Advances in Differential Equations and Control Processes
© 2022 Pushpa Publishing House, Prayagraj, India
http://www.pphmj.com
http://dx.doi.org/10.17654/0974324322002
Volume 26, 2022, Pages 21-38 **ISSN: 0974-3243**

SECOND-ORDER ITERATIVE METHOD FOR OPTIMAL CONTROL PROBLEMS OF MULTISTAGE PROCESSES

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Abstract

The paper proposes a second-order strong improvement method for optimal control problems with non-fixed stage time intervals. The technique of inference algorithms is based on the theory of V. F. Krotov. Conditions are given for the control to be improvable, which are closely related to the necessary and sufficient conditions for a strong local minimum.

Received: October 11, 2021; Accepted: November 12, 2021
2020 Mathematics Subject Classification: 49M05.
Keywords and phrases: optimal control, improvement method, inference algorithms, theory of Krotov, sufficient conditions, strong local minimum.

How to cite this article: V. A. Baturin, V. N. Sizykh and A. V. Daneev, Second-order iterative method for optimal control problems of multistage processes, Advances in Differential Equations and Control Processes 26 (2022), 21-38.
DOI: 10.17654/0974324322002

Published Online: December 20, 2021

0. Introduction

Mathematical models describing multistage processes are quite common in practice: technological processes, for example, the processes of obtaining gasoline and kerosene from oil, assembling various machines, chemical-technological production and metal production both. So the flights of medium-range missiles consist of three stages: the first - the main engine takes the object into airless space, in the second stage, the rocket is controlled by other engines to achieve a given position in space, the third consists in choosing a maneuver to hit the target. All these systems are controllable, and optimal control problems are relevant. These problems have been studied for more than fifty years, and so far they have not lost their relevance. We list some of them: [1, 3-24]. The article is structured as follows: first, the problem of optimal control of a multistage process is posed in general form, then a formula for the functional increment for a two-stage process is derived, and an algorithm for successive improvements is formulated, which can be easily generalized to the case of many stages. The case is specially investigated when the initial approximation satisfies the necessary optimality conditions, but does not provide the functional with a strong local minimum. Using the necessary and sufficient conditions for the optimality of a strong local minimum, a special improvement algorithm is constructed.

1. Statement of the Problem

A controlled process is considered, consisting of several stages, and the moment of the end of the previous stage is the moment of the beginning of the next stage. Each of the stages is described by its own system of differential equations:

$$\frac{dx^i}{dt} = f^i(t,\ x^i(t),\ u^i(t)),\quad i = \overline{0,\ p}, \tag{1}$$

$t \in [\tau_i,\ \tau_{i+1}],\ \ \tau_i$ are not fixed. The initial conditions are determined from the relations:

$$x_0(\tau_0) = \kappa^0,$$

$$x^i(\tau_i) = \kappa^i(\tau_i,\ x^{i-1}(\tau_i)), \quad i = \overline{1,\ p}. \tag{2}$$

The functions $x^i(t)$ are piecewise differentiable and take values in the Euclidean space $R^{n(i)}$; $u^i(t) \in U_i \subset R^{m(i)}$ are piecewise continuous; κ^i are given functions.

Let $x = (x^0(t), x^1(t), ..., x^p(t)),$ $u = (u^0(t), u^1(t), ..., u^p(t)),$ prime means transposition. The set of triplets $(x(t), u(t), \tau)$, satisfying the listed conditions, as well as differential constraints (1) and initial conditions (2), will be called the *set of admissible ones* and denoted by *D*. It is assumed that $D \neq \varnothing$.

We define the functional

$$I(x,\ u,\ \tau) = F(x^p(\tau_{p+1}),\ \tau_{p+1}) \to \text{min}.$$

Let us pose the problem of finding a minimizing sequence $\{(x_s,\ u_s,\ \tau_s)\} \subset D$ on which

$$I(x_s,\ u_s,\ \tau_s) \to \inf_D I,\ s \to \infty.$$

This sequence will be called *minimizing*. Let us introduce the Lagrange functional in accordance with Krotov's theory [11] on sufficient conditions for optimality. Introduce the following constructions. Let $\varphi^i(t,\ x^i)$ be functions continuously differentiable with respect to their arguments, $t \in [\tau_i,\ \tau_{i+1}]$,

$$R^i(t,\ x^i,\ u^i) = \varphi'^i_{x^i}(t,\ x^i)\, f^i(t,\ x^i,\ u^i) + \varphi^i_{x^i}(t,\ x^i),$$

$$G^0(x^0(\tau_0),\ x^0(\tau_1)) = \varphi^0(\tau_1,\ x^0(\tau_1)) - \varphi^0(\tau_0,\ x^0(\tau_0)),$$

$$G^i(x^i(\tau_i),\ x^i(\tau_{i+1})) = \varphi^i(\tau_{i+1},\ x^i(\tau_{i+1})) - \varphi^i(\tau_i,\ \kappa^i(\tau_i,\ x^{i-1}(\tau_i))),$$

$$G^p(x^p(\tau_p),\ x^p(\tau_{p+1})) = F(x^p(\tau_{p+1})) + \varphi^p(\tau_{p+1},\ x^p(\tau_{p+1}))$$

$$- \varphi^p(\tau_p,\ \kappa^p(\tau_p),\ x^{p-1}(\tau_p))),$$

$$L(x,\ u,\ \tau) = \sum_{i=0}^{p} \left[G^i(x^i(\tau_i),\ x^i(\tau_{i+1})) - \int_{\tau_i}^{\tau_{i+1}} R^i(t,\ x^i,\ u^i)dt \right].$$

The functional L is defined on a wider set E, which is obtained from D by discarding the differential connection (1).

2. The Problem of Improvement

Let controls $u_I^i(t)$, moments of time τ_I^i and corresponding states $x_I^i(t)$ be given. It is required to find $u_{II}^i(t)$, $x_{II}^i(t)$, τ_{II}^i such that

$$I(x_{II},\ u_{II},\ \tau_{II}) < I(x_I,\ u_I,\ \tau_I).$$

This problem is called an *improvement problem*. Solving successively the improvement problems, we obtain an iterative method that ensures the monotonicity of the functional over iterations. If improvement algorithms use a constraint on control variation, then such methods are called *weak improvement methods*, and if only state variation, then the *strong improvement methods*. In the future, we will carry out the presentation for two stages, which makes it possible to more clearly demonstrate the research. The obtained improvement methods can be easily carried over to the case of p stages.

3. Two-step Process: Functional Increment Formula

Let the systems of differential equations and the functional to be minimized be given on time intervals $[\tau_0,\ \tau_1]$ and $[\tau_1,\ \tau_2]$:

$$\frac{dx}{dt} = f(t,\ x,\ u), \quad x(\tau_0) = x_0, \quad t \in [\tau_0,\ \tau_1], \tag{3}$$

$$\frac{dy}{dt} = g(t,\ y,\ \upsilon), \quad y(\tau_1) = \kappa(x(\tau_1)), \quad t \in [\tau_1,\ \tau_2], \tag{4}$$

$u(t) \in U \subset R^{m_1}$, $v(t) \in V \subset R^{m_2}$, $x(t) \in R^{n_1}$, $y(t) \in R^{n_2}$, moments in time τ_0, τ_2 are fixed, τ_1 is not fixed,

$$I(\tau_1, x, u, y, \upsilon) = F(y(\tau_2)).$$

We introduce the functions

$$G^0(\tau_1) = \varphi^0(\tau_1, x) - \varphi^0(\tau_0, x_0),$$

$$R^0(t, x, u) = \varphi_x^0(t, x)' f(t, x, u) + \varphi_t^0(t, x), \quad t \in [\tau_0, \tau_1],$$

$$G^1(\tau_1, y, x) = \varphi^I(\tau_2, y) - \varphi_x^I(\tau_1, \kappa^i(x(\tau_1))) + F(y).$$

Let us compose the Lagrange functional

$$L(\tau_1, x, u, y, \upsilon) = G^0(\tau_1, x) + G^1(\tau_1, y, x(\tau_1)) + F(y)$$
$$- \left[\int_{\tau_2}^{\tau_1} R^0(t, x, u)dt + \int_{\tau_1}^{\tau_2} R^1(t, y, \upsilon)dt \right].$$

We introduce the following functions:

$$H^0(t, x, p^0, u) = p^{0'} f(t, x, u),$$

$$H^1(t, y, p^1, \upsilon) = p^{1'} g(t, y, \upsilon),$$

$$H^0(t, x, p^0) = \max_{u \in U} H^0(t, x, p^0, u), \; H^1(t, y, p^1) = \max_{u \in V} H^1(t, y, p^1, \upsilon),$$

$$P^0(t, x) = \max_{u \in U} R^0(t, x, u), \; P^1(t, y) = \max_{\upsilon \in V} R^1(t, y, \upsilon),$$

$$\tilde{u}(t, x, p^0) = \arg\max_{u \in U} H^0(t, x, p^0, u), \tag{5}$$

$$\tilde{\upsilon}(t, y, p^1) = \arg\max_{\upsilon \in V} H^1(t, y, p^1, \upsilon) \tag{6}$$

and the functional

$$\mathfrak{M}(\tau_1, x, y) = \min_{\substack{u(\cdot):\{u(t)\in U\}\\ v(\cdot):\{v(t)\in U\}}} L(\tau_1, x, y, u, \upsilon).$$

Then

$$P^0(t, x) = H^0(t, x, \varphi_x^0) + \phi_t^0, \quad P^1(t, y) = H^1(t, y_y^1) + \varphi_t^1,$$

$$\mathfrak{M}(\tau_1, x, y) = G^0 + G^1 - \int_{\tau_0}^{\tau_1} P^0(t, x)dt - \int_{\tau_1}^{\tau_2} P^1(t, y)dt.$$

Let it be given that $(\bar{\tau}_1, \bar{x}(t), \bar{u}(t), \bar{y}(t), \bar{\upsilon}(t))$. Introduce the functional

$$\mathfrak{M}_a(\tau_1, x, y) = G^0(\tau_1, x) + G_a^1(\tau_1, y, x(\tau_1))$$

$$- \int_{\tau_0}^{\tau_1} P^0(t, x)dt - \int_{\tau_1}^{\tau_2} P^1(t, y)dt$$

$$+ \frac{(1-\alpha)}{2}\left[\int_{\tau_0}^{\tau_1}(x - \bar{x}(t))^2 dt + \int_{\tau_1}^{\tau_2}(y - \bar{y}(t))^2 dt\right],$$

where $G_a^1(\tau_1, y, x) = \varphi^1(\tau_2, y) - \varphi^1(\tau_1, \kappa(x)) + \alpha F(y)$.

Consider the increment of the functional M_a:

$$\mathfrak{M}_a(\tau_1, x, y) - \mathfrak{M}_a(\bar{\tau}_1, \bar{x}, \bar{y})$$

$$= G^0(\tau_1, x) - G^0(\bar{\tau}_1, \bar{x}) + G_a^1(\tau_1, y, x) - G_a^1(\bar{\tau}_1, \bar{y}, \bar{x})$$

$$- \left[\int_{\tau_0}^{\tau_1+\Delta_\tau} P^0(t, x)dt - \int_{\tau_0}^{\tau_1} P^0(t, \bar{x})dt + \int_{\tau_1+\Delta_\tau}^{\tau_2} P^1(t, y)d\tau - \int_{\tau_1}^{\tau_2} P^1(t, \bar{y})d\tau\right]$$

$$+ \frac{(1-\alpha)}{2}\left[\int_{\tau_0}^{\tau_1}(x - \bar{x}(t))^2 dt + \int_{\tau_1}^{\tau_2}(y - \bar{y}(t))^2 dt\right].$$

We expand the increment in a Taylor series in the vicinity of the point $(\bar{\tau}_1, \bar{x}(t), \bar{y}(t))$ up to terms of the second order in Δx and Δy and up to terms of the first order in $\Delta\tau$, where $x = x - \bar{x}(t)$, $y = y - \bar{y}(t)$, $\tau = \tau_1 - \bar{\tau}_1$. We have

$$\Delta\mathfrak{M}_a = (G_x^0 + G_{ax}^1)'\Delta x + G_{ay}^{1'}\Delta y + (G_{\tau_1}^0 + G_{\tau_1}^1)\Delta\tau$$

$$+ \frac{1}{2}[\Delta x'(G_{xx}^0 + G_{axx}^1)\Delta x + \Delta y' G_{ayx}^1 \Delta x + \Delta x' G_{axy}^1 \Delta y + \Delta y' G_{ayy}^1 \Delta y]$$

$$-\left[\int_{\tau_0}^{\bar{\tau}_1}\left(dP^0 + \frac{1}{2}d^2P\right)dt + \int_{\bar{\tau}_1}^{\tau_2}\left(dP^1 + \frac{1}{2}d^2P^1\right)dt\right.$$

$$\left. + P^0(\bar{\tau}_1, \bar{x}(\bar{\tau}_1)) - P^1(\bar{\tau}_1, \bar{y}(\tau_1))\right]$$

$$+ \frac{(1-a)}{2}\left[\int_{\tau_0}^{\bar{\tau}_1}(x - \bar{x}(t))^2 dt + \int_{\bar{\tau}_1}^{\tau_2}(y - \bar{y}(t))^2 dt\right] + o(\cdot).$$

The functions $\varphi^0(t, x)$ and $\varphi^0(t, \mathcal{H})$ are set in linear-quadratic form

$$\varphi^0(t, x) = \psi^0(t)'(x - \bar{x}(t)) + \frac{1}{2}(x - \bar{x}(t))'\sigma^0(t)'(x - \bar{x}(t)),$$

$$\varphi^1(t, y) = \psi^1(t)'(y - \bar{y}(t)) + \frac{1}{2}y'\sigma^1(t)'y,$$

where $\psi^0(t), \psi^1(t) - n_1, n_2$ are vectors, respectively, $\sigma^0(t), \sigma^1(t) - n_1 \times n_2$, $n_2 \times n_2$ are symmetric matrices. Functions φ^0, φ^1 set so that $\Delta\mathfrak{M}_a$ it does not depend on x, y, x^2, y^2, we obtain the following relations:

$$\psi^0 = -H_x^0 - \sigma^0(H_{\psi^0}^0 - H_{\psi^0}^0), \tag{7}$$

$$\sigma^0 = -H_{xx}^0 + (1-a)E^{(n_1)} - \sigma^0 H_{\psi^0 x}^0 - H_{x\psi^0}^0\sigma^0 - \sigma^0 H_{\psi^0\psi^0}^0\sigma^0, \tag{8}$$

$$\psi^1 = -H_y^1 - \sigma^1(H_{\psi^1}^1 - H_{\psi^1}^1), \tag{9}$$

$$\sigma^1 = -H_{yy}^1 + (1-a)E^{(n_2)} - \sigma^1 H_{\psi^1 y}^1 - H_{y\psi^1}^1\sigma^1 - \sigma^1 H_{\psi^1\psi^1}^1\sigma^1, \tag{10}$$

$$\psi^0(\bar{\tau}_1) = \psi^1(\bar{\tau}_1)\kappa_x(\bar{x}(\bar{\tau}_1)),$$

$$\sigma^0(\bar{\tau}_1) = \psi^1(\bar{\tau}_1)\kappa_x(\bar{x}(\bar{\tau}_1))_x + \kappa'_x(\bar{x}(\bar{\tau}_1))\sigma^1(\bar{\tau}_1)\kappa_x(\bar{x}(\bar{\tau}_1)),$$

$$\psi^1(\bar{\tau}_2) = -aF_y(\bar{y}(\bar{\tau}_2)),$$

$$\sigma^1(\bar{\tau}_2) = -aF_{yy}(\bar{y}(\bar{\tau}_2))$$

and the expression for $\Delta\mathfrak{M}_a$ becomes

$$\Delta\mathfrak{M}_a = (H^1(\bar{\tau}_1,\ \bar{y}(\bar{\tau}_1),\ \psi^1(\bar{\tau}_1)) - H^0(\bar{\tau}_1,\ \bar{y}(\bar{\tau}_1),\ \psi^0(\bar{\tau}_1)))\Delta\tau + o(\cdot).$$

The derivatives of the functions H^0 and H^1 are calculated along $(t,\ \bar{x}(t),\ \psi^0(t))$ and $(t,\ \bar{y}(t),\ \psi^1(t))$, respectively, $E^{(n_1)},\ E^{(n_2)}$ are the unit matrices.

4. Method of Improvement

The resulting constructions define the method of successive improvements.

Algorithm.

(1) At each stage, the initial controls $u^I(t)$ and $\upsilon^I(t)$ and the moment τ_1^I of the stage change are set, from equations (3)-(4) which are determined $x^I(t),\ y^I(t)$.

(2) Parameters $\alpha \in [0, 1]$, $\beta \geq 0$ are set and from the system (7)-(10), we find $\psi^0(t),\ \psi^1(t),\ \sigma^0(t),\ \sigma^1(t)$.

(3) New trajectories $x^{II}(t),\ y^{II}(t)$ are determined from equations (3)-(4) at $u = \tilde{u}(t,\ x,\ \psi^0 + \sigma^0(x - \bar{x}(t))),\ u = \tilde{u}(t,\ y,\ \psi^1 + \sigma^1(y - \bar{y}(t)))$, where $\tilde{u}$ and $\tilde{\upsilon}$ are given by formulas (5)-(6), $\tau_1^{II} = \tau_1^I + \beta\tau_1$, $\Delta\tau_1 = H^0 - H^1$, thereby obtaining new controls and a new value τ_1.

(4) The values of the functionals I^I and I^{II} are compared. If there is no improvement, then the parameters α and β decrease and the process is repeated, starting from item 2.

If $\psi^0(t)$ and $\psi^1(t)$ are given, the equations with respect to σ^0 and σ^1 turn into matrix Riccati equations, which may not have a solution on the intervals $[\tau_0, \tau_1]$ and $[\tau_1, \tau_2]$. In [2], such systems were investigated in detail; it was shown that there exists an α^* such that for all $\alpha < \alpha^*$, the system of equations (7)-(10) will have a solution.

The algorithm can be easily extended to the case when the functional I includes the integral terms $I_1 = \int_{\tau_0}^{\tau_1} f^0(t, x, u)dt$ and $I_2 = \int_{\tau_1}^{\tau_2} g^0(t, x, u)dt$. In this case, the functions H^0 and H^1 are written in the form $H^0 = \psi^{0'} f - af^0$ and $H^1 = \psi^{1'} g - ag^0$, $a \in [0, 1]$.

Example 1.

Stage 1	Stage 2
$\dot{x} = u,$	$\dot{y} = \upsilon,$
$x(0) = x_0,$	$y(\tau) = x(\tau),$
$t \in [0, \tau],$	$t \in [\tau, T],$
$I_1 = \frac{1}{2}\int_0^{\tau} u^2 dt.$	$I_2 = \frac{1}{2} y^2(T) + \frac{1}{2}\int_{\tau}^{T} \upsilon^2(t)dt.$

Functional

$$I = I_1 + I_2.$$

T is fixed, τ is not fixed.

Let us choose the parameter $\alpha = 1$. Write the functions $H^0 = p^0 u - \frac{1}{2}u^2$ and $H^0 = \frac{p^{0^2}}{2}$. Find the derivatives

$$H^0_{p^0} = u, \quad H^0_x = 0, \quad H^0_{xx} = 0, \quad H^0_{xp^0} = 0,$$

$$H^0_{p^0} = p^0, \quad H^0_{p^0 p^0} = 1, \quad H^1 = p^1\upsilon - \frac{1}{2}\upsilon^2,$$

$$H^1 = \frac{p^{1^2}}{2}, \quad H^1_y = 0, \quad H^1_{yy} = 0, \quad H^1_{p^1} = p^1, \quad H^1_{p^1 p^1} = 1, \quad H^1_p = \upsilon.$$

Let us choose an initial approximation $\bar{u}(t) = 0$, $\bar{\upsilon}(t) = 0$, $\tau = \bar{\tau}$, $\bar{x}(t) = x_0$.

The system of equations (7)-(10) takes the form $\psi^0 = -\sigma^0\psi^0$, $\sigma^0 = -\sigma^{0^2}$, $\psi^1 = -\sigma^1\psi^1$, $\sigma^1 = -\sigma^{1^2}$, $\psi^0(\bar{\tau}) = \psi^1(\bar{\tau})$, $\sigma^0(\bar{\tau}) = \sigma^1(\bar{\tau})$, $\sigma^1(T) = -1$ and $\psi^1(T) = -y(T)$. The decision $\sigma^1(t) = \frac{1}{t-T-1}$, $\psi^1(t) = -\frac{x_0}{t-T-1}$, $\sigma^0(t) = \frac{1}{t-T-1}$, $\psi^0(t) = -\frac{x_0}{t-T-1}$. Following the algorithm, we solve the equations:

$$\dot{x} = (\psi^0 + \sigma^0(x - x_0)), \quad x(0) = x_0,$$

$$\dot{y} = (\psi^1 + \sigma^1(y - x_0)), \quad y(\bar{\tau}) = \bar{x}(\tau),$$

and solution takes the form

$$x(t) = \frac{x_0}{T+1}|t - T - 1|, \quad u = -\frac{x_0}{T+1}, \tag{11}$$

$$y(t) = \frac{x_0}{T+1}|t - T - 1|, \quad \upsilon = -\frac{x_0}{T+1}. \tag{12}$$

The values of functionals are given by

$$I_1 = \frac{1}{2}\frac{x_0^2}{(T+1)^2}\bar{\tau}, \quad I_2 = \frac{1}{2}\frac{x_0^2}{(T+1)^2}(T - \bar{\tau}), \quad I = \frac{x_0^2}{2(T+1)^2}.$$

Initial approximation is given by $I = \frac{x_0^2}{2}$. Note that the obtained approximation is optimal and does not depend on τ-change of stage.

5. Algorithm Properties

Let us make the following assumptions regarding the problem statement:

(1) The functions $H^0(t, x, p^0), H^1(t, y, p^1)$ are continuous and twice continuously differentiable with respect to x, p^0, y and p^1, respectively.

(2) There are continuous p^0 and p^1 twice differentiable functions with respect to $\tilde{u}(t, x, p^0)$ and $\tilde{\upsilon}(t, y, p^1)$ and such that

$$H^0(t, x, p^0, \tilde{u}(t, x, p^0)) = H^0(t, x, p^0),$$

$$H^1(t, y, p^1, \tilde{u}(t, y, p^1)) = H^1(t, y, p^1).$$

(3) The function $F(y)$ is twice continuously differentiable with respect to y.

Let us formulate the necessary optimality conditions for the two-stage optimal control problem. Let $(u^*(t), v^*(t), \tau^*)$ be the optimal process. Then the conditions of the Pontryagin maximum principle are satisfied, i.e.,

$$\frac{dx^*}{dt} = f(t, x^*(t), u^*(t)), \quad x^*(\tau_0) = x_0,$$

$$\frac{dy^*}{dt} = g(t, y^*(t), \upsilon^*(t)), \quad y^*(\tau^*) = \kappa(x^*(t^*)),$$

$$\frac{d\psi^0}{dt} = -H_x^0(t, x^*, \psi^0, u^*), \quad \psi^0(\tau^*) = \psi^1(\tau^*)\kappa(x^*(t^*)),$$

$$\frac{d\psi^1}{dt} = -H_y^1(t, y^*, \psi^1, \upsilon^*), \quad \psi^1(t_2) = -F_y(y^*(\tau_2)),$$

$$H^0(t, x^*(t), \psi^0(t), u^*(t)) = \max_{u \in U} H^0(t, x^*(t), \psi^0(t), u), \tag{13}$$

$$H^1(t, y^*(t), \psi^1(t), \upsilon^*(t)) = \max_{\upsilon \in V} H^1(t, y^*(t), \psi^1(t), \upsilon), \tag{14}$$

$$H^0(\tau^*,\ x^*(\tau^*),\ \psi^0(\tau^*),\ u^*(\tau^*)) = H^1(\tau^*,\ y^*(\tau^*),\ \psi^1(\tau^*),\ \upsilon^*(\tau^*)). \quad (15)$$

Consider two cases: (1) the initial approximation does not satisfy the required optimality conditions; (2) satisfies the conditions of optimality.

Now, we formulate an improvement theorem.

Theorem. *If the initial approximation does not satisfy the necessary optimality conditions, then it is improved by the algorithm*.

Let along the initial approximation $(x^I(t),\ u^I(t),\ y^I(t),\ \upsilon^I(t),\ \tau_1^I)$ conditions (13)-(14) are satisfied, and $H^0(\tau_1^I,\ \cdot) = H^1(\tau_1^I,\ \cdot) \neq 0$. Then the solution of equations (3)-(4), closed by the synthesis of controls $\tilde{u}(t,\ x,\ \psi^0 + \sigma^0 x)$, $\tilde{v}(t,\ y,\ \psi^1 + \sigma^1 y)$, will give a solution $u = u^I(t)$, $x = x^I(t),\ \upsilon = \upsilon^I(t),\ y = y^I(t)$. Then on the set of admissible ones, we obtain the estimate

$$aI \leq -\beta(H^1(\tau_1^I,\ \cdot)) - H^0(\tau_1^I,\ \cdot) + o(\tau_1),$$

therefore, α and β can be chosen so that $\Delta I < 0$.

Consider the second case: conditions (13)-(14) are satisfied, condition (15) is not satisfied. In this case, as shown in [2], there will also be an improvement.

Consider a special case when conditions (13)-(15) are satisfied for $\alpha = 1$. Then the system (7)-(10) is divided into the adjoint system and the matrix Riccati equations:

$$\frac{d\psi^0}{dt} = -H_x^0,\ \psi^0(\tau_1^I) = \psi^1(\tau_1^I)\kappa_x(x^I(\tau_1)), \quad t \in [\tau_0;\ \tau_1^I], \quad (16)$$

$$\frac{d\psi^1}{dt} = -H_y^0, \quad \psi^1(\tau_2^I) = F_y(y^I(\tau_2)), \quad t \in [\tau_1^I;\ \tau_2], \quad (17)$$

$$\frac{d\sigma^0}{dt} = -H_{xx}^0 - H_{x\psi^0}^0\sigma^0 - \sigma^0 H_{\psi^0 x}^0 - \sigma^0 H_{\psi^0\psi^0}^0\sigma^0,$$

$$\sigma^0(\tau_1^I) = (\psi^1(\tau_1^I)\kappa_x(x^I(\tau_1^I))_x + \kappa_x(x^I(\tau_1^I))\sigma^1(\tau_1^I)\kappa_x(x^I(\tau_1^I)),$$

$$t \in [\tau_0;\ \tau_1^I], \qquad (18)$$

$$\frac{d\sigma^1}{dt} = -H^1_{yy} - H^1_{y\psi^1}\sigma^1 - \sigma^1 H^1_{\psi^1 y} - \sigma^1 H^1_{\psi^1\psi^1}\sigma^1, \qquad (19)$$

$$\sigma^0(\tau^2) = -F_{yy}(y(\tau_2)), \quad t \in [\tau_1^I;\ \tau_2]. \qquad (20)$$

The derivatives of the functions H^0 and H^1 are calculated along $(t,\ x^I(t),\ \psi^0(t))$ and $(t,\ y^I(t),\ \psi^1(t))$, respectively. It is known that the matrix Riccati equation may not have a solution over the entire required time interval.

Let us write down a second-order algorithm for the case when the necessary optimality conditions are satisfied, the matrix Riccati equation contains a singular point $t_* \in (\tau_1^I;\ \tau_2]$.

Algorithm (Singular point case).

(1) The parameter α is selected so that the singular point coincides with τ_1^I.

(2) Find a symmetric submatrix $\overline{\sigma}$ of the matrix σ with the property

$$\lim_{t\to\tau_1^I+0} \det(\overline{\sigma}(t)) = \infty.$$

(3) The limit is found

$$\chi = \lim_{t\to t^*}(\overline{\sigma}(t))^{-1}.$$

(4) The parameter $\varepsilon > 0$ is set.

(5) The solution of the system $\chi b = 0, |\, b\,| = \varepsilon$ is determined.

(6) The system is integrated

$$\frac{dy}{dt} = g(t,\ y,\ \overline{\upsilon}(t,\ y,\ \psi^1(t) + \sigma^1(t)(x - x^I(t)))), \quad t \in [\tau_1^I;\ \tau_2],$$

$$\dot{y}(\tau_1^I) = g(\tau_1^I),\ y(\tau_1^I),\ \psi^1(\tau_1) + (b',\ 0)), \quad y(\tau_1^I) = x^I(\tau_1^I),$$

thereby finding $y^{II}(t)$ and

$$\upsilon^{II}(t) = \tilde{\upsilon}(t,\ y^{II}(t),\ \psi^1(t) + \sigma^1(t)(y^{II}(t) - y^I(t))).$$

(7) If $I^{II} \geq I^I$, then the parameter ε decreases and the process repeats from item (5).

The improvement process can be constructed in a similar way for the case of a singular point in matrix equation (18).

Let us illustrate the resulting scheme by an example.

Example 2.

Stage 1	Stage 2
$\dot{x} = u,$	$\dot{y} = \upsilon,$
$x(0) = 0,$	$y(\tau) = x(\tau),$
$t \in [0,\ \tau],$	$t \in [\tau,\ T],$
$I_1 = \frac{1}{2}\int_0^\tau (u^2(t))dt.$	$I_2 = -y^2(T) + \int_\tau^T \upsilon^2 dt.$

Functional

$$I = I_1 + I_2.$$

We take as an initial approximation $u^I(t) = 0$, $\upsilon^I(t) = 0$, accordingly $x^I(t) = 0$, $y^I(t) = 0$. The selected controls satisfy the conditions of the Pontryagin maximum principle. For this problem, we define the auxiliary functional J_α in the form

$$J_\alpha = -\alpha y^2(T) + \int_0^T \upsilon^2 dt, \quad \alpha \in [0, 1].$$

Consider functions

$$H^1(t, y, \psi^1, \upsilon) = \psi\upsilon - u^2$$

and

$$H(t, y, \psi^1) = (\psi^1)^2/4, \quad \tilde{\upsilon}(t, y, p^1) = p^1/2.$$

The equation for $\sigma(t)$ takes the form

$$\dot{\sigma}^I = -\frac{\sigma^{I^2}}{2}, \quad \sigma^I(T) = 2\alpha.$$

Its solution is determined by the following formula:

$$\sigma^I(t) = \frac{2}{T - t + 1/\alpha}.$$

Consider the three cases. For $T - \tau < 1$ and $\alpha = 1$, $\sigma^I(t)$ exists over the entire interval $[\tau, T]$, therefore, the initial approximation gives the functional I a strong local minimum. For $T - \tau = 1$ and $\alpha = 1$, $\lim_{t \to \tau + 0} \sigma^I(t) = \infty$ and the control $u(t) = 0$, $\upsilon(t) = c$, where c is an arbitrary constant, is optimal. For $T - \tau > 1$ and $\alpha = 1$, $\sigma^I(t)$ exists only on a part of the segment $[\tau, T]$, and in accordance with the algorithm, the control $\upsilon(t) = 0$ can be improved. Choosing $\alpha = 1/T$, we have

$$\sigma^I(t) = \frac{2}{t}, \quad \tilde{y} = \frac{v}{t}, \quad y(t) = c(t - \tau), \quad v(t) = c,$$

$$I = -c^2(T - \tau)^2 + c^2(T - \tau)^2 = -c^2((T - \tau)^2 - (T - \tau)) < 0$$

for any $c \neq 0$.

Let us consider another option. If τ is a fixed value and there is no singular point at $\alpha = 1$ on $[\tau, T]$, then the original controls give the functional *I* a minimum.

Another situation: τ is not fixed. Continuing the control $\upsilon^I(t) = 0$ up to $t = 0$ and examining the Riccati equation for the existence of a singular point on $(0, T)$, we get: if a singular point exists, then the initial approximation is improved, and if it does not exist, then $u^I(t) = 0$, $\upsilon^I(t) = 0$, and the functional *I* is given a minimum.

6. Conclusion

The paper proposes a second-order successive improvement method for multi-step processes. Conditions for the unimprovability of the original approximation are given. A special case of the algorithm is developed when the necessary conditions are met optimality, but there is a singular point of the Riccati equation at least at one of the stages. The operation of the algorithm is illustrated with test examples.

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