Advances in Differential Equations and Control Processes
© 2022 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324322008
Volume 26, 2022, Pages 113-130 **P-ISSN: 0974-3243**

APPLICATION OF CARTAN'S EQUIVALENCE METHOD TO DISTRIBUTION OF PLANES

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Abstract

In this paper, we apply Cartan's equivalence method to distribution of planes to give a proof of the local equivalence between two planes.

1. Introduction

Elie Cartan, in the years 1905-1910, has laid down a method for determining if two geometric structures are equivalent. The authors in [8, 9] have expanded the method, and later in the years 1997-1998 by Bryant et al. in [10], who have clarified the methodology of Cartan. The general classification problems for symplectic Monge-Ampère equations were studied in [1, 3, 4, 7, 14, 15], and others: they showed that to any differential

Received: September 20, 2021; Revised: November 17, 2021; Accepted: November 24, 2021
2020 Mathematics Subject Classification: 34C41, 34C40, 34A26, 35A15.
Keywords and phrases: equivalence method, exterior differential systems, planes, G-structures.

How to cite this article: Moheddine Imsatfia and Anouar Houmia, Application of Cartan's equivalence method to distribution of planes, Advances in Differential Equations and Control Processes 26 (2022), 113-130. DOI: 10.17654/0974324322008

Published Online: January 14, 2022

2-form ω in the manifold of 1-jets of functions on a 2-dimensional smooth manifold $\mathcal{M}$, $J^1\mathcal{M}$, we can associate a Monge-Ampère equation E_ω which is entirely determined by the sign of the Pfaffian function $\mathrm{Pf}(\omega)$ at each point of $\mathcal{D} \subset J^1\mathcal{M}$. Locally, the application of Cartan's method to study the equivalence problem to classify Monge-Ampère equations in [11, 12] in two variables leads to three non-zero orbits: a negative space, a null space and a positive space, which correspond, respectively, to three types of Monge-Ampère equations: hyperbolic, parabolic and elliptic equations. In [10], Bryant et al. applied the equivalence method to classify Monge-Ampère equations of hyperbolic type and the elliptic type in [6]. The works of Kushner et al. in [2, 13] contain results on equivalence of Monge-Ampère equation to homogeneous Laplace equation. Those results are formulated in terms of the number of coefficients in the Monge-Ampère equation and can be explicitly satisfied with just finite number of usual algebraic operations and partial differentiations.

Our aim in this work is to apply the equivalence method to distribution of two planes. The proof of the local equivalence of two planes is carried out in two steps: in the first step, we define for a smooth manifold $\mathcal{M}$ a G-structure for some subgroup G of $GL(n, \mathbb{R})$, and in the second, we define the fundamental formula for the equivalence method given in [9] by

$$d\omega = -\varphi \wedge \omega + \tau,$$

where $\tau \in \Omega^2(\mathcal{M} \times G)$ is the torsion of the pseudo-connection φ and $\omega = (\omega^\imath)_\imath$ forms a local coframe. Finally, we follow the Cartan's equivalence method to prove that the system of structural equations is involutive, which according to Cartan, gives the local equivalence.

2. Basic Definition

Let $\mathcal{M}$ be an n-dimensional smooth manifold, where $n \in \mathbb{N}$ and G be a subgroup of $GL(n, \mathbb{R})$. Then a G-structure of basis $\mathcal{M}$ is a reduction of the coframe bundle of $\mathcal{M}$. In other words, $\forall x \in \mathcal{M}$, denoted by $(e_\imath)_{1 \leq \imath \leq n}$ a basis of $T_x\mathcal{M}$ and $\mathcal{R}(\mathcal{M})$ smooth manifold contains the set of frames of $\mathcal{M}$. For all frames $R = (x, \{e_\imath\})$ and $g \in GL(n, \mathbb{R})$,

$$R \cdot g = (x, \{\overline{e}_\imath\}) \text{ with } \overline{e}_\imath = e_\imath g_j^\imath.$$

We define a right action of $GL(n, \mathbb{R})$ over $\mathcal{R}(\mathcal{M})$, a G-structure, denoted $\mathcal{G}$, is a submanifold $\mathcal{G} \subset \mathcal{R}(\mathcal{M})$ having the property:

$$\forall R \in \mathcal{G}, \quad \forall g \in GL(n, \mathbb{R}), \quad R \cdot g \in \mathcal{G} \text{ if and only if } g \in G.$$

This means that two frames of $\mathcal{G}$ are in the same bundle (two frames have the same origin x) if and only if they are deduced from one another by a transformation matrix in the group G. This group is called *structural group* of $\mathcal{G}$. For example, $\mathcal{R}(\mathcal{M})$ is a $GL(n, \mathbb{R})$-structure, while the orthonormal basis of Riemannian manifold is an $O(n, \mathbb{R})$-structure.

Each diffeomorphism $\varphi : \mathcal{M} \to \overline{\mathcal{M}}$, can be extended uniquely in a diffeomorphism $\phi : \mathcal{R}(\mathcal{M}) \to \mathcal{R}(\mathcal{M})$, for each frame $R = (x, \{e_\imath\})$, denote:

$$\phi(R) = (\varphi(x), \{\overline{e}_\imath\}) \text{ with } \overline{e}_\imath = \varphi'(x) e_\imath, \tag{1}$$

where $\varphi'(x)$ is the Jacobian matrix of $\varphi(x)$, then $\forall g \in GL(n, \mathbb{R})$, we can prove that

$$\phi(R \cdot g) = \phi(R) \cdot g.$$

Definition 2.1. Two G-structures $\mathcal{G}$ and $\overline{\mathcal{G}}$ having $\mathcal{M}$ and $\overline{\mathcal{M}}$ bases are *equivalent* if there exists a diffeomorphism $\varphi : \mathcal{M} \to \overline{\mathcal{M}}$ such that the extension $\phi : \mathcal{R}(\mathcal{M}) \to \mathcal{R}(\mathcal{M})$ defined by (1) satisfies

$$\phi(\mathcal{G}) = \overline{\mathcal{G}}.$$

Then we can write $\mathcal{G} \sim \overline{\mathcal{G}}$, the restriction of ϕ to the manifold $\mathcal{G} \subset \mathcal{R}(\mathcal{M})$ is an isomorphism of $\mathcal{G}$ into $\overline{\mathcal{G}}$.

Let us now recall some properties for local equivalence of two G-structures.

Definition 2.2. Let $\mathcal{G} \subset \mathcal{R}(\mathcal{M})$ and $\overline{\mathcal{G}} \subset \mathcal{R}(\overline{\mathcal{M}})$ be two G-structures. Then we say that $\mathcal{G}$ and $\overline{\mathcal{G}}$ are *locally equivalent* in $(x, \bar{x}) \in \mathcal{M} \times \overline{\mathcal{M}}$ if there exist neighborhoods U of x and $\overline{U}$ of $\bar{x}$ and an isomorphism $\phi : \mathcal{G}|_U \to \overline{\mathcal{G}}|_{\overline{U}}$.

Proposition 2.3 [10]. *Let* θ *and* $\overline{\theta}$ *be two canonical forms defined over two G-structures* $\mathcal{G}$ *and* $\overline{\mathcal{G}}$, *and let* $\phi : \mathcal{G} \to \overline{\mathcal{G}}$ *be a diffeomorphism between these two manifolds. Then the following conditions are equivalent*:

(1) *The diffeomorphism* ϕ *is an isomorphism of G-structures.*

(2) *The diffeomorphism* ϕ *satisfies* $\phi^*(\overline{\theta}) = \theta$.

The diffeomorphism $\phi : \mathcal{G} \to \overline{\mathcal{G}}$ is an isomorphism of G-structure if and only if $\phi^*(\overline{\theta}) = \theta$, this means that the graph of ϕ in $\mathcal{G} \times \overline{\mathcal{G}}$ satisfies the Pfaff system in [10]:

$$\begin{cases} \overline{\theta} = \theta, \\ \theta^1 \wedge \cdots \wedge \theta^n \neq 0. \end{cases}$$

Theorem 2.4 (Darboux). *Let* $(\Omega_1, \mathcal{M}_1)$ *and* $(\Omega_2, \mathcal{M}_2)$ *be symplectic manifolds of the same dimension. Then for any two points* $a \in \mathcal{M}_1$ *and* $b \in \mathcal{M}_2$, *there exist neighborhoods* $\mathcal{O}_1 \ni a$ *and* $\mathcal{O}_2 \ni b$ *and a diffeomorphism* $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ *such that* $\varphi(a) = b$ *and* $\varphi^*(\mathcal{O}_2) = \mathcal{O}_1$.

Corollary 2.5 [13]. *Let* $(\Omega, \mathcal{M})$ *be a* $2n$*-dimensional symplectic manifold. Then for any point* $a \in \mathcal{M}$, *there exist local canonical coordinates* $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ *such that* $q^\imath(a) = p_\imath(a) = 0$, *for* $\imath = 1, \ldots, n$ *and* Ω *has the following canonical form*:

$$\Omega = \sum_{\imath=1}^{n} dq^\imath \wedge dp_\imath.$$

3. Application of Equivalence Problem to Distribution of Planes

In this section, the equivalence problem is applied locally to give equivalence between two planes on contact manifold of dimension 5. Then a criterion in terms of the differential invariants is obtained for a given system to be locally equivalent to the system associated to the linear homogeneous Laplace equation or to a Euler-Lagrange system. On the contact manifold $\mathcal{M}$, one can locally find a coframing $\omega = g^{-1}\eta$, where η is a local section from $\mathcal{G} \to \mathcal{M}$ and $g \in \mathfrak{g}$. The exterior derivative of this equation is

$$d\omega = g^{-1}dg \wedge \omega + g^{-1}d\eta. \tag{2}$$

Definition 3.1. Let $G \in GL(n, \mathbb{R})$ be a subgroup and $B \cong \mathcal{M} \times G$. Then a G-structure $B \to \mathcal{M}$ is a principal subbundle of the coframe bundle $\mathcal{F}(\mathcal{M}) \to \mathcal{M}$, having G as a group of structure. A pseudo-connection in the G-structure is a $\mathfrak{g}$-valued 1-form on B whose restriction to the fiber tangent spaces $\mathcal{V}_b \subset T_bB$ equals the identification $\mathcal{V}_b \cong \mathfrak{g}$ induced by the right G-action on B.

Let us introduce a pseudo-connection $\varphi \in \Omega^1(B) \otimes \mathfrak{g}$ that satisfies the fundamental formula for the equivalence method given in [9] by:

$$d\omega = -\varphi \wedge \omega + \tau,$$

where $\tau \in \Omega^2(B)$ is the torsion of the pseudo-connection φ. A consequence of (2), for $0 \leq \imath, \jmath, k \leq 4$ is

$$\varphi = -g^{-1}dg \text{ and } \tau = g^{-1}d\eta = \frac{1}{2}T^{\imath}_{\jmath k}\omega^{\jmath} \wedge \omega^{k}.$$

3.1. Calculation of structural equations

The first step of equivalence method of Cartan is to calculate the structural equations. Let us consider $\mathcal{M} = \mathbb{R}^3$. Given two distributions of planes D and $\overline{D}$ in the Grassmannian bundle $Gr_2(\mathbb{R}^3)$ in [5] defined by:

$$D : \{\mathbb{R}^3 \ni x \mapsto P(x) \in Gr_2(\mathbb{R}^3)\},$$

$$\overline{D} : \{\mathbb{R}^3 \ni x \mapsto \overline{P}(x) \in Gr_2(\mathbb{R}^3)\}.$$

There exists a diffeomorphism $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $d(\Phi_x(P(x))) = \overline{P}(\Phi(x))$ is not always true. In this study, we tried to find some conditions such that an application can be a diffeomorphism.

Denote by (e_1, e_2, e_3) a basis of $\mathbb{R}^3$ such that (e_1, e_2) a basis of $P(x)$. Denote by $\mathcal{R}(\mathbb{R}^3)$ the bundle of frame in $\mathbb{R}^3$, denote $\mathcal{G} \in \mathcal{R}(\mathbb{R}^3)$ such that $\mathcal{G} \simeq \mathbb{R}^3 \times S$ and

$$S = \begin{pmatrix} A & B \\ 0 & b_3 \end{pmatrix} = \begin{pmatrix} a_1^1 & a_2^1 & b_1 \\ a_2^1 & a_2^2 & b_2 \\ 0 & 0 & b_3 \end{pmatrix} \in GL(3, \mathbb{R}^3)$$

with the condition $a_1^1 a_2^2 - a_2^1 a_1^2 \neq b_3 \neq 0.$

A local section σ of $\mathcal{G}$ is given by vector field, in other words, an application

$$\mathbb{R}^3 \ni x \mapsto \sigma(x) \in \mathcal{G} \text{ with } \sigma(x) = (x, (X_1(x), X_2(x), X_3(x))),$$

where $(X_1(x), X_2(x))$ is a basis of $P(x)$. Any frame $R \in \mathcal{G}$ of origin x is deduced in a single way from $\sigma(x)$ by right multiplication by a group matrix G. Let $g \in G$ such that

$$\sigma(x) = R \cdot g.$$

Then an element of $\mathcal{G}$ is given by a pair $(x, g) \in \mathbb{R}^3 \times S$. For all $x \in \mathbb{R}^3$, there exists a basis $(v_1(x), v_2(x))$ of $P(x) \subset T_x\mathbb{R}^3$. By considering $v_3(x)$, we obtain a basis $(v_1(x), v_2(x), v_3(x))$ of $T_x\mathbb{R}^3$:

$$\left(x, (v_1, v_2, v_3)(x) \begin{pmatrix} a_1^1 & a_2^1 & b_1 \\ a_2^1 & a_2^2 & b_2 \\ 0 & 0 & b_3 \end{pmatrix} \right) \in \mathcal{G}.$$

Consider the 1-form $\theta = (\theta^1, \theta^2, \theta^3)$ of coframe defined by

$$\theta^a = \sum_{\mu=1}^{3} f_\mu^a(x, A, b) dx^\mu \text{ and } dS = \left\{ dx, \begin{pmatrix} da_1^1 & da_2^1 & db_1 \\ da_2^1 & da_2^2 & db_2 \\ 0 & 0 & db_3 \end{pmatrix} \right\}.$$

Locally, we have $\theta = S\omega$ or $\theta = S_b\omega^b$, in other words, $\theta^a = S_b^a\omega^b$. Then

$$\begin{aligned} d\theta &= d(S\omega) = dS \wedge \omega + S \cdot d\omega \\ &= (dS \cdot S^{-1}) \wedge S\omega + S \cdot d\omega \\ &= (dS \cdot S^{-1}) \wedge \theta + S \cdot d\omega. \end{aligned} \tag{3}$$

$d\theta^a$ in the basis of 2-forms differential of $\mathcal{G}$:

$$d\theta^a = d(S_b^a\omega^b) = dS_b^a \wedge \omega^a + S_b^a \cdot d\omega^b.$$

Consider a coframe: $\pi = (\pi^1, ..., \pi^7)$ of the structural group G. Then the set (θ, π) forms a coframe of G-structure $\mathcal{G} \simeq \mathcal{M} \times S$. We can prove that $dS \cdot S^{-1}$ is a matrix of forms of Maurer-Cartan, then there exists a tensor A_{bc}^a such that $(dS \cdot S^{-1})_b^a = A_{bc}^a\pi^c$. We have $\theta^a = S_b^a\omega^b$, then we can find a tensor T_{bd}^a such that $S_b^a \cdot d\omega^b = T_{bc}^a\theta^b \wedge \theta^c$. Then the decomposition of $d\theta^a$ in the basis of 2-forms of the differential of $\mathcal{G}$ have the form:

$$d\theta^a = A_{bc}^a\pi^c \wedge \theta^b + T_{bc}^a\theta^b \wedge \theta^c.$$

Let us calculate the Maurer-Cartan forms of group G:

$$(dS \cdot S^{-1})_b^a = \frac{1}{a_1^1 a_2^2 - a_2^1 a_1^2} \begin{pmatrix} da_1^1 & da_2^1 & db_1 \\ da_1^2 & da_2^2 & db_2 \\ 0 & 0 & db_3 \end{pmatrix} \begin{pmatrix} a_2^2 & -a_1^2 & \dfrac{a_1^2 b_2 - a_2^2 b_1}{b_3} \\ -a_2^1 & a_1^1 & \dfrac{a_2^1 b_1 - a_1^1 b_2}{b_3} \\ 0 & 0 & \dfrac{a_1^1 a_2^2 - a_2^1 a_1^2}{b_3} \end{pmatrix}$$

which can be written as

$$dS \cdot S^{-1} = \begin{pmatrix} \pi^1 & \pi^2 & \pi^3 \\ \pi^4 & \pi^5 & \pi^6 \\ 0 & 0 & \pi^7 \end{pmatrix},$$

where $\pi^1, ..., \pi^7$ are the Maurer-Cartan forms which satisfy:

$$\begin{cases} \pi^1 = \dfrac{1}{a_1^1 a_2^2 - a_2^1 a_1^2}(a_2^2 da_1^1 - a_2^1 da_2^1), \\ \pi^2 = \dfrac{1}{a_1^1 a_2^2 - a_2^1 a_1^2}(a_1^1 da_2^1 - a_1^2 da_1^1), \\ \pi^3 = \dfrac{1}{a_1^1 a_2^2 - a_2^1 a_1^2}\left(\dfrac{a_1^2 b_2 - a_2^2 b_1}{b_3} da_1^1 + \dfrac{a_2^1 b_1 - a_1^1 b_2}{b_3} da_2^1 + \dfrac{a_1^1 a_2^2 - a_2^1 a_1^2}{b_3} db_1\right), \\ \pi^4 = \dfrac{1}{a_1^1 a_2^2 - a_2^1 a_1^2}(a_2^2 da_1^2 - a_2^1 da_2^2), \\ \pi^5 = \dfrac{1}{a_1^1 a_2^2 - a_2^1 a_1^2}(a_1^1 da_2^2 - a_1^2 da_1^2), \\ \pi^6 = \dfrac{1}{a_1^1 a_2^2 - a_2^1 a_1^2}\left(\dfrac{a_1^2 b_2 - a_2^2 b_1}{b_3} da_1^2 + \dfrac{a_2^1 b_1 - a_1^1 b_2}{b_3} da_2^2 + \dfrac{a_1^1 a_2^2 - a_2^1 a_1^2}{b_3} db_2\right), \\ \pi^7 = \dfrac{1}{b_3} db_3. \end{cases}$$

Since $d\omega^b$ is a 2-form, it can be written

$$d\omega^b = \sum\nolimits_{ac\in\{12,\,23,\,31\}} B^b_{ac}\theta^a \wedge \theta^c.$$

The calculation of structure equation gives:

$$\begin{cases} d\theta^1 = \pi^1 \wedge \theta^1 + \pi^2 \wedge \theta^2 + \pi^3 \wedge \theta^3 + T^1_{12}\theta^1 \wedge \theta^2 + T^1_{23}\theta^2 \wedge \theta^3 + T^1_{31}\theta^3 \wedge \theta^1, \\ d\theta^2 = \pi^4 \wedge \theta^1 + \pi^5 \wedge \theta^2 + \pi^6 \wedge \theta^3 + T^2_{12}\theta^1 \wedge \theta^2 + T^2_{23}\theta^2 \wedge \theta^3 + T^2_{31}\theta^3 \wedge \theta^1, \\ d\theta^3 = \pi^7 \wedge \theta^3 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{23}\theta^2 \wedge \theta^3 + T^3_{31}\theta^3 \wedge \theta^1, \end{cases}$$

where:

$$\begin{cases} T^1_{12} = a^1_1 B^1_{12} + a^1_2 B^2_{12} + b_1 B^3_{12}, \\ T^1_{23} = a^1_1 B^1_{23} + a^1_2 B^2_{23} + b_1 B^3_{23}, \\ T^1_{31} = a^1_1 B^1_{31} + a^1_2 B^2_{31} + b_1 B^3_{31}, \\ T^2_{12} = a^2_1 B^1_{12} + a^2_2 B^2_{12} + b_2 B^3_{12}, \\ T^2_{23} = a^2_1 B^1_{23} + a^2_2 B^2_{23} + b_2 B^3_{23}, \\ T^2_{31} = a^2_1 B^1_{31} + a^2_2 B^2_{31} + b_2 B^3_{31}, \\ T^3_{12} = b_3 B^3_{12}, \\ T^3_{23} = b_3 B^3_{23}, \\ T^3_{31} = b_3 B^3_{31}. \end{cases}$$

3.2. Absorption of the torsion

The second step of Cartan's algorithm consists to simplify the maximum of coefficients T^b_{ac} in structure equations. We change the forms, for $\imath = 1, \ldots, 7$,

$$\pi'^{\imath} = \pi^{\imath} + \lambda^{\imath}_a \theta^a.$$

We obtain some relations in the form:

$$T'^{b}_{ac} = T^{b}_{ac} + A^{b}_{c\imath}\lambda^{\imath}_{a} - A^{b}_{a\imath}\lambda^{\imath}_{c},$$

$\pi' = \pi + \Lambda\theta$, where $\Lambda = (\lambda^{a}_{b})$ is a matrix of functions "unknowns", we have $T' = T + L(\Lambda)$, where L is a linear operator which depends only on the tensor A^{a}_{bc}. We start by changing the form $\pi'^{\imath} = \pi^{\imath} + \lambda^{\imath}_{a}\theta^{a}$.

We obtain:

$$\begin{cases} T'^{1}_{12} = T^{1}_{12} - \lambda^{1}_{2} + \lambda^{2}_{1}, \\ T'^{1}_{23} = T^{1}_{23} - \lambda^{2}_{3} + \lambda^{3}_{2}, \\ T'^{1}_{31} = T^{1}_{31} - \lambda^{3}_{1} + \lambda^{1}_{3}, \\ T'^{2}_{12} = T^{2}_{12} - \lambda^{4}_{2} + \lambda^{5}_{1}, \\ T'^{2}_{23} = T^{2}_{23} - \lambda^{5}_{3} + \lambda^{6}_{2}, \\ T'^{2}_{31} = T^{2}_{31} - \lambda^{6}_{1} + \lambda^{4}_{3}, \\ T'^{3}_{12} = T^{3}_{12}, \\ T'^{3}_{23} = T^{3}_{23} + \lambda^{7}_{2}, \\ T'^{3}_{31} = T^{3}_{31} - \lambda^{7}_{1}. \end{cases} \tag{4}$$

Then we can choose the parameters $\lambda^{1}_{2}, \lambda^{2}_{3}, \lambda^{3}_{1}, \lambda^{4}_{2}, \lambda^{5}_{3}, \lambda^{6}_{1}, \lambda^{7}_{1}$ and λ^{7}_{2} such that the new coefficients T' are zero, except $T'^{3}_{12} = T^{3}_{12}$ which is invariant. Then to absorb the torsion, we should suppose:

$$\begin{cases} \pi^1 \to \pi^1 - T^1_{12}\theta^2, \\ \pi^2 \to \pi^2 - T^1_{23}\theta^3, \\ \pi^3 \to \pi^3 - T^1_{31}\theta^1, \\ \pi^4 \to \pi^4 - T^2_{12}\theta^2, \\ \pi^5 \to \pi^5 - T^2_{23}\theta^3, \\ \pi^6 \to \pi^6 - T^2_{31}\theta^1, \\ \pi^7 \to \pi^7 - T^3_{31}\theta^1 + T^3_{23}\theta^2. \end{cases}$$

This means $\pi = \pi + \Lambda^{(1)}\theta,$ where

$$\Lambda^{(1)} = \begin{pmatrix} 0 & -T^1_{12} & 0 \\ 0 & 0 & -T^1_{23} \\ -T^1_{31} & 0 & 0 \\ 0 & -T^2_{12} & 0 \\ 0 & 0 & -T^2_{23} \\ -T^2_{31} & 0 & 0 \\ -T^3_{31} & T^3_{23} & 0 \end{pmatrix}.$$

The variables $\lambda^1_1, \lambda^1_3, \lambda^2_1, \lambda^2_2, \lambda^3_2, \lambda^3_3, \lambda^4_1, \lambda^4_3, \lambda^5_1, \lambda^5_2, \lambda^6_2, \lambda^6_3$ and λ^7_3 are arbitrary. Write

$$\Lambda^{(2)} = \begin{pmatrix} \lambda^1_1 & 0 & \lambda^1_3 \\ \lambda^2_1 & \lambda^2_2 & 0 \\ 0 & \lambda^3_2 & \lambda^3_3 \\ \lambda^4_1 & 0 & \lambda^4_3 \\ \lambda^5_1 & \lambda^5_2 & 0 \\ 0 & \lambda^6_2 & \lambda^6_3 \\ 0 & 0 & \lambda^7_3 \end{pmatrix}.$$

Then, if we change the form $\pi \to \pi + \Lambda\theta$ with $\Lambda = \Lambda^{(1)} + \Lambda^2$, the structure equations become

$$\begin{cases} d\theta^1 = \pi^1 \wedge \theta^1 + \pi^2 \wedge \theta^2 + \pi^3 \wedge \theta^3, \\ d\theta^2 = \pi^4 \wedge \theta^1 + \pi^5 \wedge \theta^2 + \pi^6 \wedge \theta^3, \\ d\theta^3 = \pi^7 \wedge \theta^3 + T^3_{12}\theta^1 \wedge \theta^2, \end{cases}$$

with $T^3_{12} = b_3 B^3_{12}$ is an invariant.

3.3. Normalisation

Suppose that $\lambda = T^3_{12}$. Then we can write:

$$d(d\theta^3) = d\pi^7 \wedge \theta^3 - \pi^7 \wedge d\theta^3 + d\lambda \wedge \theta^1 \wedge \theta^2 - \lambda d\theta^1 \wedge \theta^2 + \lambda\theta^1 \wedge d\theta^2$$

$$= d\pi^7 \wedge \theta^3 - \lambda\pi^7 \wedge \theta^1 \wedge \theta^2 + d\lambda \wedge \theta^1 \wedge \theta^2 + \lambda\pi^1 \wedge \theta^1 \wedge \theta^2$$

$$+ \lambda\pi^3 \wedge \theta^3 \wedge \theta^2,$$

$$-\lambda\theta^1 \wedge \pi^5 \wedge \theta^2 - \lambda\theta^1 \wedge \pi^6 \wedge \theta^3 = (d\lambda - \lambda(\pi^7 - (\pi^1 + \pi^5)))$$

$$\wedge \theta^1 \wedge \theta^2 \bmod \theta^3.$$

Horizontally, according to $\theta^1 \wedge \theta^2 \wedge \theta^3$, we have

$$d\lambda = \lambda(\pi^7 - (\pi^1 + \pi^5)).$$

Consider the sub G_1-structure $\mathcal{G}_1 \subset \mathcal{G}$, such that the subgroup $G_1 \subset G$ acts on

$$S = \begin{pmatrix} a^1_1 & a^1_2 & b_1 \\ a^1_2 & a^2_2 & b_2 \\ 0 & 0 & b_3 \end{pmatrix} \in GL(3, \mathbb{R}^3), \text{ where } \frac{a^1_1 a^2_2 - a^1_2 a^2_1}{b_3} = 1.$$

We can normalize G_1 to the subgroup

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix} \in GL(3, \mathbb{R}^3), \text{ with } \rho > 0 \text{ and } \rho^{-1}\lambda = 1.$$

Then we have the fundamental formula of equivalence method

$$d\theta = \pi \wedge \theta + \tau,$$

where

$$\pi = \begin{pmatrix} \pi^1 & \pi^2 & \pi^3 \\ \pi^4 & \pi^5 & \pi^6 \\ 0 & 0 & \pi^7 \end{pmatrix}.$$

Recalculate after the structure equations in the new subgroup with condition $\pi^1 + \pi^5 = \pi^7$. Now, it is not possible to change the form. After new absorption of torsion, we can find the new structure equation in the form:

$$\begin{cases} d\theta^1 = \pi^1 \wedge \theta^1 + \pi^2 \wedge \theta^2 + \pi^3 \wedge \theta^3, \\ d\theta^2 = \pi^4 \wedge \theta^1 + \pi^5 \wedge \theta^2 + \pi^6 \wedge \theta^3, \\ d\theta^3 = (\pi^1 + \pi^5) \wedge \theta^3 + \lambda\theta^1 \wedge \theta^2. \end{cases}$$

Then

$$0 \equiv d(d\theta^3) = d\lambda \wedge \theta^1 \wedge \theta^2 \bmod \theta^3.$$

Then, on $\mathcal{G}_1$, normalize $\lambda = 1$. In fact, denote the change of form:

$$\begin{cases} \theta^1 \leftarrow \theta^1, \\ \theta^2 \leftarrow \theta^2, \\ \theta^3 \leftarrow \tilde{\theta}^3 = \rho^{-1}\theta^3. \end{cases}$$

Then

$$d\widetilde{\theta}^3 = d(\rho^{-1}\theta^3) = d\rho^{-1} \wedge \theta^3 + \rho^{-1}((\pi^1 + \pi^5) \wedge \theta^3 + \lambda\theta^1 \wedge \theta^2)$$

$$= d\rho^{-1} \wedge \theta^3 + (\pi^1 + \pi^5) \wedge \widetilde{\theta}^3 + \theta^1 \wedge \theta^2.$$

We have $d\rho^{-1}$ is semi-basis, then there exist functions α and β such that $d\rho^{-1} = \alpha\theta^1 + \beta\theta^2$, then absorb in π^7 without changing the expression $\pi^1 + \pi^5 = \pi^7$. The equation of structure can be written:

$$\begin{cases} d\theta^1 = \pi^1 \wedge \theta^1 + \pi^2 \wedge \theta^2 + \pi^3 \wedge \theta^3, \\ d\theta^2 = \pi^4 \wedge \theta^1 + \pi^5 \wedge \theta^2 + \pi^6 \wedge \theta^3, \\ d\theta^3 = (\pi^1 + \pi^5) \wedge \theta^3 + \theta^1 \wedge \theta^2. \end{cases} \quad (5)$$

3.4. Involution test (test of Cartan)

Assume that we have an equivalence problem in the basis $(\theta^1, ..., \theta^n)$. Denote by r the dimension of group G_1. The equation of structure can be written as:

$$d\theta^i = \sum_{j=1}^{n}\sum_{k=1}^{r} A^i_{jk}\pi^k \wedge \theta^j + \sum_{j,l=1}^{n} T^i_{jl}\theta^j \wedge \theta^l, \quad i = 1, ..., n, \quad (6)$$

where π^k are the Maurer-Cartan forms and A^i_{jk} are the coefficients of structure. We can write $\pi^k = \lambda^k_j\theta^j$, then we obtain the following system, where the unknowns are λ^k_j:

$$\sum_{k=1}^{r}(A^i_{jk}\lambda^k_l - A^i_{lk}\lambda^k_j) = T^i_{jk}, \quad i, j, l = 1, ..., n, \quad j < k. \quad (7)$$

Definition 3.2. Denote by $r^{(1)}$ the number of independent variables in the linear system:

$$\sum_{k=1}^{r}(A_{jk}^{i}\lambda_{l}^{k}-A_{lk}^{i}\lambda_{j}^{k})=0,\quad i,\ j,\ l=1,\ ...,\ n,\quad j<k. \tag{8}$$

In other words, $r^{(1)}$ is the dimension of the solutions of the system (8).

In the example of p-plane, we have $r=6$ and $n=3$. In (5), replacing π^k by $\lambda_1^k\theta^1+\lambda_2^k\theta^2+\lambda_3^k\theta^3$, we obtain the following linear system:

$$\begin{cases}\lambda_3^1-\lambda_1^3=\lambda_3^2-\lambda_2^3=0,\\ \lambda_3^4-\lambda_1^6=\lambda_2^6-\lambda_3^5=0,\\ \lambda_1^2=\lambda_2^1=-\lambda_2^5,\\ \lambda_2^4=\lambda_1^5=-\lambda_1^1.\end{cases} \tag{9}$$

In equations (9), the ten variables $\lambda_2^1,\ \lambda_3^1,\ \lambda_3^2,\ \lambda_1^5,\ \lambda_3^4,\ \lambda_2^6,\ \lambda_2^2,\ \lambda_3^3,\ \lambda_3^6,\ \lambda_1^4$ can be selected arbitrarily. We have

$$r^{(1)}=10.$$

To continue the description of the involution test of Cartan, we define the reduced characters of Cartan.

Definition 3.3. Denoting $X=(x^1,\ ...,\ x^n)\in\mathbb{R}^n$ and the matrix M of dimension $n\times r$ defined by

$$M(X):=M_k^i(X):=\sum_{j=1}^{n}A_{jk}^{i}x^{j},\quad i=1,\ ...,\ n,\quad k=1,\ ...,\ r.$$

In this definition, A_{jk}^i are defined in (7). Then, if we denote by $s_1',\ ...,\ s_{n-1}',\ s_n'$, the reduced characters of Cartan, one defines:

$$s_1'+\cdots+s_k'=\max_{X_1,\ldots,X_k\in\mathbb{R}^n}\operatorname{rg}\begin{pmatrix}M(X_1)\\ \vdots\\ M(X_k)\end{pmatrix},\quad k=1,\ ...,\ n-1,$$

and s_n' is defined by the equation $s_1'+\cdots+s_{n-1}'+s_n'=r$.

Definition 3.4. Denote by θ a basis of cotangent bundle $T_x^*\mathbb{R}$ and $s'_1, \ldots, s'_n$ the reduced characters of Cartan, denote by $r^{(1)}$ defined in (3.2). Then we say that θ is *involutive* if Cartan test satisfies:

$$s'_1 + 2s'_2 + \cdots + ns'_n = r^{(1)}.$$

Remark 3.5. One has always:

$$r^{(1)} \leq s'_1 + 2s'_2 + \cdots + ns'_n.$$

In our problem, we have in equations (5), $n = 3$ and $r = 6$. Then

$$(A_{2k}^{\imath} + A_{3k}^{\imath} + A_{1k}^{\imath})_{\substack{1 \leq \imath \leq 3 \\ 1 \leq k \leq 6}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, for $X = (x^1, x^2, x^3) \in \mathbb{R}^3$, we find

$$M(X) = \begin{pmatrix} x^1 & x^2 & x^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^1 & x^2 & x^3 \\ x^3 & 0 & 0 & 0 & x^3 & 0 \end{pmatrix}. \tag{10}$$

For $X = (0, 0, 1)$, we have $\mathrm{rg} M(0, 0, 1) = 3$, then $s'_1 = 3$. And we have:

$$s'_1 + s'_2 = \max_{X, Y \in \mathbb{R}^3} \mathrm{rg} \begin{pmatrix} x^1 & x^2 & x^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & x^1 & x^2 & x^3 \\ x^3 & 0 & 0 & 0 & x^3 & 0 \\ y^1 & y^2 & y^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^1 & y^2 & y^3 \\ y^3 & 0 & 0 & 0 & y^3 & 0 \end{pmatrix}.$$

For $X = (0, 0, 1)$ and $Y = (1, 1, 0)$, then $s'_1 + s'_2 = 5$, which gives

$$s'_2 = 2 \text{ and } s'_3 = 6 - 3 - 2 = 1.$$

We have $s'_1 + 2s'_2 + 3s'_3 = 10 = r^{(1)}$, then the system (5) is involutive.

Conclusion 3.6. The equivalence method of Cartan is a crucial tool to prove the local equivalence between two G-structures. Hence, we defined a structure of distribution of planes over a manifold $\mathcal{M}$ and used this method to prove the local equivalence between two planes.

Acknowledgements

The authors extend their appreciation to the Deanship of Scientific Research at King Khaled University for funding this work through research group program [grant number $R.G.P.$1/129/40].

Also, the authors thank the anonymous referees for their valuable suggestions and constructive criticism which improved the presentation of the paper.

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