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STRONGLY GENERALIZED SOLUTION OF A FRACTIONAL PROBLEM OF PARABOLIC EVOLUTION OF ORDER-TWO IN A PLATE WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract

The aim of this article is to prove uniqueness of solution to mix a fractional problem of parabolic evolution of order-two in a plate with integral boundary conditions:

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$$\begin{cases} D_t^{\alpha} v - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x}\right) - \frac{1}{x^2}\frac{\partial^2 v}{\partial y^2} = F(x, y, t) \\ v(x, y, 0) = \varphi(x, y) \\ v(\ell_1, y, t) = \frac{\partial v}{\partial y}(x, \ell_2, t) = 0 \\ \int_0^{\ell_1} xv(x, y, t)dx = 0 \\ \int_0^{\ell_2} v(x, y, t)dy = 0. \end{cases}$$

A functional analysis method is used. The proof is based on an energy inequality and on a priori estimates established in non-classical function spaces.

1. Posing of the Problem

In the plate $\Omega = (0, \ell_1) \times (0, \ell_2) \times (0, T)$, where $\ell_1 < +\infty$, $\ell_2 < +\infty$ and $T < +\infty$, we determine a solution *u* of the fractional differential equation:

$$D_t^{\alpha} v - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x}\right) - \frac{1}{x^2}\frac{\partial^2 v}{\partial y^2} = f(x, y, t) \tag{1.1}$$

satisfying the initial condition

$$v(x, y, 0) = \varphi(x, y), \tag{1.2}$$

the Neumann conditions

$$v(\ell_1, y, t) = \frac{\partial u}{\partial y}(x, \ell_2, t) = 0 \tag{1.3}$$

and the nonlocal conditions

$$\int_0^{\ell_1} xv(x, y, t)dx = 0, \tag{1.4}$$

$$\int_0^{\ell_2} v(x, y, t)dy = 0. \tag{1.5}$$

For the consistency, it follows that

$$\int_0^{\ell_1} x\varphi(x, y, t)dx = \int_0^{\ell_2} u(x, y, t)dy = 0,$$

$$\varphi(\ell_2, y) = \frac{\partial\varphi}{\partial y}(x, \ell_2) = 0,$$

where $0 < \alpha < 1$, $n \in \mathbb{N}^*$. The left Caputo derivative D_t^α and the gamma function Γ are, respectively, defined as

$$D_t^\alpha v(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial v}{\partial \tau} \frac{d\tau}{(t-\tau)^\alpha},$$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{1-\alpha} e^{-x} dx.$$

Fractional differential equations have generated a lot of interest to engineers and scientists in recent years. It is because FDEs have memory, nonlocal relations in space and time and complex phenomena can be modeled by using these equations. Indeed, we can find numerous applications in viscoelasticity, electro-chemistry, control theory, porous media, fluid flow, rheology, diffusive transport, electrical network, electromagnetic theory, probability, signal processing, and many other physical processes.

Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. The existence and uniqueness of solutions to initial and boundary-value problems for fractional differential equations has been extensively studied by many authors; see for example [1-4, 8, 10-15]. Some of the existence and uniqueness results have been obtained by using Laplace transform method, Fourier's method and energy-integral method [1, 2, 6, 7, 9, 14].

Motivated by this, we extend and generalize the study for PDEs with integral conditions to the study of two-dimensional fractional PDEs with integral conditions.

In this paper, we extend an energy-integral method to the study of mixed-type two-dimensional fractional differential equations.

This paper is outlined as follows: after this introductory section, in Section 2, we present abstract formulations of the posed problems and make precise the concept of solution of the problems. Finally, we establish a priori estimates which are derived to show the uniqueness and continuous dependence of the solution upon the data in Section 3.

2. Preliminaries

We introduce now a new function $u(x, y, t) = v(x, y, t) - \varphi(x, y)$. Then the problem can be formulated as:

$$D_t^{\alpha} u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x^2}\frac{\partial^2 u}{\partial y^2} = f(x, y, t), \tag{2.1}$$

$$u(x, y, 0) = \varphi(x, y), \tag{2.2}$$

$$u(\ell_1, y, t) = \frac{\partial u}{\partial y}(x, \ell_2, t) = 0, \tag{2.3}$$

$$\int_0^{\ell_1} xu(x, y, t)dx = 0, \tag{2.4}$$

$$\int_0^{\ell_2} u(x, y, t)dy = 0, \tag{2.5}$$

where

$$f(x, t) = F(x, t) + \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \varphi}{\partial x}\right) + \frac{1}{x^2}\frac{\partial^2 \varphi}{\partial y^2}.$$

Instead of searching for the function v, we search for the function u. So the solution of the problem (1.1), (1.2), (1.3), (1.4) and (1.5) will be given by $v(x, t) = u(x, t) + \varphi(x, y)$.

In this article, we establish an energy inequality which is derived to show the uniqueness and continuous dependence of the solution upon the data (2.1), (2.2), (2.3), (2.4) and (2.5). For this, we consider the problem (2.1)-(2.5) as a solution of the operator equation:

$$Lu = \mathcal{F} = f, \tag{2.6}$$

with domain of definition $D(L)$ consisting of functions $v \in L_2(\Omega)$ such that

$$D_t^\alpha u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial t} \in L_2(\Omega)$$

and v satisfies conditions (2.3), (2.4) and (2.5). The operator L is considered from E to F, where E is the Banach space consisting of functions $u \in L_2(\Omega)$, satisfying (2.3), (2.4) and (2.5) with the finite norm:

$$\int_\Omega x^3 (\Im_y D_t^\alpha u)^2 dxdydt + 2\int_\Omega x^3 (u_x)^2 dxdydt$$

$$+ \int_\Omega x(\Im_y u_x)^2 dxdydt + \int_\Omega x(u_y)^2 dxdydt, \tag{2.7}$$

where

$$\Im_x u = \int_0^x u(\xi, y, t) d\xi, \ \ \Im_y v = \int_0^y v(x, \eta, t) d\eta,$$

$$J_{xy} v = \Im_x (\Im_y v), \ \ \Im_y^2 v = \Im_{yy} v = \Im_y (J_y v).$$

Here F is the Hilbert space of vector-valued functions $\mathcal{F} = f$ obtained by completing of the space $L_2(\Omega)$ with respect to the norm

$$\| \mathcal{F} \|_F^2 = \int_\Omega f^2(x, y, t) dxdydt. \tag{2.8}$$

Definition 1. A solution of the operator $\overline{L}u = \mathcal{F}$ is called a *strong solution* of the problem (1.1), (1.2), (1.3), (1.4) and (1.5).

3. An Energy Inequality and its Applications

Theorem 3.1. *For any function* $v \in E$, *there is the a priori estimate*

$$\| u \|_E \leq c \| Lu \|_{\mathbb{F}}, \tag{3.1}$$

where c is a constant which may depend on T but not on v.

Proof. Consider the inner product in $L_2(\Omega)$, and the linear operator

$$Mu = -x^3 \Im_y^2 D_t^\alpha u + 2x^2 \Im_y^2 \frac{\partial u}{\partial x} + x^3 \Im_y \frac{\partial u}{\partial y}.$$

In light of the initial condition (2.2), the boundary conditions (2.3)-(2.5), and the standard integration by parts of the equation (2.1), we obtain

$$\int_\Omega x^3 (\Im_y D_t^\alpha u)^2 dxdydt - 2\int_\Omega x^2 \Im_y D_t^\alpha u \Im_y u_x dxdydt$$

$$- \int_\Omega x^3 \Im_y D_t^\alpha u u_y dxdydt$$

$$2\int_\Omega x^2 \Im_y u_x \Im_y D_t^\alpha u dxdydt + \int_\Omega x^3 \Im_y u_x \Im_y D_t^\alpha u_x dxdydt$$

$$+ a^2 \int_0^b \int_0^\tau x(\Im_y^2 u_x) dydt$$

$$- 2\int_\Omega x^2 u_y \Im_y u_x dxdydt + 2\int_\Omega x^3 (u_x)^2 dxdydt + \int_\Omega x^2 (D_t^{\frac{\alpha}{2}} u)^2 dxdydt$$

$$+ 2\int_\Omega u_y \Im_y u_x dxdydt + \int_\Omega x(u_y)^2 dxdydt$$

$$= -\int_\Omega x^3 f \Im_y^2 D_t^\alpha u dxdydt + 2\int_\Omega x^2 f \Im_y^2 u_x dxdydt$$

$$+ \int_{\Omega^\tau} x^3 f \Im_y u_y dxdydt. \tag{3.2}$$

Applying the elementary inequality and the Cauchy ε-inequality on certain terms of (2.3), we have

$$2\int_{\Omega} x^2 \Im_y D_t^{\alpha} u \Im_y u_x dxdydt \leq \varepsilon_1 \int_{\Omega} x^2 (\Im_y D_t^{\alpha} u)^2 dxdydt$$

$$+\frac{1}{\varepsilon_1}\int_{\Omega^{\tau}} x^2 (\Im_y u_x)^2 dxdydt, \tag{3.3}$$

$$-\int_{\Omega} x^3 u_y \Im_y D_t^{\alpha} udxdydt \leq \frac{\varepsilon_2}{2}\int_{\Omega} x^3 (\Im_y D_t^{\alpha} u)^2 dxdydt$$

$$+\frac{1}{2\varepsilon_2}\int_{\Omega} x^3 (u_y)^2 dxdydt, \tag{3.4}$$

$$2\int_{\Omega} x^2 u_y \Im_y u_x dxdydt \leq \varepsilon_3 \int_{\Omega} x^3 (u_y)^2 dxdydt$$

$$+\frac{1}{\varepsilon_3}\int_{\Omega} x^2 (J_y u_x)^2 dxdydt, \tag{3.5}$$

$$-2\int_{\Omega} u_y \Im_y u_x dxdydt \leq \varepsilon_4 \int_{\Omega} (u_y)^2 dxdydt$$

$$+\frac{1}{\varepsilon_4}\int_{\Omega} x^2 (J_y u_x)^2 dxdydt, \tag{3.6}$$

$$-2\int_{\Omega} \Im_y u_x \Im_y D_t^{\alpha} udxdydt \leq \varepsilon_5 \int_{\Omega} (\Im_y u_x)^2 dxdydt$$

$$+\frac{1}{\varepsilon_5}\int_{\Omega} (\Im_y D_t^{\alpha} u)^2 dxdydt, \tag{3.7}$$

$$-2\int_{\Omega} \Im_y u_x \Im_y D_t^{\alpha} u_x dxdydt \leq \varepsilon_6 \int_{\Omega} (\Im_y u_x)^2 dxdydt$$

$$+\frac{1}{\varepsilon_6}\int_{\Omega} (\Im_y D_t^{\alpha} u_x)^2 dxdydt, \tag{3.8}$$

$$-\int_{\Omega} x^3 f(\Im_{yy} D_t^{\alpha} u) dxdydt \leq \frac{\varepsilon_7}{2}\int_{\Omega^{\tau}} x^3 f^2 dxdydt$$

$$+\frac{1}{2\varepsilon_7}\int_{\Omega} x^3 (\Im_{yy} D_t^{\alpha} u)^2 dxdydt, \tag{3.9}$$

$$2\int_{\Omega} x^2 f(\Im_{yy}u_x)dxdydt \le \varepsilon_8 \int_{\Omega} x^2 f^2 dxdydt$$

$$+ \frac{b^2}{2\varepsilon_8}\int_{\Omega} x^2 (\Im_y u_x)^2 dxdydt, \tag{3.10}$$

$$\int_{\Omega} x^3 f(\Im_y u_y)dxdydt \le \frac{\varepsilon_9}{2}\int_{\Omega} x^3 f^2 dxdydt$$

$$+ \frac{b^2}{4\varepsilon_9}\int_{\Omega} x^3 (u_y)^2 dxdydt. \tag{3.11}$$

Substituting (3.3)-(3.11) in (3.2), it follows that

$$\int_{\Omega} x^3(\Im_y D_t^{\alpha}u)^2 dxdydt + 2\int_{\Omega} x^3 (u_x)^2 dxdydt + \ell_1^2 \int_0^{\ell_2}\int_0^T x(\Im_y^2 u_x)dydt$$

$$+ \int_{\Omega} x^2 (D_t^{\frac{\alpha}{2}} u)^2 dxdydt + \int_{\Omega} x(u_y)^2 dxdydt$$

$$\le \int_{\Omega}\left(\varepsilon_1 x^2 + \frac{\varepsilon_2}{2} + \frac{1}{\varepsilon_5} + \frac{b^2x^3}{4\varepsilon_7}\right)(\Im_y D_t^{\alpha}u)^2 dxdydt$$

$$+ \int_{\Omega}\left(\frac{x^2}{\varepsilon_1} + \frac{1}{\varepsilon_4} + \frac{x^2}{\varepsilon_3} + \varepsilon_5 + \varepsilon_6 + \frac{b^2x^2}{2\varepsilon_8}\right)(\Im_y u_x)^2 dxdydt$$

$$+ \int_{\Omega}\left(\frac{1}{2\varepsilon_1} + \varepsilon_3 + \varepsilon_4 + \frac{b^2}{4\varepsilon_9}\right)(u_y)^2 dxdydt + \frac{b^2}{2\varepsilon_6}\int_{\Omega} (\Im_y D_t^{\alpha} u)^2 dxdydt$$

$$+ \left(\frac{\ell_1^3\varepsilon_7}{2} + \frac{\ell_1^3\varepsilon_9}{2} + \ell_1^2\varepsilon_8\right)\int_{\Omega} (f)^2 dxdydt. \tag{3.12}$$

By virtue of *f*, and the elementary inequality:

$$\int_0^{\ell_1} (\Im_x u)^2 dx \le \frac{\ell_1^2}{2}\int_0^{\ell_1} u^2 dx,$$

from (3.12), it follows that

$$\left(1-\varepsilon_1\ell^2-\frac{\varepsilon_2}{2}-\frac{1}{\varepsilon_5}-\frac{\ell_2^2\ell_1^3}{4\varepsilon_7}-\frac{b^2}{2\varepsilon_6}\right)\int_\Omega x^3(\Im_y D_t^\alpha u)^2\,dxdydt$$

$$+\,2\int_\Omega x^3(u_x)^2\,dxdydt$$

$$+\left(1-\frac{\ell_1^2}{\varepsilon_1}-\frac{1}{\varepsilon_4}-\frac{\ell_1^2}{\varepsilon_3}-\varepsilon_5-\varepsilon_6-\frac{\ell_2^2\ell_1^2}{2\varepsilon_8}\right)\int_\Omega x(\Im_y u_x)^2\,dxdydt$$

$$+\left(1-\frac{1}{2\varepsilon_1}-\varepsilon_3-\varepsilon_4-\frac{\ell_2^2}{4\varepsilon_9}\right)\int_\Omega x(u_y)^2\,dxdydt$$

$$\leq\left(\frac{\ell_1^3\varepsilon_7}{2}+\frac{\ell_1^3\varepsilon_9}{2}+\ell_1^2\varepsilon_8\right)\int_\Omega (f)^2\,dxdydt. \tag{3.13}$$

Hence, if

$$\Delta_1=1-\varepsilon_1\ell^2-\frac{\varepsilon_2}{2}-\frac{1}{\varepsilon_5}-\frac{\ell_2^2\ell_1^3}{4\varepsilon_7}-\frac{b^2}{2\varepsilon_6}>0;$$

$$\Delta_2=1-\frac{\ell_1^2}{\varepsilon_1}-\frac{1}{\varepsilon_4}-\frac{\ell_1^2}{\varepsilon_3}-\varepsilon_5-\varepsilon_6+\frac{\ell_2^2\ell_1^2}{2\varepsilon_8}>0;$$

$$\Delta_3=1-\frac{1}{2\varepsilon_1}-\varepsilon_3-\varepsilon_4-\frac{\ell_2^2}{4\varepsilon_9}>0,$$

it follows from estimation (3.14) that

$$\int_\Omega x^3(\Im_y D_t^\alpha u)^2\,dxdydt+2\int_\Omega x^3(u_x)^2\,dxdydt+\int_\Omega x(\Im_y u_x)^2\,dxdydt$$

$$+\int_\Omega x(u_y)^2\,dxdydt\leq c\int_\Omega (f)^2\,dxdydt, \tag{3.14}$$

where $c=\dfrac{\frac{\ell_1^3\varepsilon_7}{2}+\frac{\ell_1^3\varepsilon_9}{2}+\ell_1^2\varepsilon_8}{\min(2,\,\Delta_1,\,\Delta_2,\,\Delta_3)}$. This completes the proof.

Proposition 3.1 [5]**.** *The operator L from E to F admits a closure.*

Corollary 1. *Under the conditions of Theorem* 3.1, *there is a constant* $C > 0$ *independent of v such that*

$$\| v \|_E \leq \| \overline{L} v \|_F, \quad v \in D'(\Omega). \tag{3.15}$$

Corollary 2. *If a strong solution exists*, *it is unique*, *which depends continuously on f*, *if v is considered in the topology of E and f is considered in the topology of F*.

Corollary 3. *The rang* $R(\overline{L})$ *of the operator* $\overline{L}$ *is closed in F and* $R(\overline{L}) = \overline{R(L)}$, *where* $R(L)$ *is the range of L*.

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