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IDENTIFICATION OF TWO PARAMETERS IN AN ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract

This paper concerns an inverse problem which consists in determining two coefficients b and c in the equation $-b(x)u'' + c(x)u' = f$, $x \in]0, 1[$, knowing the solution function u and the right-hand side function f. The questions of uniqueness and stability are investigated. This problem is solved by using the nonlinear least squares method. We present some numerical examples to illustrate our algorithm.

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1. Introduction

Consider the boundary value problem for $1D$ elliptic equation with the homogeneous Dirichlet conditions:

$$\begin{cases} Lu := -b(x)u''(x) + c(x)u'(x) = f(x), & x \in \,]0, 1[, \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where the coefficients (b, c) satisfy the conditions: $b \in C^1[0, 1]$; $b(x) \geq b_0 > 0$, $c \in C[0, 1]$ and the right-hand side $f \in L^2(0, 1)$. *The direct problem* is well-posed, i.e., there exists one solution $u \in H^2(0, 1) \cap H_0^1(0, 1)$ of (1) and is stable relative to the data (b, c, f). For fixed right-hand f and parameters $p = (b, c)$, we denote the solution u by $\Phi(p)$, Φ is called the *forward operator*.

The inverse problem is set as follows: given (u, f), determine the pair of parameters $p = (b, c)$. Such problem is formulated as a nonlinear equation $\Phi(p) = u$. It is well-known that such problem is ill-posed [3], that is the solution may be not unique and not stable. Many articles have studied the parameter identification problems. In [4], the author gives a condition that ensures the uniqueness in the problem of transmissivity parameter identification. In [2, 3], the authors developed an abstract framework for nonlinear ill-posed problems. They generalized the Newton-Kantorovich method for nonlinear equation $F(x) = y$, when $F : H_1 \to H_2$ acts between two Hilbert spaces and the derivative is not invertible or ill-conditioned. In [9], the approximate solution's stability for nonlinear ill-posed problems is established by giving conditions to improve the convergence rate.

Several numerical approaches have been proposed. The most common approach is to reformulate the inverse problem as a least squares problem which is solved by optimization methods using the gradient of the objective function. In the book [6], the author explores the problem of parameters estimation by setting it as a minimization problem involving several techniques to compute the gradient such as the adjoint state method and sensitivity functions. In [7], Knowles has identified the parameter p in the equation $-\nabla \cdot (p(x)\nabla u) = f$ in a bounded domain. The problem of identifying the parameter has many engineering applications like hydrology, geology and ecology. We must cite the paper [10] which is concerned with an inverse problem in groundwater hydrology, the author formulates the problem as a constrained minimization problem.

The paper is organized as follows: In Section 2, we solve the direct problem with finite element method. In Section 3, we consider the inverse problem. First, we prove a result on the stability of the solution. Next, we solve the inverse problem in the sense of least squares problem, we apply the Levenberg-Marquardt algorithm to recover the pair of parameters (b, c). In Section 4, we show some numerical examples to validate our algorithm. Finally, in Section 5, we give the conclusion.

2. Direct Problem

2.1. Well-posedness

We show that the direct problem is well-posed.

Let $V = H_0^1(0, 1)$ and $H = L^2(0, 1)$ be the Sobolev spaces. The weak form of problem (1) is: find $u \in V$, such that

$$a(u, v) = (f, v), \quad \forall v \in V, \tag{2}$$

where

$$a(u, v) = \int_0^1 [u'(x)v(x)(b'(x) + c(x)) + u'(x)v'(x)b(x)]dx$$

and

$$(f, v) = \int_0^1 f(x)v(x)dx.$$

The bilinear form $a(\cdot, \cdot)$ is continuous and H-coercive: There exist two constants $C_1 > 0$ and $C_2 > 0$ such that

$$| a(u, v) | \leq C_1 \| u \|_V, \quad \forall u, v \in V, \tag{3}$$

and

$$a(u, u) + \frac{C_2}{2} \| u \|_H^2 \geq b_0 \| u \|_V^2, \quad \forall u \in V. \tag{4}$$

Let A be the operator defined by the form $a(u, v)$. By Lax-Milgram's theorem, the operator $A_\lambda = A + \lambda I$ is invertible for $\lambda \geq \frac{C_2}{2}$, moreover A_λ^{-1} is a compact operator. Now we show that A is injective, that is the homogeneous problem

$$Lu = 0, \quad u(0) = u(1) = 0, \tag{5}$$

has only the trivial solution. For this, we put $u'(x) = v(x)$ in (1), then the solution of equation (5) is given by

$$u(x) = \int_0^x k_1 \exp\left(\int_0^s \frac{c(t)}{b(t)} dt \right) ds + k_2. \tag{6}$$

Using the boundary conditions, we get $u(x) = 0, \forall x \in (0, 1)$.

From the Fredholm's alternative, there exists a unique solution $u \in V$ of problem (2), which depends continuously on the data f, i.e.,

$$\| u \|_V \leq M \| f \|_H. \tag{7}$$

2.2. Discretization

We approximate the space V by a finite dimensional subspace $V_h \subset V$ defined by $V_h = \{v_h \in V \cap C[0, 1]$, such that $v_h |_{[x_i, x_{i+1}]} \in P_1$, $\forall i = 1 \cdots n - 1\}$. In our study, we consider the approximate solution u_h obtained by finite element method, that is, u_h is the solution of the variational problem: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{8}$$

The basis of V_h is given by the functions

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x \in [x_{i-1}, x_i], \\ \dfrac{x_{i+1} - x}{h} & \text{if } x \in [x_i, x_{i+1}], \\ 0 & \text{otherwise,} \end{cases}$$

where $\{x_i = ih\}, i = 1 \cdots n-1,$ is a subdivision of $[0, 1]$.

We denote by $X = (u_h(x_i))_{1 \le i \le n-1}$, $B = \left(\int_0^1 f(x) \phi_j(x) dx \right)_{1 \le j \le n-1}$ and the three-diagonal stiffness matrix $M = (M_{ij})_{1 \le i,\, j \le n-1}$, such that

$$M_{ij} = \begin{cases} -\dfrac{1}{h} b(x_i) - \dfrac{1}{6}(c(x_{i-1}) + 2c(x_i)), & \text{if } j = i - 1, \\ \dfrac{2}{h} b(x_i) + \dfrac{1}{6}(c(x_{i-1}) - c(x_{i+1})), & \text{if } j = i, \\ -\dfrac{1}{h} b(x_i) + \dfrac{1}{6}(2c(x_i) + c(x_{i+1})), & \text{if } j = i + 1. \end{cases} \tag{9}$$

Then, the weak formulation (8) leads to solving the linear system $MX = B$.

3. Inverse Problem

3.1. Uniqueness of the solution

For one observation (u, f), there exists more than one solution $p = (b, c)$. We can furnish the following example: $u(x) = x(1 - x)$, $f(x) = 2$.

- For $b_1 = 1, c_1 = 0,$ we have $L(u) = -u'' = f$.
- For $b_2 = \frac{1}{2}\left(x + \frac{3}{2} \right), c_2 = \frac{1}{2},$ we have $L(u) = -b_2 u'' + c_2 u' = f$.

In this paragraph, we consider the case of two observations (u_1, f_1) and (u_2, f_2). Our aim is to give a condition for which the solution of the inverse problem is unique. The equation $\Phi(p) = (u_1, u_2)$ has a unique solution

$p = (b, c)$ if and only if the linear system

$$\begin{cases} Lu_1 = -b(x)u_1''(x) + c(x)u_1'(x) = f_1, \\ Lu_2 = -b(x)u_2''(x) + c(x)u_2'(x) = f_2 \end{cases} \tag{10}$$

has a unique solution with respect to (b, c). The determinant of system (10) is given by

$$\Delta(x) = \frac{1}{b(x)}(f_1(x)u_2'(x) - f_2(x)u_1'(x)). \tag{11}$$

As a consequence, we have the following proposition:

Proposition 3.1. *Assume that* $f_1(x)u_2'(x) - f_2(x)u_1'(x) \neq 0$, $\forall x \in (0, 1)$. *Then, the equation* $\Phi(p) = (u_1, u_2)$ *has a unique solution* $p = (b, c)$.

3.2. Stability of the solution

First, we introduce some notations that will be used throughout this paper.

In what follows, we provide a condition for the stability of the inverse problem. For that, we introduce some notations.

(1) The parameter space:

$$M_{ad} = \{(b, c); b \in C^1[0, 1], b(x) \geq b_0 > 0, c \in C[0, 1]\}.$$

(2) Consider two pairs $p_1 = (b_1, c_1)$, $p_2 = (b_2, c_2) \in M_{ad}$.

(3) We assume that $f_1, f_2 \in L^\infty(0, 1)$.

(4) The forward operator $\Phi : M_{ad} \to Y \times Y$, where $Y = H^2(0, 1)$, is defined by $\Phi(p_1) = (u_1, u_2)$, $\Phi(p_2) = (v_1, v_2)$, with u_j (resp. v_j) solution of $-b_1 u_j'' + c_1 u_j' = f_j$ (resp. $-b_2 v_j'' + c_2 v_j' = f_j$), $j = 1, 2$.

(5) We set

$$\Delta_1(x) = f_1(x)u_2'(x) - f_2(x)u_1'(x) \quad \text{and} \quad \Delta_2(x) = f_1(x)v_2'(x) - f_2(x)v_1'(x).$$

Theorem 3.1. *Assume that there exists* $\alpha_1 > 0$ *such that*

$$|\Delta_1(x)| \geq \alpha_1 > 0, \quad \forall x \in [0, 1]. \tag{12}$$

Then we have the following estimate:

$$\| p_1 - p_2 \|_H \leq M(p_1) \| \Phi(p_1) - \Phi(p_2) \|_{Y \times Y}. \tag{13}$$

Proof. We set $w_1 = u_1 - v_1$ and $w_2 = u_2 - v_2$, such that $\| w_1 \|_{H^2} + \| w_2 \|_{H^2} \leq \eta$, then

$$\begin{aligned} |\Delta_2(x)| &= | f_1(x) v_2'(x) - f_2(x) v_1'(x) | = | f_2 w_1' - f_1 w_2' + \Delta_1(x) | \\ &\geq |\Delta_1(x)| - | f_2(x) || w_1'(x) | - | f_1(x) || w_2'(x) | \\ &\geq \alpha_1 - (\| f_1 \|_\infty + \| f_2 \|_\infty)(\| w_1' \|_\infty + \| w_2' \|_\infty) \\ &\geq \alpha_1 - C\eta(\| f_1 \|_\infty + \| f_2 \|_\infty) \text{ (since } \| w_j' \|_\infty \leq C \| w_j \|_{H^2}) \\ &\geq \alpha_2 > 0 \text{ (for } \eta > 0 \text{ small enough).} \end{aligned} \tag{14}$$

In this case, the equation $\Phi(p_2) = (v_1, v_2)$ has a unique solution.

From the system

$$\begin{cases} \Phi(p_1) = (u_1, u_2), \\ \Phi(p_2) = (v_1, v_2), \end{cases} \tag{15}$$

we obtain the following systems:

$$\begin{cases} -b_1 u_1'' + c_1 u_1' = f_1, \\ -b_1 u_2'' + c_1 u_2' = f_2; \end{cases} \quad \begin{cases} -b_2 v_1'' + c_2 v_1' = f_1, \\ -b_2 v_2'' + c_2 v_2' = f_2. \end{cases}$$

Combining these equations, we obtain the system

$$\begin{cases} (b_2 - b_1) v_1'' + (c_1 - c_2) v_1' = b_1 w_1'' - c_1 w_1', \\ (b_2 - b_1) v_2'' + (c_1 - c_2) v_2' = b_1 w_2'' - c_1 w_2'. \end{cases} \tag{16}$$

The determinant of system (16) is $-\dfrac{\Delta_2(x)}{b_2}$ (does not vanish by (14)). Therefore, we obtain the inversion formulas:

$$\begin{cases} \Delta b := b_2 - b_1 = -\dfrac{b_2}{\Delta_2(x)}[v_2'(b_1 w_1'' - c_1 w_1') - v_1'(b_1 w_2'' - c_1 w_2')], \\ \Delta c := c_2 - c_1 = \dfrac{b_2}{\Delta_2(x)}[-v_2''(b_1 w_1'' - c_1 w_1') + v_1''(b_1 w_2'' - c_1 w_2')]. \end{cases}$$

Using the equations $\Delta b w_1'' = \Delta c w_1'$ and $\Delta b w_2'' = \Delta c w_2'$, we deduce the system:

$$\begin{cases} (\Delta b)^2 = \dfrac{-b_2}{\Delta_2}(b_1 \Delta c - c_1 \Delta b)(v_2' w_1' - w_2' v_1'), \\ (\Delta c)^2 = \dfrac{-b_2}{\Delta_2}(b_1 \Delta c - c_1 \Delta b)(v_2'' w_1'' - v_1'' w_2''). \end{cases} \tag{17}$$

From (14) and (17), we obtain the estimations

$$\begin{cases} |\Delta b|^2 \le \dfrac{b_2}{\alpha_2}\sqrt{b_1^2 + c_1^2}\sqrt{\Delta c^2 + \Delta b^2}(|v_2'||w_1'| + |w_2'||v_1'|), \\ |\Delta c|^2 \le \dfrac{b_2}{\alpha_2}\sqrt{b_1^2 + c_1^2}\sqrt{\Delta c^2 + \Delta b^2}(|v_2''||w_1''| + |w_2''||v_1''|). \end{cases} \tag{18}$$

Therefore,

$$\begin{aligned}(|\Delta b|^2 + |\Delta c|^2)^{\frac{1}{2}} \le \frac{b_2}{\alpha_2}\sqrt{b_1^2 + c_1^2}\Big(&\sqrt{|v_2'|^2 + |v_1'|^2}\sqrt{|w_2'|^2 + |w_1'|^2} \\ &+ \sqrt{|v_2''|^2 + |v_1''|^2}\sqrt{|w_2''|^2 + |w_1''|^2}\Big).\end{aligned} \tag{19}$$

From the stability of the direct problem, we have the estimation

$$\| v_j \|_{H^2(0,1)} \le C(p_2)\| f_j \|_\infty, \quad j = 1, 2,$$

which leads to

$$\begin{aligned}&\| \Delta b \|_H^2 + \| \Delta c \|_H^2 \\ &\le M(p_1)(\| f_1 \|_\infty + \| f_2 \|_\infty)^2(\| w_1 \|_{H^2(0,1)} + \| w_2 \|_{H^2(0,1)})^2.\end{aligned}$$

Remark. Theorem 3.1 means that the operator $\Phi : p \mapsto u = (u_1, u_2)$ from $M_{ad} \subset H \times H$ to $Y \times Y$ is invertible in a neighborhood of p_1. Moreover, Φ^{-1} is continuous (locally Lipschitz).

Remark. When $p = (b, c) \in \mathbb{R}_+^* \times \mathbb{R}$, one observation suffices to have uniqueness. We can prove that $\Phi(p)$ is injective: If $-b_1 u'' + c_1 u' = -b_2 u'' + c_2 u' = f$, then $(b_1 - b_2) u'' = (c_1 - c_2) u'$ and from $u(0) = u(1) = 0$, it follows that $b_1 = b_2$ and $c_1 = c_2$.

3.3. Fundamental spaces

(1) Model space:

$$M_{ad} = \{p = (b, c) \in C^1[0, 1] \times C[0, 1], b(x) \geq b_0 > 0\}.$$

(2) State space: $V = H_0^1(0, 1)$.

(3) Data space: We consider two possibilities of the measurements:

3.1. One observation

We choose one distributed observation denoted by d, in other words, the state u is measured at all points of $(0, 1)$. Then $D = H$ and the observation operator is given by $Ku = d$. In this case, the inverse problem is formulated by the equation

$$\Phi(b, c) = d. \tag{20}$$

3.2. Two observations

We choose two distributed observations d_1, d_2, with $d_j = u_j$, such that u_j is the solution of the system $-bu_j'' + cu_j' = f_j$, $j = 1, 2$. Then $D = H \times H$ and the observation operator is given by $Ku = (d_1, d_2)$. In this case, the inverse problem is formulated by the system

$$\Phi_1(b, c) = d_1, \quad \Phi_2(b, c) = d_2. \tag{21}$$

$\Phi_j(p)$ is associated to f_j.

Remark. In the case of constant parameters, we can consider the observation at the boundary $d = (u'(0), u'(1))$. Indeed, the couple (b, c) must satisfy the following system:

$$\begin{cases} \dfrac{1}{b(\exp(\lambda)-1)}\displaystyle\int_0^1 [\exp(\lambda(1-y))-1]f(y)dy = u'(0), \\ -\dfrac{1}{b}\exp(\lambda)\displaystyle\int_0^1 \exp(-\lambda y)f(y)dy + \exp(\lambda)u'(0) = u'(1), \end{cases}$$

with $\lambda = \dfrac{c}{b}$. We do not study the resolution of this system here.

3.4. Least squares formulation

One of the most commonly used approaches for solving the inverse problem is by setting it as a least squares problem [1], whose solution is an approximation of the parameter $p = (b, c)$. In our work, we consider two cases:

(1) One observation: The cost functional is

$$J_1(p) = \frac{1}{2}\| \Phi(p) - d \|_H^2, \text{ for } p \in M_{ad}. \tag{22}$$

(2) Two observations: The cost functional is

$$J_2(p) = \frac{1}{2}(\| \Phi_1(p) - d_1 \|_H^2 + \| \Phi_2(p) - d_2 \|_H^2), \text{ for } p \in M_{ad}. \tag{23}$$

3.5. Derivative of $\Phi(p)$

For $p \in M_{ad}$, we define the unbounded operator $A(p) : \mathcal{D}(A(p)) \to H$:

$$\begin{cases} \mathcal{D}(A(p)) = H^2(0, 1) \cap H_0^1(0, 1), \\ A(p)\varphi = L\varphi, \quad L\varphi = -b(x)\varphi'' + c(x)\varphi'. \end{cases} \tag{24}$$

The operator $A(p)$ is invertible (the direct problem admits a unique solution (Subsection 2.1)) and $A(p)^{-1} \in L(H)$ is a compact operator.

Theorem 3.2. *The operator* Φ *is Fréchet-differentiable*, *the partial derivatives are given by*:

$$\begin{cases} \dfrac{\partial \Phi}{\partial b}(p; h) = A(p)^{-1}(hu''), \\ \dfrac{\partial \Phi}{\partial c}(p; k) = -A(p)^{-1}(ku'). \end{cases} \tag{25}$$

Proof. We suppose that $\Phi(p) = u(p)$ for one observation.

(1) Let h be an increment of b and δu be an increment of u. Then

$$-(b+h)(u+\delta u)'' + c(u+\delta u)' = f. \tag{26}$$

By developing equation (26), we obtain

$$-bu'' + cu' - b(\delta u)'' + c(\delta u)' - hu'' - h(\delta u)'' = f,$$

since $\|(\delta u)''\| = O(1)$, hence

$$L(\delta u) = hu'' + O(\|h\|),$$

which implies that

$$\delta u = A(p)^{-1}(hu'') + O(\|h\|).$$

Thus,

$$\frac{\partial \Phi}{\partial b}(p;\, h) = A(p)^{-1}(hu''). \tag{27}$$

(2) Let k be an increment of c and δu be an increment of u. Then

$$-b(u+\delta u)'' + (c+k)(u+\delta u)' = f. \tag{28}$$

Similarly, from (28), we obtain

$$(-bu'' + cu') + (-b(\delta u)'' + c(\delta u)') + ku' + k(\delta u)' = f.$$

It follows that

$$L(\delta u) = -ku' - k(\delta u)'.$$

Consequently,

$$\delta u = -A(p)^{-1}(ku') + O(\|k\|).$$

From this, we conclude that

$$\frac{\partial \Phi}{\partial c}(p;\, k) = -A(p)^{-1}(ku'), \tag{29}$$

which completes the proof.

The full derivative is

$$\Phi'(p) : H \times H \to H \times H$$

$$(h, k) \mapsto \Phi'(p)(h, k) = (A(p)^{-1}(hu''), -A(p)^{-1}(ku')).$$

3.6. Reconstruction algorithm

To solve least squares problem (23), we apply Newton's method, which consists of iterating the procedure:

(1) p_0 : initial approximation,

(2) $p_{n+1} = p_n + h_n$, where h_n is the solution of the linearized equation:

$$\Phi'(p_n)h_n = d - \Phi(p_n). \tag{30}$$

The operator $\Phi'(p) = \left(\frac{\partial \Phi}{\partial b}, \frac{\partial \Phi}{\partial c}\right)$ is compact from $H \times H$ to $H \times H$. Since $A(p)^{-1}$ is compact, equation (30) is ill-posed, which needs a strategy of regularization. For this, we use the Levenberg-Marquardt scheme [5, 8]:

$$\Phi'^{*}(p_n)\Phi'(p_n)h_n + \alpha_n h_n = \Phi'^{*}(p_n)(d - \Phi(p_n)), \tag{31}$$

where $\Phi'(p)$ is the Fréchet-derivative of Φ and $\Phi'^{*}(p)$ is the adjoint operator of $\Phi'(p)$.

4. Numerical Examples

In this section, we present numerical examples to illustrate the effectiveness of the recovery algorithm. To simulate noised data, we perturb the exact data u with the noise level δ, such that:

$$u^{\delta}(x) = u(x) + \delta\sigma(x),$$

where $\sigma(x)$ is the Gaussian random distribution.

In the first example (Test 1), we used $n = 80$ points (in the interval $[0, 1]$) and $n = 60$ for the second example (Test 2). All results are obtained after 10 iterations.

4.1. Two observations

Test 1. Let $b(x) = 1 + 0.5\sin(\pi x)$ and $c(x) = 1 + x - x^2$.

Test 1-a. We take two observations associated to the right-hand sides:

$$f_1(x) = \cos(\pi x) \quad \text{and} \quad f_2(x) = x - x^2.$$

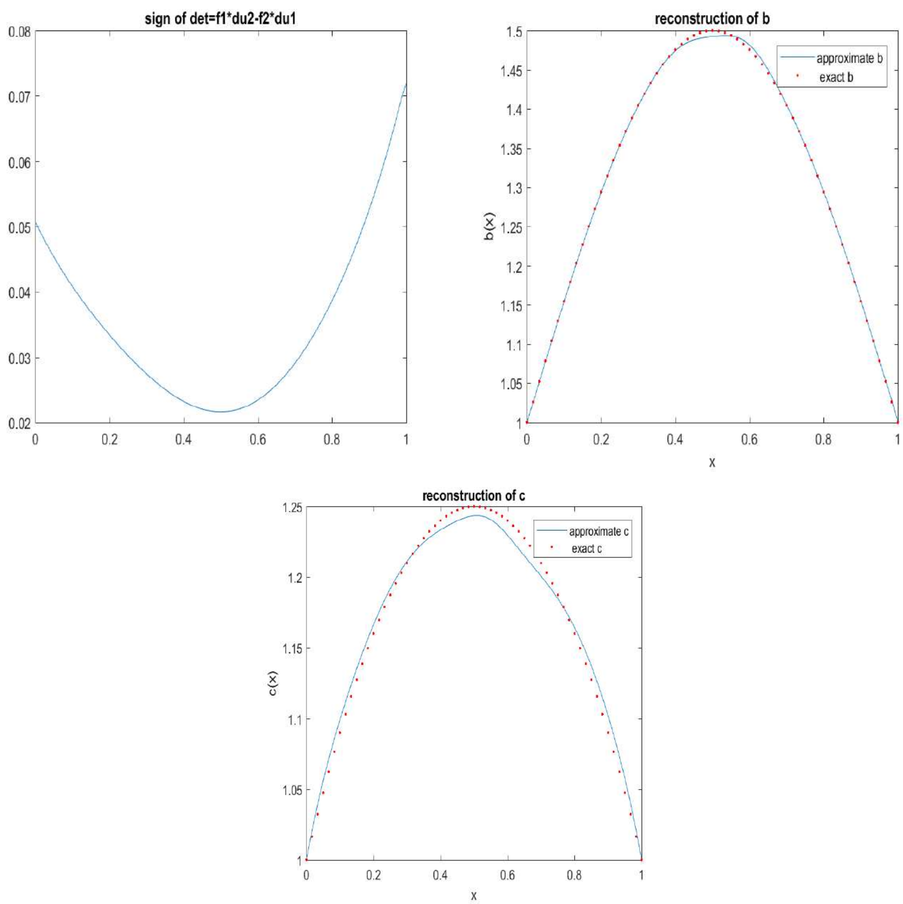

Figure 1. Test 1-a. Variation of $\Delta_1(x)$ and reconstruction of *b* and *c* without noise, initial guess $p_0 = (1, 1)$.

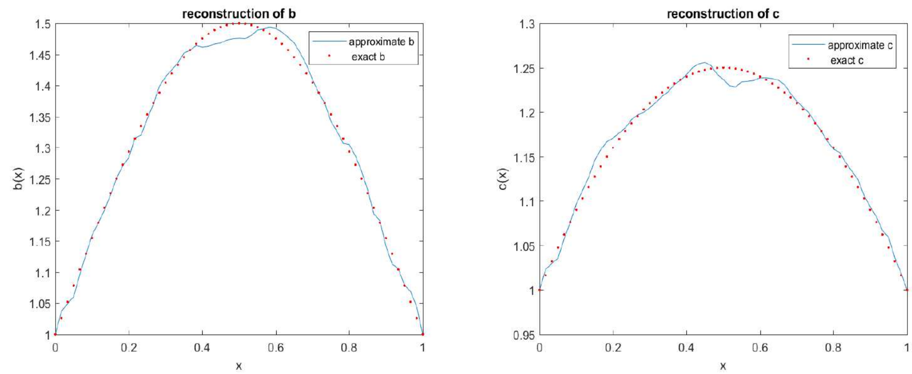

Figure 2. Test 1-a. Reconstruction of b and c with level noise $\delta = 10^{-5}$, $p_0 = (1, 1)$.

In this example (Test 1-a), we have uniqueness and stability which agree with Theorem 3.1.

Test 1-b. Now we take two observations with

$$f_1(x) = \begin{cases} x, & \text{if } x \leq 0.5, \\ 0.5, & \text{if } x > 0.5 \end{cases} \quad \text{and} \quad f_2 = x - x^2.$$

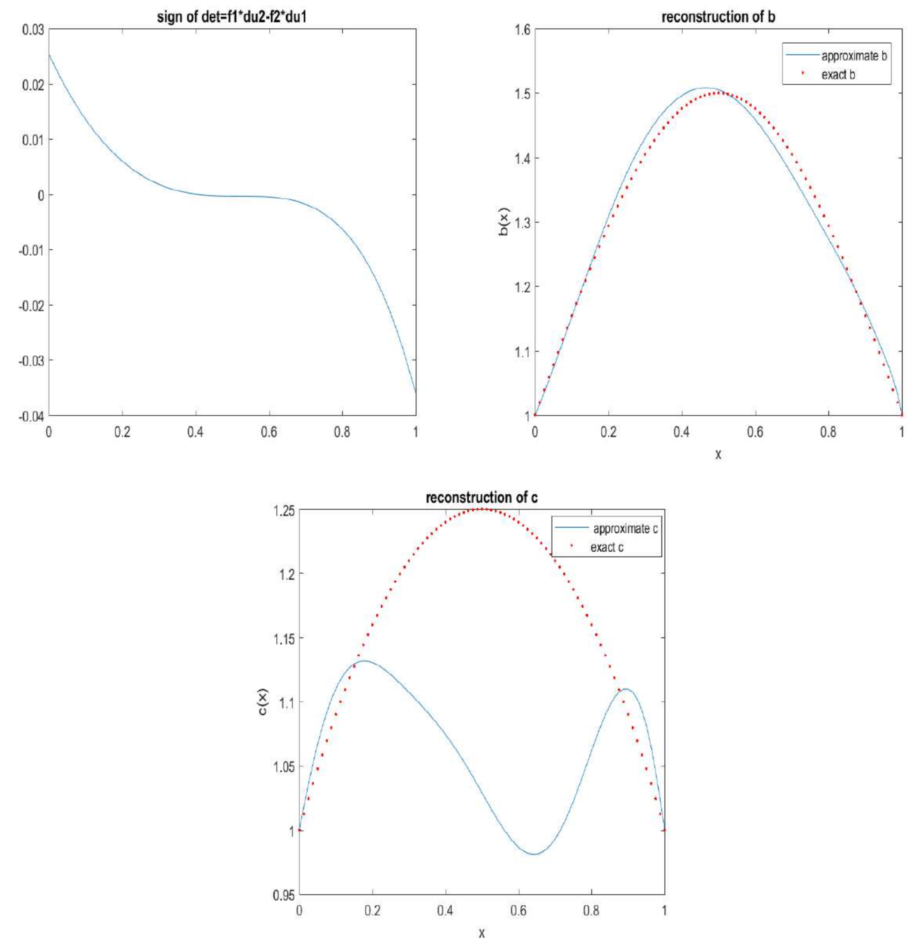

Figure 3. Test 1-b. Variation of $\Delta_1(x)$ and reconstruction of b and c, without noise, $p_0 = (1, 1)$.

In this example (Test 1-b), we lose the stability (of the parameter c), since $\Delta_1(x)$ vanishes at $x = 0.5$ and changes the sign.

4.2. One observation

Test 2. Let $b(x) = 1 + 0.5\sin(\pi x)$ and $c(x) = 1 + x - x^2$.

Test 2-a. We consider one observation with $f(x) = x - x^2 + \exp(x^2)$ and initial guess $p_0 = (1, 1)$.

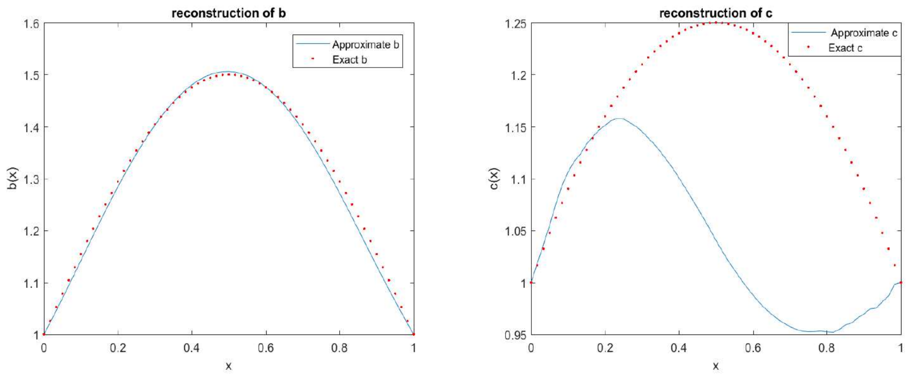

Figure 4. Test 2-a. Exact and regularized solution without noise.

In this example (Test 2-a), the solution is not unique $(c_{ap}$ approaches to another solution$)$. The approximate solution depends upon the initial guess (see [2, Theorem 2.4. source condition (iii)]).

Test 2-b. Now we consider the previous example $f(x) = x - x^2 + \exp(x^2)$, with initial guess $b_0 = 1$ and c_0 satisfying the source condition

$$c_0 = c^\star - \left(\frac{\partial\Phi(c^\star)}{\partial c}\right)^*(w), \tag{32}$$

where $c^\star$ is the exact solution c_{ex} to be estimated, and $w \in H$ is adequate.

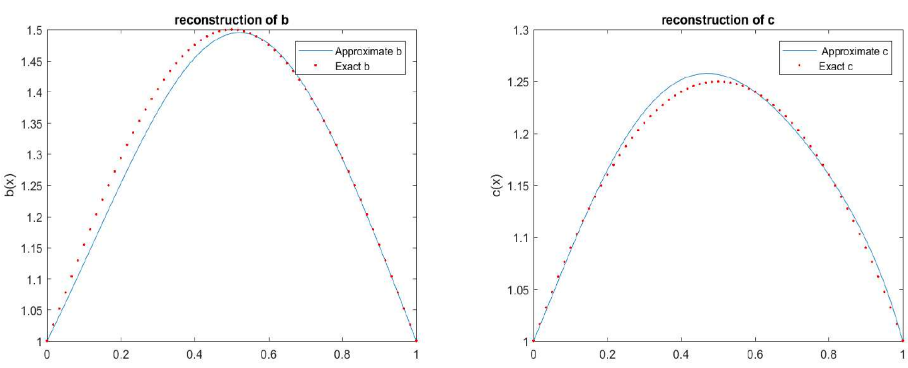

Figure 5. Test 2-b. Exact and regularized solution without noise.

This example (Test 2-b) confirms the convergence forced by the source condition. However, we have used the knowledge of the exact solution which is not possible in application. This question is not well studied from the numerical aspect.

4.3. Commentaries

(1) In Figures 1 and 2, Δ_1 does not change the sign $(\Delta_1(x) \geq 0.02)$, which confirms the numerical stability (in accordance with Theorem 3.1).

(2) In Figure 3, Δ_1 changes the sign, it vanishes at $x = 0.5$. For the parameter $c(x)$, the algorithm converges to another solution (lack of uniqueness). However, the parameter b is relatively stable (stable for a small noise-level $\delta \leq 10^{-5}$).

(3) In Figure 4, considering one observation and the initial guess $p_0 = (1, 1)$, the algorithm converges in a stable way to another solution $(b, \hat{c})$.

(4) In Figure 5, we consider one observation such that $b_0 = 1$ and c_0 is well-chosen using the formula (32), the algorithm converges to the exact solution in the case of exact data, otherwise it diverges.

5. Conclusion

In this paper, we have studied the uniqueness and the stability of an inverse problem which consists of identifying two parameters in elliptic boundary value problem. We have established a Lipschitz stability estimate under a suitable condition on the data $(u_1, f_1), (u_2, f_2)$ (two observations). The numerical experiments confirm the theoretical results. In the case of one observation, the algorithm converges but to another solution. Finally, we could not have good a priori estimates on the parameters. Thus, it is hard to find a good initial guess required by a locally convergent algorithm.

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