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STABILITY OF NONLINEAR HYBRID FRACTIONAL DIFFERENTIAL EQUATION WITH ATANGANA-BALEANU OPERATOR

S. Britto Jacob and A. George Maria Selvam

Department of Mathematics
Sacred Heart College (Autonomous)
Tirupattur - 635601, Tamil Nadu
India

Abstract

Fractional calculus is a dynamic research field for mathematicians, engineers and physicists. Analysis of qualitative behavior of fractional order differential equations is an advanced topic and it has significant growth due to its applications in real world problems. Study on fractional order differential equations with non-singular kernel is an emerging area in fractional calculus and it gives impressive results. This paper aims to study the Hyers-Ulam stability of nonlinear hybrid fractional order differential equation with Atangana-Baleanu-Caputo operator. From the defined hypotheses and standard fixed point theorem, the existence of solutions is obtained. Sufficient condition

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which ensures the Hyers-Ulam stability of the nonlinear hybrid fractional differential equation is established. An example with numerical illustration is given to support the theoretical outcomes.

1. Introduction

Theory of derivatives and integrals of arbitrary order is known as *fractional calculus* which started in a conversation between L'Hospital and Leibniz in 1695. Fractional Calculus (FC) is used to describe and model the physical phenomena to analyze the qualitative behavior of dynamical system in terms of long duration where both past and future data are needed. In this case, the fractional order differential equations (FODEs) with retardatory and anticipatory arguments on the unbounded intervals of real line are considered. FC has a number of applications related with real world problems in the field of science and engineering, such as fluid flow, heat conduction, artificial intelligence, control theory, image processing, fractals theory, population dynamics, data fitting, finance, chemistry and viscoelasticity [1, 7].

In modelling natural phenomena, fractional order systems give accurate and exciting results comparing to integer order systems because of its memory effect. Academicians analyzed the fractional differential inequalities related to hybrid fractional order differential equation (HFODE) and examined the qualitative properties of the solution [5, 8, 13-15]. Also, they derived the solutions of maximal and minimal. In [4], the authors have investigated the analysis of first order HFODE. In [16], they have established the existence results of HFODE using Riemann-Liouville derivative. Nonlinear HFODE and integro-HFODE with boundary conditions are studied in [11, 18]. Stability of HFODE with $\mathcal{P}$ Laplacian operator is discussed in [6]. In [17], researchers studied the results on positive solutions and stability of HFODE with singularity.

The study of fractional differential operators in the absence of singular kernel is an interesting topic. Caputo and Fabrizio introduced a new fractional operator with exponential kernel [10]. Later, Atangana-Baleanu presented a fractional operator containing Mittag-Leffler function as its kernel in the sense of Caputo [2]. These operators have been provided to demonstrate the fractional order model in different aspects. Also, some fractional derivatives (FDs) such as Caputo FD, Riemann-Liouville FD, the conformable FD, the Hadamard and Hilfer FD are used to discuss the FODE. Analysis of FODE using Atangana-Baleanu-Caputo (ABC) operator is an advanced topic, where the ABC operator is nonlocal and has a non-singular kernel. Stability of nonlinear FODE using ABC operator is analyzed in [3]. In [9], Kucche and Sutar studied the nonlinear FODE with ABC operator. Also, they extended their work to analyze the hybrid FODE with ABC operator in [12].

Inspired by the above mentioned analysis of hybrid FODE with ABC operator, we propose to study the stability of the following nonlinear hybrid fractional order differential equation (NHFODE) with of Atangana-Baleanu-Caputo (ABC) operator in the following manner:

$$ {}_{0}^{\mathcal{ABC}}D_{\kappa}^{\varsigma}\left[\frac{\upsilon(\kappa)-g(\kappa,\,\upsilon(\zeta(\kappa)))}{f(\kappa,\,\upsilon(\zeta(\kappa)))}\right]=h(\kappa,\,\upsilon(\eta(\kappa))), \tag{1} $$

$$ \upsilon(0)=\upsilon_0, $$

where $\kappa\in\mathcal{U}:\mathcal{U}=[0,\mathcal{T}]$, $\mathcal{T}>0$ and $0<\varsigma<1$, ${}_{0}^{\mathcal{ABC}}D_{\kappa}^{\varsigma}$ is an ABC operator, $f\in\mathcal{C}(\mathcal{U}\times\mathbb{R}\to\mathbb{R}\backslash\{0\})$, $g\in\mathcal{C}(\mathcal{U}\times\mathbb{R}\to\mathbb{R})$ and $h:\mathcal{U}\times\mathbb{R}\to\mathbb{R}$, ζ and η are functions from $\mathcal{U}$ into itself.

The significance of analysis of NHFODE is in consisting of various dynamic systems as special cases. The core objective of the paper is to establish the stability of the solution of hybrid fractional order differential equation (1) using ABC derivative. This research article is coordinated as: The basic properties of ABC derivative are presented in Section 2. Section 3 deals with the existence of the solutions, and new sufficient condition ensuring Hyers-Ulam stability is determined in Section 4. To support the analytical results, an example with numerical illustration is provided in Section 5. Finally, Section 6 presents the conclusion of the paper.

2. Prerequisites

This section recollects some important properties that are essential to derive the main results.

Definition 2.1 [12]**.** Let $\upsilon \in \mathcal{AC}(\mathcal{U}, \mathbb{R})$ be a solution of ABC-NHFODE (1) if the mapping $\upsilon \to \frac{\upsilon}{f(\kappa, \upsilon)}$ is completely continuous for every $\upsilon \in \mathbb{R}$ and υ satisfies equation (1), where $\mathcal{AC}(\mathcal{U}, \mathbb{R}) = \{z \,|\, z : \mathcal{U} \to \mathbb{R}\}$ is completely continuous.

Definition 2.2 [2]**.** Suppose $\upsilon \in H^1(0, \mathcal{T})$ and $\varsigma \in [0, 1]$. Then the left ABC operator of υ of order ς is defined by

$$^{\mathcal{ABC}}D_{\kappa}^{\varsigma}\upsilon(\kappa) = \frac{\mathcal{B}(\varsigma)}{1-\varsigma}\int_0^{\kappa} \mathbb{E}_{\varsigma}\left[-\frac{\varsigma}{1-\varsigma}(\kappa-\chi)^{\varsigma}\right]\upsilon'(\chi)d\chi.$$

Here the normalization function is $\mathcal{B}(\varsigma)$ with $\mathcal{B}(0) = \mathcal{B}(1) = 1$ and $\mathbb{E}_{\varsigma}$ is the Mittag-Leffler function (one parameter) defined by $\mathbb{E}_{\varsigma}(h) = \sum_{\nu=0}^{\nu=\infty} \frac{h^{\nu}}{\Gamma(\nu\varsigma+1)}$.

The corresponding fractional integral is defined by

$$_0^{\mathcal{AB}}\mathcal{I}_{\kappa}^{\varsigma}\upsilon(\kappa) = \frac{1-\varsigma}{\mathcal{B}(\varsigma)}\upsilon(\kappa) + \frac{\varsigma}{\mathcal{B}(\varsigma)}{}_0\mathcal{I}_{\kappa}^{\varsigma}\upsilon(\kappa).$$

Here ${}_0\mathcal{I}_{\kappa}^{\varsigma}\upsilon(\kappa) = \frac{1}{\Gamma(\varsigma)}\int_0^{\kappa}(\kappa-\chi)^{\chi-1}\upsilon(\chi)d\chi$ is a Riemann-Liouville fractional integral of υ with fractional order ς.

Lemma 2.1 [12]**.** *If* $0 < \varsigma < 1$, *then* $_0^{\mathcal{AB}}\mathcal{I}_{\kappa}^{\varsigma}(_0^{\mathcal{ABC}}\mathcal{D}_{\kappa}^{\varsigma}\upsilon(\kappa)) = \upsilon(\kappa) - \upsilon(0)$.

Lemma 2.2 [12]**.** *Consider the ABC-FODE as follows*:

$$_0^{\mathcal{ABC}}D_{\kappa}^{\varsigma}\upsilon(\kappa) = f(\kappa, \upsilon(\kappa)), \quad \kappa \in \mathcal{U} = [0, \mathcal{T}], \quad \mathcal{T} > 0,$$

$$\upsilon(0) = \upsilon_0. \tag{2}$$

The integral equation corresponding to (2) *is*

$$\upsilon(\kappa) = \upsilon_0 + \frac{1-\varsigma}{\mathcal{B}(\varsigma)} f(\kappa, \upsilon(\kappa)) + \frac{\varsigma}{\mathcal{B}(\varsigma)\Gamma(\varsigma)} \int_0^{\kappa} (\kappa - \chi)^{\varsigma - 1} f(\chi, \upsilon(\chi)) d\chi.$$

Lemma 2.3 [4]**.** *Let* $\mathcal{P}$ *be a subset of Banach algebra* Φ *and let* $\mathcal{X}_1 : \Phi \to \Phi$, $\mathcal{X}_2 : \Phi \to \Phi$ *and* $\mathcal{X}_3 : \mathcal{P} \to \Phi$ *be the operators such that*

(1) $\mathcal{X}_1$ *and* $\mathcal{X}_2$ *are Lipschitzians with Lipschitz constants* λ *and* γ, *respectively*,

(2) $\mathcal{X}_3$ *is completely continuous*,

(3) $\upsilon = \mathcal{X}_2\upsilon\mathcal{X}_3\nu + \mathcal{X}_1\upsilon \Rightarrow \upsilon \in \mathcal{P}, \forall \nu \in \mathcal{P}$,

(4) $\lambda\gamma\mathcal{R} < 1$, *where* $\mathcal{R} = \sup\{\|\mathcal{X}_3(\upsilon)\| : \upsilon \in \mathcal{P}\}$.

Then the operator $\upsilon = \mathcal{X}_2\upsilon\mathcal{X}_3\upsilon + \mathcal{X}_1\upsilon$ *has a solution in* $\mathcal{P}$.

3. Existence of Solutions for Equation (1)

This section focuses on establishing the existence of solution of equation (1).

Theorem 3.1. *Let* $h \in \mathcal{C}(\mathcal{U} \times \mathbb{R}, \mathbb{R})$ *and suppose* $\upsilon \to \frac{\upsilon - g(\kappa, \upsilon(\zeta(\kappa)))}{f(\kappa, \upsilon(\zeta(\kappa)))}$ *is increasing in* $\mathbb{R}$ *almost everywhere for each* $\kappa \in \mathcal{U}$. *Then* $\upsilon \in \mathcal{AC}(\mathcal{U}, \mathbb{R})$ *is a solution of ABC-NHFODE* (1) *iff* υ *is a solution of fractional integral equation*

$$\upsilon(\kappa) = f(\kappa, \upsilon(\zeta(\kappa))) \begin{bmatrix} \frac{\upsilon_0}{f(0, \upsilon_0)} + \frac{1-\varsigma}{\mathcal{B}(\varsigma)} h(\kappa, \upsilon(\eta(\kappa))) \\ + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)} \int_0^{\varsigma} (\upsilon - \chi)^{\chi - 1} h(\chi, \upsilon(\eta(\chi))) d\chi \end{bmatrix}$$

$$+ g(\kappa, \upsilon(\zeta(\kappa))), \quad \kappa \in \mathcal{U}. \tag{3}$$

Proof. From Lemma 2.2, suppose υ is a solution of ABC-NHFODE (1). Then υ satisfies the following equation:

$$\frac{\upsilon(\kappa)-g(\kappa,\,\upsilon(\zeta(\kappa)))}{f(\kappa,\,\upsilon(\zeta(\kappa)))}$$

$$=\frac{\upsilon_0}{f(0,\,\upsilon_0)}+\frac{1-\varsigma}{\mathcal{B}(\varsigma)}h(\kappa,\,\upsilon(\eta(\kappa)))$$

$$+\frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}\int_0^{\varsigma}(\upsilon-\chi)^{\chi-1}h(\chi,\,\upsilon(\eta(\chi)))d\chi,\ \kappa\in\mathcal{U}, \tag{4}$$

which leads to equation (3).

On the contrary, suppose υ satisfies equation (3). Then equation (3) can be written as equation (4). Employing the operator ${}^{\mathcal{ABR}}D_{\kappa}^{\varsigma}$ to equation (4), we have

$${}^{\mathcal{ABR}}D_{\kappa}^{\varsigma}\left(\frac{\upsilon(\kappa)-g(\kappa,\,\upsilon(\zeta(\kappa)))}{f(\kappa,\,\upsilon(\zeta(\kappa)))}\right)={}^{\mathcal{ABR}}D_{\kappa}^{\varsigma}\left[\frac{\upsilon_0}{f(0,\,\upsilon_0)}+{}^{\mathcal{AB}}\mathcal{I}_{\kappa}^{\varsigma}h(\kappa,\,\upsilon(\eta(\kappa)))\right]$$

$$=\frac{\upsilon_0}{f(0,\,\upsilon_0)}\,{}^{\mathcal{ABR}}D_{\kappa}^{\varsigma}(1)+h(\kappa,\,\upsilon(\eta(\kappa)))$$

$$=\frac{\upsilon_0}{f(0,\,\upsilon_0)}\mathcal{E}_{\varsigma}\left[-\frac{\varsigma}{1-\varsigma}\kappa^{\varsigma}\right]+h(\kappa,\,\upsilon(\eta(\kappa))).$$

Hence, we obtain

$${}^{\mathcal{ABR}}D_{\kappa}^{\varsigma}\left(\frac{\upsilon(\kappa)-g(\kappa,\,\upsilon(\zeta(\kappa)))}{f(\kappa,\,\upsilon(\zeta(\kappa)))}\right)-\frac{\upsilon_0}{f(0,\,\upsilon_0)}\mathcal{E}_{\varsigma}\left[-\frac{\varsigma}{1-\varsigma}\kappa^{\varsigma}\right]=h(\kappa,\,\upsilon(\eta(\kappa))).$$

From Theorem 1 in [2], the relation between the fractional differential derivatives ${}^{\mathcal{ABR}}D_{\kappa}^{\varsigma}$ and ${}^{\mathcal{ABC}}D_{\kappa}^{\varsigma}$ is as follows:

$${}^{\mathcal{ABC}}D_{\kappa}^{\varsigma}\left(\frac{\upsilon(\kappa)-g(\kappa,\,\upsilon(\zeta(\kappa)))}{f(\kappa,\,\upsilon(\zeta(\kappa)))}\right)=h(\kappa,\,\upsilon(\eta(\kappa))),\ \kappa\in\mathcal{U}.$$

Now, taking $\kappa=0$ in equation (4) and using the fact that $g(0,\,\upsilon(0))=0$ and $h(0,\,\upsilon(0))=0,$ we have

$$\frac{\upsilon_0}{f(0,\,\upsilon(0))}=\frac{\upsilon_0}{f(0,\,\upsilon_0)}. \tag{5}$$

For every $\kappa \in \mathcal{U}$, let $z_\kappa : \mathbb{R} \to \mathbb{R}$ be the mapping defined as

$$z_\kappa(\sigma) = \frac{\sigma}{f(\kappa, \sigma)}, \ \sigma \in \mathbb{R}.$$

Since z_κ is an increasing function, also it is injective. From the definition of z_κ, equation (5) can be written as $z_0(\upsilon(0)) = z_0(\upsilon_0)$. Since z_0 is injective, we have $\upsilon(0) = \upsilon_0$. This proves the theorem.

In order to derive the existence of solutions of ABC-NHFODE (1), we define the following hypotheses:

($H1$) Let $g \in \mathcal{C}(\mathcal{U} \times \mathbb{R}, \mathbb{R})$ be continuous and $\exists\lambda > 0$ such that

$$| g(\kappa, \upsilon(\zeta(\kappa))) - g(\kappa, \nu(\zeta(\kappa))) | \leq \lambda | \upsilon(\zeta(\kappa)) - \nu(\zeta(\kappa)) |.$$

($H2$) Let $f \in \mathcal{C}(\mathcal{U} \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ be continuous and $\exists\gamma > 0$ such that

(i) $| f(\kappa, \upsilon(\zeta(\kappa))) - f(\kappa, \nu(\zeta(\kappa))) | \leq \gamma | \upsilon(\zeta(\kappa)) - \nu(\zeta(\kappa)) |$.

(ii) The mapping $\upsilon \to \dfrac{\upsilon - g(\kappa, \upsilon)}{f(\kappa, \upsilon)}$, $\kappa \in \mathcal{U}$ is increasing in $\mathbb{R}$ almost everywhere.

($H3$) Let $h : \mathcal{C}(\mathcal{U} \times \mathbb{R} \to \mathbb{R})$, such that $| h(\kappa, \upsilon(\eta(\kappa))) | \leq z(\kappa)$ a.e.

Here $\kappa \in \mathcal{U}$, $z \in \mathcal{C}(\mathcal{U}, \mathbb{R}^+)$.

Theorem 3.2. *Suppose that the hypotheses* ($H1$)-($H3$) *hold. Then ABC-NHFODE* (1) *has a solution if*

$$\lambda\gamma\left(\left| \frac{\upsilon_0}{f(0, \upsilon_0)} \right| + \left[1 - \varsigma + \frac{\kappa^\varsigma}{1 - \varsigma} \right] \frac{\| z \|}{\mathcal{B}(\varsigma)} \right) < 1. \tag{6}$$

Proof. Let $\Phi = (C(\mathcal{U}, \mathbb{R}), \| \cdot \|)$, here $\| \upsilon \| = Sup_{\kappa \in \mathcal{U}} | \upsilon(\kappa) |$. Then Φ is a Banach algebra with multiplication defined by $(\upsilon\nu)(\kappa) = \upsilon(\kappa)\nu(\kappa)$, $\upsilon, \nu \in \Phi$, $\kappa \in \mathcal{U}$. Define

$$\mathcal{M} = \frac{\mathcal{R}_f \lambda\left(\left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left[1 - \varsigma + \frac{\kappa^{\varsigma}}{1 - \varsigma}\right]\frac{\| z \|}{\mathcal{B}(\varsigma)}\right) + \gamma|\upsilon(\zeta(\kappa))|}{1 - \lambda\left(\left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left[1 - \varsigma + \frac{\kappa^{\varsigma}}{1 - \varsigma}\right]\frac{\| z \|}{\mathcal{B}(\varsigma)}\right) + \gamma|\upsilon(\zeta(\kappa))|}. \tag{7}$$

Here $\mathcal{R}_f = \underset{\kappa \in \mathcal{U}}{Sup} |f(\kappa, 0)|$, from the inequality (6), $\mathcal{M} > 0$.

Consider the set $\mathcal{P} = \{\upsilon \in \Phi : \| \upsilon \| \leq \mathcal{M}\}$. Here $\mathcal{P}$ is a closed, convex and bounded subset of Banach algebra Φ.

Let $\mathcal{X}_1 : \Phi \to \Phi$, $\mathcal{X}_2 : \Phi \to \Phi$ and $\mathcal{X}_3 : \mathcal{P} \to \Phi$ be the operators defined by

(C1) $(\mathcal{X}_1\upsilon)(\kappa) = g(\kappa, \upsilon(\zeta(\kappa)))$, $\kappa \in \mathcal{U}$.

(C2) $(\mathcal{X}_2\upsilon)(\kappa) = f(\kappa, \upsilon(\zeta(\kappa)))$, $\kappa \in \mathcal{U}$.

(C3)

$$(\mathcal{X}_3\upsilon)(\kappa) = \frac{\upsilon_0}{f(0, \upsilon_0)} + \frac{1 - \varsigma}{\mathcal{B}(\varsigma)} h(\kappa, \upsilon(\eta(\kappa))) + \frac{\varsigma}{\mathcal{B}(\varsigma)(1 - \varsigma)} \int_0^{\kappa} (\kappa - \chi)^{\chi - 1} h(\chi, \upsilon(\eta(\chi))) d\chi.$$

From the defined operators, equation (3) can be written as

$$\upsilon = \mathcal{X}_2\upsilon\mathcal{X}_3\upsilon + \mathcal{X}_1\upsilon, \quad \upsilon \in \Phi.$$

Now, we show that the operators $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$ satisfy the conditions of Lemma 2.3. The proof contains five claims.

Claim I. $\mathcal{X}_1$ is Lipschitz.

From the Lipschitz condition on g, for any $\upsilon, \nu \in \Phi$ and $\kappa \in \mathcal{U}$, we have

$$|(\mathcal{X}_1\upsilon)(\kappa) - (\mathcal{X}_1\nu)(\kappa)| = |g(\kappa, \upsilon(\zeta(\kappa))) - g(\kappa, \nu(\zeta(\kappa)))| \leq \lambda|\upsilon(\zeta(\kappa)) - \nu(\zeta(\kappa))|.$$

This leads to $\| \mathcal{X}_1\upsilon - \mathcal{X}_1\nu \| \leq \lambda\| \upsilon - \nu \|$.

Claim II. $\mathcal{X}_2$ is Lipschitz.

From the Lipschitz condition on *f*, for any $\upsilon, \nu \in \Phi$ and $\kappa \in \mathcal{U}$, we have

$$|(\mathcal{X}_2\upsilon)(\kappa) - (\mathcal{X}_2\nu)(\kappa)| = |f(\kappa, \upsilon(\zeta(\kappa))) - f(\kappa, \nu(\zeta(\kappa)))| \\ \leq \gamma|\upsilon(\zeta(\kappa)) - \nu(\zeta(\kappa))|$$

which yields $\|\mathcal{X}_2\upsilon - \mathcal{X}_2\nu\| \leq \gamma\|\upsilon - \nu\|$.

Claim III. $\mathcal{X}_3$ is completely continuous.

We prove that $\mathcal{X}_3 : \mathcal{P} \to \Phi$ is a compact and continuous operator on $\mathcal{P}$ into Φ. Firstly, we establish $\mathcal{X}_3$ is continuous on $\mathcal{P}$. Let $\{\upsilon_n\}$ be a sequence in $\mathcal{P}$ converging to a point $\upsilon \in \mathcal{P}$. From Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty}(\mathcal{X}_3\upsilon_n)(\kappa) &= \lim_{n\to\infty}\left[\begin{aligned}&\frac{\upsilon_0}{f(0, \upsilon_0)} + \frac{1-\varsigma}{\mathcal{B}(\varsigma)}h(\kappa, \upsilon_n(\eta(\kappa))) \\ &+ \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}\int_0^\kappa (\kappa-\chi)^{\varsigma-1}h(\chi, \upsilon_n(\eta(\chi)))d\chi\end{aligned}\right] \\ &= \frac{\upsilon_0}{f(0, \upsilon_0)} + \frac{1-\varsigma}{\mathcal{B}(\varsigma)}\lim_{n\to\infty} h(\kappa, \upsilon_n(\eta(\kappa))) \\ &\quad + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}\int_0^\kappa (\kappa-\chi)^{\varsigma-1}\{\lim_{n\to\infty} h(\chi, \upsilon_n(\eta(\chi)))d\chi\} \\ &= \frac{\upsilon_0}{f(0, \upsilon_0)} + \frac{1-\varsigma}{\mathcal{B}(\varsigma)}h(\kappa, \upsilon(\eta(\kappa))) \\ &\quad + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}\int_0^\kappa (\kappa-\chi)^{\varsigma-1}h(\chi, \upsilon(\eta(\chi)))d\chi \\ &= (\mathcal{X}_3\upsilon)(\kappa), \ \forall \kappa \in \mathcal{U}. \end{aligned}$$

It proves that $\mathcal{X}_3$ is a continuous operator on $\mathcal{P}$.

From the hypothesis $(H3)$ for any $\upsilon \in \mathcal{P}$ and $\kappa \in \mathcal{U}$, we have

$$|(\mathcal{X}_3\upsilon)(\kappa)| \leq \left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \frac{1-\varsigma}{\mathcal{B}(\varsigma)}|h(\kappa, \upsilon(\eta(\kappa)))|$$

$$+ \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}\int_0^{\kappa}(\kappa-\chi)^{\varsigma-1}|h(\chi, \upsilon(\eta(\chi)))|d\chi$$

$$\leq \left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \frac{1-\varsigma}{\mathcal{B}(\varsigma)}|z(\kappa)| + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}\int_0^{\kappa}(\kappa-\chi)^{\varsigma-1}|z(\chi)|d\chi$$

$$\leq \left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \frac{1-\varsigma}{\mathcal{B}(\varsigma)}\|z\| + \frac{\|z\|\kappa^{\varsigma}}{\mathcal{B}(\varsigma)(1-\varsigma)}.$$

This leads to

$$|(\mathcal{X}_3\upsilon)(\kappa)| \leq \left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left(1-\varsigma+\frac{\kappa^{\varsigma}}{1-\varsigma}\right)\frac{\|z\|}{\mathcal{B}(\varsigma)}, \quad \upsilon \in \mathcal{P}, \kappa \in \mathcal{U}. \tag{8}$$

From equation (8), $\mathcal{X}_3$ is uniformly bounded on $\mathcal{U}$.

Now, we show that $\mathcal{X}_3(\mathcal{P})$ is an equicontinuous set in Φ. Let $\upsilon \in \mathcal{P}$ and $0 \leq \kappa_1 < \kappa_2 \leq \mathcal{T}$. Then

$$|\mathcal{X}_3\upsilon(\kappa_1) - \mathcal{X}_3\upsilon(\kappa_2)| \leq \frac{1-\varsigma}{\mathcal{B}(\varsigma)}|h(\kappa_1, \upsilon(\eta(\kappa_1))) - h(\kappa_2, \upsilon(\eta(\kappa_2)))|$$

$$+ \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}$$

$$\times\left|\begin{array}{l}\int_0^{\kappa_1}(\kappa_1-\chi)^{\varsigma-1}h(\chi, \upsilon(\eta(\chi)))d\chi \\ -\int_0^{\kappa_2}(\kappa_2-\chi)^{\varsigma-1}h(\chi, \upsilon(\eta(\chi)))d\chi\end{array}\right|. \tag{9}$$

Since $h(\kappa, \upsilon(\eta(\kappa)))$ is continuous on the compact set $\mathcal{U} \times [-R, R]$, it is uniformly continuous, and therefore

$$|h(\kappa_1, \upsilon(\eta(\kappa_1))) - h(\kappa_2, \upsilon(\eta(\kappa_2)))| \to 0 \text{ as } |\kappa_1 - \kappa_2| \to 0 \text{ for every } \upsilon \in \mathcal{P}. \tag{10}$$

From $(H3)$,

$$\left|\begin{array}{l}\int_0^{\kappa_1}(\kappa_1-\chi)^{\varsigma-1}h(\chi,\upsilon(\eta(\chi)))d\chi\\ -\int_{\kappa_1}^{\kappa_2}(\kappa_2-\chi)^{\varsigma-1}h(\chi,\upsilon(\eta(\chi)))d\chi\end{array}\right|$$

$$\leq \int_0^{\kappa_1}\{(\kappa_1-\chi)^{\varsigma-1}-(\kappa_2-\chi)^{\varsigma-1}\}|h(\chi,\upsilon(\eta(\chi)))|d\chi$$

$$+\int_{\kappa_1}^{\kappa_2}(\kappa_2-\chi)^{\varsigma-1}|h(\chi,\upsilon(\eta(\chi)))|d\chi$$

$$\leq \int_0^{\kappa_1}\{(\kappa_1-\chi)^{\varsigma-1}-(\kappa_2-\chi)^{\varsigma-1}\}|z(\chi)|d\chi$$

$$+\int_{\kappa_1}^{\kappa_2}(\kappa_2-\chi)^{\varsigma-1}|z(\chi)|d\chi$$

$$\leq \|z\|\left(\begin{array}{l}\int_0^{\kappa_1}\{(\kappa_1-\chi)^{\varsigma-1}-(\kappa_2-\chi)^{\varsigma-1}\}d\chi\\ +\int_{\kappa_1}^{\kappa_2}(\kappa_2-\chi)^{\varsigma-1}d\chi\end{array}\right)$$

$$\leq \|z\|(\kappa_1^{\varsigma}+(\kappa_2-\kappa_1)^{\varsigma}-\kappa_2^{\varsigma}+(\kappa_2-\kappa_1)^{\varsigma})$$

$$\leq \|z\|(\kappa_2-\kappa_1)^{\varsigma}.$$

Hence

$$\left|\begin{array}{l}\int_0^{\kappa_1}(\kappa_1-\chi)^{\varsigma-1}h(\chi,\upsilon(\eta(\chi)))d\chi\\ -\int_{\kappa_1}^{\kappa_2}(\kappa_2-\chi)^{\varsigma-1}h(\chi,\upsilon(\eta(\chi)))d\chi\end{array}\right| \to 0 \text{ as } |\kappa_2-\kappa_1|\to 0,\ \upsilon\in\mathcal{P}. \quad (11)$$

From equations (9), (10) and (11), we obtain that

$$|\mathcal{X}_3\upsilon(\kappa_1)-\mathcal{X}_3\upsilon(\kappa_2)|\to 0 \text{ as } |\kappa_2-\kappa_1|\to 0 \text{ for each } \upsilon\in\mathcal{P}.$$

Hence $\mathcal{X}_3(\mathcal{P})$ is an equicontinuous set in Φ. Since $\mathcal{X}_3$ is a uniformly bounded and equicontinuous set in Φ, from Ascoli-Arzela fixed point theorem, $\mathcal{X}_3$ is completely continuous.

Claim IV. Let $\upsilon \in \Phi$. For any $\nu \in \mathcal{P}$, consider the equation in the form of operators $\upsilon = \mathcal{X}_2\upsilon\mathcal{X}_3\upsilon + \mathcal{X}_1\upsilon$. Let us prove that $\upsilon \in \mathcal{P}$. From the assumptions (H1), (H2), (H3) and from the inequality (8), we have

$$\begin{aligned}
|\upsilon(\kappa)| &= |\mathcal{X}_2\upsilon\mathcal{X}_3\upsilon + \mathcal{X}_1\upsilon| \\
&\leq |\mathcal{X}_2\upsilon\mathcal{X}_3\upsilon| + |\mathcal{X}_1\upsilon| \\
&\leq |\mathcal{X}_2\upsilon||\mathcal{X}_3\upsilon| + |\mathcal{X}_1\upsilon| \\
&\leq (|f(\kappa, \upsilon(\zeta(\kappa))) - f(\kappa, 0) + f(\kappa, 0)|) \\
&\quad \times\left(\left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left(1 - \varsigma + \frac{\kappa^{\varsigma}}{1-\varsigma}\right)\frac{\|z\|}{\mathcal{B}(\varsigma)}\right) \\
&\quad + |g(\kappa, \upsilon(\zeta(\kappa))) - g(\kappa, 0) + g(\kappa, 0)| \\
&\leq \{\lambda|\upsilon(\kappa)| + \mathcal{R}_f\}\left(\left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left(1 - \varsigma + \frac{\kappa^{\varsigma}}{1-\varsigma}\right)\frac{\|z\|}{\mathcal{B}(\varsigma)}\right) + \gamma|\upsilon(\kappa)|
\end{aligned}$$

which gives

$$|\upsilon(\kappa)| \leq \frac{\mathcal{R}_f\lambda\left(\left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left(1 - \varsigma + \frac{\kappa^{\varsigma}}{1-\varsigma}\right)\frac{\|z\|}{\mathcal{B}(\varsigma)}\right) + \gamma|\upsilon(\kappa)|}{1 - \lambda\left(\left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left(1 - \varsigma + \frac{\kappa^{\varsigma}}{1-\varsigma}\right)\frac{\|z\|}{\mathcal{B}(\varsigma)}\right) + \gamma|\upsilon(\kappa)|}.$$

Therefore, $\|\upsilon\| \leq \mathcal{M}$.

Claim V. The constants u, v and $\mathcal{R}$ of Lemma 2.3 corresponding to the operators $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$ defined in (C1), (C2), (C3), respectively, are

$$u = \lambda,\ v = \gamma \ \text{ and } \ \mathcal{R} = \left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left(1 - \varsigma + \frac{\kappa^{\varsigma}}{1-\varsigma}\right)\frac{\|z\|}{\mathcal{B}(\varsigma)}.$$

From inequality (6), it follows that

$$uv\mathcal{R} = \lambda\gamma\left(\left|\frac{\upsilon_0}{f(0, \upsilon_0)}\right| + \left(1 - \varsigma + \frac{\kappa^{\varsigma}}{1-\varsigma}\right)\frac{\| z \|}{\mathcal{B}(\varsigma)}\right) < 1.$$

From Claim I to Claim V, it is evident that the conditions given in Lemma 2.3 are satisfied. Therefore, $\upsilon = \mathcal{X}_2\upsilon\mathcal{X}_3\upsilon + \mathcal{X}_1\upsilon$ has a fixed point in $\mathcal{P}$, which is a solution of ABC-NHFODE (1). This concludes the proof.

Theorem 3.3. *Under* (H1)-(H3), *the zero solution of* (1) *is attractive*.

Proof. From Theorem 3.2, the solution of (1) exists which lies in $\mathcal{P}$. All functions of $\upsilon(\kappa)$ tend to 0 as $\kappa \to \infty$. Then the solution of $(1) \to 0$ as $\kappa \to \infty$ which concludes the proof.

4. Hyers-Ulam Stability

The Hyers-Ulam stability (HUS) of integral equation (3) is presented in this section.

Definition 4.1. The integral equation (3) is HUS if there exists a nonnegative constant Δ satisfying the following:

For any $\delta > 0$, suppose

$$\left|\upsilon(\kappa) - f(\kappa, \upsilon(\zeta(\kappa)))\left[\begin{array}{l}\frac{\upsilon_0}{f(0, \upsilon_0)} + \frac{1-\varsigma}{\mathcal{B}(\varsigma)}h(\kappa, \upsilon(\eta(\kappa))) \\ + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}\int_0^{\kappa}(\kappa - \chi)^{\chi-1}h(\chi, \upsilon(\eta(\chi)))d\chi\end{array}\right] + g(\kappa, \upsilon(\zeta(\kappa)))\right| \leq \delta.$$

Then there exists $\nu(\kappa)$ satisfying

$$\nu(\kappa) = f(\kappa, \nu(\zeta(\kappa)))\left[\begin{array}{l}\frac{\nu_0}{f(0, \nu_0)} + \frac{1-\varsigma}{\mathcal{B}(\varsigma)}h(\kappa, \nu(\eta(\kappa))) \\ + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)}\int_0^{\kappa}(\kappa - \chi)^{\chi-1}h(\chi, \nu(\eta(\chi)))d\chi\end{array}\right]$$

$$+ g(\kappa, \nu(\zeta(\kappa)))$$

such that

$$|\upsilon(\kappa) - \nu(\kappa)| \leq \Delta\delta. \tag{12}$$

Theorem 4.1. *Under the hypotheses* ($H1$)-($H3$), *the ABC-NHFODE is Hyers-Ulam stability*.

Proof. From Theorem 3.2 and Definition 4.1, let $\upsilon(\kappa)$ be the solution of (3) and $\nu(\kappa)$ be the approximate solution satisfying (12).

Here $\upsilon(0) = \nu(0) = \upsilon_0$. Then we have

$$\begin{aligned}
&|\upsilon(\kappa) - \nu(\kappa)| \\
&= \left| \begin{array}{l} f(\kappa, \upsilon(\zeta(\kappa))) \left[\begin{array}{l} \frac{1-\varsigma}{\mathcal{B}(\varsigma)} h(\kappa, \upsilon(\eta(\kappa))) \\ + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)} \int_0^{\kappa} (\kappa - \chi)^{\chi-1} h(\chi, \upsilon(\eta(\chi))) d\chi \end{array} \right] \\ + g(\kappa, \upsilon(\zeta(\kappa))) \end{array} \right| \\
&\quad - \left| \begin{array}{l} f(\kappa, \nu(\zeta(\kappa))) \left[\begin{array}{l} \frac{1-\varsigma}{\mathcal{B}(\varsigma)} h(\kappa, \nu(\eta(\kappa))) \\ + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)} \int_0^{\kappa} (\kappa - \chi)^{\chi-1} h(\chi, \nu(\eta(\chi))) d\chi \end{array} \right] \\ + g(\kappa, \nu(\zeta(\kappa))) \end{array} \right| \\
&\leq | f(\kappa, \upsilon(\zeta(\kappa))) - f(\kappa, \nu(\zeta(\kappa)))| \\
&\quad + \frac{1-\varsigma}{\mathcal{B}(\varsigma)} | h(\kappa, \upsilon(\eta(\kappa))) - h(\kappa, \nu(\eta(\kappa)))| \\
&\quad + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)} \int_0^{\kappa} (\kappa - \chi)^{\chi-1} | h(\chi, \upsilon(\eta(\chi))) - h(\chi, \nu(\eta(\chi)))| d\chi \\
&\quad + | g(\kappa, \upsilon(\zeta(\kappa))) - g(\kappa, \nu(\zeta(\kappa)))| \\
&\leq \gamma | \upsilon(\kappa) - \nu(\kappa)| + \left(\frac{1-\varsigma}{\mathcal{B}(\varsigma)} + \frac{\varsigma}{\mathcal{B}(\varsigma)(1-\varsigma)} \right)
\end{aligned}$$

$$\times \| z \| | \upsilon(\kappa) - \nu(\kappa) | + \lambda | \upsilon(\kappa) - \nu(\kappa) |$$

$$\leq \left[\gamma + \lambda + \| z \| \left(\frac{1-\varsigma}{\mathcal{B}(\varsigma)} + \frac{\kappa^{\varsigma}}{\mathcal{B}(\varsigma)(1-\varsigma)} \right) \right] | \upsilon - \nu |$$

$$| \upsilon - \nu | \leq \Delta \| \upsilon - \nu \|.$$

Therefore, from Theorem 3.2 and Definition 4.1, we conclude that the integral equation is HUS with constant $\Delta = \gamma + \lambda + \frac{\| z \|}{\mathcal{B}(\varsigma)} \left(1 - \varsigma + \frac{\kappa^{\varsigma}}{(1-\varsigma)} \right).$ This completes the proof.

5. Numerical Example

Example 5.1. Consider the following ABC-hybrid fractional order differential equation

$$D^{\frac{1}{4}} \left[\frac{\upsilon(\mu) - g(\mu, \upsilon(\zeta(\mu)))}{f(\mu, \upsilon(\zeta(\mu)))} \right] = h(\mu, \upsilon(\eta(\mu))),$$

$$g(\mu, \upsilon(\zeta(\mu))) = \frac{1}{18} e^{1-\mu} \left(\frac{1}{2} \sin \upsilon(\zeta(\mu)) + \frac{\upsilon^2(\mu) + 4 | \upsilon(\zeta(\mu)) |}{2 + | \upsilon(\zeta(\mu)) |} \right). \tag{13}$$

Here

$$f(\mu, \upsilon(\zeta(\mu))) = \frac{1}{18} \mu \left(\frac{1}{22} \cos \upsilon(\zeta(\mu)) + \frac{\upsilon^2(\mu) + | \upsilon(\zeta(\mu)) |}{1 + | \upsilon(\zeta(\mu)) |} \right) + \frac{1}{4},$$

$$h(\mu, \upsilon(\eta(\mu))) = \frac{\mu | \eta(\mu) |}{(7 + e^{\mu})(1 + | \eta(\mu) |)}.$$

From the calculations, we have

$$| g(\mu, \upsilon(\zeta(\mu))) - g(\mu, \nu(\zeta(\mu))) | \leq \frac{e^{1-\mu}}{8} | \upsilon(\zeta(\mu)) - \nu(\zeta(\mu)) |,$$

$$|f(\mu, \upsilon(\zeta(\mu))) - f(\mu, \nu(\zeta(\mu)))| \leq \frac{\mu}{22}|\upsilon(\zeta(\mu)) - \nu(\zeta(\mu))|,$$

$$|h(\mu, \upsilon(\zeta(\mu))) - h(\mu, \nu(\zeta(\mu)))| \leq \frac{\mu}{(7 + e^{\mu})}.$$

Therefore, $\gamma = \frac{1}{22}\mu$, $\lambda = \frac{1}{8}e^{1-\mu}$ which imply that

$$\|\gamma\| = 0.045, \quad \|\lambda\| = 0.339, \quad \|z\| = 0.102.$$

Hence

$$\gamma + \lambda + \|z\|\left(\frac{1-\varsigma}{\mathcal{B}(\varsigma)} + \frac{\mu^{\varsigma}}{\mathcal{B}(\varsigma)(1-\varsigma)}\right) = 0.5625 < 1.$$

Therefore, the corresponding ABC-NHFODE (13) is Hyers-Ulam stability. For different values of ς, the values of Δ are calculated and presented in Table 1. The stability of the system (13) is graphically shown in Figure 1. From Figure 1, we observe that when the fractional order $\varsigma = 0.1$, the value of Δ attains the maximum, as the value of fractional order increases, the value of Δ decreases and it becomes stable. The surface plot corresponding to Figure 1 is plotted in Figure 2. Surface plot shows the functional relationship between the variables λ, ς and Z-axis.

Table 1. Different values of Δ for $\varsigma \in (0, 1]$

ς	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Δ	0.573	0.559	0.547	0.536	0.525	0.516	0.507	0.499	0.492	0.486

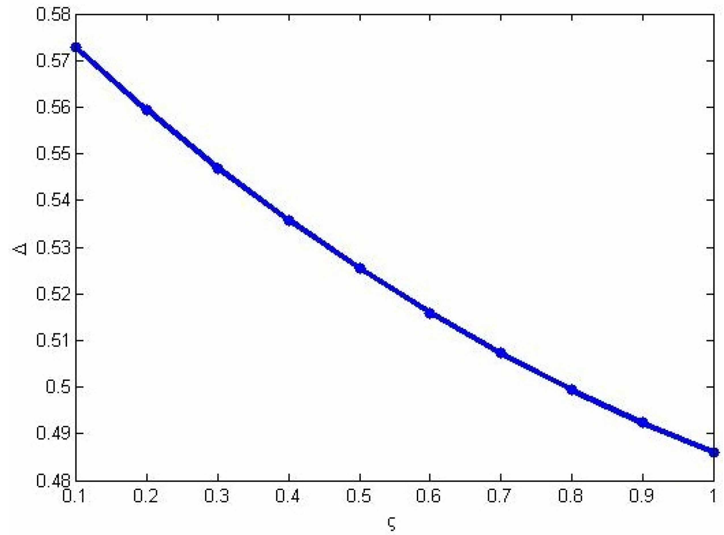

Figure 1. Graphical representation for Example 5.1 for different values of fractional order ς.

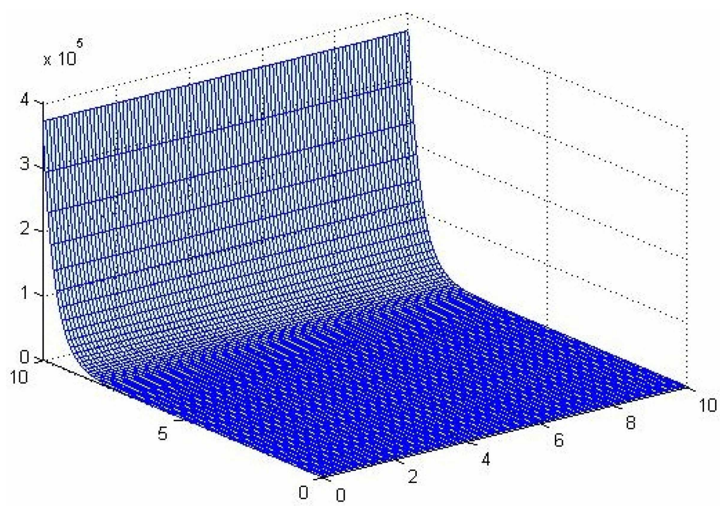

Figure 2. Surface plot corresponding to Figure 1.

6. Conclusion

This article discussed the Hyers-Ulam stability (HUS) of nonlinear hybrid fractional order differential equation using Atangana-Baleanu-Caputo operator. Utilizing the defined hypotheses and Arzela-Ascoli fixed point theorem, the existence of solutions is established. Sufficient condition which ensures the Hyers-Ulam stability of ABC-NHFODE is derived with the help of Definition 4.1. The analytical results presented in Sections 3 and 4 are supported with an example in Section 5. The graphical representation of Example 5.1 exhibits the stability of equation (13). In this paper, based on the fractional integral inequalities and comparison results in [12], the stability of the hybrid nonlinear fractional order differential equation is analyzed.

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