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COMMUTATIVITY OF HIGH-ORDER LINEAR TIME-VARYING SYSTEMS

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Abstract

This paper presents the commutativity of high-order linear time-varying systems (LTVSs). Explicit conditions for the commutativity of high-order LTVSs are derived. The feedback conjugate pairs for high-order LTVSs are considered. The effects of sensitivity and disturbance on sixth-order LTVSs have been investigated. Example is given to support the results.

1. Introduction

Differential equation is one of the important fields of applied

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mathematics that arises in network design, wave motion, fluid dynamics, telecommunications, electro-magnetic, wave distribution, electronic dynamics and many other branches of engineering and sciences. Marshall investigated the commutativity condition of first-order [1]. In the past and present, so many works have been done on commutative theories and conditions. Among the few ones that appeared in the literature are [2-6] which represent commutativity for second, third, fourth, fifth, sixth-order systems, respectively. As part of the application of commutativity, the authors in [7-11] studied and presented the decomposition of fourth-order LTVSs.

This paper introduces the explicit commutativity for high-order LTVSs together with its feedback conjugate pairs. The sensitivity of the system toward changes in parameters or initial conditions is also investigated by considering an example to support the results.

2. Preliminaries

Commutativity: Consider two LTVSs A and B of sixth-order defined by

$$A : a_6(t)y_A^{(6)}(t) + a_5(t)y_A^{(5)}(t) + a_4(t)y_A^{(4)}(t) + a_3(t)y_A^{(3)}(t) + a_2(t)y''_A(t) + a_1(t)y'_A(t) + a_0(t)y_A(t) = x_A(t), \tag{1}$$

$$B : b_6(t)y_B^{6}(t) + b_5(t)y_B^{(5)}(t) + b_4(t)y_B^{(4)}(t) + b_3(t)y_B^{(3)}(t) + b_2(t)y''_B(t) + b_1(t)y'_B(t) + b_0(t)y_B(t) = x_B(t), \tag{2}$$

where $x_A(t), y_A(t)$ and $x_B(t), y_B(t)$ are the input and output of systems A and B, respectively; $a_i(t), b_i(t)$ are the time-varying coefficients, and $y_A^{(i)}(t_0)$ and $y_B^{(i)}(t_0)$ are the initial condition up to fifth-order.

3. Main Results

Theorem [6]**.** *The commutative formulas for sixth-order LTVS in the case of zero-initial conditions are that*

(i) *The coefficients of the LTVS describing B must be expressed in terms of the coefficients of the LTVS describing system A*

$$\begin{bmatrix} b_6 \\ b_5 \\ b_4 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_5 & f_{5,5} & 0 & 0 & 0 & 0 & 0 \\ a_4 & f_{4,5} & f_{4,4} & 0 & 0 & 0 & 0 \\ a_3 & f_{3,5} & f_{3,4} & f_{3,3} & 0 & 0 & 0 \\ a_2 & f_{2,5} & f_{2,4} & f_{2,3} & f_{2,2} & 0 & 0 \\ a_1 & f_{1,5} & f_{1,4} & f_{1,3} & f_{1,2} & f_{1,1} & 0 \\ a_0 & f_{0,5} & f_{0,4} & f_{0,3} & f_{0,2} & f_{0,1} & 1 \end{bmatrix} \begin{bmatrix} c_6 \\ c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}, \tag{3}$$

where $c_6, c_5, c_4, c_3, c_2, c_1, c_0$ *are constants and the* $f_{i,j}$ *are nonlinear algebraic equations*:

$$f_{5,5} = a_6^{5/6},\ f_{4,4} = a_6^{4/6},\ f_{3,3} = a_6^{3/6},\ f_{4,5} = \frac{5}{12a_6^{1/6}}(2a_5 - a_6'),$$

$$f_{3,4} = \frac{8}{12a_6^{2/6}}(a_5 - a_6'),\ f_{1,1} = a_6^{1/6},\ f_{2,3} = \frac{3}{12a_6^{3/6}}(2a_5 - 3a_6'),$$

$$f_{1,2} = \frac{4}{12a_6^{4/6}}(a_5 - 2a_6'),\ f_{0,1} = \frac{1}{12a_6^{5/6}}(2a_5 - 5a_6'),\ f_{2,2} = a_6^{2/6},$$

$$f_{3,5} = \frac{5}{72a_6^{1/6}}(12a_4 - 6a_5' - a_6'') - \frac{5}{864a_6^{7/6}}[12a - 24a_5a_6' - 35(a_6')^2],$$

$$f_{2,4} = \frac{1}{9a_6^{2/6}}(6a_4 - 6a_5' + a_6'') + \frac{1}{27a_6^{8/6}}[-3a_5^2 + 9a_5a_6' + 4(a_6')^2],$$

$$f_{1,3} = \frac{1}{8a_6^{3/6}}(4a_4 - 6a_5' + 3a_6'') + \frac{1}{32a_6^{9/6}}[-4a_5^2 + 16a_5a_6' - 3(a_6')^2],$$

$$f_{0,2} = \frac{1}{9a_6^{4/6}}(3a_4 - 6a_5' + 5a_6'') + \frac{1}{27a_6^{10/6}}[-3a_5^2 + 15a_5a_6' - 10(a_6')^2],$$

$$\begin{aligned} f_{2,5} &= \frac{1}{144 a_6^{1/6}}\left(120a_3 - 60a_4' - 10a_5'' + 45a_6^{(3)}\right) + \frac{1}{144 a_6^{7/6}} \\ &\quad \times \left(-20a_4a_5 + 20a_5a_5' + 30a_4a_6' + 35a_5'a_6' + 5a_5a_6'' - 105a_6'a_6''\right) \\ &\quad + \frac{1}{10368 a_6^{13/6}}\left[280a_5^3 - 1260a_5^2a_6' - 1470a_5(a_6')^2 + 4095(a_6')^3\right], \end{aligned}$$

$$\begin{aligned} f_{1,4} &= \frac{1}{9 a_6^{2/6}}\left(6a_3 - 6a_4' + a_5'' + 4a_6^{(3)}\right) + \frac{1}{27 a_6^{8/6}} \\ &\quad \times \left(-6a_4a_5 + 9a_5a_5' + 12a_4a_6' - 2a_5a_6'' - 32a_6'a_6''\right) \\ &\quad + \frac{1}{18 a_6^{14/6}}\left[4a_5^3 - 24a_5^2a_6' + 4a_5(a_6')^2 + 56(a_6')^3\right], \end{aligned}$$

$$\begin{aligned} f_{2,5} &= \frac{1}{144 a_6^{1/6}}\left(120a_3 - 60a_4' - 10a_5'' + 45a_6^{(3)}\right) + \frac{1}{144 a_6^{7/6}} \\ &\quad \times \left(-20a_4a_5 + 20a_5a_5' + 30a_4a_6' + 35a_5'a_6' + 5a_5a_6'' - 105a_6'a_6''\right) \\ &\quad + \frac{1}{10368 a_6^{13/6}}\left[280a_5^3 - 1260a_5^2a_6' - 1470a_5(a_6')^2 + 4095(a_6')^3\right], \end{aligned}$$

$$\begin{aligned} f_{2,5} &= \frac{1}{144 a_6^{1/6}}\left(120a_3 - 60a_4' - 10a_5'' + 45a_6^{(3)}\right) + \frac{1}{144 a_6^{7/6}} \\ &\quad \times \left(-20a_4a_5 + 20a_5a_5' + 30a_4a_6' + 35a_5'a_6' + 5a_5a_6'' - 105a_6'a_6''\right) \\ &\quad + \frac{1}{10368 a_6^{13/6}}\left[280a_5^3 - 1260a_5^2a_6' - 1470a_5(a_6')^2 + 4095(a_6')^3\right], \end{aligned}$$

$$\begin{aligned} f_{1,4} &= \frac{1}{9 a_6^{2/6}}\left(6a_3 - 6a_4' + a_5'' + 4a_6^{(3)}\right) + \frac{1}{27 a_6^{8/6}} \\ &\quad \times \left(-6a_4a_5 + 9a_5a_5' + 12a_4a_6' - 2a_5a_6'' - 32a_6'a_6''\right) \\ &\quad + \frac{1}{18 a_6^{14/6}}\left[4a_5^3 - 24a_5^2a_6' + 4a_5(a_6')^2 + 56(a_6')^3\right], \end{aligned}$$

$$f_{0,3} = \frac{1}{16a_6^{3/6}}(8a_3 - 12a_4' + 6a_5'' + 5a_6^{(3)}) + \frac{1}{16a_6^{9/6}}$$

$$\times(-4a_4a_5 + 8a_5a_5' + 10a_4a_6' - 9a_5'a_6' - 5a_5a_6'' - 15a_6'a_6'')$$

$$+ \frac{1}{128a_6^{15/6}}[8a_5^3 - 60a_5^2a_6' + 70a_5(a_6')^2 + 75(a_6')^3],$$

$$f_{1,5} = \frac{1}{864a_6^{\frac{1}{6}}}(720a_2 - 360a_3' - 60a_4'' + 270a_5^{(3)} - 151a_6^{(4)}) + \frac{1}{10368a_6^{7/6}}$$

$$\times[-720a_4^2 - 1440a_3a_5 + 1440a_5a_4' + 2160a_4a_5' + 420(a_5')^2$$

$$+ 2880a_3a_6' + 840a_4'a_6' + 360a_5a_5'' - 6300a_6'a_5'' - 120a_4a_6''$$

$$- 7980a_5'a_6'' + 3997(a_6'')^2 - 2160a_5a_6^{(3)} + 3276a_6'a_6^{(3)}]$$

$$+ \frac{1}{62208a_6^{13/6}}[5040a_4a_5^2 - 7560a_5^2a_5' - 20160a_4a_5a_6'$$

$$- 4200a_5a_5'a_6' + 420a_4(a_6')^2 + 68250a_5'(a_6')^2 - 420a_5^2a_6''$$

$$+ 70560a_5a_6'a_6'' - 53417(a_6')^2a_6'']$$

$$+ \frac{1}{1492992a_6^{19/6}}[-21840a_5^4 + 174720a_5^3a_6' + 10920a_5^2(a_6')^2$$

$$- 1441440a_5(a_6')^3 + 489307(a_6')^4],$$

$$f_{0,4} = \frac{1}{27a_6^{2/6}}(18a_2 - 18a_3' + 3a_4'' + 12a_5^{(3)} - 13a_6^{(4)}) + \frac{1}{81a_6^{4/3}}$$

$$\times[-9a_4^2 - 18a_3a_5 + 27a_5a_4' + 36a_4a_5' - 12(a_5')^2 + 45a_3a_6'$$

$$- 24a_4'a_6' - 6a_5a_5'' - 72a_6'a_5'' - 18a_4a_6'' - 84a_5'a_6'' + 89(a_6'')^2$$

$$- 30a_5a_6^{(3)} + 102a_6'a_6^{(3)}] + \frac{1}{243a_6^{7/3}}[36a_4a_5^2 - 72a_5^2a_5'$$

$$-180a_4a_5a_6' + 132a_5a_5'a_6' + 96a_4(a_6')^2 + 378a_5'(a_6')^2$$

$$+30a_5^2a_6'' + 420a_5a_6'a_6'' - 812(a_6')^2a_6''] + \frac{1}{729a_6^{20/6}}[-21a_5^4$$

$$+210a_5^3a_6' - 315a_5^2(a_6')^2 - 1050a_5(a_6')^3 - 1064(a_6')^4],$$

$$f_{0,5} = \frac{1}{17915904a_6^{25/6}}[165984a_5^5 - 2074800a_5^4a_6' + 3458000a_5^3(a_6')^2$$

$$+25935000a_5^2(a_6')^3 - 46077850a_5(a_6')^4 - 36092875(a_6')^5]$$

$$+\frac{1}{373248a_6^{19/6}}(-21840a_4a_5^3 + 43680a_5^3a_5' + 163800a_4a_5^2a_6'$$

$$-92820a_5^2a_5'a_6' - 147420a_4a_5(a_6')^2 - 879060a_5a_5'(a_6')^2$$

$$-313950a_4(a_6')^3 + 980343a_5'(a_6')^3 - 9100a_5^3a_6'' - 518700a_5^2a_6'a_6''$$

$$+1659385a_5(a_6')^2a_6'' + 2022930(a_6')^3a_6'') + \frac{1}{124416a_6^{13/6}}(10080a_4^2a_5$$

$$+10080a_3a_5^2 - 15120a_5^2a_4' - 40320a_4a_5a_5' + 9240a_5(a_5')^2$$

$$-25200a_4^2a_6' - 50400a_3a_5a_6' + 18480a_5a_4'a_6' + 31920a_4a_4'a_6'$$

$$+114660(a_5')^2a_6' + 26040a_3(a_6')^2 + 103740a_4'(a_6')^2 - 840a_5^2a_5''$$

$$+118440a_5a_6'a_5'' - 158158(a_6')^2a_5'' + 11760a_4a_5a_6'' + 131880a_5a_5'a_6''$$

$$+142800a_4a_6'a_6'' - 412776a_5'a_6'a_6'' - 112630a_5(a_6'')^2$$

$$-371735a_6'(a_6'')^2 + 27300a_5^2a_6^{(3)} - 139440a_5a_6'a_6^{(3)}$$

$$-246155(a_6')^2a_6^{(3)}) + \frac{1}{10368a_6^{7/6}}(-1440a_3a_4 - 1440a_2a_5$$

$$+ 1440 a_5 a_3' + 2160 a_4 a_4' + 2880 a_3 a_5' - 840 a_4' a_5' + 3600 a_2 a_6'$$

$$- 840 a_3' a_6' + 360 a_4'' - 5040 a_6' a_4'' - 120 a_4 a_5'' - 6720 a_5' a_5''$$

$$- 1320 a_3 a_6'' - 6720 a_4' a_6'' + 10052 a_5'' a_6'' - 2160 a_5 a_5^{(3)}$$

$$+ 4228 a_6' a_5^{(3)} - 3300 a_4 a_6^{(3)} + 7938 a_6^{(3)} + 11725 a_6'' a_6^{(3)}$$

$$+ 1510 a_5 a_6^{(4)} + 6055 a_6' a_6^{(4)})$$

$$+ \frac{(1440 a_1 - 720 a_2' - 120 a_3'' + 540 a_4^{(3)} - 302 a_5^{(4)} - 265 a_6^{(5)})}{1728 a_6^{1/6}}.$$

(ii) *Also, maintaining the same constant* c_i'*s, the coefficients of the first system must satisfy the following five nonlinear algebraic equations*:

$$\begin{bmatrix} F_{5,5} & F_{5,4} & F_{5,3} & F_{5,2} & F_{5,1} \\ F_{4,5} & F_{4,4} & F_{4,3} & F_{4,2} & F_{4,1} \\ F_{3,5} & F_{3,4} & F_{3,3} & F_{3,2} & F_{3,1} \\ F_{2,5} & F_{2,4} & F_{2,3} & F_{2,2} & F_{2,1} \\ F_{1,5} & F_{1,4} & F_{1,3} & F_{1,2} & F_{1,1} \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4}$$

where $F_{i,j}$ *are the non-linear algebraic equations which involve derivative details of the* $f_{i,j}$.

4. Example

An example is shown to validate the effectiveness of our work toward sensitivity and the feedback response. Consider the sixth-order LTVS as

$$A : t^6 y_1^{(6)} + y_1^{(5)} + y_1 = x_1. \tag{5}$$

The coefficients of equation (5) are used in equation (3) by computing all the $f_{i,j}$s, the LTVS B is obtained in matrix form as:

$$\begin{bmatrix} b_6 \\ b_5 \\ b_4 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2t}{3} & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{5t^2}{72} - \frac{5}{12} & \frac{2t}{3} & 1 & 0 & 0 & 0 \\ 0 & \frac{35t^3}{1296} + \frac{5t}{36} & -\frac{t^2}{9} - \frac{2}{3} & \frac{t}{2} & 1 & 0 & 0 \\ 0 & -\frac{455t^4}{31104} - \frac{35t^2}{288} + \frac{35}{864} & \frac{4t^3}{81} + \frac{t}{3} & -\frac{t^2}{8} - \frac{3}{4} & \frac{5t}{6} & 1 & 0 \\ 1 & \frac{1729t^5}{186624} + \frac{455t^3}{3888} + \frac{385t}{5184} & -\frac{7t^4}{243} - \frac{8t^2}{27} - \frac{4}{27} & \frac{t^3}{16} + \frac{t}{2} & -\frac{t^2}{9} - \frac{2}{3} & \frac{t}{6} & 1 \end{bmatrix} \begin{bmatrix} c_6 \\ c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix}. \tag{6}$$

Considering the second commutative condition in equation (4), the $F_{i,j}$s are obtained in matrix form as:

$$\begin{bmatrix} \frac{5t}{6} & -\frac{10t^2}{9} - \frac{10}{3} & \frac{15t^3}{16} + \frac{45t}{8} & -\frac{140t^4}{243} - \frac{140t^2}{27} - \frac{80}{27} & \frac{43225t^5}{186624} + \frac{43225t^3}{15552} + \frac{15925t}{5184} \\ 0 & -\frac{20t}{9} & \frac{15t^2}{4} + \frac{15}{2} & -\frac{280t^3}{81} - \frac{1360t}{81} & \frac{43225t^4}{23328} + \frac{56875t^2}{3888} + \frac{11375}{1296} \\ 0 & 0 & \frac{15t}{4} & -\frac{560t^2}{81} - \frac{280}{27} & \frac{43225t^3}{7776} + \frac{56875t}{2592} \\ 0 & 0 & 0 & -\frac{280t}{81} - 4 & -\frac{581}{216} \\ 0 & 0 & 0 & 0 & \frac{8645t}{7776} \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{7}$$

Solving for the following unknown constants in equation (7) indicates that the system generates commutative constant feedback conjugate pairs $c_5 = c_4 = c_3 = c_2 = c_1 = 0$. By considering $c_5 = c_4 = c_3 = c_2 = c_1 = 0$, the commutative constant feedback conjugate pair of A is given by

$$B : c_6[y_2^{(6)} + ty_2^{(5)} + y_2] + c_0 y_2 = x_2. \tag{8}$$

We consider the commutative with nonzero initial conditions. We analogously make use of Theorem 2 in [6], together with the computed result and we come up with the following:

$$(c_0 + c_6 - 1) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 & 0 & 0 \\ t^2 - 2 & -t & 1 & 0 & 0 & 0 \\ -t^3 + 5t & t^2 - 3 & -t & 1 & 0 & 0 \\ 8 - 9t^2 + t^4 & -t^3 + 7t & t^2 - 4 & -t & 1 & 0 \\ -33t + 14t^3 - t^5 & 15 - 12t^2 + t^4 & -t^3 + 9t & t^2 - 5 & -t & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_1' \\ y_1'' \\ y_1^{(3)} \\ y_1^{(4)} \\ y_1^{(5)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

Solving equation (9) leads to:

$$c_0 = 1 - c_6. \tag{10}$$

Observe that, from equation (9), the initial conditions are set to arbitrary.

Left: For $c_6 = 0.25$ and $c_0 = 0.75$ with Amplitude 24, Frequency 13 and Bias −3.3 as the input, we obtained the same response $AB = BA$ (solid blue curve) with non-zero initial conditions $(y_j(1) = -0.5,\ \ y_j'(1) = 0.5,\ \ j = 1, 2$ all others are zero$)$, which was due to $c_0 = 1 - c_6$ in equation (10), hence commutativity holds. For the sensitivity of *AB* and *BA* due to initial conditions, (switching $y_1(1) = -0.5$ to −1), the commutativity for *ABI* (long dashed-red curve) and *BAI* (dot dashed-green curve) is spoil.

Right: Maintaining the same input, initial state and initial condition, we obtained the same response $AB = BA$ (solid blue curve). Investigating the effects of sensitivity of *AB* and *BA* toward parameters, (switching $c_0 = 0.75$ to $c_0 = 0.25$), the commutativity for *ABI* (long dashed-red curve) and *BAI* (dot dashed-green curve) is no longer valid as a result of altering with equation (10) $(c_0 = 1 - c_6)$.

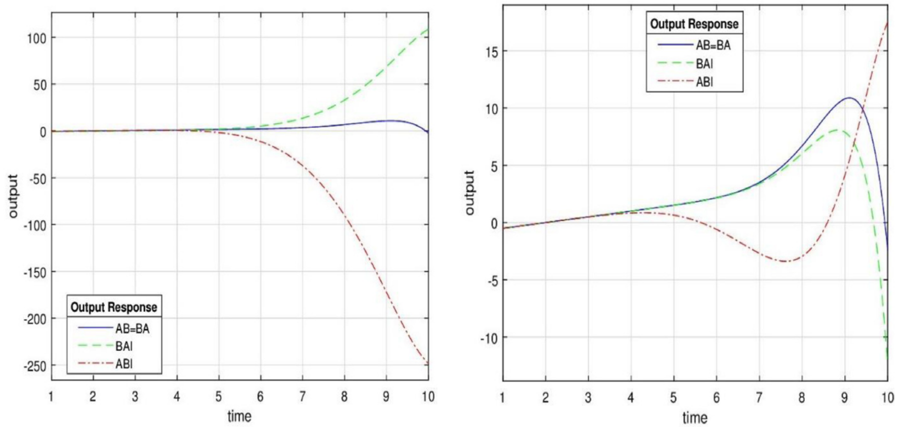

Figure 1. Simulation results for the example.

5. Conclusions

This paper studied the commutativity of high-order LTVSs. Commutative formulas for high-order are presented. The results show that the LTVSs *A* in equation (5) and *B* in equation (8) are commutative under certain conditions, however, the commutative systems *AB* and *BA* are sensible toward changes in initial conditions and parameters. Commutativity theories are very important in science and engineering. This study can be extended to nonlinear, partial, and fractional differential equation.

References

[1] E. Marshall, Commutativity of time-varying systems, Electro Lett. 18 (1977), 539-540.

[2] S. V. Saleh, Comments on ‘Commutativity of second-order time-varying systems’, International Journal of Control 37 (1983), 1195-1196.

[3] M. Koksal, General conditions for commutativity of time-varying systems, Telecommunication and Control (1984), 225-229.

[4] M. Koksal, Commutativity of 4th order systems and Euler systems time-varying systems, Proceedings of the National Congress of Electric Engineering, 1985, BI-6.

[5] M. E. Koksal and M. Koksal, Commutativity of linear time-varying differential systems with nonzero initial conditions: A Review. Mathematical Problems in Engineering (2011), 1-25.

[6] S. Ibrahim and M. E. Koksal, Commutativity of sixth-order time-varying linear systems, Circuits Syst. Signal Process. 40(10) (2021), 1-34. https://doi.org/10.1007/s00034-021-01709-6

[7] S. Ibrahim and M. E. Koksal, Realization of a fourth-order linear time-varying differential system with nonzero initial conditions by cascaded two second-order commutative pairs, Circuits Syst. Signal Process, 2021. https://doi.org/10.1007/s00034-020-01617-1

[8] S. Ibrahim and M. E. Koksal, Decomposition of fourth-order linear time-varying differential system into its twin second-order commutative pairs, 3rd International Symposium on Multidisciplinary Studies and Innovative Technologies, Ankara 2019, pp. 224-226.

[9] S. Ibrahim and M. E. Koksal, Decomposition of fourth-order linear time-varying system, 4th International Symposium on Innovative Approaches in Engineering and Natural Sciences, Samsun 4(6) (2019), pp. 139-141.

[10] S. Ibrahim, Explicit commutativity and stability for the Heun’s linear time-varying differential systems, Authorea 2021. https://doi.org/10.22541/au.162566323.35099726/v1

[11] S. Ibrahim and A. Rababah, Decomposition of fourth-order Euler-type linear time-varying differential systems, into Cascaded Two Second-Order Euler Commutative Pairs, Complexity, Vol. 2022, Article ID 3690019, 9 pages. https://doi.org/10.1155/2022/3690019