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REGULARIZATION OF A SYSTEM OF THE FIRST KIND VOLTERRA INCORRECT TWO-DIMENSIONAL EQUATIONS

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Abstract

In this paper, we study a system of the first kind Volterra incorrect integral equations. On the basis of the developed method of asymptotic nature with a singular function with respect to a small parameter, the regularizability and uniqueness of the solution of the original system in the introduced space are proved.

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We note that this system is degenerated in many ill-posed inverse problems of mathematical physics, for example, in problems of integral geometry, thermal conductivity, moisture transfer in soil, hereditary environment, filtration [2-4, 9], etc., that is the relevance of this paper.

1. Introduction

In the theory of the first kind integral equations (IE-1), various cases of the regularization method (RM) are connected with kernel of given equations [1, 5-8, 10] and others.

RM variants that allow to construct special solutions in certain spaces, where special functions are taken into account, which have singularities with a relatively small parameter [5, 7, 8, and others] are of great importance. However, investigations on nonlinear IE-1 do not have general methods yet.

Therefore, we study the IVE-1 system with a special solution, and in order to prove regularizability in the introduced space $Z_n^2(D_1)$, the RM is used in the proposed one in the work.

2. Formulation of the Problem

Let IVE-1 be given by the vector-matrix notation:

$$H\theta \equiv \int_0^x \int_0^b K(x, y, \tau, \nu)\theta^2(\tau, \nu)d\nu d\tau = F(x, y), \tag{1}$$

where

$$\begin{cases} C_{n\times n}(D_0) \ni K(x, y, \tau, \nu): \\ \| K(.)\| \le C_{01}, (D_0 = [0, X]\times[0, b]\times\{0 \le \tau \le x \le X, 0 \le v \le y \le b\}; K(.) \ge 0, \\ \int_0^b K(0, y, 0, \nu)d\nu \ne 0, \forall y \in [0, b]; (x, y) \in \overline{D}_1, (D_1 = (0, X)\times(0, b)), \\ C_n(\overline{D}_1) \ni F(x, y); F(0, y) \ne 0, \| F(0, y)\| \le C_{02}, \forall y \in [0, b], (y_0 \in [0, b]), \end{cases} \tag{2}$$

with known F, K, i.e., n-dimensional vector-functions (column), and an

$n \times n$ matrix function. For an $n \times n$-dimensional matrix A, the norm is defined as

$$\| A \| = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} | a_{ij} |^2 \right\}^{\frac{1}{2}},$$

while for an n-dimensional vector $u \in R^n$ as

$$\| u \| = \left\{ \sum_{i=1}^{n} | u_i |^2 \right\}^{\frac{1}{2}}.$$

Then under the above conditions, θ is an unknown n-dimensional vector function on $Z_n^2(D_1)$.

Here $Z_n^2(D_1)$ is a space of all n-dimensional vector-functions with components in $Z^2(D_1)$, where $Z^2(D_1)$ is the space whose elements are all piecewise continuous functions with a finite number of the first kind discontinuities and square-summarizable in $\overline{D}_1$ functions, as well as generalized functions $z(x, y)$ concentrated at the origin by argument x on $[0, X]$, with condition:

$$\sup_{y \in [0, b]} \int_0^X z^2(\tau, y) d\tau < \infty.$$

It is known that under the condition (2), the system (1) is Adamar incorrect [9], i.e., has no solution in $C_n(\overline{D}_1)$.

3. Regularizing Algorithm for the System (1)

In order to determine a unique solvability and regularizability of the system (1) in $Z_n^2(D_1)$, first, we transform (1) into the form:

$$(H\theta)(x, y_0) \equiv \int_0^x \int_0^b K(x, \tau, y_0, \nu) \theta^2(\tau, \nu) d\nu d\tau = F(x, y_0) \qquad (3)$$

and assuming that conditions (2) and (3) are satisfied, we suppose that

$H_0(x, y_0)$ is a diagonal matrix, where

$$\begin{cases} H_0(x, y_0) \equiv G(x)F(x, y_0);\ G(x) \equiv \gamma + \frac{1}{\alpha}\lambda(x), \\ H_0(x, y_0) = diag(H_{01}(x, y_0), \ldots, H_{0n}(x, y_0)), \\ H_{0i}(x, y_0) \equiv \left[\gamma + \frac{1}{\alpha}\lambda_i(x)\right]F_i(x, y_0),\ (i = \overline{1, n};\ 1 < \gamma = const), \\ 0 < \lambda_0(x) = \min\limits_{1 \le i \le n} \lambda_i(x) \in L^1(0, X);\ \phi(x) = \int_0^x \lambda_0(\tau)d\tau, \\ \min\limits_{1 \le i \le n} F_i(x, y_0) = \tilde{F}(x, y_0) \ge \alpha > 0,\ \forall x \in [0, X], \\ F_0(x, y) \equiv F(x, y) - F(0, y);\ \| F_0(x, y) \| \le C_{03},\ \forall (x, y) \in \overline{D}_1, \\ \| F_0(x, y) - F_0(\tau, y) \| \le L_{F_0} | x - \tau |;\ \| H_0(x, y_0) \| \le C_{04}h(x), \\ \| G(x) \| \le C_{05}h(x);\ h_0(x) \equiv \gamma + \frac{1}{\alpha}\lambda_0(x);\ h(x) \equiv h_0(x)\tilde{F}(x, y_0), \\ (0 < \max(C_{01}\sqrt{n}, C_{02}\sqrt{n}, C_{0j}) = \tilde{C}_1 = const,\ j = 3, 4, 5), \\ C_1 = \max(1, \sqrt{n^m} C_{01}\tilde{C}_1^k),\ (m = \overline{0, 3};\ k = \overline{1, 5}), \\ \phi_0(x) = \int_0^x h(\tau)d\tau = \int_0^x \left[\gamma + \frac{1}{\alpha}\lambda_0(\tau)\right]\tilde{F}(\tau, y_0)d\tau. \end{cases} \tag{4}$$

Here $0 < L_{F_0}$ is a Lipschitz coefficient of the function F_0. Then, carrying out mathematical operations with respect to the system (1) based on the operator H_0 given by the formula:

$$H_0\theta \equiv \int_0^x H_0(\tau, y_0)\theta(\tau, y)d\tau, \tag{$*$}$$

the specified system is equivalently converted into the form:

$$\begin{cases} \int_0^x H_0(\tau, y_0)\theta(\tau, y)d\tau = (Q\theta)(x, y) + F(x, y), \\ Q\theta \equiv (\tilde{Q}\theta)(x, y) + (H\theta)(x, y), \\ \tilde{Q}\theta \equiv \int_0^x G(\tau)(Q_0\theta)(\tau, y)d\tau, \end{cases} \tag{5}$$

with the vector-function

$$Q_0\theta = colon\{Q_{01}\theta, \ldots, Q_{0n}\theta\};\ Q_{0i}\theta \equiv \theta_i(x, y)(H_i\theta)(x, y_0),\ (i = \overline{1, n}),$$

$$(H\theta)\,(x,\ y_0) \equiv \int_0^x\int_0^b K(x,\ \tau,\ y_0,\ \nu)\theta^2(\tau,\ \nu)d\nu d\tau;\ \theta \in R^n.$$

From the system (5), it is evident that for the matrix function $H_0(x,\ y_0)$, by means of (4), there exist eigenvalues $\lambda_i(x)$, such that

$$0 < \lambda_0(x) = \min\{\lambda_i(x) | i = 1, ..., n\}.$$

Further, since in the IVE-1 theory, one of the possible methods is RM, we introduce a singular system in the form:

$$\begin{cases} \varepsilon\theta_\varepsilon(x,\ y) + (\Phi\theta_\varepsilon)(x,\ y) = F_\varepsilon(x,\ y), \\ (\Phi\theta_\varepsilon)(x,\ y) \equiv \int_0^x H_0(\tau,\ y_0)\theta_\varepsilon(\tau,\ y)d\tau - (Q\theta_\varepsilon)(x,\ y), \end{cases} \tag{6}$$

having the feature of a relatively small parameter, where the condition is allowed to be:

$$\begin{cases} \theta_\varepsilon(0,\ y) = \dfrac{1}{\varepsilon}F(0,\ y), \\ C_n(\overline{D}_1) \ni F_\varepsilon(x,\ y) : \| F_\varepsilon(x,\ y) - F(x,\ y) \|_{C_n} \le \Delta_0(\varepsilon), \\ F_\varepsilon(0,\ y) \equiv F(0,\ y). \end{cases} \tag{7}$$

We are looking for a solution to this system according to the rule:

$$\begin{cases} \theta_\varepsilon(x,\ y) = \dfrac{1}{\varepsilon}\prod_\varepsilon(x,\ y) + \upsilon(x,\ y) + \xi_\varepsilon(x,\ y), \\ \prod_\varepsilon(0,\ y) = F(0,\ y),\ \upsilon(0,\ y) = 0,\ \xi_\varepsilon(0,\ y) = 0. \end{cases} \tag{8}$$

In this case, with respect to unknown vector functions, respectively, we obtain the following systems:

$$\prod_\varepsilon(x,\ y) = -\frac{1}{\varepsilon}\int_0^x H_0(\tau,\ y_0)\prod_\varepsilon(\tau,\ y)d\tau + F(0,\ y), \tag{9}$$

$$\begin{cases} \int_0^x H_0(\tau,\ y_0)\upsilon(\tau,\ y)d\tau = (Q\upsilon)(x,\ y) + F_0(x,\ y), \\ F_0 \equiv F(x,\ y) - F(0,\ y), \end{cases} \tag{10}$$

$$\varepsilon\xi_\varepsilon + \int_0^x H_0(\tau,\ y_0)\xi_\varepsilon(\tau,\ y)d\tau = \left(Q\left[\frac{1}{\varepsilon}\prod\nolimits_\varepsilon + \upsilon + \xi_\varepsilon\right]\right)(x,\ y)$$
$$- (Q\upsilon)(x,\ y) + F_\varepsilon(x,\ y)$$
$$- F(x,\ y) - \varepsilon\upsilon(x,\ y). \quad (11)$$

Here

(a) $\prod_\varepsilon(x,\ y)$ is a solution of the system (9), which redefines a special vector function $\Omega_\varepsilon(x,\ y)$ with the condition

$$\| \Omega_\varepsilon(x,\ y) \| \xrightarrow[\varepsilon\to 0]{} \begin{cases} 0, & x \neq 0, \\ \infty, & x = 0; \end{cases} \quad (**)$$

(b) $\upsilon(x,\ y)$ is the solution of modified degenerate system (10), where the free term at the beginning of the segment $[0,\ X]$ vanishes. Moreover, the system (10) is regularizable in $C_n(\overline{D}_1)$;

(c) $\xi_\varepsilon(x,\ y)$ is defined uniquely from the system (11), which converges to zero in the sense of $C_n(\overline{D}_1)$, when small parameter $\varepsilon \to 0$.

In order to show in what sense the system (5) is regularizable, we first prove the conditions of items (a, b, c).

(a) Indeed, since the matrix Cauchy function $W(x,\ y_0,\ 0,\ \varepsilon)$ of the system

$$U_x(x,\ y) = -\frac{1}{\varepsilon}H_0(x,\ y_0)U(x,\ y),$$

and, by virtue of the Vazhevsky inequality, satisfies the estimate:

$$\| W(x,\ y_0,\ 0,\ \varepsilon) \| \leq \sqrt{n}\exp\left(-\frac{1}{\varepsilon}\phi_0(x)\right), \quad (12)$$

from the system (9), by (12), it follows that

$$\begin{cases} \prod_{\varepsilon}(x,\ y) = W(x,\ y_0,\ 0,\ \varepsilon)F(0,\ y), \\ \left\| \prod_{\varepsilon}(x,\ y) \right\| \leq C_{02}\sqrt{n}\exp\left(-\frac{1}{\varepsilon}\phi_0(x)\right) \leq C_1\exp\left(-\frac{1}{\varepsilon}\phi_0(x)\right). \end{cases} \tag{13}$$

It means that for the vector-function $\Omega_{\varepsilon}(x,\ y),\ (**)$ takes place.

(b) On the other hand, since the vector function $\upsilon(x,\ y)$ is a solution of the system (10), it means that the approximation to this solution under certain conditions can be the solution of the following system with a small parameter of the form:

$$\delta\upsilon_{\delta}(x,\ y) + \int_0^x h(\tau)\upsilon_{\delta}(\tau,\ y)d\tau = (Q\upsilon_{\delta})(x,\ y) + F_0(x,\ y). \tag{14}$$

Lemma 1. *Under the assumed conditions*, *the system* (10) *has a solution with conditions* (2), (4) *and* (8), *while the solution of the system* (14) *converges uniformly to the solution* (10), *when* $\delta \to 0$, *i.e.*,

$$\| \upsilon_{\delta}(x,\ y) - \upsilon(x,\ y) \|_{C_n} = \| \mu_{\delta}(x,\ y) \|_{C_n} \leq (1 - L_P)^{-1}\beta\delta. \tag{15}$$

Remark 1. The proof of Lemma 1 is based on the following conditions:

$$\begin{cases} 0 < L_P = \sqrt{n}C_1\Big\{(2+e^{-1})\dfrac{1}{\alpha}bX(\tilde{r}_1 + r_2)(2\tilde{r}_1 + r_2) + (2+e^{-1}) \\ \qquad \times \dfrac{1}{\alpha}bX\tilde{r}_1^{\,2} + \left[\dfrac{1}{\alpha\gamma}2(1+L_K X) + 4e^{-2}M_2\right]b(2\tilde{r}_1 + r_2)\Big\} < 1, \\ \upsilon_\delta(x,\ y) = \mu_\delta(x,\ y) + \upsilon(x,\ y), \\ \mu_\delta(x,\ y) = -\dfrac{1}{\delta^2}\displaystyle\int_0^x W(x,\ y_0,\ \tau,\ \varepsilon)H_0(\tau,\ y_0)\{(Q[\upsilon+\mu_\delta])(\tau,\ y) \\ \qquad - (Q\upsilon)(\tau,\ y) - (Q[\upsilon+\mu_\delta])(x,\ y) + (Q\upsilon)(x,\ y)\}d\tau \\ \qquad + \dfrac{1}{\delta}W(x,\ y_0,\ 0,\ \varepsilon)\{(Q[\upsilon+\mu_\delta])(x,\ y) - (Q\upsilon)(x,\ y)\} + \Delta(\delta,\ \upsilon) \\ \quad \equiv (P\mu_\delta)(x,\ y) + \Delta(\delta,\ \upsilon), \\ \Delta(\delta,\ \upsilon) \equiv -\dfrac{1}{\delta}\displaystyle\int_0^x W(x,\ y_0,\ \tau,\ \varepsilon)H_0(\tau,\ y_0)[-\upsilon(\tau,\ y) + \upsilon(x,\ y)]d\tau \\ \qquad - W(x,\ y_0,\ 0,\ \varepsilon)\upsilon(x,\ y), \\ \|\Delta(\delta,\ \upsilon)\|_{C_n} \le L_\upsilon\sqrt{n}\Big\{C_{04}\displaystyle\int_0^x (x-\tau)\exp\Big(-\dfrac{1}{\delta}(\phi_0(x) - \phi_0(\tau))\Big) \\ \qquad \times d\Big[-\dfrac{1}{\delta}(\phi_0(x) - \phi_0(\tau))\Big] + x\exp\Big(-\dfrac{1}{\delta}\phi_0(x)\Big)\Big\} \le \beta\delta, \\ L_\upsilon\dfrac{1}{\gamma\alpha}\sqrt{n}C_1\Big\{\displaystyle\int_0^\infty e^{-z}zdz + e^{-1}\Big\} \le 2C_1\sqrt{n}L_\upsilon\dfrac{1}{\gamma\alpha} = \beta,\ (0 < L_\upsilon), \\ \|\upsilon\| \le \tilde{r}_1,\ \forall(x,\ y) \in \overline{D}_1;\ \|\upsilon(x,\ y) - \upsilon(\bar{x},\ y)\| \le L_\upsilon|x - \bar{x}|, \\ S_{r_1}(0) = \{\upsilon_\delta(x,\ y) \in C_n(\overline{D}_1) : \|\upsilon_\delta(x,\ y)\| \le r_1,\ \forall(x,\ y) \in \overline{D}_1\}, \\ S_{r_2}(0) = \{\mu_\delta(x,\ y) \in C_n(\overline{D}_1) : \|\mu_\delta(x,\ y)\| \le r_2,\ \forall(x,\ y) \in \overline{D}_1\}, \\ \rho \equiv \dfrac{1}{\delta}\phi_0(x);\ \chi(\rho) = \rho^k\exp(-\rho),\ \Big(k = 1,\ 2,\ \dfrac{7}{2}\Big), \\ \sup\limits_{\rho\ge 0}\chi(\rho) = k^k\exp(-k);\ \rho = 0 : \chi(0) = 0,\ \rho \to \infty : \chi \to 0, \end{cases}$$

$$\begin{cases} \text{for example, } \lambda(x) \equiv \frac{1}{4\sqrt[4]{x^3}};\ h(x) = \left[\gamma + \frac{1}{\alpha}\lambda_0(x)\right]\tilde{F}(x,\ y_0): \\ x - \tau \le \frac{1}{\gamma\alpha}\int_\tau^x \left[\gamma + \frac{1}{\alpha}\lambda(\tau)\right]\tilde{F}(\tau,\ y_0)d\tau = M_0(\phi_0(x) - \phi_0(\tau)),\ (\tau \le x), \\ x \in [0,\ X]:\ x = (\sqrt[4]{x})^{\frac{7}{2}}(\sqrt[4]{x})^{\frac{1}{2}} \le M_1(\phi_0(x))^{\frac{7}{2}}, \\ \text{or}:\ x \le M_2(\phi_0(x))^2,\ (M_1 = X^{\frac{1}{8}};\ M_2 = X^{\frac{1}{8}}(\sqrt[4]{X})^{\frac{3}{2}} = X^{\frac{1}{2}}), \\ \gamma > 1;\ M_0 = \frac{1}{\gamma\alpha};\ \chi \equiv \rho^k \exp(-\rho). \end{cases} \tag{16}$$

(c) In order to define the vector-function at the beginning, the system (11) is converted to the form:

$$\begin{cases} \xi_\varepsilon(x,\ y) = -\frac{1}{\varepsilon^2}\int_0^x W(x,\ y_0,\ \tau,\ \varepsilon)H_0(\tau,\ y_0) \\ \qquad \times\left\{\left(Q\left[\upsilon + \xi_\varepsilon + \frac{1}{\varepsilon}\prod\nolimits_\varepsilon\right]\right)(\tau,\ y) - (Q\upsilon)(\tau,\ y)\right. \\ \qquad \left. -\left(Q\left[\upsilon + \xi_\varepsilon + \frac{1}{\varepsilon}\prod\nolimits_\varepsilon\right]\right)(x,\ y) + (Q\upsilon)(x,\ y)\right\}d\tau \\ \qquad + \frac{1}{\varepsilon}W(x,\ y_0,\ 0,\ \varepsilon)\left\{\left(Q\left[\upsilon + \xi_\varepsilon + \frac{1}{\varepsilon}\prod\nolimits_\varepsilon\right]\right)(x,\ y) - (Q\upsilon)(x,\ y)\right\} \\ \qquad + \Delta_1(\varepsilon,\ F_\varepsilon,\ F) + \Delta(\varepsilon,\ \upsilon) \equiv P_0\xi_\varepsilon + \Delta_0(\varepsilon,\ F_\varepsilon,\ F) + \Delta(\varepsilon,\ \upsilon), \end{cases}$$

(17)

where

$$\begin{cases} \Delta(\varepsilon, \upsilon) \equiv -\frac{1}{\varepsilon}\int_0^x W(x, y_0, \tau, \varepsilon) H_0(\tau, y_0) \\ \qquad \times [-\upsilon(\tau, y) + \upsilon(x, y)] d\tau - W(x, y_0, 0, \varepsilon)\upsilon, \\ \Delta_1(\varepsilon, F_\varepsilon, F) \equiv -\frac{1}{\varepsilon^2}\int_0^x W(x, y_0, \tau, \varepsilon) H_0(\tau, y_0)(F_\varepsilon(\tau, y)) \\ \qquad - F(\tau, y)) d\tau + \frac{1}{\varepsilon}[F_\varepsilon(x, y) - F(x, y)], \\ \| \Delta(\varepsilon, \upsilon) \|_{C_n} \leq \sqrt{n} L_\upsilon \frac{1}{\gamma\alpha} \varepsilon C_1 \left\{ \int_0^\infty e^{-z} z dz + e^{-1} \right\} \leq \beta\varepsilon \text{ (see (16))}. \end{cases} \tag{18}$$

Further, since there are estimates of the form:

$$\begin{cases} (\mathrm{a}_4)\ \| \Delta_1(\varepsilon,\ F_\varepsilon,\ F) \| \leq C_1(\sqrt{n}+1)\frac{1}{\varepsilon}\Delta_0(\varepsilon);\ \left(\frac{1}{\varepsilon}\Delta_0(\varepsilon)\xrightarrow[\varepsilon\to 0]{} 0\right), \\ \left\| \frac{1}{\varepsilon^2}\int_0^x W(x,\ y_0,\ \tau,\ \varepsilon)H_0(\tau,\ y_0)\left\{\left(Q\left[\upsilon+\xi_\varepsilon+\frac{1}{\varepsilon}\prod_\varepsilon\right]\right)(\tau,\ y)\right.\right. \\ \left.\left. -(Q\upsilon)(\tau,\ y)-\left(Q\left[\upsilon+\xi_\varepsilon+\frac{1}{\varepsilon}\prod_\varepsilon\right]\right)(x,\ y)+(Q\upsilon)(x,\ y)\right\}d\tau\right\| \\ \leq \sqrt{n}C_1\left\{\frac{1}{\alpha}(\tilde{r}_1+\tilde{r}_2)\left[bX(2\tilde{r}_1+\tilde{r}_2)+2b\int_0^x\frac{1}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right]\right. \\ \times\|\xi_c\|_{C_n}+\frac{1}{\alpha}(\tilde{r}_1+\|\xi_\varepsilon\|_{C_n})\left[2\tilde{r}_1 b\int_0^x\frac{1}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}(\phi_0(\bar{\tau}))d\bar{\tau}\right.\right. \\ \left.+b\int_0^x\frac{1}{\varepsilon^2}\exp\left(-\frac{2}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right] \\ +\frac{1}{\alpha}b\left[2\tilde{r}_1\int_0^x\frac{1}{\varepsilon^2}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}+\int_0^x\frac{1}{\varepsilon^3}\exp\left(-\frac{2}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right] \\ +\frac{1}{\alpha}b\left[2\int_0^x\frac{1}{\varepsilon^2}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}+(2\tilde{r}_1+\tilde{r}_2)\int_0^x\frac{1}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right] \\ \times\|\xi_\varepsilon\|_{C_n}+\frac{1}{\alpha}b\tilde{r}_1^2\left[X\|\xi_\varepsilon\|_C+\int_0^x\frac{1}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right] \\ +L_K\frac{1}{\gamma\alpha}b\left\{\left[X(2\tilde{r}_1+\tilde{r}_2)+2\int_0^x\frac{1}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right]\|\xi_\varepsilon\|_{C_n}\right. \\ \left.+2\tilde{r}_1\int_0^x\frac{1}{\varepsilon}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}+\int_0^x\frac{1}{\varepsilon^3}\exp\left(-\frac{2}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right\} \\ +b\left\{\frac{1}{\gamma\alpha}(2\tilde{r}_1+\tilde{r}_2)\|\xi_\varepsilon\|_{C_n}+2\tilde{r}_1\int_0^x\frac{1}{\varepsilon^2}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right. \\ \left.\left.+2\int_0^x\frac{1}{\varepsilon^2}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\|\xi_\varepsilon\|_{C_n}+\int_0^x\frac{1}{\varepsilon^3}\exp\left(-\frac{1}{\varepsilon}\phi_0(\bar{\tau})\right)d\bar{\tau}\right\}\right\} \\ \leq T_3\sqrt{\varepsilon}+\tilde{q}_1\|\xi_\varepsilon\|_{C_n}, \end{cases} \tag{19}$$

and

$$\begin{cases} (a_5) \left\| \frac{1}{\varepsilon} W(x, y_0, 0, \varepsilon) \left\{ \left(Q\left[\upsilon + \xi_\varepsilon + \frac{1}{\varepsilon} \prod_\varepsilon \right] \right)(x, y) - (Q\upsilon)(x, y) \right\} \right\| \\ \leq \sqrt{n} C_1 \left\{ \frac{1}{\alpha} (\tilde{r}_1 + \tilde{r}_2) \left[bX(2\tilde{r}_1 + \tilde{r}_2) + 2b \int_0^x \frac{1}{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \right] \| \xi_\varepsilon \|_{C_n} \right. \\ + \left[2\tilde{r}_1 \int_0^x \frac{1}{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} + \int_0^x \frac{1}{\varepsilon^2} \exp\left(-\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \right] \\ \times \frac{b}{\alpha} (\| \xi_\varepsilon \|_{C_n} + \tilde{r}_1) + \frac{b}{\alpha \varepsilon^2} \left[2\tilde{r}_1 \int_0^x \exp\left(-\frac{1}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} + \int_0^x \frac{1}{\varepsilon} \exp\left(-\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \right] \\ + \frac{b}{\alpha} \left[\int_0^x \left(2\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} (2\tilde{r}_1 + \tilde{r}_2) \right) \exp\left(-\frac{1}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \right] \| \xi_\varepsilon \|_{C_n} \\ + \frac{1}{\alpha} b\tilde{r}_1^2 \left[X \| \xi_\varepsilon \|_{C_n} + \int_0^x \frac{1}{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \right] + \left[(2\tilde{r}_1 + \tilde{r}_2) \varepsilon^{\frac{5}{2}} \left(\frac{1}{\varepsilon} \phi_0(x) \right)^{\frac{7}{2}} \right. \\ \left. \times \exp\left(-\frac{1}{\varepsilon} \phi_0(x) \right) + 2\varepsilon^{\frac{3}{2}} \left(\frac{1}{\varepsilon} \phi_0(x) \right)^{\frac{7}{2}} \exp\left(-\frac{1}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \right] bM_1 \| \xi_\varepsilon \|_{C_n} \\ \left. + bM_1 \sqrt{\varepsilon} \left(\frac{1}{\varepsilon} \phi_0(x) \right)^{\frac{7}{2}} \exp\left(-\frac{1}{\varepsilon} \phi_0(x) \right) \right\} \leq \tilde{T}_3 \sqrt{\varepsilon} + \tilde{q}_2 \| \xi_\varepsilon \|_{C_n}, \\ \| \xi_\varepsilon(x, y) \| \leq \tilde{r}_2, \ \forall (x, y) \in \overline{D}_1, \end{cases} \tag{20}$$

where in the obtained estimates, the following facts were taken into account:

$$\begin{cases} \int_0^x \frac{1}{\varepsilon^3} \exp\left(-\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} = \frac{1}{\varepsilon^3} \bar{\tau} \exp\left(-\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) \Big|_0^x + \int_0^x \frac{1}{\varepsilon^3} \bar{\tau} \\ \times \exp\left(-\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) d\left(\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) \leq \frac{1}{\varepsilon^3} x \exp\left(-\frac{2}{\varepsilon} \phi_0(x) \right) + M_1 \frac{1}{\sqrt{2^7}} \sqrt{\varepsilon} \\ \times \int_0^x \left(\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right)^{\frac{7}{2}} \exp\left(-\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) d\left(\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \\ \leq M_1 \frac{1}{\sqrt{2^7}} \sqrt{\varepsilon} \left[\left(\frac{7}{2} \right)^{\frac{7}{2}} e^{-\frac{7}{2}} + \frac{105\sqrt{\pi}}{16} \right] = T_1 \sqrt{\varepsilon}, \\ \text{analogically:} \\ \int_0^x \frac{1}{\varepsilon} \exp\left(-\frac{1}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \leq M_1 \varepsilon^{\frac{5}{2}} \left[\left(\frac{7}{2} \right)^{\frac{7}{2}} e^{-\frac{7}{2}} + \frac{105\sqrt{\pi}}{16} \right] = T_2 \varepsilon^{\frac{5}{2}}, \\ \int_0^x \frac{1}{\varepsilon^2} \exp\left(-\frac{1}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \leq T_2 \varepsilon^{\frac{3}{2}}; \int_0^x \frac{1}{\varepsilon^2} \exp\left(-\frac{2}{\varepsilon} \phi_0(\bar{\tau}) \right) d\bar{\tau} \leq T_1 \varepsilon^{\frac{3}{2}}. \end{cases}$$

From the estimate of the system (17), it follows that:

$$\begin{cases} \| \xi_\varepsilon(x, y) \|_{C_m} \leq (1 - L_{P_0})^{-1} \left[\beta\varepsilon + \frac{2}{\varepsilon} \Delta_0(\varepsilon) + T_0 \sqrt{\varepsilon} \right] = \Delta_2(\varepsilon), \\ \frac{1}{\varepsilon} \Delta_0(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0; L_{P_0} = \tilde{q}_1 + \tilde{q}_2 < 1; T_0 = T_3 + \tilde{T}_3. \end{cases} \tag{21}$$

Lemma 2. *Under the conditions of Lemma* 1 *and* (4), (7), (13), (18), (21), *the system* (17) *is solvable in* $C_n(\overline{D}_1)$, *and as* $\varepsilon \to 0$, *it converges to zero in sense of* $C_n(\overline{D}_1)$.

Theorem 1. *If the conditions of Lemmas* 1 *and* 2 *are satisfied*, *then the solution of the system* (6) *can be uniquely represented in the form* (8). *Moreover*,

(a) $\left\| \prod_\varepsilon \right\|_{Z_n^2(D_1)} \leq \gamma_1 \varepsilon^{\frac{7}{4}}, \left\{ \gamma_1 = C_1 2^{-\frac{7}{4}} \sqrt{M_1 b \left[(2^{-1} \cdot 7)^{\frac{7}{2}} e^{-\frac{7}{2}} + \frac{105}{16} \sqrt{\pi} \right]} \right\},$

(b) $\| \theta_\varepsilon - \upsilon \|_{Z_n^2(D_1)} \leq 2[\Delta_2(\varepsilon) \sqrt{Xb} + \gamma_1 \varepsilon^{\frac{3}{4}}] = \tilde{M}_0(\varepsilon),$

(c) $\| \Phi\theta_\varepsilon - F \|_{Z_n^2(D_1)} \leq 4[\Delta_0(\varepsilon) \sqrt{Xb} + \varepsilon \tilde{M}_0(\varepsilon) + \varepsilon \tilde{r}_1 \sqrt{Xb}] = \tilde{M}(\varepsilon),$

where $\tilde{M}_0(\varepsilon), \tilde{M}(\varepsilon) \to 0$ *when* $\varepsilon \to 0$.

Proposition 1. *Under the conditions of Theorem* 1, *the system* (1) *is regularized according to the rule* (8) *in* $Z_n^2(D_1)$ *in a generalized sense*.

4. Conclusion

In this paper, we investigated a nonlinear IVE-1 system with a special solution in $Z_n^2(D_1)$. The solution to the original system is constructed using a special perturbation method, after transforming it on the basis of a modification of the method of integral operators with weighted functions [7]. At the same time, sufficient conditions for the solvability and regularizability of the system under study were revealed in $Z_n^2(D_1)$.

The results of the work can be used to inverse problems of mathematical physics, where nonlinear ill-posed IVE-1 systems of the specified class degenerate.

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