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MATHEMATICAL BASIS OF EXTREMAL THEORY OF DIMENSIONS

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Abstract

The paper offers the full theoretical basis of "extremal theory of dimensions". This simple theory, using only the notion "singular extremum" on the set of dimensional physical parameters, permits to find unknown laws of nature and very complex differential equations including many arbitrary additive terms. Note that almost all known equations of physics and their some generalizations already were found in the framework of this theory.

Introduction

"Extremal theory of dimensions" [1-4] is based only on some set of dimensional parameters $A = (A_1, \ldots, A_m)$ and on the arbitrary expansions

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$X = CA_1^{\alpha_1} \cdots A_m^{\alpha_m}$. The base of this theory is very simple. We introduce into consideration a new notion of "singular extremum" of the function *X* of many variables α_k. Notice that the serious difficulties in the classic analysis of dimensions [5] arise when the number of unknown values, satisfying the system of linear algebraic equations, turns out to be more than the number of equations equal to the number of basic units of dimensions. Unfortunately, such a situation has place almost always and therefore the use of the classical analysis of dimensions in practice almost makes no sense. If, however, we supplement the classical analysis of dimensions by means of a new notion "singular extremal", then it opens the great possibilities.

The classical analysis of dimensions stopped before the problem searching of the sole solution of system of *n* linear algebraic equations with *m* unknown quantities $(m > n)$. Our proposal consists in that it follows to search "singular extremum" of the function *X* about $m - n$ of the parameters α_k. It is reduced to the searching of extremum of some linear function $f(x) = \sum_{k=1}^{m-n} a_k x_k$ on non-limited set of values x_k, but it could seem as a complete absurdity. Equaling to zero the partial derivatives of the linear function $f(x) = \sum_{k=1}^{m-n} a_k x_k$, we obtain $a_k = 0$, $k = 1, ..., m - n$. Hence, we have $f = 0$. We shall name such extremum as "singular". But what can be obtained from absolute zero? It is astonishing, but from zero, one can find all existing laws in nature and equations (algebraic, differential, partial derivatives, and others).

Extremal Theory of Dimensions

It is necessary to find dependence of the dimensional parameter (or function) *X* upon other dimensional parameters (or functions) $A_1, ..., A_m$ and to find all algebraic and differential equations by means of which these parameters are connected with each other. This problem can be considered also as follows: for a given set of dimensional parameters, find all laws and equations which these parameters satisfy.

In accordance with this theory, the parameter X is presented in the form of production (expansion, decomposition):

$$X = CA_1^{\alpha_1} \cdots A_m^{\alpha_m}, \tag{1}$$

where C is a dimensionless parameter, A_i are the dimensional parameters, and α_i are unknown powers which must be found. Let there be n basic units of dimensions (for example, centimeter, gram, second, i.e., $n = 3$). By equating the basic dimensions on both the sides of the afore-cited equality (for X), we obtain the linear algebraic system of n equations with m unknown powers $\alpha_1, ..., \alpha_m$. As a rule, $m > n$. By solving this linear system for any n powers and substituting the result in the initial expansion (1) for X, we obtain the new expansion depending on $m - n$ powers. By equating to zero the partial derivatives with respect to these $m - n$ powers, we obtain $m - n$ extremal equations. By substituting these extremals in the new expansion, we obtain as well common representation of solution for X. The extremals and the common solution define mathematical models of our world. Consider this problem in every detail.

Before giving the common theoretical basis of the extremal theory of dimensions, we consider in the beginning some elementary examples.

Example 1. We shall attempt to find all solutions (by means of singular extremals) of Newton's equation $\frac{d^2x}{dt^2} = g$ in the constant gravitational field (g). Let $x(t)$ be a solution, g be an acceleration of gravity on the surface of Earth, $\dot{x} \equiv dx/dt$ be the first derivation (of position $x(t)$ of moving body), $\ddot{x}$ be second derivation. We shall search for a solution in the form of the following decomposition (expansion):

$$x(t) = C\ddot{x}^k \dot{x}^l g^m t^n. \tag{2}$$

By rewriting this equality in the basic dimensions of Gaussian system, let $[L]$ be the dimension of length (centimeter), $[T]$ be the dimension of time (second), and $[M]$ be the dimension of mass (gram), we obtain

$$[L] = \left[\frac{L}{T^2}\right]^k \left[\frac{L}{T}\right]^l \left[\frac{L}{T^2}\right]^m [T]^n.$$

By equating the dimensions $[L]$ and $[T]$ on both the sides of this relation, we obtain a system of two algebraic equations with four unknown powers:

$$1 = k + l + m, \quad 0 = 2k + l + 2m - n.$$

The classical analysis of dimensions further loses any sense. However, by expressing from this system any two powers across the other two $(\text{for example, } k = n - m - 1, \ l = 2 - n)$ and substituting ones into initial decomposition (2), we obtain the other form of decomposition (2):

$$x(t) = C\ddot{x}^{n-m-1}\dot{x}^{2-n}g^m \cdot t^n, \tag{3}$$

defining dependence $x(t)$ only on two variables *m* and *n*. Further, we shall search extremum of this function $x(t)$ from conditions: $\frac{\partial x}{\partial m} = 0, \frac{\partial x}{\partial n} = 0.$ In essence, it means that we search an envelope of the parametrical set (3). However, before to find extremum of this function, it is comfortable firstly to take the logarithm of *x* (which does not change the position of extremum of function *x*):

$$\lg x = \lg C + (n - m - 1)\lg \ddot{x} + (2 - n)\lg \dot{x} + m\lg g + n\lg t.$$

By equating to zero the partial derivatives of the function $\lg x$ (linear on *m* and *n*) with respect to *m* and *n*, we obtain the following "singular extremals":

$$\frac{\partial \lg x}{\partial m} = -\lg \ddot{x} + \lg g = 0,$$

$$\frac{\partial \lg x}{\partial n} = \lg \ddot{x} - \lg \dot{x} + \lg t = 0.$$

Hence, we find the extremal relations $\ddot{x} + g$ and $\dot{x} = \ddot{x}t = gt$, describing the envelope of the set (3). Substitution of these relations (in any form, for example, $g = \ddot{x}$ and $t = \dot{x}/\ddot{x}$) in the decomposition (3) permits to eliminate the powers *m* and *n* in the decomposition (3) and to bring it to the following common form of solution $x = C\left(\frac{\dot{x}^2}{\ddot{x}}\right) = Cgt^2$ (defining, essentially, a common envelope of the considered parametric set). And what is more, substitution of this solution into extremals permits to find also $C = 1/2$, $\ddot{x} = g$, $x = gt^2/2$ and $\dot{x} = gt$. This means that in this example exists the common envelope of the set (2). Notice that, in common case, we cannot always find non-dimensional parameter *C*. But this new theory always permits to find a functional form of all additive members of any unknown differential equation.

However, if in the initial expansion (2), there is no common envelope, then it follows to search some partial envelopes, i.e., some particular solutions, each of which can be obtained as a result of the successive substitutions of some extremal relations into the common form of the solution.

The following simple example demonstrates how to search all solutions in the case when exists only the particular envelopes.

It follows to take into account that, in the common case, the arbitrary expansion $X = CA_1^{\alpha_1} \cdots A_m^{\alpha_m}$ permits to obtain a solution not only of the problem which is interesting for us this moment but also of many other adjacent problems.

Example 2. The expansion $\dot{x} = C\ddot{x}^k x^l g^m t^n$, in Gauss's system, has extremals $x = \ddot{x}t^2$ and $\ddot{x} = g$ and also common form of solution $\dot{x} = C\ddot{x}t$.

If these three equations are incompatible for any C, then the common envelope of considering parametrical function $\dot{x}$ does not exist. However, the pair, including the common form of solution and second extremal, has the known classic solution $x = gt^2/2$ for $C = 1$ and $C = 1 + C_1/t$, where $C_1 = const$, i.e., in this case, the particular envelope exists. Other particular envelope exists also in the case of the pair, including the common form of solution and the first extremal (it corresponds, essentially, to case $m = 0$). In this case, we have the solution $x = t^C$, where $C = \frac{1}{2}(1 \pm \sqrt{5})$, undoubtedly corresponding to some real process (for example, probably, this solution defines growing and droop of plants).

Common Theoretical Basis of the Extremal Theory of Dimensions

In the common case, it is required to find dependence of the dimensional parameter (or function) X upon other dimensional parameters (or functions) $A_1, ..., A_m$ and to find nature's laws, and also all algebraic and the differential equations by means of which these parameters are connected. Consider this problem in detail.

Assumptions 1. Choose some system of units with basic units $B_1, ..., B_n$ (for example, let $B_1 = L$ be a unit of length, $B_2 = M$ be a unit of mass, and $B_3 = T$ be a unit of time in the Gaussian system of units: centimeter, gram, second), and let it be given k known extremal basic constants $A_1, ..., A_k$ (for example, the charge of electron $A_1 = e$, the light velocity $A_2 = c$ in vacuum, and the gravitational constant $A_3 = G$) and $m - k$ arbitrarily chosen dimensional parameters $A_{k+1}, ..., A_m$, which have dimensions $[A_i] = [B_1]^{\alpha_{i1}}, ..., [B_n]^{\alpha_{in}}$, $i = 1, ..., m.$ Suppose that the problem is to represent, via the parameters A_i, an arbitrary parameter X that has dimension $[X] = [B_1]^{\beta_1}, ..., [B_n]^{\beta_n}$ and is written out in the following form, more convenient for forthcoming references:

$$R = X - CA_1^{\alpha_1} \cdots A_m^{\alpha_m} = 0, \tag{4}$$

where C is an arbitrary dimensionless quantity. In the arguments of dimensions, this relation takes the form:

$$[X] = [B_1]^{\beta_1} \ldots [B_n]^{\beta_n} = ([B_1]^{\alpha_{11}} \ldots [B_n]^{\alpha_{1n}})^{\alpha_1} \cdots ([B_1]^{\alpha_{m1}} \ldots [B_n]^{\alpha_{mn}})^{\alpha_m}.$$

The matching of dimensions on both the sides of this relation leads to the linear system of n equations with m unknowns α_i:

$$\beta_j = \sum_{i=1}^{m} \alpha_{ij}\alpha_i, \quad j = 1, \ldots, n. \tag{5}$$

If the resulting system is inconsistent, then this implies that some parameters A_i are chosen unsuccessfully and they must be replaced. But if the system is consistent, then depending on the rank n_0 of the matrix $\{\alpha_{ij}\}$, there is a possibility to express n_0 parameters α_i $(n_0 \le n)$ via the remaining parameters (their number is equal to $m - n_0$) and to substitute these α_i's into (4).

Affirmation 1. Under the above-formulated assumptions, the problem of the representation of the parameter X via the parameters A_i, $i = 1, \ldots, m$, permits to find $m - n_0$ extremal formulas relating the parameters A_i with the help of singular extremals from the conditions $\partial lnX/\partial\alpha_j = 0$, $j = n_0 + 1, \ldots, m$, and to find n_0 of the extremal formulas expressing n_0 of basic units via the parameters A_i, $i = 1, \ldots, m$, from the conditions $\partial lnR/\partial\alpha_j = 0$ in which the parameters B_j, $j = 1, \ldots, n_0$, are substituted successively instead of X.

Affirmation 2. Suppose that for the arbitrary set of m parameters and variables A_i, $i = 1, \ldots, m$, and for any selected set of the base dimension unit n_0, we found all $(m - n_0)$ possible singular extremals. Substituting these extremals in (4), we obtain a common representation of solution X. If, for some $C \neq 0,$ this solution is consistent with all singular extremals, then there exists a common envelope of the parametric family of extremals and the solution is quite determined (especially, if there is a possibility to find the value of dimensionless parameter *C*). If the common solution is inconsistent with all extremals simultaneously, then it is necessary to search this solution in form of sums of additive terms, every one of which can be found from the condition of compatibility of this solution with the different extremals (or groups of extremals). In this case, almost before every member of this sum is taken the different dimensionless parameters C_k. Thus, the dimensionless parameter *C* can take the same or different values before the different additive terms of the searched final decision.

Affirmation 3. On the basis of the same decomposition and their extremals, it is possible also to find the solutions of many problems simultaneously which are absolutely not connected with each other.

Proof. If the decomposition (4) is chosen to find the solution of some concrete problem and the common form of the solution and if all extremals in this problem are found, then these results can be used for searching solutions to many other problems based on the same parameters. Indeed, if some extremal includes the parameter (or function) that we consider as solution in the new problem, then expressing this parameter, in this extremal equation, in terms of the remaining parameters and introducing in this equation a new dimensionless parameter C_1, we find a common form of the solution of the new problem, in which the old common solution turns out to be the extremal equation with $C = 1$.

In most cases, all solutions to problem (4) can be found considering only one arbitrary case of the maximum rank in (5) (equal to $n_0 = n$). If there is doubt about completeness of received results, then we can study also other cases of maximum rank in (5) that number does not exceed $C_m^n = \dfrac{m!}{n!(m-n)!}$. It allows to find all possible solutions for decomposition

(4). As a rule, in practice, it is sufficient to consider not only the large number of cases to find all extremals in the problem (4).

Definition 1. A dimensional physical parameter X is said to belong to the class of "*extremal fundamental physical parameters*" if it was found by means of "extremal theory of dimensions".

In this case, each extremal formula, by means of which this parameter can be defined, gives the same numerical value for this parameter.

Corollary 1. *If some parameter can be expressed via extremal base parameters by some formula, then this parameter is also an extremal base parameter.*

Moreover, there exists a set of other, principally different, formulas for their representations via the extremal base parameters. And all these formulas define the same numerical value of the above-mentioned parameter.

This is an important difference of the latter from a non-extremal parameter.

Corollary 2. *Only the extremal parameters form a class of the fundamental physical constants which include, for example, the constants* (e, c, G) *(charge of electron* e*, velocity of light in vacuum* c*, gravitational constant* G*) but do not include, for example, Planck's constant* $\hbar$*.*

Only production $\hbar\alpha = \hat{h}$ *turns out fundamental physical constant. Also, the constants* ε_0 *and* μ_0 *in the electrodynamics are not fundamental constants in "International System of Units SI" but their production* $\varepsilon_0\mu_0 = 1/c^2$ *is the fundamental (extremal) physical constant.*

Corollary 3. *Before* 21*st century, in physics, it was not proposed any strict mathematical definition of the fundamental physical constants. It was only intuitive understanding of fundamentalness founded on empirical tests. Physics made the great step forward only after the discovery of three fundamental constants* (e, c, G).

All extremal parameters belong to the class of the fundamental parameters (constants) which can be represented by the following simple formula:

$$X = e^{(k+l+m)} c^{-(3k+2l)} G^{\frac{1}{2}(k+l-m)}, \tag{6}$$

where the parameter X has dimension $[T]^k [L]^l [M]^m$ *and the constants* (e, c, G) *are the basic extremal fundamental physical constants*.

Due to the fact that the numerical values of these constants are known with the same great accuracy, this formula guarantees that the set of any other extremal fundamental parameters can be defined with the same accuracy. As to all Planck's constants (including $\hbar$), they do not satisfy Definition 1 and cannot be considered as fundamental. Formula (6) defines all dimensional extremal fundamental physical parameters of the world known to us now. The set of these parameters is closed in itself, i.e., any extremal fundamental parameter can be defined by a large number of formulas via other extremal physical parameters, and all these formulas define the same numerical value for this parameter. Notice that the constants (e, c, G) are not expressed one across the other, i.e., they are mutually independent, and other extremal fundamental constants are expressed via the three by means of the formula (6). Then it is natural to recognize these three constants as the main extremal fundamental constants.

Corollary 4. *In the theory of dimensions*, *it does not mean what is some value Y*: *Is it a constant or a function*?

For example, the velocity of light *c* can be considered as some velocity $v(t)$.

Remark 1. All Planck's constants (including $\hbar$) do not satisfy Definition 1 and cannot be considered as fundamental in the sense of this definition. Consider only one Planck's constant of time *T*. If formula representing this constant includes in itself some non-extremal parameter (for example, classical Planck's constant $\hbar$), then every one of these formulae gives different numerical values for this parameter T. Consider, for example, some formulas for representation of the parameter T via the set of other physical parameters: $(e, c, G, h, \ldots)$: $T_1 = (e\sqrt{G})/c^3$, $T_2 = (h^3\sqrt{G})/e^5$, $T_3 = \sqrt{(Gh)/c^5}, \ldots$. If these formulas are based on the extremal constants, then we have $T_1 = T_2 = T_3 = \cdots = 4{,}6 \cdot 10^{-45}\text{s}$. And, if they include the non-extremal constants (for example, Planck's constant $\hbar = 1{,}055 \cdot 10^{-27}$), then they give different numerical values for T: $T_1 = 4{,}6 \cdot 10^{-45}\text{s}$, $T_2 = 1{,}2 \cdot 10^{-38}\text{s}$, $T_3 = 5{,}4 \cdot 10^{-44}\text{s}, \ldots$. So, it is impossible to agree with that Planck's constants are fundamental in the sense of Definition 1 (only production $\hbar\alpha = \hat{h}$ is turned out fundamental physical constant).

Remark 2. In conclusion, we shall point out some circumstances giving ground for doubts about justice of "quark-theory" based on the assumption that there exists the stable fractional (relative of charge of electron) electric charge $(e/3)$. The extremal formula (6) explains that, in case of existence of such stable charge, velocity of light c would be far less. Indeed, in the case of $(e/3)$, from the experimental equality $\alpha\hbar = \hat{h} = e^2/c$, it follows that the velocity of light must be equal to $c/9$, but this contradicts reality.

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