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MODERNIZING CLASSICAL OPTIMAL CONTROL: HARNESSING DIRECT AND INDIRECT OPTIMIZATION

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Abstract

An introduction to optimal control, a fundamental concept in engineering and science disciplines, is a process of finding ways of controlling dynamic systems in such a way that they achieve certain goals while being subjected to given state limitations. The conventional approaches developed in the past have been integral to solving optimal control problems, including the maximum principle belonging to Pontryagin and the dynamic programming method. While relatively straightforward, these methods are not always amenable to higher-dimensional scenarios, interacting forces, or other non-trivial constraints. This paper presents a new methodology to extend classical optimal control, considering both direct and indirect optimization techniques. The direct methods, Euler and Runge-Kutta, Trapezoidal, and Hermite-Simpson, do not require the explicit derivation of the analytical control laws to execute control trajectories. Semi-analytical techniques, such as the shooting method, are based the control laws on proposed adjoint differential equations. This paper presents the basic outline of the classical optimal control problem and discusses the direct and indirect optimization methods. It is illustrated by referencing various examples, like the fixed-rate royalty payment approach. After outlining each framework, we identify the positive and negative aspects of the approach in questions of consistency and performance. Finally, the problems on the synergy of direct and indirect methods are considered further, and areas of further development of the presented methods and their integration with the more sophisticated tools, such as machine learning are identified. It can be concluded that the approach of using both direct and indirect optimization methods presents great potential in modernizing the classical optimal control, which challenges the conventional techniques and indicates potential for further development of the control system optimization.

1. Introduction

It is also a cornerstone of disciplines like engineering, economics, and science since it provides a framework for engineering ways to manage dynamic systems and achieve goals while avoiding certain outcomes [9]. For these control problems, the necessary tools have been available for a long time within the classical approaches of optimal control specified by the Pontryagin maximum principle, dynamic programming, and the calculus of variations.

However, as technology and system developments increase, the problems that will require solutions become evident. This has resulted in the search for innovative ways of modernizing the approach to optimization problems, such as the combination of direct and indirect optimisation methods that, if advanced, could further extend the realm of control system optimization beyond more classic approaches.

To address these concerns, this paper discusses the integration of direct and indirect methods in reformulating classical optimal control. The direct optimization methods, in which no attempt is made to formulate the control laws analytically, but direct attempts are made to optimize the control trajectories, are pronounced as a shift. In the placement of predicting and calculating the value of a function, Euler, Runge-Kutta, Trapezoidal, and Hermite-Simpson methods seem to hold great promise in the precision of the control of output and the proper and efficient handling of constraints.

Conversely, indirect optimization procedures include the shooting method, and inferred control laws using related first-order adjoint differential equations. Although these methods have been integral in optimal control theory, they fail in nonlinear dynamical systems with high dimensions. Unlike the direct method, which only focuses on achieving specific objectives and has certain drawbacks that the indirect method incorporates into its system, a combination of the two methods provides a way out of the drawbacks as a new and more efficient method for optimal control.

This paper provides comprehensive details of these modernization endeavours with actual life examples illustrating how the modernisation agenda has been implemented, including fixed-rate royalty payment structures. In comparative analysis, the advantages and disadvantages of direct as well as indirect optimization methods are identified.

This research contributes to the existing body of knowledge on the best methods of controlling dynamical systems by proposing a solution to update the classical methods. Thus, the possibilities for reformulating the optimal control problems using the advantages of both direct and indirect approaches are shown.

2. Classical Optimal Control Problem

The classical optimal control theory, although very important in a theoretical setting, always encounters some difficulties regarding realistic scenarios of system behaviour, large state and control space, and tight constraints.

Direct optimization methods can be considered as an approach that is non-related to deriving analytical control laws when the primary interest is the direct optimization of control trajectories. Several approaches, such as Euler, Runge-Kutta, Trapezoidal, and Hermite-Simpson methods, help transform optimal control issues into finite-dimensional ones. This approach allows the control inputs to be chosen efficiently while considering such dynamics and limitations. Direct optimization methods can be applied to a large variety of situations since they do not require the appearance of special expressions. In this research, the program construction was carried out using the AMPL programming language [4].

On the other hand, indirect optimization methods like the shooting method find the control laws by solving the adjoint differential equations associated with the given optimal control problem. Although these methods have been useful in analyzing historical systems, they have been poorly suited to address nonlinear dynamics and complex systems with many variables. Integrating indirect methods into modern approaches seeks to unlock their potential by combining them with the efficiency and adaptability of direct optimization. The shooting method routine was implemented using C++ programming [10].

This research also touches on the fixed-rate royalty payment issue, a topic studied by many previous researchers. For instance, Cai et al. [2] examine the impact of fixed royalty payments on sustainable fashion brand franchising, using game theory to model interactions between franchisors and franchisees. Horal et al. [5] discuss defining and justifying Ukraine's distribution ratio of oil and gas royalties under decentralization. Meanwhile, Yahya and Habbal [12] propose a blockchain-based music royalty payment scheme, arguing that blockchain's decentralized nature can provide a secure and transparent platform for royalty distribution in the music industry.

3. Direct Optimization Method

Direct optimization methods handle complex dynamics, high-dimensional state and control spaces, and intricate constraints. Since direct methods give rise to systems of equations, these methods naturally incorporate various constraints such as state and control constraints, path constraints, and boundary conditions. In terms of adaptability to complex dynamics, direct methods are based on solving systems of algebraic equations; the presented direct methods can be used to effectively control understood complex system dynamics and possess the ability to work in high-dimensional states and control spaces. In the numerical precision aspect, direct optimization methods do not involve forming an indirect program and, therefore, can provide a great level of exactness and, therefore, solutions of high numeric accuracy that are useful in solving other complex optimal control problems.

When implementing methods like Euler, Runge-Kutta, Trapezoidal, or Hermite-Simpson in an optimization context using the AMPL programming language with the MINOS solver [4], the following steps are typically followed:

(1) Problem formulation: The formulation of the basic optimal control problem is presented, including the definition of the objective function as well as the dynamic model representing state variables.

(2) Discretization of time: The time domain is discretized into a finite number of time intervals or time steps, often represented by discrete time points such as the initial and final times.

(3) Control parameterization: A parameterization for the control inputs over the time intervals is selected. In the case of the Euler method, control inputs may take on a piecewise constant nature over each time step.

(4) State and control variables: State and control variables are defined as decision variables within the AMPL model.

(5) Initialization: State variables are initialized at the initial time.

(6) Objective function formulation: The objective function, representing the performance to be maximized, is composed. This frequently involves discretizing an integral or sum over the time intervals using the Euler approximation.

(7) Dynamics discretization: Continuous-time dynamics are discretized using the Euler method. Within each time interval, the state transition is approximated.

(8) Constraint formulation: If state and control constraints exist, they are formulated as constraints within the AMPL model.

(9) Optimization problem setup: The optimal control problem is formulated as a nonlinear optimization problem within AMPL. The objective function is designed for maximization, and constraints are introduced to represent dynamics and existing constraints.

(10) AMPL scripting: An AMPL script is authored to specify the optimization problem, encompassing decision variables, the objective function, constraints, and any settings specific to the solver.

(11) Solver selection: The MINOS solver is chosen to solve the optimization problem. Care is taken to ensure that the AMPL installation is configured correctly for using the MINOS solver.

(12) Optimization problem solving: The AMPL script is executed to solve the optimization problem utilizing the MINOS solver. This leads to choosing the most appropriate control analysis which results in attaining best control inputs within limitations enhancing the performance function.

(13) Results retrieval and analysis: After the optimization process is done the optimal control inputs and the state trajectories are extracted from the solution of the solver and are analyzed.

(14) Post-processing: The results are post-processed as necessary to facilitate the visualization of the optimal control strategy, the analysis of state trajectories, or the implementation of the control strategy within a real-world system.

(15) Iterative refinement: Based on the problem size and the quality of the outcome, there is always a possibility that the solution may need to be refined by altering various factors such as discretization, size of time step or any other factors.

4. Indirect Optimization Method

Either uses a concept of indirect methods involving optimizing the Hamiltonian, which comprises a system's dynamical equations and its objective function, as described by Kirk [6]. These methods work using the concept of obtaining the so-called adjoint differential equations, which help to determine how the objective function is affected by the changes of state and control variables. Indirect optimization methods offer several benefits for solving optimal control problems.

The indirect methods help in obtaining regular information about the inherent dynamics of the system and consequently develop easily understandable and effective control mechanisms. In addition, the indirect methods often lead to analytical control laws and often supply closed-form solutions which help in comprehending system behaviour. Substituting the shooting method, an indirect method of solving the optimal control problem stated above using C++ entails some steps [10]. The process involves the following steps:

(1) Problem formulation: The problem of control is formulated, including the criterion to be optimized, which is the objective function dynamic system.

(2) Initialization: State variables and control inputs are initialized at the initial time.

(3) Guess control inputs: An initial guess is made for the control inputs over the entire time horizon.

(4) Dynamics integration: Using the estimated control inputs, the dynamic system is numerically integrated forward in time. Numerical integration techniques like the Newton and Golden Section Search are used.

(5) Solve adjoint equations: The adjoint differential equations associated with the optimal control problem are formulated and solved. These equations provide sensitivity information regarding how changes in the control inputs impact the objective function.

(6) Update control inputs: The guessed control inputs are updated using the information derived from the adjoint equations. This update is aimed at aligning the control inputs with the optimality conditions.

(7) Convergence check: A check for convergence is performed by comparing the updated control inputs with the previous guess. If the convergence criteria are met, the process proceeds to the next step; otherwise, it returns to step (4), and the process is repeated.

(8) Results analysis: Upon achieving convergence, an analysis of the final control inputs, state trajectories, and the value of the objective function is conducted. These results constitute the representation of the optimal control strategy.

(9) Iterative refinement: Depending on the complexity of the problem and the quality of the results, there may be a need to iterate and enhance the solution by adjusting the initial guess, integration method, or other parameters.

(10) C++ implementation: The steps outlined above are implemented in the C++ programming language. As needed, libraries and tools are used to solve adjoint equations and perform numerical integration.

(11) Compile and run: To address the optimal control problem, the C++ code is compiled and run.

(12) Visualisation of results: To obtain insights into the best control approach, the results which include control trajectories, state trajectories, costate information, and the objective function are displayed.

5. Comparative Analysis: Direct and Indirect Method

It is noteworthy that direct methods are superior in dealing with various dynamical features and constraints as they directly solve for control inputs. Due to that, they can establish precise, detailed control conditions and approaches; hence, they are suitable for complex process circumstances. However, there is a disadvantage of this approach, which is that it becomes inevitable to use numerical approximations when discretizing, which may influence the quality of the solutions to be obtained.

However, the indirect methods offer an understanding of the system sensitivities and almost always lead to analytical control laws; the solution is precise and easily interpretable. They are especially useful where it is necessary to obtain an analytical solution for performing stability analysis and making decisions [1].

Direct methods normally put the optimal control problem in a finite-dimensional figure by using methods that have a fast convergence and can be easily computed. This is especially useful for high-state and control spaces, as is normally the case with most problems [7, 8]. On the other hand, the indirect methods require the solution of the differential equations governing state and the adjoint variables that may be computationally expensive particularly for complex systems. Some of the indirect methods are iterative in form, and this also causes slower convergence as compared to others [7, 8].

Constraints are incorporated in the optimization process by direct methods that lead efficiently to formulations of overall solutions without complications of path constraints and boundary conditions. Though indirect approaches to optimization can also manage constraints, their use can make the derivation of equations of adjoint more cumbersome and affect the optimality conditions.

A major difference between these two methods is that the direct methods are numerical and do not involve the analytical aspect that is found in indirect methods. This is because the indirect methods provide one with analytical control laws and system sensitivities and, therefore, provide a deeper insight into the behaviour of the system.

Another advantage of the direct methods is that they enable the solution of large-scale systems, thus making them useful in solving very large problems. But as we move upwards, we might come across a problem of scalability, particularly as the numbers of features increase. The indirect method may fail in a high-dimensional system by reason of the formulation of adjoint equations for each of the state variables.

6. Formulation of Classical Optimal Control Problem

The classical optimal control problem is the mathematical model applied to identify the best control policy to be employed by a dynamical system within an applied time. It is concerned with finding the control variables or inputs which yield the minimum, maximum or desired value of an objective function while satisfying the system of equations and constraints. On the other hand, a non-classical optimal control problem can also be described as a sort of optimization problem that is different from a standard or conventional optimal control problem in one or more aspects. Nonetheless, non-classical optimal control problems add difficulties or deviations from the standard structure of classic optimal control issues, including a dynamical system, control inputs, an objective function, constraints, and time horizon.

Cruz et al. [3] have introduced a new category of variational problems called non-classical (or non-standard) variational problems. As opposed to the conventional variational problems, these are personalized by the notion of the objective function, which is a measure of the disparity between two integrals. This makes way for approaching similar great optimization problems that are hard to solve with classical approaches.

Another relevant study is Zinober and Sufahani [14], in which the authors analyze a non-standard optimal control (OC) problem in an economic context. This problem concerns the royalty payment scheme represented with a two-stage piecewise function. According to the authors' findings, a solution must be made under the framework of Pontryagin's maximum principle, which is the most basic idea of optimal control.

Let us consider a dynamic system described by a set of state variables denoted as $y(t)$ and a set of control inputs denoted as $u(t)$,

$$\dot{y}(t) = u(t). \tag{1}$$

The goal is to maximize the objective function while satisfying system dynamics and constraints over a given time horizon $[t_i, t_f]$. The objective function discussed by Spence [11] is rooted in the concept of the learning curve. This concept suggests that as firms gain experience through repeated production, they become more productive and can lower their costs. Essentially, the more a company produces, the better it gets at production, leading to improved efficiency and cost savings over time:

$$J = \int_{t_i}^{t_f} (e^{0.025t} u^{0.5} - (\rho + 1 + e^{-0.12y})u) e^{-0.1t} dt \tag{2}$$

subject to

$$\rho = 1.0. \tag{3}$$

The cost function in this context involves the royalty payment function, denoted by ρ. It combines state variables, control inputs, and auxiliary variables, each weighted by specific coefficients. The goal is to optimize this cost function to meet performance criteria while adhering to constraints. Now, the cost function to be maximized can be expressed as follows:

$$\text{Maximize } J = \int_{t_i}^{t_f} (e^{0.025t}u^{0.5} - (2 + e^{-0.12y})u)e^{-0.1t}dt. \tag{4}$$

Zinober and Kaivanto [13] attempted to solve the cost function using matrix formulation, considering current demand values and discount rates for optimizing royalty payments. However, they encountered challenges, particularly when the royalty payment level changed, which complicated the differentiation process needed to compute the optimal objective function. As systems grow more complex and constraints become more detailed, the limitations of traditional methods become apparent. This has led to exploring modernization strategies incorporating direct and indirect optimization methods.

7. Experimental Results and Discussion

The objective function described by equation (4) is optimized using both direct and indirect methods. The direct approach uses software tools like AMPL, which helps formulate and solve the optimization problem efficiently. On the other hand, the indirect method involves programming in C++, where a custom solution is developed to address the optimization problem through detailed control of the computational process.

The data in Table 1 shows that the optimal objective function values are very similar across methods, aligning up to two decimal places between direct methods and up to one decimal place with the shooting method. The final state values are also closely aligned, with Euler, Runge-Kutta, and Trapezoidal discretization methods matching the shooting method upto two decimal places. However, the Hermite-Simpson approximation only matches the shooting method upto one decimal place for the final state value. All methods show a close match upto three decimal places for the costate values at the initial time. Additionally, the final costate value computed by all methods is zero.

Table 1. Optimal solution

Methods	Final state value	Initial costate value	Final costate value	Objective function
Shooting	0.369646	0.026387	0.000000	0.659773
Euler	0.367532	0.026514	-	0.662962
Runge-Kutta	0.369485	0.026515	-	0.662983
Trapezoidal	0.364828	0.026515	-	0.662983
Hermite-Simpson	0.375801	0.026518	-	0.663049

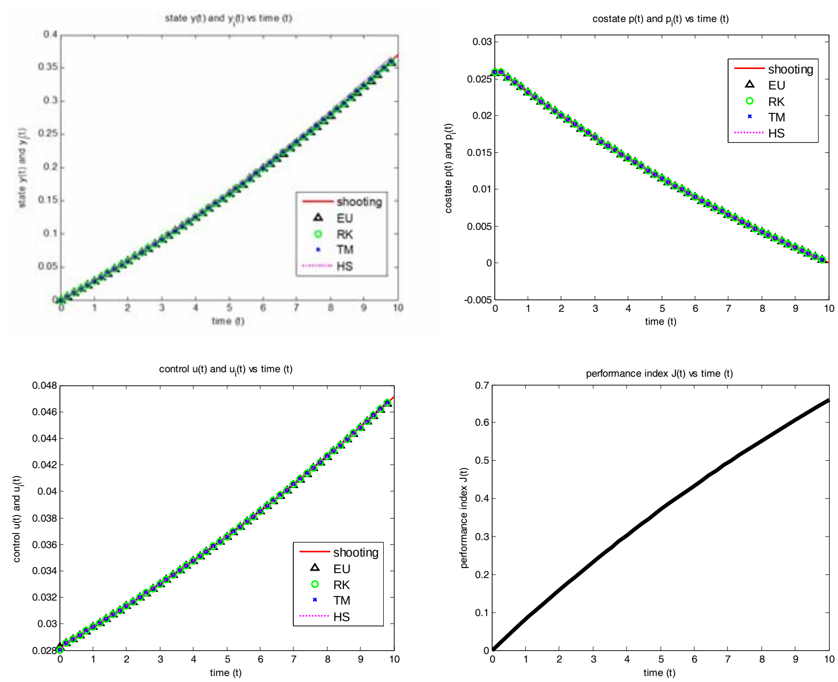

Figure 1. Optimal plot for the variables and cost function.

Figure 1 illustrates that the cost function from the shooting method shows an upward trend, which aligns with the increasing state value. Overall, the plots for state, costate, and control values reveal a strong similarity between the shooting and discretization methods.

8. Challenges and Future Research Direction

Integrating direct and indirect optimization methods to modernize classical optimal control brings new challenges and opens avenues for future research.

In the numerical stability context, comparing direct and indirect methods can introduce numerical challenges, requiring careful consideration of stability and accuracy. In the dimensionality scaling aspect, managing high dimensionality entails effective algorithms and approaches to decrease the relevance of computational costs. In terms of constraint handling, the synthesis of methods that may employ different techniques of constraint handling calls for harmonious solution approaches that uphold the constraint accord and optimality.

For future research directions, it is apparent that there is an advantage in creating new algorithms that incorporate both direct and indirect control methods and can switch from one to another depending on the current situation in the system and the environment. For data-driven optimization, investigating real-time data to improve optimization decisions yields optimum control data. Referring to earning-based optimization, researching the application of reinforcement learning and neural networks with optimization can lead to the development of control strategies that develop over time.

9. Conclusion

In this paper, the goal that has been pursued is to update classical optimal control using direct and indirect optimization approaches. This integration was planned to deal with issues like complicated structures and further dimensions, complicated constraints and new control methodologies in various areas. In this way, we were able to illustrate in detail how the fusion of these methods alters classical optimal control. The case studies here demonstrated the benefits of each method, where direct methods provided adaptability and speed in computations while indirect methods provided exactness in analysis and significant information. We have, therefore, taken the two and compared them, emphasizing how they are particularly relevant and have influenced modern controls. The given analysis has shown that the selection of direct and indirect approaches is based on the problem's peculiarities and objectives. Whereas direct methods are more computationally effective, more interpretive power is provided by indirect methods. State and applicational complexities were also discussed, along with the directions that can be taken to merge these approaches in the future, including hybrid algorithms and data-driven optimization. The application of machine learning was also suggested as an idea that can be addressed in the future. These methods can then be combined and incorporated into the existing classical optimal control, which is considered a considerable enhancement in achieving a higher level of accurate control schemes, sensitivity flexibility and understanding of the overall control processes. This work continues the development of control theory. It can act as a guide for researchers and those who are interested in implementing the current improvements in optimal control and who want to analyze the potential of the synergy of the direct and indirect optimization approaches.

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