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ON THE EXACT SOLUTION OF THE FUNCTIONAL DIFFERENTIAL EQUATION $y'(t) = ay(t) + by(-t)$

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Abstract

This paper focuses on obtaining the exact solution of the functional differential equation: $y'(t) = ay(t) + by(-t)$ subject to the initial

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condition $y(0) = \lambda.$ The standard series approach is applied to obtain the solution in a power series form. The convergence issue is addressed. In addition, the exact solution is established in terms of elementary functions such as hyperbolic and trigonometric functions. The exact solutions of some special cases, at particular choices of *a* and *b*, are determined. The obtained results may be introduced for the first time regarding the solution of the current problem.

1. Introduction

In this paper, we consider a special kind of a delay differential equation (DDE) in the form:

$$y'(t) = ay(t) + by(-t), \quad y(0) = \lambda, \tag{1}$$

where $a, b, \lambda \in \mathbb{R}.$ The initial value problem (IVP) (1) is a special case of the pantograph delay equation [1-10]. The pantograph is a particular device which collects the current in electric trains. In fact, the IVP (1) can be solved via several approaches such as the Adomian decomposition method (ADM) [11], the homotopy perturbation method (HPM) [12], and the Laplace transform method [13-17]. However, the standard series method is preferred in this work for its simplicity when compared with the above approaches. The standard series method was successfully applied to solve the delay differential equation describing the light absorption in interstellar physics (known as the Ambartsumian model) under different situations [18-20]. So, the objective of this work is to extend the application of the standard series method to deal with the present model. The convergence issue will be addressed. In addition, it will be shown that the power series solution has other equivalent forms in terms of elementary functions such as the hyperbolic functions, the Mittag-Leffler functions, and the trigonometric functions.

Also, the IVP (1) enjoys several interesting special cases of the constants a and b such as $a = 0, a + b = 0,$ and $a - b = 0.$ The exact solutions of such special cases will also be constructed in a direct manner. Moreover, the special case $b = 0$ is not considered here. This is because it is a trivial case in which the corresponding exact solution is already well known as $y(t) = \lambda e^{at}$.

2. The Standard Series Method

Here, we apply the standard series method in the form:

$$y(t) = \sum_{n=0}^{\infty} \omega_n t^n, \tag{2}$$

to search for a solution of equation (1). Substituting (2) into (1), we have

$$\sum_{n=1}^{\infty} n\omega_n t^{n-1} = a\sum_{n=0}^{\infty} \omega_n t^n + b\sum_{n=0}^{\infty} \omega_n (-1)^n t^n, \tag{3}$$

i.e.,

$$\sum_{n=0}^{\infty} (n+1)\omega_{n+1} t^n = \sum_{n=0}^{\infty} (a + (-1)^n b)\omega_n t^n. \tag{4}$$

Thus

$$\sum_{n=0}^{\infty} [(n+1)\omega_{n+1} - (a + (-1)^n b)\omega_n] t^n = 0, \tag{5}$$

which leads to

$$(n+1)\omega_{n+1} - (a + (-1)^n b)\omega_n = 0. \tag{6}$$

Therefore,

$$\omega_{n+1} = \left[\frac{a + (-1)^n b}{n+1}\right]\omega_n, \quad n \geq 0. \tag{7}$$

Accordingly,

$$\omega_1 = \frac{1}{1!}(a+b)\omega_0,$$

$$\omega_2 = \frac{1}{2}(a-b)\omega_1 = \frac{1}{2!}(a^2-b^2)\omega_0,$$

$$\omega_3 = \frac{1}{3}(a+b)\omega_2 = \frac{1}{3!}(a^2-b^2)(a+b)\omega_0,$$

$$\omega_4 = \frac{1}{4}(a-b)\omega_3 = \frac{1}{4!}(a^2-b^2)^2\omega_0,$$

$$\omega_5 = \frac{1}{5}(a+b)\omega_4 = \frac{1}{5!}(a^2-b^2)^2(a+b)\omega_0,$$

$$\omega_6 = \frac{1}{6}(a-b)\omega_5 = \frac{1}{6!}(a^2-b^2)^3\omega_0,$$

$$\omega_7 = \frac{1}{7}(a+b)\omega_6 = \frac{1}{7!}(a^2-b^2)^3(a+b)\omega_0,$$

$$\vdots \tag{8}$$

Applying the initial condition $y(0) = \lambda$ on the series (3) gives $\omega_0 = \lambda$. In view of (8), the coefficients ω_n can be compacted in the form

$$\omega_n = \begin{cases} \frac{\lambda}{n!}(a^2-b^2)^{\frac{n}{2}}, & \text{if } n = 2, 4, 6, \ldots \text{ (even)}, \\ \frac{\lambda}{n!}(a+b)^{\frac{n+1}{2}}(a-b)^{\frac{n-1}{2}}, & \text{if } n = 1, 3, 5, \ldots \text{ (odd)}. \end{cases} \tag{9}$$

2.1. Convergence

Theorem 1. *For all* $a, b \in \mathbb{R}$, *the series*

$$y(t) = \sum_{n=0}^{\infty} \omega_n t^n, \tag{10}$$

with the ω_n *defined by* (8) *or* (9) *has an infinite radius of convergence and hence the series converges*, $\forall t \in \mathbb{R}$.

Proof. Assume that ρ is the radius of convergence. Then the ratio test leads to

$$\frac{1}{\rho} = \lim_{n\to\infty}\left|\frac{\omega_{n+1}t^{n+1}}{\omega_n t^n}\right| = \lim_{n\to\infty}\left|\frac{\omega_{n+1}}{\omega_n}\right||t|$$

$$= \lim_{n\to\infty}\left|\frac{a+(-1)^n b}{n+1}\right||t|. \tag{11}$$

The term $(-1)^n = \pm 1$ according to n is even or odd, hence, the limit (11) becomes

$$\frac{1}{\rho} = \lim_{n\to\infty}\left|\frac{a \pm b}{n+1}\right||t| = 0, \quad \forall a, b, t \in \mathbb{R}, \tag{12}$$

which completes the proof. □

3. The Exact Solution

In this section, the solution of the IVP (1) is to be expressed in exact forms in terms of some elementary functions such as the Mittag-Leffler, the hyperbolic, and the trigonometric functions. First of all, the series solution (2) can be written as

$$y(t) = \omega_0 + \sum_{n=1}^{\infty}\omega_n t^n$$

$$= \lambda + \sum_{n=1}^{\infty}\omega_{2n-1}t^{2n-1} + \sum_{n=1}^{\infty}\omega_{2n}t^{2n}, \tag{13}$$

where ω_{2n-1} and ω_{2n} are obtained from (9) as

$$\omega_{2n-1} = \frac{\lambda}{(2n-1)!}(a+b)^n(a-b)^{n-1}, \quad \omega_{2n} = \frac{\lambda}{(2n)!}(a^2-b^2)^n. \tag{14}$$

Accordingly, equation (13) becomes

$$y(t) = \lambda\left[1 + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!}(a+b)^n (a-b)^{n-1} + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}(a^2 - b^2)^n\right], \quad (15)$$

i.e.,

$$y(t) = \lambda\left[1 + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}(a+b)^{n+1} (a-b)^{n} + \sum_{n=0}^{\infty} \frac{t^{2n+2}}{(2n+2)!}(a^2 - b^2)^{n+1}\right]. \quad (16)$$

3.1. Solution in terms of hyperbolic trigonometric functions $(a > b)$

We have from (16) that

$$y(t) = \lambda\left[1 + (a+b)\sum_{n=0}^{\infty} \frac{(\sqrt{a^2 - b^2})^{2n} t^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(\sqrt{a^2 - b^2})^{2n+2} t^{2n+2}}{(2n+2)!}\right]$$

$$= \lambda\left[1 + \frac{a+b}{\sqrt{a^2 - b^2}} \sum_{n=0}^{\infty} \frac{(\sqrt{a^2 - b^2})^{2n+1} t^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(\sqrt{a^2 - b^2}\,t)^{2n+2}}{(2n+2)!}\right]$$

$$= \lambda\left[1 + \sqrt{\frac{a+b}{a-b}} \sum_{n=0}^{\infty} \frac{(\sqrt{a^2 - b^2}\,t)^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{(\sqrt{a^2 - b^2}\,t)^{2n+2}}{(2n+2)!}\right]$$

$$= \lambda\left[1 + \sqrt{\frac{a+b}{a-b}} \sinh(\sqrt{a^2 - b^2}\,t) + \sum_{n=1}^{\infty} \frac{(\sqrt{a^2 - b^2}\,t)^{2n}}{(2n)!}\right]$$

$$= \lambda\left[\sqrt{\frac{a+b}{a-b}} \sinh(\sqrt{a^2 - b^2}\,t) + \cosh(\sqrt{a^2 - b^2}\,t)\right], \quad a > b. \quad (17)$$

3.2. Solution in terms of Mittag-Leffler functions

Here, it is noted that equation (16) can be written using the Gamma function in the form:

$$y(t) = \lambda\left[1 + \sum_{n=0}^{\infty}\frac{t^{2n+1}}{\Gamma(2n+2)}(a+b)^{n+1}(a-b)^{n} + \sum_{n=0}^{\infty}\frac{t^{2n+2}}{\Gamma(2n+3)}(a^2-b^2)^{n+1}\right] \tag{18}$$

or

$$y(t) = \lambda\left[1 + (a+b)t\sum_{n=0}^{\infty}\frac{((a^2-b^2)t^2)^n}{\Gamma(2n+2)} + (a^2-b^2)t^2\sum_{n=0}^{\infty}\frac{((a^2-b^2)t^2)^n}{\Gamma(2n+3)}\right]. \tag{19}$$

Using the definition of the two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \tag{20}$$

equation (19) takes the following final form:

$$y(t) = \lambda[1 + (a+b)tE_{2,2}((a^2-b^2)t^2) + (a^2-b^2)t^2E_{2,3}((a^2-b^2)t^2)]. \tag{21}$$

This last form can also be used to establish the solution in terms of trigonometric functions via some properties of the Mittag-Leffler functions. This point is declared below.

3.3. Solution in terms of trigonometric functions $(b > a)$

Suppose that $b > a$. Then we can rewrite (21) as

$$y(t) = \lambda[1 + (a+b)tE_{2,2}(-(\sqrt{b^2-a^2}\,t)^2)$$
$$- (b^2-a^2)t^2E_{2,3}(-(\sqrt{b^2-a^2}\,t)^2)]. \tag{22}$$

Applying the following properties:

$$E_{2,2}(-z^2) = \frac{\sin(z)}{z}, \quad E_{2,3}(-z^2) = \frac{1-\cos(z)}{z^2}, \tag{23}$$

for $z = \sqrt{b^2 - a^2}t$, we have

$$E_{2,2}(-(\sqrt{b^2 - a^2}t)^2) = \frac{\sin(\sqrt{b^2 - a^2}t)}{\sqrt{b^2 - a^2}t}, \tag{24}$$

$$E_{2,3}(-(\sqrt{b^2 - a^2}t)^2) = \frac{1 - \cos(\sqrt{b^2 - a^2}t)}{(b^2 - a^2)t^2}. \tag{25}$$

Substituting (24) and (25) into (33) and simplifying, we obtain

$$y(t) = \lambda\left[\sqrt{\frac{b + a}{b - a}}\sin(\sqrt{b^2 - a^2}t) + \cos(\sqrt{b^2 - a^2}t)\right], \quad b > a. \tag{26}$$

It is easy to show that the solution (26) is periodic with a period equals $\frac{2\pi}{\sqrt{b^2 - a^2}}$ provided $b > a$.

4. Special Cases

In this section, we aim to derive the exact solutions of the IVP (1) at some interesting special cases for the constants a and b such as $a = 0$, $a = -b$, and $a = b$.

4.1. $a = 0$

In this case, the IVP (1) reduces to

$$y'(t) = by(-t), \quad y(0) = \lambda. \tag{27}$$

The solution in terms of trigonometric functions can be directly obtained by setting $a = 0$ into equation (26), and this gives

$$y(t) = \lambda[\sin(bt) + \cos(bt)], \quad b > 0. \tag{28}$$

4.2. $a = -b$

Let $a = -b = \sigma$. Then the IVP (1) becomes

$$y'(t) = \sigma(y(t) - y(-t)), \quad y(0) = \lambda. \tag{29}$$

The solution of this case is directly obtained by substituting $a = \sigma$ and $b = -\sigma$ into equation (26), thus $y(t) = \lambda$. This constant solution satisfies the IVP (29), $\forall \sigma \in \mathbb{R}$.

4.3. $a = b$

Let $a = b = \mu$. Then the IVP (1) becomes

$$y'(t) = \mu(y(t) + y(-t)), \quad y(0) = \lambda. \tag{30}$$

On substituting $a = \mu$ and $b = \mu$ into equation (26), it is noted that the first term in the right hand side equals $\frac{0}{0}$ which forces us to use L'Hospital's rule. This can be done as follows. Equation (26) can be rewritten as

$$\begin{aligned} y(t) &= \lambda\left[\sqrt{\frac{b+a}{b-a}}\sin(\sqrt{(b+a)(b-a)}\,t) + \cos(0)\right], \quad b > a \\ &= \lambda\left[\sqrt{2\mu}\lim_{b-a\to 0}\frac{\sin(\sqrt{2\mu}\sqrt{(b-a)}\,t)}{\sqrt{b-a}} + 1\right] \\ &= \lambda\left[\sqrt{2\mu}\lim_{\varepsilon\to 0}\frac{\sin(\sqrt{2\mu}\sqrt{\varepsilon}\,t)}{\sqrt{\varepsilon}} + 1\right], \quad \varepsilon = b - a \to 0 \\ &= \lambda(1 + 2\mu t), \end{aligned} \tag{31}$$

where $\lim_{\varepsilon\to 0}\frac{\sin(\sqrt{2\mu}\sqrt{\varepsilon}\,t)}{\sqrt{\varepsilon}} = \sqrt{2\mu}t.$

5. Conclusion

In this paper, exact solutions were obtained for the functional differential equation $y'(t) = ay(t) + by(-t)$ under the condition $y(0) = \lambda$. The standard series approach was applied to obtain the power series solution. The convergence of the series solution was addressed and proved. The obtained power series was successfully modified via some algebras and accordingly the exact solution was established. The exact solution was expressed in terms of elementary functions such as Mittag-Leffler functions, hyperbolic functions, and trigonometric functions. In conclusion, the current approach can be further applied on the full pantograph model $y'(t) = ay(t) + by(ct), \ c \in \mathbb{R}$.

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