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HIGHER REGULARITY FOR PARABOLIC EQUATIONS BASED ON MAXIMAL L_p-L_q SPACES

Naoto Kajiwara

Applied Physics Course
Department of Electrical, Electronic
and Computer Engineering
Gifu University
1-1 Yanagido Gifu
Gifu, 501-1193, Japan
e-mail: kajiwara@gifu-u.ac.jp

Abstract

In this paper, we prove higher regularity for $2m$th order parabolic equations with general boundary conditions. This is a kind of maximal L_p-L_q regularity with differentiability, i.e., the main theorem is isomorphism between the solution space and the data space using Besov and Triebel-Lizorkin spaces. The key is compatibility condition for the initial data. As a corollary, we are able to get a unique smooth solution if the data satisfying compatibility conditions are smooth.

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1. Introduction

We consider the following parabolic evolution equations:

$$\begin{cases} \partial_t u + \omega u + \mathcal{A}(x, D)u = f(t, x) & (t \in \mathbb{R}_+, x \in G), \\ \mathcal{B}_j(x, D)u = g_j(t, x) & (t \in \mathbb{R}_+, x \in \Gamma, j = 1, ..., m), \\ u(0, x) = u_0(x) & (x \in G). \end{cases} \tag{1.1}$$

The operators are $\mathcal{A}(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$ and $\mathcal{B}_j(x, D) = \sum_{|\beta| \leq m_j} b_{j\beta} D^\beta$ with $0 \leq m_j < 2m$, $m \in \mathbb{N}$, where $D = -i\partial_x$. The domain G is bounded or exterior domain with a smooth boundary $\Gamma := \partial G$. For given f, $\{g_j\}_{j=1}^m$ and u_0, we show the unique solvability of u in the point of maximal L_p-L_q regularity, which means the isomorphism between them. One character of our paper is to treat higher regularity of data and solution. Our main theorem is that the solution belongs to a suitable regularity class when the data are L_p in time and L_q in space with differentiability and vice versa. The proof is based on the induction argument. To use this argument, we need to focus on the compatibility conditions of the initial data. When we require the regularity of the solutions, we should take care of the initial boundary data. This compatibility assumption is essential because we construct the theorem by sufficient and necessary. There are a lot of results on maximal regularity and relevant operator theory. We refer to [1, 2, 7] for parabolic and elliptic type, [3] for mixed order type, [4] for relaxation type and [6] for quasi-steady type. In particular, we refer to the comprehensive book [10]. This book lists various methods, results and even higher order maximal L_p-L_q regularity for parabolic and elliptic equations. However, only the space regularity for the higher regularity is considered in parabolic case, i.e., base space is L_p in time and H_q^s in space. Therefore, the compatibility conditions became a little complicated. In fact, they stated the case $s = 1$, while they avoided other case s. On the other hand, there is a higher regularity based on maximal L_2-L_2 regularity in the book [5] for the second order equation with zero Dirichlet boundary conditions. The main result of our paper is a generalization to the L_p-L_q settings and 2*m*th order equation with general boundary conditions. The key is to set the base space $H_p^k(\mathbb{R}_+; L_q(G)) \cap L_p(\mathbb{R}_+; H_q^{2mk}(G))$. This is a natural space to consider higher order differentiability, which is a different point from the book [10]. The function spaces of initial data and boundary data are used Besov and Triebel-Lizorkin spaces. This is similar to the classical case $k = 0$, but we give the proof again. The compatibility conditions are required according to the differentiability index *k*. By considering *k* inductively, we are able to get the smooth solution as a corollary of the main theorem.

This paper is organized as follows. In Section 2, we consider the equation more precisely. After setting the situation and assumptions, we state our main theorem. In Section 3, we prepare for the definitions and lemmas to prove the necessity of the theorem. In Section 4, we prove the main theorem. The sufficient conditions are proved by the induction argument as we can see the book [5]. The necessity conditions are almost similar to the results on [2, 10] which is explained in Section 3.

2. Settings and Main Theorem

2.1. Some settings

At first, we define some function spaces to state our main theorems. Few terminologies are given in Section 3. For almost all of definitions and notations in this section, we can find in [1, 3, 9-11].

Let *X* be a Banach space, $\mathbb{N} := \mathbb{N} \cup \{0\}$, $r \geq 0$ and $1 < p, q < \infty$. Then we define the vector-valued Besov and Triebel-Lizorkin spaces by

$$B^r_{p,q}(\mathbb{R}^n; X)$$

$$:= \{u \in \mathcal{S}'(\mathbb{R}^n; X) | \|u\|_{B^r_{p,q}(\mathbb{R}^n; X)} := \|(2^{rj}\|\mathcal{F}^{-1}[\phi_j \mathcal{F}u]\|_{L_p(\mathbb{R}^n; X)})_{j \in \mathbb{N}_0}\|_{l_q} < \infty\},$$

$F^r_{p,q}(\mathbb{R}^n; X)$

$:= \{u \in \mathcal{S}'(\mathbb{R}^n; X) | \|u\|_{F^r_{p,q}(\mathbb{R}^n; X)} := \| \|(2^{rj}\mathcal{F}^{-1}[\phi_j \mathcal{F}u])_{j\in\mathbb{N}_0}\|_{l_q(X)}\|_{L_p(\mathbb{R}^n)} < \infty\}$,

where $\{\phi_j\}_{j\in\mathbb{N}}$ is Littlewood-Paley smooth dyadic decomposition. Moreover, for a domain $\Omega \subset \mathbb{R}^n$ we define $B^r_{p,q}(\Omega; X)$ and $F^r_{p,q}(\Omega; X)$ by a restriction of $B^r_{p,q}(\mathbb{R}^n; X)$ and $F^r_{p,q}(\mathbb{R}^n; X)$ to Ω.

Note that

$$B^r_{p,q}(\Omega; X) = (H^{k_0}_p(\Omega; X), H^{k_1}_p(\Omega; X))_{\theta,q}$$

with $k_0, k_1 \in \mathbb{N}_0$, $\theta \in (0, 1)$, $r = (1-\theta)k_0 + \theta k_1$, where $(\cdot, \cdot)_{\theta,q}$ is a real interpolation. We use this relation later.

Definition 2.1. (a) A Banach space X is said to be of *class* $\mathcal{HT}$ if the Hilbert transform H defined by

$$Hf(t) := \frac{1}{\pi}\lim_{R\to\infty}\int_{R^{-1}\le|s|\le R} f(t-s)\frac{ds}{s}$$

is bounded linear operator on $L_p(\mathbb{R}; X)$ for some $p \in (1, \infty)$.

(b) A Banach space X is said to have *property* (α) if there exists $C > 0$ such that

$$\left|\sum_{i,j=1}^{N} \alpha_{i,j}\varepsilon_i\varepsilon'_j x_{ij}\right|_{L_2(\Omega\times\Omega'; X)} \le C\left|\sum_{i,j=1}^{N} \varepsilon_i\varepsilon'_j x_{ij}\right|_{L_2(\Omega\times\Omega'; X)}$$

for all $\alpha \in \{-1, 1\}$, $\{x_{ij}\}_{i,j=1}^N \subset X$, $N \in \mathbb{N}$, and all symmetric independent $\{-1, 1\}$-valued random variables ε_i (respectively ε'_j) on a probability space $(\Omega, \mathcal{A}, \mu)$ (respectively $(\Omega', \mathcal{A}', \mu')$).

(c) $\mathcal{HT}(\alpha)$ denotes the class of Banach spaces which belong to $\mathcal{HT}$ having property (α).

We consider the function $u(x, t) \in E \in \mathcal{HT}(\alpha)$ and note that $\mathbb{C}^{N\times N}$ is a Banach space of class $\mathcal{HT}(\alpha)$, so we are able to consider a system of usual parabolic equations.

We prepare for function spaces of f and $\{g_j\}_{j=1}^m$. Let $k \in \mathbb{N}_0$, $\kappa_j = 1 - \frac{m_j}{2m} - \frac{1}{2mq}$ and

$$\mathbb{F}_1^k := H_p^k(\mathbb{R}_+; L_q(G; E)) \cap L_p(\mathbb{R}_+; H_q^{2mk}(G; E)),$$

$$\mathbb{F}_2^{k, m_j} := F_{p,q}^{k+\kappa_j}(\mathbb{R}_+; L_q(\Gamma; E)) \cap L_p(\mathbb{R}_+; B_{q,q}^{2m(k+\kappa_j)}(\Gamma; E)).$$

Then we need to consider the operators $(\mathcal{A}(x, D), \mathcal{B}_1(x, D), ..., \mathcal{B}_m(x, D))$. For the regularity of the operators, we assume the following (R_k):

$(R_k)_1$ For each $|\alpha| = 2m$, the highest order coefficients $|\alpha|$ are bounded continuous, and $\lim_{|x|\to\infty} a_\alpha$ exists if G is unbounded.

$(R_k)_2$ $a_\alpha \in H_{r_l}^k(G; \mathcal{B}(E)) + H_\infty^k(G; \mathcal{B}(E))$ for each $|\alpha| = l \leq 2m$, with $r_l \geq q$ and $2m + k - l > \frac{n}{r_l}$.

$(R_k)_3$ $b_{j\beta} \in B_{r_{jl},q}^{2m\kappa_j+k}(\Gamma; \mathcal{B}(E))$ for each $|\beta| = l \leq m_j$, with $r_{jl} \geq q$ and $2m\kappa_j + k > \frac{n-1}{r_{jl}}$.

Above regularity assumption is the same used in higher order elliptic problems, see [10].

Definition 2.2. We call the system $(\mathcal{A}(x, D), \mathcal{B}_1(x, D), ..., \mathcal{B}_m(x, D))$ uniformly normally elliptic if

(i) The operator $\mathcal{A}(x, D)$ is normally elliptic in the sense that infimum of angle $\phi \in [0, \pi)$ such that $\sigma(\mathcal{A}(x, \xi)) \subset \Sigma_\phi$ for all $\xi \in \mathbb{R}^n$, $|\xi| = 1$ is less than $\pi/2$, for each $x \in \overline{G}$, and $x = \infty$ if G is unbounded.

(ii) The following Lopatinskii-Shapiro condition (LS) holds for each $x \in \Gamma$:

(LS) For all $(\lambda, \xi') \in (\Sigma_\theta \times \mathbb{R}^{n-1}) \backslash \{(0, 0)\} (\theta > \pi/2)$ and $\{g_j\}_{j=1}^m \in E^m$, the ODEs on the half line $\mathbb{R}_+$ given by

$$\begin{cases} \lambda v(y) + \mathcal{A}_\#(x, \xi', D_y) v(y) = 0 & (y > 0), \\ \mathcal{B}_{j\#}(x, \xi', D_y) v(0) = g_j & (j = 1, \ldots, m) \end{cases} \tag{2.1}$$

admit a unique solution $v \in C_0^\infty(\mathbb{R}_+; E)$, where the subscript # denotes the principal part of the corresponding operator

$$\mathcal{A}_\#(x, D) = \sum_{|\alpha| = 2m} a_\alpha(x) D^\alpha,$$

$$\mathcal{B}_{j\#}(x, D) = \sum_{|\beta| = m_j} b_{j\beta}(x) D^\beta$$

and

$$\sum_\phi := \{z \in \mathbb{C} \backslash \{0\} \,|\, |\arg z| < \phi\},$$

$$C_0^{2m}(\mathbb{R}_+; E) := \{v \in C^{2m}(\mathbb{R}_+; E) \,|\, \lim_{y \to \infty} v(y) = 0\}.$$

2.2. Main theorem

Theorem 2.3. *Let* $G \subset \mathbb{R}^n$ *be open with compact boundary of class* $C^{2m(k+1)}$, $k \in \mathbb{N}_0$, $1 < p, q < \infty$, *and let E be a Banach space of class* $\mathcal{HT}(\alpha)$. *Assume that* $(\mathcal{A}(x, D), \mathcal{B}_1(x, D), \ldots, \mathcal{B}_m(x, D))$ *is uniformly normally elliptic and satisfies* (R_{2mk}). *Let* $\kappa_j \neq 1/p$ *for all j. Then there is* $\omega_0 \in \mathbb{R}$ *such that for each* $\omega > \omega_0$, *equations* (1.1) *admit a unique solution u in the class*

$$u \in \mathbb{E}^k := H_p^{k+1}(\mathbb{R}_+; L_q(G; E)) \cap L_p(\mathbb{R}_+; H_q^{2m(k+1)}(G; E)),$$

if and only if the data are subject to the following conditions:

$$(f, \{g_j\}_{j=1}^m) \in \mathbb{F}_1^k \times \prod_{j=1}^m \mathbb{F}_2^{k, m_j},$$

$$v_0 := u_0 \in B_{q,p}^{2m\left(k+1-\frac{1}{p}\right)}(G) \text{ with } \mathcal{B}_j v_0 \in g_j|_{t=0},$$

$$v_1 := f|_{t=0} - (\omega + \mathcal{A}(x, D))v_0 \in B_{q,p}^{2m\left(k-\frac{1}{p}\right)}(G) \text{ with } \mathcal{B}_j v_1 = (\partial_t g_j)|_{t=0},$$

$$v_2 := (\partial_t f)|_{t=0} - (\omega + \mathcal{A}(x, D))v_1 \in B_{q,p}^{2m\left(k-1-\frac{1}{p}\right)}(G)$$

$$\text{with } \mathcal{B}_j v_2 = (\partial_t^2 g_j)|_{t=0},$$

$$\vdots$$

$$v_{k-1} := (\partial_t^{k-2} f)|_{t=0} - (\omega + \mathcal{A}(x, D))v_{k-2} \in B_{q,p}^{2m\left(2-\frac{1}{p}\right)}(G)$$

$$\text{with } \mathcal{B}_j v_{k-1} = (\partial_t^{k-1} g_j)|_{t=0},$$

$$v_k := (\partial_t^{k-1} f)|_{t=0} - (\omega + \mathcal{A}(x, D))v_{k-1} \in B_{q,p}^{2m\left(1-\frac{1}{p}\right)}(G)$$

$$\text{with } \mathcal{B}_j v_k = (\partial_t^k g_j)|_{t=0} \text{ if } \kappa_j > \frac{1}{p}.$$

The solution depends continuously on the data in the corresponding spaces, *i.e., there is* $C > 0$ *such that*

$$\| u \|_{\mathbb{E}^k} \le C\left(\| f \|_{\mathbb{F}_1^k} + \sum_{j=1}^m \| g_j \|_{\mathbb{F}_2^{k, m_j}} + \| u_0 \|_{B_{q,p}^{2m\left(k+1-\frac{1}{p}\right)}(G)} \right)$$

for all $(f, \{g_j\}, u_0)$ *satisfying conditions*.

Remark 2.4. Since we would like to construct the global estimate, i.e., time interval $\mathbb{R}_+$, we need the term ωu in equations (1.1). If we replace the time interval $[0, T]$ for some $T \in (0, \infty)$, then we are able to set $\omega = 0$.

Remark 2.5. For the case $\kappa_j = 1/p$, there is a unique solution $u \in E^k$ under the above conditions including $\mathcal{B}_j v_k = (\partial_t^k g_j)|_{t=0}$. However, it is a delicate problem of the boundary initial trace of v_k. We expect the last compatibility conditions as neither its strong sense (like $\kappa_j > 1/p)$ nor nothing (like $\kappa_j < 1/p$).

Corollary 2.6. *Let* $T \in (0, \infty)$. *Assume* G *is an open with compact boundary of class* C^∞, $f \in C^\infty([0, T] \times \overline{G})$, $g_j \in C^\infty([0, T] \times \Gamma)$ *for all* j, $u_0 \in C^\infty(\overline{G})$ *and the kth order compatibility conditions hold for all* $k \in \mathbb{N}_0$. *Then the parabolic equations* (1.1) *have a unique solution* $u \in C^\infty([0, T] \times \overline{G})$.

Proof. Apply Theorem 2.3 for k inductively and use embedding theorem.

3. Preliminaries

3.1. Trace to the initial data

In this subsection, we consider the trace to the initial data. This needs the necessity of the maximal regularity. For the classical $k = 0$ case, the characterization of the initial space has shown, e.g. [10] by the real interpolation theory. However, we are not able to find the case of $k \neq 0$ which is the higher order regularity. So, we consider the trace space. The method is almost same as before, i.e., the real interpolation.

Definition 3.1. *Let* A *be a closed linear operator in* X. *Then the operator* A *is called sectorial if the following two conditions are satisfied*:

(i) $\overline{D(A)} = \overline{R(A)} = X$, $(-\infty, 0) \in \rho(A)$.

(ii) *There exists* $M > 0$ *such that* $|t(t + A)^{-1}| \leq M$ *for all* $t > 0$.

If $(-\infty, 0) \subset \rho(A)$ *and* (ii) *hold, then the operator* A *is said to be pseudo-sectorial. For the pseudo-sectorial operator, spectral angle is given by*

$$\phi_A := \inf\{\phi \,|\, \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda \in \Sigma_{\pi-\phi}} |\lambda(\lambda + A)^{-1}| < \infty\}.$$

Definition 3.2. Let A be a densely defined pseudo-sectorial operator with spectral angle $\phi_A < \pi/2$ in X. Let $\alpha \in (0, 1)$ and $p \in [1, \infty)$. Then the space $D_A(\alpha, p)$ is defined by means of

$$D_A(\alpha, p) := \left\{ x \in X \,|\, [x]_{\alpha, p} := \left(\int_0^\infty |t^{1-\alpha} A e^{-tA} x|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty \right\}.$$

When equipped with the norm

$$|x|_{\alpha, p} := |x| + [x]_{\alpha, p}, \quad x \in D_A(\alpha, p),$$

the space $D_A(\alpha, p)$ becomes a Banach space. For $k \in \mathbb{N}$, the space $D_A(k + \alpha, p)$ is defined by

$$D_A(k + \alpha, p) := \{x \in D(A^k) \,|\, A^k x \in D_A(\alpha, p)\}, \; |x|_{k+\alpha, p} := |x| + [A^k x]_{\alpha, p}.$$

Lemma 3.3. *Suppose A is a densely defined invertible sectorial operator in X with spectral angle* $\phi_A < \pi/2$, $p \in (1, \infty)$ *and* $k \in \mathbb{N} \cup \{0\}$. *Then for the function* $u(t) := e^{-tA}x$, *the following assertions are equivalent*:

(a) $u(t) \in D(A^{k+1})$ *for a.e.* $t > 0$ *and* $u \in L_p(\mathbb{R}_+; D(A^{k+1}))$,

(b) $u \in H_p^{k+1}(\mathbb{R}_+; X)$,

(c) $x \in D_A(k + 1 - 1/p, p)$.

In this case, there is a constant C depending only on k, p and A such that

$$|u|_{H_p^k(\mathbb{R}_+;\, X)} + |u|_{L_p(\mathbb{R}_+;\, D(A^k))} \le C|x|_{k+1-1/p,\, p}.$$

Proof. By a standard semigroup theory, we have $e^{-tA}X \subset D(A^{k+1})$ for $t > 0$ and

$$|e^{-tA}| + t|Ae^{-tA}| \le Me^{-\omega t},\ t > 0$$

for some $\omega > 0$. By definition, $x \in D_A(k+1-1/p,\ p)$ implies $A^k x \in D_A(1-1/p,\ p)$, i.e., $Ae^{-tA}(A^k x) \in L_p(\mathbb{R}_+;\ X)$. This and commutativity of A and e^{-tA} mean $e^{-tA}x \in L_p(\mathbb{R}_+;\ D(A^{k+1}))$, hence (c) implies (a). Since e^{-tA} is holomorphic and $\frac{d}{dt}e^{-tA} = -Ae^{-tA}$ for $t > 0$, $\frac{d^{k+1}}{dt^{k+1}}u = (-A)^{k+1}u \in L_p(\mathbb{R}_+;\ X)$ if we assume (a), hence (a) implies (b). On the other hand, (b) yields $A^{k+1}u = (-1)^{k+1}\frac{d^{k+1}}{dt^{k+1}}u \in L_p(\mathbb{R}_+;\ X)$ and

$$[A^k x]_{1-1/p}^p = |A^{k+1}u|_{L_p(\mathbb{R}_+;\, X)}^p.$$

This shows (b) implies (c).

3.2. Trace to the boundary data

In this subsection, we consider the trace to the boundary data. This also needs the necessity of the maximal regularity. We cite the results on Denk et al. [2], which characterize the boundary trace by Triebel-Lizorkin space, see also [10].

Let $L_0 := \omega + \partial_t + (-\Delta_x)^m$ in the space $X_0 := L_p(\mathbb{R}_+;\ L_q(\mathbb{R}^{n-1};\ E))$ with domain

$$D(L_0) = {}_0H_p^1(\mathbb{R}_+;\ L_q(\mathbb{R}^{n-1};\ E)) \cap L_p(\mathbb{R}_+;\ H_q^{2m}(\mathbb{R}^{n-1};\ E)),$$

where the index 0 means that initial trace of the function takes value 0. This operator is a sectorial operator with angle $\pi/2$ and $-L_0^{1/2m}$ is the generator of an analytic C_0-semigroup $e^{-yL_0^{1/2m}}$. Let L be the canonical extension of L_0 to the space $L_p(\mathbb{R}_+; L_q(\mathbb{R}_+^n; E))$.

Lemma 3.4. *Let* $1 < p, q < \infty$ *and* E *be a Banach space with property* $\mathcal{HT}(\alpha)$. *Moreover, let* L_0 *and* L *be defined as above, and let* $u(y) := e^{-yL_0^{1/2m}} g,\ g \in X_0,\ y > 0.$ *Then the following assertions are equivalent*:

(a) $u \in {}_0H_p^{1/2m}(\mathbb{R}_+; L_q(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E)) \cap L_p(\mathbb{R}_+; H_q^1(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E))$,

(b) $L^{\frac{1}{2m}} u \in L_p(\mathbb{R}_+; L_q(\mathbb{R}_+ \times \mathbb{R}^{n-1}; E))$,

(c) $g \in {}_0F_{p,q}^{1/2m-1/(2mq)}(\mathbb{R}_+; L_q(\mathbb{R}^{n-1}; E)) \cap L_p(\mathbb{R}_+; B_{q,q}^{1-1/q}(\mathbb{R}^{n-1}; E))$.

Similar statements are valid on $\mathbb{R}$.

3.3. Higher regularity for the elliptic equations

In this subsection, we collect the results on higher regularity for the elliptic equations. The result is written in [10, p. 279]. We do not need the compatibility conditions, which is different from parabolic problems.

Theorem 3.5. *For* $f \in H_q^k(G; E)$ *and* $g_j \in B_{q,q}^{2m\kappa_j + k}(\Gamma; E)$, *the elliptic problem*

$$\begin{cases} (\omega + \mathcal{A}(x, D))u = f & (x \in G), \\ \mathcal{B}_j(x, D)u = g_j & (x \in \Gamma,\ j = 1, \dots, m) \end{cases}$$

has a unique solution in $H_q^{2m}(G; E)$, *provided* $\mathcal{A}(x, D)$ *is normally elliptic, Lopatinskii-Shapiro condition* (*LS*) *holds*, $\omega > s(-A) := \sup \Re\sigma(-A)$, $\Gamma \in C^{2m+k}$, *and the coefficients satisfy the regularity conditions* (R_k).

4. Proof of Theorem 2.3

Proof of a sufficiency of the main theorem. The proof is based on an induction on k. For the sake of simplicity of notation, we prove the theorem up to the case $k+1$. The case $k=0$ is just a result of the paper [2]. Assume that a sufficient condition of the theorem is valid for some nonnegative integer k, and suppose that the data $(f, \{g_j\}_{j=1}^m, u_0)$ satisfy the $k+1$ st compatibility conditions. By differentiate the equation with respect to t, we consider, by setting $\tilde{u} := \partial_t u$, $\tilde{f} := \partial_t f$ and $\widetilde{g_j} := \partial_t g_j$,

$$\begin{cases} \partial_t \tilde{u} + \omega \tilde{u} + \mathcal{A}(x, D)\tilde{u} = \tilde{f}(t, x) & (t \in J, x \in G), \\ \mathcal{B}_j(x, D)\tilde{u} = \widetilde{g_j}(t, x) & (t \in J, x \in \Gamma, j = 1, \ldots, m), \\ \tilde{u}(0, x) = f|_{t=0} - (\omega + \mathcal{A}(x, D))u_0 =: \widetilde{u_0} & (x \in G). \end{cases}$$

Note that

$$(\tilde{f}, \{\widetilde{g_j}\}_{j=1}^m) \in \mathbb{F}_1^k \times \prod_{j=1}^m \mathbb{F}_2^{k, m_j},$$

$$\widetilde{v_0} := \widetilde{u_0} \in B_{q,p}^{2m\left(k+1-\frac{1}{p}\right)}(G) \text{ with } \mathcal{B}_j v_0 \in g_j|_{t=0},$$

$$\widetilde{v_1} := \tilde{f}|_{t=0} - (\omega + \mathcal{A}(x, D))\widetilde{v_0} \in B_{q,p}^{2m\left(k-\frac{1}{p}\right)}(G) \text{ with } \mathcal{B}_j \widetilde{v_1} = (\partial_t \widetilde{g_j})|_{t=0},$$

$$\widetilde{v_2} := (\partial_t \tilde{f})|_{t=0} - (\omega + \mathcal{A}(x, D))\widetilde{v_1} \in B_{q,p}^{2m\left(k-1-\frac{1}{p}\right)}(G)$$

$$\text{with } \mathcal{B}_j \widetilde{v_2} = (\partial_t^2 \widetilde{g_j})|_{t=0},$$

$$\vdots$$

$$\widetilde{v_{k-1}} := (\partial_t^{k-2} \tilde{f})|_{t=0} - (\omega + \mathcal{A}(x, D))\widetilde{v_{k-2}} \in B_{q,p}^{2m\left(2-\frac{1}{p}\right)}(G)$$

$$\text{with } \mathcal{B}_j \widetilde{v_{k-1}} = (\partial_t^{k-1} \widetilde{g_j})|_{t=0},$$

$$\widetilde{v_k} := (\partial_t^{k-1} \tilde{f})|_{t=0} - (\omega + \mathcal{A}(x, D))\widetilde{v_{k-1}} \in B_{q,p}^{2m\left(1-\frac{1}{p}\right)}(G)$$

$$\text{with } \mathcal{B}_j \widetilde{v_k} = (\partial_t^k \widetilde{g_j})|_{t=0} \text{ if } \kappa_j > \frac{1}{p}.$$

Namely, $(\tilde{f}, \{\widetilde{g_j}\}_{j=1}^m, \widetilde{u_0})$ satisfies the kth compatibility. Thus, applying the induction assumption, we deduce $\tilde{u} \in \mathbb{E}^k$ and

$$\| \tilde{u} \|_{\mathbb{E}^k} \leq C\left(\| \tilde{f} \|_{\mathbb{F}_1^k} + \sum_{j=1}^m \| \widetilde{g_j} \|_{\mathbb{F}_2^{k, m_j}} + \| \widetilde{u_0} \|_{B_q^{2m(k+1-1/p)}} \right).$$

This implies

$$\partial_t u \in \mathbb{E}^k = H_p^{k+1}(J; L_q(G; E)) \cap L_p(J; H_q^{2m(k+1)}(G; E))$$

and

$$\| \partial_t u \|_{\mathbb{E}^k}$$

$$\leq C\left(\| \partial_t f \|_{\mathbb{F}_1^k} + \sum_{j=1}^m \| \partial_t g_j \|_{\mathbb{F}_2^{k, m_j}} + \| f|_{t=0} - (\omega + \mathcal{A}(x, D))u_0 \|_{B_{q,p}^{2m\left(k+1-\frac{1}{p}\right)}(G)} \right)$$

$$\leq C\left(\| f \|_{\mathbb{F}_1^{k+1}} + \sum_{j=1}^m \| g_j \|_{\mathbb{F}_2^{k+1, m_j}} + \| u_0 \|_{B_{q,p}^{2m\left(k+2-\frac{1}{p}\right)}(G)} \right),$$

where we used the estimate

$$\| f|_{t=0} \|_{B_{q,p}^{2m\left(k+1-\frac{1}{p}\right)}} \leq C \| f \|_{\mathbb{F}_1^{k+1}}.$$

Here and hereafter we use the letter C to denote any constant that is not important, which may be different from line to line.

We next consider the following the higher order elliptic problems for each fixed t:

$$\begin{cases} (\omega + \mathcal{A}(x, D))u = f - \partial_t u & (x \in G), \\ \mathcal{B}_j(x, D)u = g_j & (x \in \Gamma,\ j = 1, \ldots, m). \end{cases}$$

Under the assumptions on the regularity of the coefficients (R_{k+1}), we have $u(t) \in H_q^{2m(k+2)}(G)$ by Theorem 3.5 and

$$\| u(t) \|_{H_q^{2m(k+2)}(G)}$$

$$\le C\left(\| f(t) - \partial_t u(t) \|_{H_q^{2m(k+1)}(G)} + \sum_{j=1}^{m} \|g_j(t)\|_{B_{q,q}^{2m(k+2)-m_j-\frac{1}{q}}(\Gamma)} + \|u(t)\|_{L_q(G)} \right)$$

$$\le C\left(\| f \|_{H_q^{2m(k+1)}(G)} + \| \partial_t u(t) \|_{H_q^{2m(k+1)}(G)} + \sum_{j=1}^{m} \|g_j(t)\|_{B_{q,q}^{2m(k+1)+\kappa_j}(\Gamma)} + \|u(t)\|_{L_q(G)} \right),$$

for a.e. t. We take $L_p(\mathbb{R}_+)$-norm in time, then we have

$$\| u \|_{L_p(\mathbb{R}_+; H_q^{2m(k+2)}(G))}$$

$$\le C\left(\| f \|_{L_p(\mathbb{R}_+; H_q^{2m(k+1)}(G))} + \| \partial_t u \|_{L_p(\mathbb{R}_+; H_q^{2m(k+1)}(G))} \right.$$

$$\left. + \sum_{j=1}^{m} \| g_j \|_{L_p(\mathbb{R}_+;\, B_{q,q}^{2m(k+1)+\kappa_j}(\Gamma))} + \|u\|_{L_p(\mathbb{R}_+; L_q(G))} \right)$$

$$\le C\left(\| f \|_{\mathbb{F}_1^{k+1}} + \sum_{j=1}^{m} \| g_j \|_{\mathbb{F}_2^{k+1, m_j}} + \|u_0\|_{B_{q,p}^{2m\left(k+2-\frac{1}{p}\right)}(G)} + \| u \|_{L_p(\mathbb{R}_+; L_q(G))} \right),$$

by using the estimate (4.2). We again consider the estimate (4.2), and deduce

$$\| u \|_{\mathbb{E}^{k+1}} \le C\left(\| f \|_{\mathbb{F}_1^{k}} + \sum_{j=1}^{m} \| g_j \|_{\mathbb{F}_2^{k+1, m_j}} + \|u_0\|_{B_{q,p}^{2m\left(k+2-\frac{1}{p}\right)}(G)} \right)$$

since we have the estimate

$$\| u \|_{L_p(\mathbb{R}_+; L_q(G))} \le C\left(\| f \|_{\mathbb{F}_1^{k+1}} + \sum\nolimits_{j=1}^{m} \| g_j \|_{\mathbb{F}_2^{k+1, m_j}} + \|u_0\|_{B_{q,p}^{2m\left(k+2-\frac{1}{p}\right)}(G)} \right).$$

This proves the sufficient condition to get the higher order maximal L_p-L_q regularity solutions.

Proof of a necessity of the main theorem. For the sake of simplicity, we prove the case that $G = \mathbb{R}^n_+$. The desired case is straightforward by coordinate transformation. Let $u \in \mathbb{E}^k$ be the solution of (1.1). Then we have $f \in \mathbb{F}^k_1$. We extend the function u in space so that $E_x u \in H^{k+1}_p(\mathbb{R}_+; L_q(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; H^{2m(k+1)}_q(\mathbb{R}^n))$, where E_x is an extension operator from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. From Lemma 3.3, we know

$$E_x u|_{t=0} \in B^{2m\left(k+1-\frac{1}{p}\right)}_{q,\,p}(\mathbb{R}^n;\, E),$$

which implies $u_0 \in B^{2m\left(k+1-\frac{1}{p}\right)}_{q,\,p}(\mathbb{R}^n_+;\, E)$. Moreover, $\partial_t u = f - (\omega + \mathcal{A})u$ $=: v_1$ at

$$\begin{aligned}
\partial_t^2 u &= \partial_t(f - (\omega + \mathcal{A})u) \\
&= \partial_t f - (\omega + \mathcal{A})v_1 =: v_2 \\
\partial_t^3 u &= \partial_t(\partial_t f - (\omega + \mathcal{A})v_1) \\
&= \partial_t^2 f - (\omega + \mathcal{A})v_2 =: v_3 \\
&\;\;\vdots
\end{aligned}$$

at $t = 0$ derives the compatibility conditions whose regularity is same as the theorem.

As a next step, we consider the tangential part of $y := x_n = 0$. We extend the function u in time so that $E_t u =: v \in H^{k+1}_p(\mathbb{R}; L_q(\mathbb{R}^n_+))$ $\cap\, L_p(\mathbb{R}; H^{2m(k+1)}_q(\mathbb{R}^n_+))$, where E_t is an extension operator from $\mathbb{R}_+$ to

$\mathbb{R}$. Hence, $w := (\omega + \partial_t)^{\alpha} \partial_y^l D_x^{\beta} v$ belongs $H_p^{1/2m}(\mathbb{R}; L_q(\mathbb{R}_+^n; E)) \cap L_p(\mathbb{R}; H_q^1(\mathbb{R}_+^n; E))$ if $2m\alpha + l + |\beta| = 2m(k+1) - 1$. Define the function $\overline{w}$ by the solution of

$$\partial_y \overline{w} + L_0^{1/2m} \overline{w} = \partial_y w + L_0^{1/2m} w, \; y > 0, \; \overline{w}(0) = 0.$$

This function also belongs to $H_p^{1/2m}(\mathbb{R}; L_q(\mathbb{R}_+^n; E)) \cap L_p(\mathbb{R}; H_q^1(\mathbb{R}_+^n; E))$ since L_0 satisfies the maximal regularity, see [2, 10]. Hence, $w - \overline{w} = e^{-yL_0^{1/2m}} w|_{y=0}$ has same regularity as well. Then we are able to use Lemma 3.4 to get

$$w|_{y=0} \in F_{p,q}^{1/2m - 1/(2mq)}(\mathbb{R}; L_q(\mathbb{R}^{n-1}; E)) \cap L_p(\mathbb{R}; B_{q,q}^{1-1/q}(\mathbb{R}^{n-1}; E)).$$

By a definition of w and proper choices of β and l, these yield

$$\mathcal{B}_j(x, D)v \in F_{p,q}^{k+\kappa_j}(\mathbb{R}; L_q(\mathbb{R}^{n-1}; E)) \cap L_p(\mathbb{R}; B_{q,q}^{2mk+2m\kappa_j}(\mathbb{R}^{n-1}; E)),$$

by restriction to $t > 0$. We finally obtain $g_j \in \mathbb{F}_2^{k, m_j}$. This completes the necessity of the main theorem.

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References

[1] R. Denk, M. Hieber and J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166(788) (2003), 114.

[2] R. Denk, M. Hieber and J. Prüss, Optimal L_p-L_q-estimate for parabolic problems with inhomogeneous boundary data, Math. Z. 257(1) (2007), 193-224.

[3] R. Denk and M. Kaip, General parabolic mixed order systems in L_p and applications, Operator Theory: Advances and Applications, Vol. 239, Birkhäuser/Splinger Cham, 2013.

[4] R. Denk, J. Prüss and R. Zacher, Maximal L_p-regularity of parabolic problems with boundary dynamics of relaxation style, J. Funct. Anal. 255(11) (2008), 3149-3187.

[5] L. C. Evans, Partial differential equations, 2nd ed., Graduate Studies in Mathematics 19, Amer. Math. Soc., Providence, RI., 2010.

[6] K. Furukawa and N. Kajiwara, Maximal L_p-L_q regularity for the quasi-steady elliptic problems, J. Evol. Eq. 21 (2021), 1601-1625.

[7] P. C. Kunstmann and L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, Functional Analytic Methods for Evolution Equations, Vol. 1855 of Lecture Notes in Math. Springer, Berlin, 2004, pp. 65-311.

[8] A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Progress in Nonlinear Differential Equations and their Applications, Vol. 16, Birkhäuser Verlag, Basel, 1995.

[9] A. Lunardi, Interpolation theory, 2nd ed., Appunti, Scoula Normale Superiore di Pisa, Edizioni della Normale, Pisa, 2009.

[10] J. Prüss and G. Simonett, Moving interfaces and Quasilinear Parabolic Evolution Equations, Vol. 105, Birkhäuser/Splinger, Cham., 2016.

[11] H. Triebel, Theory of Function Spaces, Monographs in Mathematics, Vol. 78, Birkhäuser Verlag, Basel, 1983.