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UNIQUENESS OF THE SOLUTIONS OF NONLOCAL PLURIPARABOLIC FRACTIONAL PROBLEMS WITH WEIGHTED INTEGRAL BOUNDARY CONDITIONS

DJIBIBE Moussa Zakari[1], SOAMPA Bangan[2] and TCHARIE Kokou[1]

[1]Laboratoire d'Analyse, de Modélisation Mathématiques
and Applications (LAMMA)
Département de Mathématiques
Université de Lomé
PB 1515 Lomé, Togo
e-mail: zakari.djibibe@gmail.com
tkokou09@yahoo.fr

[2]Département de Mathématiques
Faculté des Sciences and Techniques
Université de Kara
PB 404 Kara, Togo
e-mail: bangansoampa@gmail.com

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Abstract

The aim of this article is to prove the uniqueness of solutions to mixed problems for pluriparabolic equations with nonlocal boundary conditions. The proofs are based on a priori estimates established in non-classical function spaces.

1. Introduction

This paper deals with a priori estimates and continuous dependence of the solution of a following class of pluriparabolic fractional equations

$$\sum_{i=1}^{n} D_{t_i}^{\alpha} u - \frac{1}{a(x)}\frac{\partial}{\partial x}\left(b(J_x\alpha, t)\frac{\partial u}{\partial x}\right) = F(x, t), \quad (x, t) \in \Omega \tag{1.1}$$

satisfying the initial condition

$$u(x, t_{i,0}) = \varphi_i(x), \quad x \in (0, \ell), \quad t_{i,0} = (t_1, t_2, ..., t_{i-1}, 0, t_{i+1}, ..., t_n), \tag{1.2}$$

and weighted integral boundary conditions

$$\int_0^{\ell} a(x)u(x, t)dx = E(t), \quad t = (t_1, t_2, ..., t_n) \in I, \tag{1.3}$$

$$\int_0^{\ell} xa(x)u(x, t)dx = G(t), \quad t = (t_1, t_2, ..., t_n) \in I \tag{1.4}$$

in the domain $\Omega = (0, \ell) \times I$, where $\ell < +\infty$, $I = \prod_{i=1}^{n}(0, T_i)$ and $T_i < \infty$, for $i = 1, 2, ..., n$,

For the consistency, we get

$$\int_0^{\ell} a(x)\varphi_i(x)dx = \xi(t_{i,0})$$

and

$$\int_0^{\ell} xa(x)\varphi_i(x)dx = \eta(t_{i,0}),$$

where F, φ_i, ψ, E and G are the known functions, $0 < \alpha < 1$, $n \in \mathbb{N}^*$. The left Caputo derivative D_t^{α} and the gamma function Γ are, respectively, defined as

$$D_t^{\alpha} u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u}{\partial \tau} \frac{d\tau}{(t-\tau)^{\alpha}},$$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{1-\alpha} e^{-x} dx.$$

Next, we formulate the main conditions:

Condition 1. There are non-negative constants c_0, c_1, c_2, c_3 such that

$$c_0 \leq b(J_x a, t) \leq c_2,$$

$$\left| \frac{\partial b(J_x a, t)}{\partial t_i} \right| \leq c_3, \quad i = 1, 2, ..., n, \text{ for } (a, t) \in \Omega,$$

$$\left| \frac{\partial b(J_x a, t)}{\partial J_x a} \right| \leq c_4, \text{ for } (a, t) \in \Omega.$$

Condition 2. (1) The function a is positive and continuous on $(0, \ell)$, such that $a(x) \leq c_5$, where c_5 is a positive constant.

(2) $F \in C(\Omega, \mathbb{R}, \psi, E, G \in C^1(I, \mathbb{R}))$.

The existence and uniqueness of solutions to initial and boundary-value problems for fractional differential equations have been extensively studied by many authors; see for example, [1-4, 7, 9-12, 14-16]. Some of the existence and uniqueness results have been obtained by using the well-known Lax-Milgram theorem, by fixed point theorem and energy-integral method [1, 2, 5, 6, 8, 15].

A suitable variational formulation is the starting point of many numerical methods, such as finite element methods, spectral methods, and Laplace transform method [7, 16]. Thus, the construction of a variational formulation is essential, and relies strongly on the choice of spaces and their norms.

Motivated by this, we extend and generalize the study for PDEs with integral conditions to the study of fractional PDEs with integral conditions.

In this paper, we extend an energy-integral method to the study of a mixed-type fractional differential equation.

This paper is outlined as follows: After this introductory section, in Section 2, we present abstract formulation of the posed problem and make precise the concept of solution of the problem. Finally, we establish a priori estimates which are derived to show the uniqueness and continuous dependence of the solution upon the data in Section 3.

2. Preliminaries

We introduce now a new function $v(x, t) = u(x, t) - w(x, t) - \varphi_i(x)$. Then the problem (1.1)-(1.4) can be formulated as

$$(\mathcal{L}v)(x, t) = \sum_{i=1}^{n} D_{t_i}^{\alpha} v - \frac{1}{a(x)} \frac{\partial}{\partial x}\left(b(J_x \alpha, t) \frac{\partial v}{\partial x}\right) = f(x, t), \; (x, t) \in \Omega, \quad (2.1)$$

$$v(x, t_{i,0}) = 0, \; x \in (0, \ell), \; t_{i,0} = (t_1, t_2, ..., t_{i-1}, 0, t_{i+1}, ..., t_n), \quad (2.2)$$

$$\int_0^{\ell} a(x) v(x, t) dx = 0, \quad t \in I, \quad (2.3)$$

$$\int_0^{\ell} x a(x) v(x, t) dx = 0, \quad t \in I, \quad (2.4)$$

where

$$w(x, t) = \frac{2(-3x + 2\ell)}{\ell^2 a(x)} E(t) + \frac{6(2x - \ell)}{\ell^3 a(x)} G(t),$$

$$f(x,t) = F(x,t) - \frac{2(-3x+2\ell)}{\ell^2 a(x)} \sum_{i=1}^{n} D_{t_i}^{\alpha} E(t) + \frac{6(2x-\ell)}{\ell^3 a(x)} \sum_{i=1}^{n} D_{t_i}^{\alpha} G(t)$$

$$- \frac{2}{\ell^2 a(x)} \frac{\partial}{\partial x}\left(b(J_x a, t) \frac{3a(x) + (2\ell - 3x)a'(x)}{a^2(x)} \right) E(t)$$

$$+ \frac{6}{\ell^3 a(x)} \frac{\partial}{\partial x}\left(b(J_x a, t) \frac{2a(x) - (2x - \ell)a'(x)}{a^2(x)} \right) G(t)$$

$$+ \frac{1}{a(x)} \frac{\partial}{\partial x}(b(J_x a, t)\varphi_i'(x)).$$

Instead of searching for the function u, we search for the function v. So the solution of problem (2.1), (2.2), (2.3) and (2.4) will be given by $u(x,t) = v(x,t) + w(x,t) + \varphi_i(x)$.

In this paper, we establish a priori estimates which are derived to show the uniqueness and continuous dependence of the solution upon the data (2.1), (2.2), (2.3) and (2.4). For this, we consider the problem (2.1)-(2.4) as a solution of the operator equation

$$Lv = \mathcal{F} = f, \tag{2.5}$$

with domain of definition $D(L)$ consisting of function $v \in L_2(\Omega)$ such that

$$D_{t_i}^{\alpha} v, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2} \in L_2(\Omega),$$

where v satisfies conditions (2.3) and (2.4). The operator L is considered from E to F, where E is the Banach space consisting of functions $v \in L_2(\Omega)$, satisfying (2.3) and (2.4) with the finite norm

$$\| v \|_E = \sum_{i=1}^{n} \int_{\Omega} \left(D_{t_i}^{\alpha} \int_0^x a(\xi) v(\xi, t) d\xi \right)^2 dxdt + \int_{\Omega} \left(\frac{\partial v}{\partial x} \right)^2 dxdt. \tag{2.6}$$

Here F is the Hilbert space of vector-valued functions $\mathcal{F} = f$ obtained by completing of the space $L_2(\Omega)$ with respect to the norm

$$\| \mathcal{F} \|_F^2 = \int_\Omega \left(\int_0^x a(\xi) f(\xi, t) d\xi \right)^2 dxdt. \tag{2.7}$$

Definition 1. A solution of the operator $\overline{L}v = \mathcal{F}$ is called a *strong solution* of the problem (2.1), (2.2), (2.3) and (2.4).

3. A Priori Estimates and its Applications

Theorem 3.1. *Let conditions* (1) *and* (2) *be fulfilled. Then for any function* $v \in E$, *there is the a priori estimate*

$$\| v \|_E \leq c \| Lv \|_{\mathbb{F}}, \tag{3.1}$$

where c is a constant which may depend on T but not depend on v.

Proof. Applying the linear operator $M_x v = \int_0^x v(\xi, t) d\xi$, on $a\mathcal{L}v$ of (2.1), we get

$$M_x(\mathcal{L}v) = \sum_{i=1}^n D_{t_i}^\alpha \int_0^x a(\xi) v(\xi, t) d\xi - 2\int_0^x \frac{\partial}{\partial \xi}\left(b(J_\xi a, t) \frac{\partial v}{\partial \xi} \right) d\xi$$

$$= \int_0^x a(\xi) f(\xi, t) d\xi. \tag{3.2}$$

Taking the square of the norm in the space $L_2(\Omega)$ of $M_x(a\mathcal{L}v)$, we obtain:

$$\sum_{i=1}^n \int_\Omega \left(D_{t_i}^\alpha \int_0^x a(\xi) v(\xi, t) d\xi \right)^2 dxdt + \int_\Omega \left(\int_0^x \frac{\partial}{\partial \xi}\left(b(J_\xi a, t) \frac{\partial v}{\partial \xi} \right) d\xi \right)^2 dxdt$$

$$- 2\sum_{i=1}^n \int_\Omega \left(D_{t_i}^\alpha \int_0^x a(\xi) v(\xi, t) d\xi \right) \left(\int_0^x \frac{\partial}{\partial \xi}\left(b(J_\xi a, t) \frac{\partial v}{\partial \xi} \right) d\xi \right) dxdt$$

$$= \int_\Omega \left(\int_0^x a(\xi) f(\xi, t) d\xi \right)^2 dxdt. \tag{3.3}$$

Integrating by parts the two last terms on the left-hand side in (3.3) with the use of boundary conditions, we get

$$\int_\Omega \left(\int_0^x \frac{\partial}{\partial \xi}\left(b(J_\xi a, t) \frac{\partial v}{\partial \xi} \right) d\xi \right)^2 dxdt = \int_\Omega b^2(J_x a, t) \left(\frac{\partial v}{\partial x} \right)^2 dxdt, \quad (3.4)$$

$$-2\sum_{i=1}^{n} \int_\Omega \left(D_{t_i}^{\alpha} \int_0^x a(\xi) v(\xi, t) d\xi \right) \left(\int_0^x \frac{\partial}{\partial \xi}\left(b(J_\xi a, t) \frac{\partial v}{\partial \xi} \right) d\xi \right) dxdt$$

$$= -2\sum_{i=1}^{n} \int_\Omega \left(D_{t_i}^{\alpha} \int_0^x a(\xi) v(\xi, t) d\xi \right) b(J_x a, t) \frac{\partial v}{\partial x} dxdt$$

$$= 2\sum_{i=1}^{n} \int_\Omega (D_{t_i}^{\alpha} a(x) v(x, t)) \left(\int_0^x b(J_\xi a, t) \frac{\partial v}{\partial \xi} \right) dxdt. \quad (3.5)$$

Applying an elementary inequality to (3.5), we have

$$2\sum_{i=1}^{n} \int_\Omega (D_{t_i}^{\alpha} a(x) v(x, t)) \left(\int_0^x \frac{\partial}{\partial \xi}\left(b(J_\xi a, t) \frac{\partial v}{\partial \xi} \right) d\xi \right) dxdt$$

$$\leq \varepsilon_1 \sum_{i=1}^{n} \int_\Omega (D_{t_i}^{\alpha} a(x) v(x, t))^2 dxdt + \frac{n}{\varepsilon_1} \int_\Omega \left(\int_0^x b(J_\xi a, t) \frac{\partial v}{\partial \xi} d\xi \right)^2 dxdt. \quad (3.6)$$

Substituting (3.4) and (3.6) in (3.3), it follows that

$$\sum_{i=1}^{n} \int_\Omega \left(D_{t_i}^{\alpha} \int_0^x a(\xi) v(\xi, t) d\xi \right)^2 dxdt + \int_\Omega b^2(J_x a, t) \left(\frac{\partial v}{\partial x} \right)^2 dxdt$$

$$\leq \varepsilon_1 \sum_{i=1}^{n} \int_\Omega (D_{t_i}^{\alpha} a(x) v(x, t))^2 dxdt + \frac{n}{\varepsilon_1} \int_\Omega \left(\int_0^x b(J_\xi a, t) \frac{\partial v}{\partial \xi} d\xi \right)^2 dxdt$$

$$+ \int_\Omega \left(\int_0^x a(\xi) f(\xi, t) d\xi \right)^2 dxdt. \quad (3.7)$$

By virtue of the condition (1), from (3.7), we observe that

$$(1 - \ell^2\varepsilon_1)\sum_{i=1}^{n}\int_{\Omega}\left(D_{t_i}^{\alpha}\int_0^x a(\xi)v(\xi, t)d\xi\right)^2 dxdt$$

$$+\left(c_0^2 - \frac{c_1^2\ell^2 n}{\varepsilon_1}\right)\int_{\Omega}\left(\frac{\partial v}{\partial x}\right)^2 dxdt \le \int_{\Omega}\left(\int_0^x a(\xi) f(\xi, t)d\xi\right)^2 dxdt. \tag{3.8}$$

Hence, if $\varepsilon_1 > 0$ satisfies $1 - \ell^2\varepsilon_1 > 0$ and $c_0^2 - \frac{c_1^2\ell^2 n}{\varepsilon_1} > 0,$ it follows from estimation (3.8) that

$$\sum_{i=1}^{n}\int_{\Omega}\left(D_{t_i}^{\alpha}\int_0^x a(\xi)v(\xi, t)d\xi\right)^2 dxdt + \int_{\Omega}\left(\frac{\partial v}{\partial x}\right)^2 dxdt$$

$$\le c\int_{\Omega}\left(\int_0^x a(\xi) f(\xi, t)d\xi\right)^2 dxdt, \tag{3.9}$$

where $c = \frac{1}{\min\left(1 - \ell^2\varepsilon_1;\ c_0^2 - \frac{c_1^2\ell^2 n}{\varepsilon_1}\right)}$. The proof of Theorem 3.1 is complete.

Proposition 3.1. *The operator L from E to F admits a closure.*

Proof. Suppose that $v_n \in D(L)$ is a sequence such that

$$v_n \xrightarrow{n\to+\infty} 0 \text{ in } E, \tag{3.10}$$

$$Lv_n \xrightarrow{n\to+\infty} g \text{ in } F. \tag{3.11}$$

Then, we must show that $g \equiv 0.$ Equation (3.10) implies that

$$v_n \xrightarrow{n\to+\infty} 0 \text{ in } D'(\Omega). \tag{3.12}$$

By virtue of the continuity of the derivation of $D'(\Omega)$ in $D'(\Omega)$, we have

$$\mathcal{L}v_n \xrightarrow{n\to+\infty} 0 \text{ in } D'(\Omega). \tag{3.13}$$

We see via (3.11) that

$$\mathcal{L}v_n \xrightarrow{n\to+\infty} g \text{ in } L_2(\Omega), \tag{3.14}$$

then

$$\mathcal{L}v_n \xrightarrow{n\to+\infty} g \text{ in } D'(\Omega). \tag{3.15}$$

By virtue of the uniqueness of the limit in $D'(\Omega)$, (3.13) and (3.15) imply that $g \equiv 0$.

Corollary 1. *Under the conditions of Theorem* 3.1, *there is a constant* $C > 0$ *independent of v such that*

$$\| v \|_E \leq \| \overline{L}v \|_F, \quad v \in D'(\Omega). \tag{3.16}$$

Corollary 2. *If a strong solution exists, it is unique and depends continuously on f, provided v is considered in the topology of E and f is considered in the topology of F.*

Corollary 3. *The range* $R(\overline{L})$ *of the operator* $\overline{L}$ *is closed in F and* $R(\overline{L}) = \overline{R(L)}$, *where* $R(L)$ *is the range of L.*

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