angles of the polygon and A, B complex constants. These constants specify the range, location and orientation of the polygon at W-plane and B is defined by the coordinates of the top W_i which is assumed to be the principal top of the polygon. This selection is arbitrary, provided that the final top is the point $z_0 \in \{z \in \mathbb{C} : Imz > 0\}$. Let us select the following points: $x_1 = -1 : F(x_1) = W_1$, $x_2 = 1 : F(x_2) = W_2$, $x_3 = -\frac{1}{\lambda} : F(x_3) = W_3$, $x_4 = \frac{1}{\lambda} : F(x_4) = W_4$, where $\lambda \in (0, 1)$. Hence $\frac{1}{\lambda} \in (1, +\infty)$. The value x_4, is not defined arbitrarily. The polygon is an orthogonal parallelogram. Thus $\Phi_1 = \Phi_2 = \Phi_3 = \Phi_4 = \frac{\pi}{2}$. The final form of F reads as

$$F(z) = W_1 + A \cdot \int_{-1}^{z} (z^2 - 1)^{-0,5} \cdot (z^2 - 1/\lambda^2)^{-0,5} dz. \qquad (36)$$

On the other hand, it is valid that: $-i(z^2 - 1)^{0,5} = i \cdot (1 - z^2)^{0,5}$ and also: $(z^2 - 1/\lambda^2)^{0,5} = i/\lambda \cdot (1 - z^2\lambda^2)^{0,5}$. By setting: $A' \leftrightarrow -A/\lambda$, the function $F(z)$ becomes

$$F(z) = W_1 + A' \cdot \int_{-1}^{z} \frac{dz}{\sqrt{(1 - z^2) \cdot (1 - z^2\lambda^2)}} = W_1 + A' \int_{-1}^{z} \frac{dz}{\pi}. \qquad (36a)$$

Here, the following relations hold:

$$F(x_1) = F(-1) = W_1 = 9 \cdot K$$
$$\wedge$$
$$F(x_2) = F(1) = W_2 = \frac{3}{2} W_1 = 13.5 \cdot K. \qquad (37a,b)$$

Thus, we may deduce that

$$F(1) = W_1 + \frac{W_1}{2} \Rightarrow F(1) = W_1 + A' \int_{-1}^{1} \frac{dZ}{\pi} \Rightarrow W_1 = 2A' \int_{-1}^{1} \frac{dz}{\pi}. \qquad (38)$$

According to Schwarz reflection principle, we infer

$$\int_{-1}^{1} \frac{dz}{P} = 2 \int_{-1}^{0} \frac{dz}{\pi}. \qquad (39)$$

Equation (38) can be combined with (39) to yield

$$W_1 = 4A' \int_{-1}^{0} \frac{dz}{\pi}. \tag{40}$$

Thus, we may derive the following relationships:

$$W_1 = -4A' \int_{0}^{-1} \frac{dz}{\pi} = -5A' \int_{0}^{-1} \frac{dz}{\pi} + A' \int_{0}^{-1} \frac{dz}{\pi}$$

$$\Rightarrow W_1 = -5A' \int_{0}^{-1} \frac{dz}{\sqrt{(1 - Z^2)(1 - Z^2 \lambda^2)}}$$

$$+ A' \int_{0}^{-1} \frac{dz}{\sqrt{(1 - Z^2)(1 - Z^2 \lambda^2)}} \tag{41}$$

and

$$F(z) = \frac{5}{4} W_1 + A' \int_{0}^{z} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2 \lambda^2)}}$$

$$\Rightarrow F(z) = 11.25K + A' \int_{0}^{z} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2 \lambda^2)}} = G(z, \lambda). \tag{42}$$

One may observe that the function $G(z, \lambda)$ constitutes an elliptic integral of the first kind. In the sequel, we may write out:

$$G\left(\frac{1}{\lambda}, \lambda\right) = 11.25\text{K} + A' \int_{0}^{1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2 \lambda^2)}}$$

$$\Rightarrow G\left(\frac{1}{\lambda}, \lambda\right) = 11.25K + A' \int_{0}^{1} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2 \lambda^2)}}$$

$$+ A' \int_{1}^{1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2 \lambda^2)}}$$

$$\Rightarrow G\left(\frac{1}{\lambda}, \lambda\right) = 11.25K + \frac{9}{4} K + A' \int_{1}^{1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2 \lambda^2)}}$$

$$\Rightarrow G\left(\frac{1}{\lambda}, \lambda\right) = 13.5K + A' \int_{1}^{1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2 \lambda^2)}} \tag{43}$$

Nonetheless the following relationship holds:

$$G\left(\frac{1}{\lambda}, \lambda\right) = F\left(\frac{1}{\lambda}\right) \Rightarrow G\left(\frac{1}{\lambda}, \lambda\right) = F(x_4) = W_4 = 9\text{K} + i \cdot 4.5\text{K}. \tag{44}$$

Hence

$$A' \int_1^{1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2\lambda^2)}} = -4.5\text{K} + i \cdot 4.5\text{K}. \tag{45}$$

Besides

$$G\left(-\frac{1}{\lambda}, \lambda\right) = 11.25\text{K} + A' \int_0^{-1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2\lambda^2)}}$$

$$\Rightarrow G\left(-\frac{1}{\lambda}, \lambda\right) = 11.25\text{K} + A' \int_0^{1} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2\lambda^2)}}$$

$$+ A' \int_1^{-1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2\lambda^2)}}$$

$$\Rightarrow G\left(-\frac{1}{\lambda}, \lambda\right) = 13.5\text{K} + A' \int_1^{-1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2\lambda^2)}}. \tag{46}$$

Concurrently, the following relationship holds:

$$G\left(-\frac{1}{\lambda}, \lambda\right) = F\left(-\frac{1}{\lambda}\right) \Rightarrow G\left(-\frac{1}{\lambda}, \lambda\right) = F(x_3) = W_3 = 13.5\text{K} + i \cdot 4.5\text{K}. \tag{47}$$

Then

$$A' \int_1^{-1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2\lambda^2)}} = i \cdot 4.5\text{K} \tag{48}$$

and therefore

$$\int_1^{1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2\lambda^2)}} = (i + 1) \cdot \int_1^{-1/\lambda} \frac{dz}{\sqrt{(1 - z^2)(1 - z^2\lambda^2)}}. \tag{49}$$

According to equation (49), the parameter λ can be evaluated via numerical approximation. Alternatively, this parameter may be estimated via the expansion of these holomorphic functions into Taylor series. Thus, we can write out:

$$\frac{1}{\sqrt{1-z^2}} = 1 + \frac{1}{2} \cdot z^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot z^6 + \cdots$$

$$\approx 1 + 0.5 \cdot z^2 + 0.375 \cdot z^4 + 0.3125 \cdot z^6 \tag{50a}$$

and

$$\frac{1}{\sqrt{1-z^2 \cdot \lambda^2}} = \frac{1}{\sqrt{1-(z \cdot \lambda)^2}} = 1 + \frac{1}{2} \cdot z^2 \cdot \lambda^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot z^4 \cdot \lambda^4$$

$$+ \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot z^6 \cdot \lambda^6 + \cdots$$

$$\Rightarrow (1 - z^2 \cdot \lambda^2)^{-0.5}$$

$$\approx (1 + 0.5 \cdot z^2 \cdot \lambda^2 + 0.375 \cdot z^4 \cdot \lambda^4 + 0.3125 z^6 \cdot \lambda^6). \tag{50b}$$

Hence, it follows:

$$\frac{1}{\sqrt{(1-z^2)(1-z^2\lambda^2)}} = \frac{1}{\sqrt{(1-z^2)(1-(z \cdot \lambda)^2)}}$$

$$= \frac{1}{\sqrt{1-z^2}} \cdot \frac{1}{\sqrt{1-(z \cdot \lambda)^2}}$$

$$\approx (1 + 0.5 \cdot z^2 + 0.375 \cdot z^4 + 0.3125 \cdot z^6)$$

$$\cdot (1 + 0.5 \cdot z^2 \cdot \lambda^2 + 0.375 \cdot z^4 \cdot \lambda^4 + 0.3125 z^6 \cdot \lambda^6). \tag{50c}$$

Conclusively

$$\int \frac{dz}{\sqrt{(1-z^2)(1-z^2\lambda^2)}}$$

$$= \int (1 + 0.5 \cdot z^2 + 0.375 \cdot z^4 + 0.3125 z^6)$$

$$\cdot (1 + 0.5 \cdot z^2 \cdot \lambda^2 + 0.375 \cdot z^4 \cdot \lambda^4 + 0.3125 z^6 \cdot \lambda^6) dz. \tag{51}$$

Thus, we may approximately estimate the integrals in equation (51) and then calculate the parameter λ.

Conclusions

A rigorous mathematical formulation of an incompressible two-dimensional groundwater creeping flow field in a porous medium, past a fully constrained impermeable obstacle was performed. The boundary of this obstacle was considered as a smooth surface from Hydrodynamics viewpoint. In this framework, an analytic calculation of the piezometric head was presented. Also, the position of velocity distribution local extrema was estimated. Nonetheless, as a future work, we may implement this method with the concurrent consideration of boundary roughness of the obstacle in the sense of Reference [17].

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