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Article

Generalized fixed-point theorem for strict almost ϕ-contractions with binary relations in b-metric spaces and its application to fractional differential equations

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Abstract: The present study is centered around establishing a generalized fixed-point theorem for strict almost ϕ-contractions in b-metric spaces in the context of binary relations. Through the introduction of an innovative lemma, we offer distinct proof methodologies that diverge from the conventional ones in metric spaces. The achieved outcomes not only fortify but also broaden the domain of prior fixed-point theorems in the pertinent literature. Moreover, as a practical exemplification, the existence and uniqueness of solutions to fractional differential equations are illustrated convincingly, thereby connecting the theoretical and applied dimensions of the research.

Keywords: fixed point; strict almost ϕ-contractions; binary relations; b-metric spaces; fractional differential equations

MSC Classification: 47H10; 54H25

1. Introduction

Fixed-point theory has been a fundamental and crucial part of mathematics, and its influence has spread across numerous fields. The traditional contraction mappings and their associated fixed-point theorems, such as the well-known Banach contraction principle in metric spaces, have been the focus of intense research and continuous evolution. These theorems have proven to be extremely valuable in establishing the existence and uniqueness of solutions (abbreviated as EUS) for a wide range of mathematical problems.

In 1998, Czerwik [1] presented the concept of b-metric space with $b \geq 1$, which serves as a significant generalization of metric spaces, and established fixed-point theorems. Numerous fixed-point results within b-metric spaces have been investigated by various authors, as noted in references such as [2–4] and others. Analyzing mappings and their fixed-points within such spaces requires a reconsideration and expansion of conventional methods. Recently, a vital aspect of research in fixed-point theory has been centered on achieving results in relational metric spaces. The structure of relational metric space was initiated by Alam and Imdad [5]. As of now, usual contractions remain stronger than relational contractions. In 2008, Babu et al. [6] introduced a strict almost contraction fixed-point theorem. In 2023, Alharbi and Khan [7] introduced a new type of strict almost ϕ-contractions under binary relations in metric spaces, and derived the corresponding fixed-point theorem. By integrating binary relations into the study of almost ϕ-contractions in b-metric spaces, we can synthesize and enhance multiple aspects of existing theories.

In the domain of fractional calculus, the necessity of ascertaining the EUS to fractional differential equations has been a major impetus for the application of fixed point theory. Fractional differential equations frequently depict complex physical and engineering phenomena that cannot be precisely modeled using traditional integer-order differential equations. For instance, in the research on anomalous diffusion in concrete [8, 9], fractional calculus offers a more precise framework for capturing non-Fickian behavior. In engineering, the utilization of fractional order derivatives in models of viscoelastic materials enables a more accurate representation of the material's memory and hereditary characteristics [10].

Ahmad et al. made remarkable achievements in the examination of Riemann-Liouville fractional integro-differential equations under fractional nonlocal multi-point and strip boundary conditions within the weighted space [11]. By employing the tools of fixed point theory, they managed to convert the initial and boundary value problems into fixed point problems, thus determining EUS. This work clearly illustrates the potency and adaptability of fixed point theory in the context of fractional differential equations.

Furthermore, the application of fixed point theory in fractional calculus extends beyond just proving existence and uniqueness. It acts as a basis for the development of numerical methods. Once the theoretical groundwork of existence and uniqueness is established, iterative algorithms based on fixed point iterations can be designed to approximate the solutions. This is of vital significance in practical computations and simulations, as it permits the quantitative analysis of systems modeled by fractional differential equations. Martin's work on the application of the variational iteration method in the context of fractional calculus for the dynamic analysis of viscoelastic beams [12] is an excellent example of this. His exploration of the stability aspects of this approach further emphasizes the importance of fixed point theory in fractional calculus.

In this study, we center our attention on devising a fixed-point theorem related to strict almost ϕ-contractions within the framework of b-metric spaces and under binary relations. By introducing an innovative lemma, we provide alternative proofs that differ from the traditional ones in metric spaces. The outcomes attained here not only reinforce but also widen the scope of existing fixed-point theorems in the relevant literature. Additionally, as an illustrative application, we convincingly exhibit EUS to fractional differential equations. Through this research, we strive to make a contribution to the continuously growing body of knowledge in fixed-point theory and its applications, especially in the area of fractional differential equations.

2. Preliminaries

Throughout the paper, let $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote sets of positive integer numbers and nonnegative real numbers respectively.

Definition 1. *Let $\mathcal{M}$ be a nonempty set, $b \geq 1$ and $\mathcal{D} : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ a function satisfying for any $\tilde{x}, \tilde{y}, \tilde{z} \in \mathcal{M}$* [1],

1) $\mathcal{D}(\tilde{x}, \tilde{y}) = 0 \iff \tilde{x} = \tilde{y}$*;*

2) $\mathcal{D}(\tilde{x}, \tilde{y}) = \mathcal{D}(\tilde{y}, \tilde{x})$;

3) $\mathcal{D}(\tilde{x}, \tilde{y}) \leq b[\mathcal{D}(\tilde{x}, \tilde{z}) + \mathcal{D}(\tilde{z}, \tilde{y})]$.

Then $\mathcal{D}$ is called a b-metric on $\mathcal{M}$ and the pair $(\mathcal{M}, \mathcal{D}, b)$ is a b-metric space.

Next, we will begin by noting that a subset of $\mathcal{M} \times \mathcal{M}$ is defined as a binary relation on the set $\mathcal{M}$. For $\tilde{x}, \tilde{y} \in \mathcal{M}$, let $\tilde{x}\mathcal{R}\tilde{y}$ denote "$(\tilde{x}, \tilde{y}) \in \mathcal{R}$" and $\tilde{x}\hat{\mathcal{R}}\tilde{y}$ denote "$\tilde{x}\mathcal{R}\tilde{y}$ or $\tilde{y}\mathcal{R}\tilde{x}$".

Definition 2. *A pair $\tilde{x}, \tilde{y} \in \mathcal{M}$ is $\mathcal{R}$-comparative if $\tilde{x}\hat{\mathcal{R}}\tilde{y}$* [5].

This is an important definition regarding the set $\mathcal{M}$ and the relation $\mathcal{R}$. It clearly identifies pairs of elements in $\mathcal{M}$ that have a specific relationship through $\mathcal{R}$. By defining when a pair $\tilde{x}, \tilde{y}$ is $\mathcal{R}$-comparative and using the notation $\tilde{x}\hat{\mathcal{R}}\tilde{y}$, the discussion of their relative positions within $\mathcal{M}$ becomes simpler. This concept will likely be foundational for further results and investigations involving $\mathcal{R}$ on $\mathcal{M}$.

Definition 3. *The inverse of relation $\mathcal{R}$ is defined as: $\mathcal{R}^{-1} := \{(\tilde{x}, \tilde{y}) \in \mathcal{M} \times \mathcal{M} : \tilde{y}\mathcal{R}\tilde{x}\}$. Additionally, $\mathcal{R}^s := \mathcal{R}^{-1} \cup \mathcal{R}$ defines a symmetric relation on $\mathcal{M}$, which is commonly known as the symmetric closure of $\mathcal{R}$.*

The introduction of the inverse relation $\mathcal{R}^{-1}$ provides a way to reverse the direction of the relationship. It allows us to consider pairs in $\mathcal{M} \times \mathcal{M}$ where the order of elements is swapped compared to those in $\mathcal{R}$ [13].

Remark 1. *The symmetric $\mathcal{R}^s$ is also significant. $\tilde{x}\mathcal{R}^s\tilde{y} \iff \tilde{x}\hat{\mathcal{R}}\tilde{y}$ (see* [10]*). By taking the union of $\mathcal{R}$ and its inverse, we obtain a new relation that is symmetric. Symmetric relations often have useful properties and can simplify the analysis of the structure and behavior of elements within the set $\mathcal{M}$ with respect to the original relation $\mathcal{R}$.*

Definition 4. *For a subset $\mathcal{A} \subseteq \mathcal{M}$, the set*

$$\mathcal{R}\big|_{\mathcal{A}} := \mathcal{R} \cap (\mathcal{A} \times \mathcal{A})$$

is referred to as the restriction of $\mathcal{R}$ to $\mathcal{A}$, and it constitutes a relation on $\mathcal{A}$ [7].

Definition 5. *$\mathcal{R}$ is T-closed, if $(T\tilde{x})\mathcal{R}(T\tilde{y}), \forall \tilde{x}, \tilde{y} \in \mathcal{M}$ satisfying $\tilde{x}\mathcal{R}\tilde{y}$* [5].

In other words, if two elements have a specific relationship, applying the transformation T to both should result in the same relationship between their images.

Definition 6. *The sequence $\{\tilde{x}_n\} \subset \mathcal{M}$ is $\mathcal{R}$-preserving, if $\tilde{x}_n\mathcal{R}\tilde{x}_{n+1}, \forall n \in \mathbb{N}$* [5].

Specifically, this indicates that for every natural number n, the relationship $\tilde{x}_n\mathcal{R}\tilde{x}_{n+1}$ is valid. Roughly speaking, each element in the sequence is connected to the following element through the relation $\mathcal{R}$.

Definition 7. *The $(\mathcal{M}, \mathcal{D}, b)$ is called $\mathcal{R}$-complete, if each $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{M}$ converges.*

In simpler terms, it means that if you have a sequence of elements that gets arbitrarily close to each other while also maintaining a specific relationship, then there is a point in the structure where this sequence will settle down or converge.

Definition 8. *The subset $\mathcal{A} \subseteq \mathcal{M}$ is $\mathcal{R}$-directed, if for every $\tilde{x}\mathcal{R}\tilde{y}$, $\exists \tilde{z} \in \mathcal{M}$ verifying $\tilde{x}\mathcal{R}\tilde{z}$ and $\tilde{z}\mathcal{R}\tilde{y}$* [14].

Definition 9. *The mapping T is $\mathcal{R}$-continuous at $\tilde{x} \in \mathcal{M}$, if for each $\mathcal{R}$-preserving* *sequence* $\{\tilde{x}_n\} \subseteq \mathcal{M}$ *with* $\lim_{n\to\infty} \mathcal{D}(\tilde{x}_n, \tilde{x}) = 0$, *verifying*

$$\lim_{n\to\infty} \mathcal{D}(T\tilde{x}_n, T\tilde{x}) = 0$$

A mapping that is continuous with respect to $\mathcal{R}$ *at every point is referred to as* $\mathcal{R}$*-continuous.*

Definition 10. $\mathcal{R}$ *is termed as* $\mathcal{D}$*-self-closed, if every* $\mathcal{R}$*-preserving* $\{\tilde{x}_n\} \subseteq \mathcal{M}$ *verifying* $\lim_{n\to\infty} \mathcal{D}(\tilde{x}_n, \tilde{x}) = 0$, *has a subsequence* $\{\tilde{x}_{n_k}\}$ *satisfying* $\tilde{x}_{n_k}\hat{\mathcal{R}}\tilde{x}$, *where* $\tilde{x} \in \mathcal{M}$.

Next, we will introduce the main theorems obtained by previous scholars, which also serve as a corollary to our main results.

For the family of all mappings from $\mathbb{R}^+$ to $\mathbb{R}^+$ as follows:

$$\Phi = \left\{\phi : \phi^{-1}(\{0\}) = \{0\}, \phi(r) < r \text{ and } \limsup_{q\to r} \phi(q) < r, \forall r > 0\right\}$$

Theorem 1. *Let* $(\mathcal{M}, \mathcal{D})$ *be a metric space,* $\mathcal{R}$ *a relation on* $\mathcal{M}$ *and* $T : \mathcal{M} \to \mathcal{M}$ *a mapping* ([7], Theorem 4). *The conditions are as follows:*

(i) $(\mathcal{M}, \mathcal{D})$ *is* $\mathcal{R}$*-complete;*

(ii) $\exists x_0 \in \mathcal{M}$ *satisfying* $x_0\mathcal{R}(Tx_0)$ *and* T*-closed;*

(iii) $\mathcal{R}$ *is locally* T*-transitive;*

(iv) $\mathcal{R}$ *is* $\mathcal{D}$*-self-closed, or* T *is* $\mathcal{R}$*-continuous;*

(v) *there exists* $\phi \in \Phi$ *and* $L \geq 0$ *verifying*

$$\mathcal{D}(T\tilde{x}, T\tilde{y}) \leq \phi(\mathcal{D}(\tilde{x}, \tilde{y})) + L\min\{\mathcal{D}(\tilde{x}, T\tilde{x}), \mathcal{D}(\tilde{x}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{x})\} \quad (1)$$

for all $x, y \in \mathcal{M}$ *with* $x\mathcal{R}y$*;*

(vi) $T(\mathcal{M})$ *is* $\mathcal{R}$*-directed.*

Then, if conditions $(i) - (v)$ *are satisfied,* T *admits a fixed-point. Additionally, if condition* (vi) *is also satisfied, then* T *admits a unique fixed-point.*

Remark 2. *In fact, Equation* (1) *is a strict almost* ϕ*-contraction.*

Following Jleli et al. [15], let Ω represent the collection of all functions $\omega : [0,\infty)^4 \to [0,\infty)$ that are continuous and satisfy the condition that $\omega(q_1, q_2, q_3, q_4) = 0$ if and only if $q_1q_2q_3q_4 = 0$.

For example, the following functions [15] belong to Ω:

(1) $\omega(q1, q2, q3, q4) = L\min\{q_1, q_2, q_3, q_4\}$;

(2) $\omega(q1, q2, q3, q4) = e^{q_1q_2q_3q_4} - 1$;

(3) $\omega(q1, q2, q3, q4) = q_1q_2q_3q_4$;

(4) $\omega(q1, q2, q3, q4) = \ln(1 + q_1q_2q_3q_4)$.

In light of the above, this paper extends conclusion to b-metric spaces, replaces L with ω, and presents results on the existence and uniqueness of fixed-points for strict almost ϕ-contractions under binary relations in b-metric spaces.

We state the following theorem, which establishes the equivalence of certain conditions related to the transformation T and the distances between elements in the structure $\mathcal{M}$.

Theorem 2. *For each* $\phi \in \Phi$ *and* $\omega \in \Omega$*, the following conditions are equivalent:*

(1) $\mathcal{D}(T\tilde{x}, T\tilde{y}) \leq \phi(\mathcal{D}(\tilde{x}, \tilde{y})) + \omega\left(\mathcal{D}(\tilde{x}, T\tilde{x}), \mathcal{D}(\tilde{x}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{x})\right)$, $\forall \tilde{x}, \tilde{y} \in \mathcal{M}$ *with* $\tilde{x}\mathcal{R}\tilde{y}$;

(2) $\mathcal{D}(T\tilde{x}, T\tilde{y}) \leq \phi(\mathcal{D}(\tilde{x}, \tilde{y})) + \omega\left(\mathcal{D}(\tilde{x}, T\tilde{x}), \mathcal{D}(\tilde{x}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{x})\right)$, $\forall \tilde{x}, \tilde{y} \in \mathcal{M}$ *with* $\tilde{x}\hat{\mathcal{R}}\tilde{y}$.

Proof. The conclusion (2) $\Rightarrow$ (1) is obviously valid. Conversely, let (1) holds. Suppose that $\tilde{x}, \tilde{y} \in \mathcal{M}$ with $\tilde{x}\hat{\mathcal{R}}\tilde{y}$. Then, in case $\tilde{x}\mathcal{R}\tilde{y}$, (1) $\Rightarrow$ (2). Otherwise, $\tilde{y}\mathcal{R}\tilde{x}$, one has

$$\begin{aligned}\mathcal{D}(T\tilde{x}, T\tilde{y}) &= \mathcal{D}(T\tilde{y}, T\tilde{x}) \\ &\leq \phi(\mathcal{D}(\tilde{y}, \tilde{x})) + \omega\left(\mathcal{D}(\tilde{y}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{x}), \mathcal{D}(\tilde{x}, T\tilde{x}), \mathcal{D}(\tilde{x}, T\tilde{y})\right), \\ &= \phi(\mathcal{D}(\tilde{x}, \tilde{y})) + \omega\left(\mathcal{D}(\tilde{x}, T\tilde{x}), \mathcal{D}(\tilde{x}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{x})\right).\end{aligned}$$

It follows that (1) $\Rightarrow$ (2). $\square$

3. Main result

Converting the fixed-point theorem from metric spaces to b-metric spaces is quite a challenging endeavor. Through the introduction of an innovative lemma, we offer distinct proof methodologies that diverge from the conventional ones in metric spaces. The following lemma is new and useful, and will be used for proving our theorems.

Lemma 1. *Let* $(\mathcal{M}, \mathcal{D}, b)$ *be a* b-MS *with* $1 \leq b$. *Let* $[\alpha, \beta]$ *be a closed interval with* $0 < \alpha < \beta \leq \epsilon$. *Suppose that* $T : \mathcal{M} \to \mathcal{M}$ *satisfies:*

$$\mathcal{D}(T\tilde{x}, T\tilde{y}) \leq \phi(\mathcal{D}(\tilde{x}, \tilde{y})),\ \forall \tilde{x}, \tilde{y} \in \mathcal{M} \tag{2}$$

where $\phi \in \Phi$. *then*

(*i*) $\inf_{\alpha \leq r \leq \beta} [r - \phi(r)] > 0$*;*

(*ii*) $\forall \tilde{x}, \tilde{y} \in \mathcal{M}$ *verifying* $\mathcal{D}(\tilde{x}, \tilde{y}) < \beta$*,* $\exists n_0 \in \mathbb{N}^+$ *such that* $\mathcal{D}(T^{n_0}\tilde{x}, T^{n_0}\tilde{y}) < \alpha$.

Proof. (*i*) Suppose not, then we have $\inf_{\alpha \leq r \leq \beta} [r - \phi(r)] = 0$. Moreover, there exists a sequence $\{r_k\} \subset [\alpha, \beta]$ such that $r_k - \phi(r_k) \to 0$ as $k \to \infty$. Since $\{r_k\}$ is bounded, we may assumed that $\{r_k\}$ converges to $r_0 \in [\alpha, \beta]$. We obtain $\lim_{k\to\infty} \phi(r_k) = \lim_{k\to\infty} r_k - [r_k - \phi(r_k)] = r_0$. By hypothesis of ϕ, we have

$$r_0 = \lim_{k\to\infty} \phi(r_k) \leq \limsup_{r\to r_0} \phi(r) < r_0$$

which is a contradiction.

(*ii*) By means of (*i*), for each $r \in [\alpha, \beta]$, we denote

$$\sigma = \inf_{\alpha \leq r \leq \beta} [r - \phi(r)] > 0 \tag{3}$$

$\forall \tilde{x}, \tilde{y} \in \mathcal{M}$ with $\mathcal{D}(\tilde{x}, \tilde{y}) < \beta$, from Equation (3) it follows that

$$\mathcal{D}(T\tilde{x}, T\tilde{y}) \leq \phi(\mathcal{D}(\tilde{x}, \tilde{y})) \leq \mathcal{D}(\tilde{x}, \tilde{y}) - \sigma < \beta - \sigma \tag{4}$$

If $\mathcal{D}(T\tilde{x}, T\tilde{y}) < \alpha$, then we take $n_0 = 1$. If $\mathcal{D}(T\tilde{x}, T\tilde{y}) \in [\alpha, \beta)$, then by Equation

(4) we have

$$\mathcal{D}(T^2\tilde{x}, T^2\tilde{y}) \leq \phi(\mathcal{D}(T\tilde{x}, T\tilde{y})) \leq \mathcal{D}(T\tilde{x}, T\tilde{y}) - \sigma < \beta - 2\sigma \tag{5}$$

Continuing this process, for $i \in \mathbb{N}$, if $\mathcal{D}(T^i\tilde{x}, T^i\tilde{y}) < \alpha$, then we take $n_0 = i$. If $\mathcal{D}(T^i\tilde{x}, T^i\tilde{y}) \in [\alpha, \beta)$, we have $\mathcal{D}(T^{i+1}\tilde{x}, T^{i+1}\tilde{y}) < \beta - (i+1)\sigma$. Therefore, $\exists n_0 \in \mathbb{Z}^+$ such that $\mathcal{D}(T^{n_0}\tilde{x}, T^{n_0}\tilde{y}) < \alpha$ for all $\tilde{x}, \tilde{y} \in \mathcal{M}$ where $\mathcal{D}(\tilde{x}, \tilde{y}) < \beta$. □

We first characterized the degree of non-linear contraction from the perspective of the decreasing amounts of contraction over local intervals. Our lemma describes the lower bound of the degree of contraction, ensuring that the sequence of points generated by the contraction mapping is indeed a Cauchy sequence. This leads us to the following technical lemma, which is the core of the fixed-point proof.

Conclusion (i) is a property of the function ϕ, derived from the nature of pure functions. Conclusion (ii) is an extension of Conclusion (i). Specifically, if the distance between two points is less than β, then after applying the operator T to these two points and iterating the function ϕ a certain number of times, the distance can be made less than α. This is because each iteration has a minimum decrease, and the amount of decrease in each step is greater than or equal to this minimum decrease. Therefore, after a sufficient number of iterations, the distance will indeed be less than α.

At this point, one proposes the following fact.

Theorem 3. *Let $(\mathcal{M}, \mathcal{D}, b)$ be a b-metric space with $1 \leq b$, $\mathcal{R}$ a relation on $\mathcal{M}$ and $T : \mathcal{M} \to \mathcal{M}$ a mapping. The conditions are as follows:*

(i) *$(\mathcal{M}, \mathcal{D}, b)$ is $\mathcal{R}$-complete;*

(ii) *$\exists \tilde{x}_0 \in \mathcal{M}$ verifying $\tilde{x}_0\mathcal{R}(T\tilde{x}_0)$, and $\mathcal{R}$ is T-closed;*

(iii) *$\mathcal{R}$ is $\mathcal{D}$-self-closed or T is $\mathcal{R}$-continuous;*

(iv) *$\exists \phi \in \Phi$ and $\omega \in \Omega$ verifying;*

$$\mathcal{D}(T\tilde{x}, T\tilde{y}) \leq \phi(\mathcal{D}(\tilde{x}, \tilde{y})) + \omega\left(\mathcal{D}(\tilde{x}, T\tilde{x}), \mathcal{D}(\tilde{x}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{y}), \mathcal{D}(\tilde{y}, T\tilde{x})\right) \tag{6}$$

$\forall \tilde{x}, \tilde{y} \in \mathcal{M}$ with $\tilde{x}\mathcal{R}\tilde{y}$;

(v) *$\mathcal{R}\big|_{T(\mathcal{M})}$ remains complete or $T(\mathcal{M})$ is $\mathcal{R}$-directed.*

Then, if conditions $(i) - (iv)$ are satisfied, T admits a fixed-point. Additionally, if condition (v) is also satisfied, then T admits a unique fixed-point.

Proof. By (ii), we take $\tilde{x}_0 \in \mathcal{M}$ satisfying $\tilde{x}_0\mathcal{R}(T\tilde{x}_0)$. We define the sequence $\{\tilde{x}_n\}_{n=0}^{\infty} \subseteq \mathcal{M}$ as $\tilde{x}_n = T\tilde{x}_{n-1} = T^n\tilde{x}_0$, where $n \in \mathbb{N}$.

We will show the outcome in several steps:

1) We claim that $\{\tilde{x}_n\}$ is $\mathcal{R}$-preserving.

By (ii), we have $(T^n\tilde{x}_0)\mathcal{R}(T^{n+1}\tilde{x}_0)$, hence, $\tilde{x}_n\mathcal{R}\tilde{x}_{n+1}, \ \forall n \in \mathbb{N}_0$.

2) We claim that $\lim\limits_{n\to\infty} \mathcal{D}(\tilde{x}_n, \tilde{x}_{n-1}) = 0$

Since $\phi \in \Phi$, we have $\phi(r) \leq r$, $\forall r \in \mathbb{R}^+$. Suppose that $\upsilon_n = \mathcal{D}(\tilde{x}_n, \tilde{x}_{n-1})$. By

$\mathcal{R}$-preserving and Equation (6), we have

$$\begin{aligned} v_{n+1} &= \mathcal{D}(T\tilde{x}_n, T\tilde{x}_{n-1}) \\ &\le \phi(\mathcal{D}(\tilde{x}_n, \tilde{x}_{n-1})) + \omega\left(\mathcal{D}(\tilde{x}_n, \tilde{x}_{n+1}), \mathcal{D}(\tilde{x}_{n-1}, \tilde{x}_n), 0, \mathcal{D}(\tilde{x}_{n-1}, \tilde{x}_{n+1})\right) \\ &= \phi(v_n) \\ &\le v_n \end{aligned}$$

Moreover, $\{v_n\}$ is nonnegative and decreasing. Therefore,

$$\lim_{n\to\infty} v_n = v \ge 0$$

In fact, if $v > 0$, then,

$$v = \lim_{n\to\infty} v_{n+1} \le \liminf_{n\to\infty} \phi(v_n) \le \limsup_{q\to v} \phi(q) < v$$

this is contradictory. As a result, $\lim_{n\to\infty} v_n = 0$.

3) We claim that $\{\tilde{x}_n\} \subset \mathcal{M}$ is a Cauchy sequence.

For closed interval $[\frac{1}{2b}, 1]$, Lemma 1 implies that for any $\tilde{x}, \tilde{y} \in \mathcal{M}$ with $\mathcal{D}(\tilde{x}, \tilde{y}) < 1$, there exists $k \in \mathbb{Z}^+$ such that $\mathcal{D}(T^k\tilde{x}, T^k\tilde{y}) < \frac{1}{2b}$. Notice that $\lim_{n\to\infty} \mathcal{D}(\tilde{x}_n, \tilde{x}_{n-1}) = 0$. Thus there exists $N \in \mathbb{N}$, such that

$$\mathcal{D}(\tilde{x}_N, \tilde{x}_{N+1}), \mathcal{D}(\tilde{x}_N, \tilde{x}_{N+2}), \ldots, \mathcal{D}(\tilde{x}_N, \tilde{x}_{N+k})$$

are not greater than $\frac{1}{2b}$. Hence, for any natural number $m > N$, if $m \le N + k$, we obtain $\mathcal{D}(\tilde{x}_N, \tilde{x}_m) \le \frac{1}{2b} < 1$; if $m = N + k + 1$, by $\mathcal{D}(\tilde{x}_N, \tilde{x}_{N+1}) < 1$ and Lemma 1, we have $\mathcal{D}(\tilde{x}_{N+k}, \tilde{x}_{N+k+1}) < \frac{1}{2b}$. Moreover,

$$\mathcal{D}(\tilde{x}_N, \tilde{x}_m) \le b[\mathcal{D}(\tilde{x}_N, \tilde{x}_{N+k}) + \mathcal{D}(\tilde{x}_{N+k}, \tilde{x}_{N+k+1})] < b[\frac{1}{2b} + \frac{1}{2b}] = 1$$

By induction, for any $m > N$, we have $\mathcal{D}(\tilde{x}_N, \tilde{x}_m) < 1$. Let $M := \sup_{m,n\in\mathbb{N}} \mathcal{D}(\tilde{x}_n, \tilde{x}_m)$.

Now, we show that $\{\tilde{x}_n\} \subset \mathcal{M}$ is a Cauchy sequence. For any $\epsilon > 0$, according to Lemma 1, there is $k \in \mathbb{Z}^+$ such that $\mathcal{D}(\tilde{x}, \tilde{y}) \le M$ and $\mathcal{D}(T^k\tilde{x}, T^k\tilde{y}) < \epsilon$, $\forall \tilde{x}, \tilde{y} \in \mathcal{M}$. Furthermore, for any $m, n \in \mathbb{N}$ with $m > n \ge k$, due to $\mathcal{D}(\tilde{x}_{n-k}, \tilde{x}_{m-k}) \le M$, by Lemma 1, we have that $\mathcal{D}(\tilde{x}_n, \tilde{x}_m) < \epsilon$. Thus $\{\tilde{x}_n\}$ is a Cauchy sequence.

Since $\{\tilde{x}_n\}$ is a $\mathcal{R}$-preserving Cauchy sequence, hence by (i), $\exists \tilde{x}^* \in \mathcal{M}$ verifying $\lim_{n\to\infty} \mathcal{D}(\tilde{x}_n, \tilde{x}^*) = 0$.

4) We claim that the above $\tilde{x}^*$ is the fixed-point of T

If $\mathcal{R}$ is $\mathcal{D}$-self-closed, then we take a subsequence $\{\tilde{x}_{n_k}\} \subset \{\tilde{x}_n\}$ verifying $\tilde{x}_{n_k}\hat{\mathcal{R}}\tilde{x}^*$, $\forall k \in \mathbb{N}$. From the hypotheses on ϕ Equation (6), Proposition 2 and $\tilde{x}_{n_k}\hat{\mathcal{R}}\tilde{x}^*$, we obtain

$$\begin{aligned}&\mathcal{D}(\tilde{x}_{n_{k+1}}, T\tilde{x}^*)\\&= \mathcal{D}(T\tilde{x}_{n_k}, T\tilde{x}^*)\\&\leq \phi(\mathcal{D}(\tilde{x}_{n_k}, \tilde{x}^*)) + \omega\left(\mathcal{D}(\tilde{x}_{n_k}, \tilde{x}_{n_{k+1}}), \mathcal{D}(\tilde{x}_{n_k}, T\tilde{x}^*), \mathcal{D}(\tilde{x}^*, T\tilde{x}^*), \mathcal{D}(\tilde{x}^*, \tilde{x}_{n_{k+1}})\right)\\&\leq \mathcal{D}(\tilde{x}_{n_k}, \tilde{x}^*) + \omega\left(\mathcal{D}(\tilde{x}_{n_k}, \tilde{x}_{n_{k+1}}), \mathcal{D}(\tilde{x}_{n_k}, T\tilde{x}^*), \mathcal{D}(\tilde{x}^*, T\tilde{x}^*), \mathcal{D}(\tilde{x}^*, \tilde{x}_{n_{k+1}})\right).\end{aligned}$$

Computing the limit of the above and by $\mathcal{D}(\tilde{x}_{n_k}, \tilde{x}^*) \to 0$ as $n \to \infty$, we derive $\mathcal{D}(\tilde{x}_{n_{k+1}}, T\tilde{x}^*) \to 0$ as $n \to \infty$, therefore, it can be concluded that $T\tilde{x}^* = \tilde{x}^*$.

If T is $\mathcal{R}$-continuous, since $\{\tilde{x}_n\}$ is $\mathcal{R}$-preserving verifying $\lim_{n\to\infty} \mathcal{D}(\tilde{x}_n, \tilde{x}^*) = 0$, then $\lim_{n\to\infty} \mathcal{D}(T\tilde{x}_n, T\tilde{x}^*) = 0$. This shows that $\lim_{n\to\infty} \mathcal{D}(\tilde{x}_{n+1}, T\tilde{x}^*) = 0$. Hence, we have $T\tilde{x}^* = \tilde{x}^*$.

5) We claim that T has a unique fixed point $\tilde{x}^* \in \mathcal{M}$.

If $\tilde{y}^* \in \mathcal{M}$ with $T\tilde{y}^* = \tilde{y}^*$ and $\tilde{x}^* \neq \tilde{y}^*$, then $\mathcal{D}(\tilde{x}^*, \tilde{y}^*) > 0$. As $\tilde{x}^*, \tilde{y}^* \in T(\mathcal{M})$, the following will be discussed in two cases:

Case 1: If $\mathcal{R}\big|_{T(\mathcal{M})}$ remains complete, then we get $\tilde{x}^*\hat{\mathcal{R}}\tilde{y}^*$. Using Equation (6) and Proposition 2, one obtains

$$\begin{aligned}\mathcal{D}(\tilde{x}^*, \tilde{y}^*) &= \mathcal{D}(T\tilde{x}^*, T\tilde{y}^*)\\&\leq \phi(\mathcal{D}(\tilde{x}^*, \tilde{y}^*)) + \omega\left(0, \mathcal{D}(\tilde{x}^*, T\tilde{y}^*), 0, \mathcal{D}(\tilde{y}^*, T\tilde{x}^*)\right)\\&< \mathcal{D}(\tilde{x}^*, \tilde{y}^*).\end{aligned}$$

This is contradictory. Hence, $\tilde{x}^* = \tilde{y}^*$.

Case 2: If $T(\mathcal{M})$ is $\mathcal{R}$-directed, then $\exists \tilde{z} \in \mathcal{M}$ satisfying $\tilde{x}^*\mathcal{R}\tilde{z}$ and $\tilde{y}^*\mathcal{R}\tilde{z}$.

Denote $u_n = \mathcal{D}(\tilde{x}^*, T^n\tilde{z})$, using $\mathcal{R}$ is T-closed, $\phi(r) \leq r$ and Equation (6), one obtains

$$\begin{aligned}&\mathcal{D}(\tilde{x}^*, T^n\tilde{z})\\&= \mathcal{D}(T\tilde{x}^*, T(T^{n-1}\tilde{z}))\\&\leq \phi(\mathcal{D}(\tilde{x}^*, T^{n-1}\tilde{z})) + \omega\left(0, \mathcal{D}(\tilde{x}^*, T^n\tilde{z}), \mathcal{D}(T^{n-1}\tilde{z}, T^n\tilde{z}), \mathcal{D}(T^{n-1}\tilde{z}, T\tilde{x}^*)\right)\\&\leq \mathcal{D}(\tilde{x}^*, T^{n-1}\tilde{z})\end{aligned}$$

So, $u_n \leq u_{n-1}$. Moreover, $\{u_n\}$ is a non-increasing and nonnegative sequence; and hence

$$\lim_{n\to\infty} u_n = u \geq 0$$

Indeed, if $u > 0$, then,

$$u = \lim_{n\to\infty} u_{n+1} \leq \liminf_{n\to\infty} \phi(u_n) \leq \limsup_{q\to u} \phi(q) < u$$

this is contradictory. Hence, $\lim_{n\to\infty} u_n = \mathcal{D}(\tilde{x}^*, T^n\tilde{z}) = 0$.

Similarly, one can find $\mathcal{D}(\tilde{y}^*, T^n\tilde{z}) = 0$.

$$\mathcal{D}(\tilde{x}^*, \tilde{y}^*) \leq b\mathcal{D}(\tilde{x}^*, T^n\tilde{z}) + b\mathcal{D}(\tilde{y}^*, T^n\tilde{z}) \to 0, \quad as \;\; n \to \infty$$

Hence, $\tilde{x}^* = \tilde{y}^*$. $\square$

Remark 3. *Theorem 3 is set in a b-metric space, which generalizes the concept of a metric space by allowing for a broader range of distance functions. Additionally, the conditions on the binary relation are weaker because the universal relation (which involves no restrictions) is a special case of a relation. The compression condition is also comparatively relaxed.*

Building on our findings, we will derive several well-known fixed point theorems by reviewing recent research. It is well known that b-metric space is a generalization of metric space. Using Theorem 3, we can obtain the fixed point theorem result of Alharbi and Khan ([7], Thorem 4) in metric space.

Under the restriction $\mathcal{R} = \mathcal{M} \times \mathcal{M}$, the universal relation, Theorem 3 deduces the following fixed point theorem.

Corollary 1. *Let* $(\mathcal{M}, \mathcal{D}, b)$ *be a complete* b-MS *with* $b \geq 1$. *Suppose that* $T : \mathcal{M} \to \mathcal{M}$ *is a mapping such that*

$$d(T\tilde{x}, T\tilde{y}) \leq \phi(d(\tilde{x}, \tilde{y})), \; for \; all \; \tilde{x}, \tilde{y} \in \mathcal{M} \tag{7}$$

where $\phi \in \Phi$. *Then* T *possesses a unique fixed point in* $\mathcal{M}$.

Remark 4. *Note that the conclusion is up to date in b-MS.*

Corollary 2. *Let* $(\mathcal{M}, \mathcal{D})$ *be a complete metric space* [16]. *Suppose that* $T : \mathcal{M} \to \mathcal{M}$ *is a mapping such that*

$$d(T\tilde{x}, T\tilde{y}) \leq \phi(d(\tilde{x}, \tilde{y})), \; for \; all \; \tilde{x}, \tilde{y} \in \mathcal{M} \tag{8}$$

where $\phi \in \Phi$. *Then* T *possesses a unique fixed point.*

4. Applications to fractional differential equation

Fractional differential equations play a crucial role in various fields, they provide a more comprehensive and refined framework for understanding and analyzing complex dynamic systems that cannot be adequately described by integer-order models. There are many definitions on fractional derivatives, the two-scale fractal derivative [17–19] is based on the idea of the two-scale fractal geometry. In the study of porous media or rough surfaces, this derivative can better capture the behavior where the properties change with different scales.

The fractional derivative in the Ji-Huan He sense offers an alternative approach to fractional calculus. They have unique properties and algorithms for handling fractional-order differentials and integrals. These derivatives have been applied in areas such as signal processing and control theory [20–22].

In general, fractional derivatives allow for a more accurate description of systems with memory, non-local effects, and anomalous diffusion. They can model processes where the rate of change is not simply proportional to the first-order derivative. For instance, in viscoelastic materials, the stress-strain relationship often exhibits fractional-order behavior. Different types of fractional derivatives provide various tools for scientists and engineers to analyze and understand the underlying dynamics of complex systems, enabling more accurate predictions and better designs in many disciplines including physics, biology, and engineering.

Now, we shall discuss the subsequent fractional differential equation

$$^{c}D^{\alpha}_{0+}\tilde{x}(s) = g(s, \tilde{x}(s)), s \in [0,1] \tag{9}$$

where

$$\tilde{x}(0) +^{c} D^{\beta}_{0+}\tilde{x}(0) = 0, \ \ \tilde{x}(1) +^{c} D^{\beta}_{0+}\tilde{x}(1) = 0 \tag{10}$$

$^{c}D^{\alpha}_{0+}$ is the Caputo fractional derivative (see [23]), $1 < \alpha \leq 2$ and $0 < \beta \leq 1$ are real number and $g : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function.

Let $\mathcal{M} := C[0,1]$ be the set of all real continuous functions on a closed interval I, and define $\mathcal{D} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$:

$$\mathcal{D}(\tilde{x}, \tilde{y}) = \begin{cases} \dfrac{1}{7} \max\limits_{s\in[0,1]} |\tilde{x}(s) - \tilde{y}(s)|, & \tilde{x}(s)\tilde{y}(s) \equiv 0, \forall s \in [0,1] \\ \max\limits_{s\in[0,1]} |\tilde{x}(s) - \tilde{y}(s)|, & \text{other} \end{cases} \tag{11}$$

where $\tilde{x}, \tilde{y} \in \mathcal{M}$.

Let $\mathcal{R}$ be a relation on $\mathcal{M}$ as

$$\tilde{x}\mathcal{R}\tilde{y} \iff \tilde{x}(s) \geq \tilde{y}(s), \ \ \forall \tilde{x}, \tilde{y} \in \mathcal{M}, \forall s \in [0,1]$$

It can be easily proved that $(\mathcal{M}, \mathcal{D}, b)$ is $\mathcal{R}$-complete with the coefficient $b = 7$ and $\mathcal{R}$ is $\mathcal{D}$-self-closed.

For simplicity, we consider the case where $\beta = 1$.

Notice that $\tilde{x} \in \mathcal{M}$ is the solution of Equation (9) $\iff$ $\tilde{x}$ solves the following integral equation,

$$\begin{aligned} \tilde{x}(s) =& \frac{1}{\Gamma(\alpha)} \int_0^1 (1-v)^{\alpha-1}(1-s)g(v, \tilde{x}(v))dv \\ &+ \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-v)^{\alpha-2}(1-s)g(v, \tilde{x}(v))dv \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^s (s-v)^{\alpha-1} g(v, \tilde{x}(v))dv \end{aligned}$$

Theorem 4. *Consider problem Equation* (9) *with* $g : [0,1] \times [0,\infty) \to [0,\infty)$ *continuous and assume the conditions:*

(i) *For any* $\tilde{x}, \tilde{y} \in \mathcal{M}$ *verifying* $\tilde{x}\mathcal{R}\tilde{y}$*, satisfies*

$$0 \leq g(s, \tilde{x}(s)) - g(s, \tilde{y}(s)) \leq \frac{1}{4}\phi\left(\tilde{x}(s) - \tilde{y}(s)\right) \tag{12}$$

where $\phi \in \Phi$ *is nondecreasing;*

(ii)

$$\sup_{s\in(0,1)} \frac{1}{4} \left| \frac{1-s}{\Gamma(\alpha+1)} + \frac{1-s}{\Gamma(\alpha)} + \frac{s^{\alpha}}{\Gamma(\alpha+1)} \right| := \theta < 1$$

(*iii*) $\exists\lambda \in \mathcal{M}$ *satisfying*

$$\begin{aligned}\lambda(s) \geq & \frac{1}{\Gamma(\alpha)}\int_0^1 (1-v)^{\alpha-1}(1-s)g(v,\lambda(v))dv \\ & + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-v)^{\alpha-2}(1-s)g(v,\lambda(v))dv \\ & + \frac{1}{\Gamma(\alpha)}\int_0^s (s-v)^{\alpha-1}g(v,\lambda(v))dv\end{aligned}$$

hold. Then there exists a unique solution for Equation (9)*.*

Proof. The integral operator $T:\mathcal{M}\to\mathcal{M}$ defined by

$$\begin{aligned}(T\tilde{x})(s) = & \frac{1}{\Gamma(\alpha)}\int_0^1 (1-v)^{\alpha-1}(1-s)g(v,\tilde{x}(v))dv \\ & + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-v)^{\alpha-2}(1-s)g(v,\tilde{x}(v))dv \\ & + \frac{1}{\Gamma(\alpha)}\int_0^s (s-v)^{\alpha-1}g(v,\tilde{x}(v))dv.\end{aligned}$$

With the help of Equation (12) it can be verified that $\mathcal{R}$ is T-closed.

Let $\tilde{x},\tilde{y}\in\mathcal{M}$ verifying $\tilde{x}\mathcal{R}\tilde{y}$ consider

$$\begin{aligned}|T\tilde{x}(s)-T\tilde{y}(s)| = & \frac{1}{\Gamma(\alpha)}\int_0^1 (1-v)^{\alpha-1}(1-s)\left[g(v,\tilde{x}(v))-g(v,\tilde{y}(v))\right]dv \\ & + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-v)^{\alpha-2}(1-s)\left[g(v,\tilde{x}(v))-g(v,\tilde{y}(v))\right]dv \\ & + \frac{1}{\Gamma(\alpha)}\int_0^s (s-v)^{\alpha-1}\left[g(v,\tilde{x}(v))-g(v,\tilde{y}(v))\right]dv \\ \leq & \frac{1}{4}\phi\left(\tilde{x}(s)-\tilde{y}(s)\right)\left[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-v)^{\alpha-1}(1-s)dv\right. \\ & + \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-v)^{\alpha-2}(1-s)dv \\ & \left. + \frac{1}{\Gamma(\alpha)}\int_0^s (s-v)^{\alpha-1}dv\right] \\ = & \frac{1}{4}\phi\left(\tilde{x}(s)-\tilde{y}(s)\right)\left[\frac{1-s}{\Gamma(\alpha+1)}+\frac{s^\alpha}{\Gamma(\alpha+1)}+\frac{1-s}{\Gamma(\alpha)}\right]\end{aligned}$$

Hence, we have

$$\begin{aligned}\mathcal{D}(T\tilde{x}(s),T\tilde{y}(s)) & \leq \phi(\mathcal{D}(\tilde{x}(s),\tilde{y}(s)))\sup_{s\in(0,1)}\frac{1}{4}\left|\frac{1-s}{\Gamma(\alpha+1)}+\frac{1-s}{\Gamma(\alpha)}+\frac{s^\alpha}{\Gamma(\alpha+1)}\right| \\ & \leq \phi\left(\mathcal{D}(\tilde{x}(s),\tilde{y}(s))\right)\end{aligned}$$

For any $\tilde{x}^*,\tilde{y}^*\in T(\mathcal{M})$, take $\tilde{z} := \min\{T\tilde{x}^*, T\tilde{y}^*\}$ implying thereby $T\tilde{x}^*\geq\tilde{z}$ and $T\tilde{y}^*\geq\tilde{z}$. This show that $T(\mathcal{M})$ is $\mathcal{R}$-directed.

From Theorem 3, we see that the Equation (9) possesses a unique solution. □

5. Conclusions

In this paper, we explored the strict almost ϕ-contraction fixed-point theorem in b-metric space with binary relations. The conditions (i)–(iv) of Theorem 3 ensure the existence of fixed-points, while condition (v) guarantees their uniqueness. Utilizing our results, we derive several corollaries applicable. Obviously, this study enhances our understanding of fixed points in b-metric spaces.

In many viscoelastic materials, the diffusion behavior may exhibit non-local and memory-dependent characteristics, necessitating a more nuanced approach, such as fractional differential equations. By incorporating the binary relation $\mathcal{R}$ as part of the proposed fixed-point theorem, we can model interactions and dependencies more flexibly, capturing how the historical states of the material directly influence the diffusion process in conjunction with its current state. This capability enables us to more accurately identify stable states or solutions, which is crucial for understanding how materials respond under various stress conditions. Moreover, we convincingly demonstrate EUS for fractional differential equations.

In the following, we will focus on the case where the boundary condition parameter $\beta \neq 1$ in Equation (10) of the research examples.

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