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NECESSARY CONDITIONS FOR OPTIMALITY IN ONE NONSMOOTH OPTIMAL CONTROL PROBLEM FOR GOURSAT-DARBOUX SYSTEMS

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Abstract

We consider a nonsmooth optimal control problem described by a system of second-order hyperbolic equations with Goursat boundary conditions. A number of necessary optimality conditions are proved in terms of directional derivatives.

1. Introduction

The main result of the theory of necessary conditions for first-order optimality, Pontryagin's maximum principle, was proven for various optimal

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control problems described by ordinary differential equations, as well as partial differential equations, under the assumption of smoothness of the right-hand sides of the equations describing the controlled process and quality criteria according to state vector [1-5].

To date, based on theoretical and practical needs, much attention has been paid to the study of various nonsmooth optimal control problems. In works [2, 6-8], various non-smooth optimal control problems described by ordinary differential equations were studied. The proposed work is devoted to the study of one problem of optimal control of Goursat-Darboux systems without the assumption of smoothness in the state vector of the quality functional and the right side of the equation describing the process under study.

2. Problem Statement

Consider a controlled process described in a rectangle $D = [t_0, t_1] \times [x_0, x_1]$ by a second-order hyperbolic system

$$z_{tx} = A(t, x)z_t + B(t, x)z_x + f(t, x, z, u) \tag{2.1}$$

with Goursat boundary conditions:

$$\begin{aligned} &z(t_0, x) = a(x), \quad x \in [x_0, x_1], \\ &z(t, x_0) = b(t), \quad t \in [t_0, t_1], \\ &a(x_0) = b(t_0). \end{aligned} \tag{2.2}$$

Here $A(t, x)$ and $B(t, x)$ are given $(n \times n)$ measurable and bounded matrix functions, $a(x)$, $b(t)$ are given absolutely continuous vector functions, and $u = u(t, x)$ is measurable and bounded vector function of control actions with values from a given non-empty and bounded set $U \subset R^r$, for almost all $(t, x) \in D$, i.e.,

$$u(t, x) = U \subset R^r, \quad (t, x) \in D, \tag{2.3}$$

where $f(t, x, z, u) = (f_1(t, x, z, u), \ldots, f_n(t, x, z, u))$ is continuous in the set of variables in a dimensional vector function that satisfies the Lipschitz condition with respect to z, and each of its components is directionally differentiable with respect to z.

Each control function that satisfies the above restrictions will be called an *admissible control*.

It is assumed that the boundary value problem (2.1)-(2.2) for a given admissible control $u(t, x)$ has a unique absolutely continuous solution $z(t, x)$ defined in D.

It is required to minimize terminal function

$$S(u) = \Phi(z(t_1, x_1)), \tag{2.4}$$

defined on the solutions of the boundary value problem (2.1)-(2.2) generated by all possible admissible controls.

Here $\Phi(z)$ is given Lipschitz and directionally differentiable scalar function.

An admissible control $u(t, x)$ that delivers a minimum value to the functional (2.4) under constraints (2.1)-(2.3) will be called an *optimal control*.

3. Optimality Conditions

Let some admissible control $u(t, x)$ be suspicious for a minimum, and $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ be an arbitrary regular point (Lebesgue point, see, for example, [1, 3, 4]) of the control, $u(t, x)$, $v \in U$ be an arbitrary vector, and $\varepsilon > 0$ be an arbitrary sufficiently small number such that $\theta + \varepsilon < t_1$, $\xi + \varepsilon < x_1$.

We determine the special increment of the admissible control $u(t,\ x)$ using formula

$$\Delta u(t,\ x;\ \varepsilon)=\begin{cases} v-u(t,x), & (t,x)\in D(\varepsilon)=[\theta,\theta+\varepsilon)\times[\xi,\xi+\varepsilon), \\ 0, & (t,x)\in D\backslash D(\varepsilon). \end{cases} \quad (3.1)$$

$\Delta z(t,\ x;\ \varepsilon)$ is a special increment of state $z(t,\ x)$, corresponding to increment (3.1) of control $u(t,\ x)$ and $z(t,\ x;\ \varepsilon)=z(t,\ x)+\Delta z(t,\ x;\ \varepsilon)$.

From the introduced notation, it is clear that $z(t,\ x;\ \varepsilon)$ is a solution to the boundary value problem

$$z_{tx}(t,\ x;\ \varepsilon)=A(t,\ x)z_t(t,\ x;\ \varepsilon)+B(t,\ x)z_x(t,\ x;\ \varepsilon)$$

$$+\ f(t,\ x,\ z(t,\ x;\ \varepsilon),\ u(t,\ x)+\Delta u(t,\ x;\ \varepsilon)), \quad (3.2)$$

$$z(t_0,\ x;\ \varepsilon)=a(x),\ \ z(t,\ x_0;\ \varepsilon)=b(t). \quad (3.3)$$

From the boundary value problem (3.2)-(3.3), it follows that the increment $\Delta z(t,\ x;\ \varepsilon)$ of the state $z(t,\ x)$ is a solution of the boundary value problem:

$$\Delta z_{tx}(t,\ x;\ \varepsilon)=A(t,\ x)\Delta z_t(t,\ x;\ \varepsilon)+B(t,\ x)\Delta z_x(t,\ x;\ \varepsilon)$$

$$+\ f(t,\ x,\ z(t,\ x;\ \varepsilon),\ u(t,\ x)+\Delta u(t,\ x;\ \varepsilon))$$

$$+\ f(t,\ x,\ z(t,\ x;\ \varepsilon),\ u(t,\ x)),$$

$$\Delta z(t_0,\ x;\ \varepsilon)=0,$$

$$\Delta z(t,\ x_0;\ \varepsilon)=0.$$

From here, passing to the integral equation, we obtain

$$\Delta z(t,\ x;\ \varepsilon)-z(t,\ x)=\int_{t_0}^{t}\int_{x_0}^{x}[A(\tau,\ s)\Delta z_\tau(\tau,\ s;\ \varepsilon)+B(\tau,\ s)\Delta z_s(\tau,\ s;\ \varepsilon)$$

$$+\ f(\tau,\ s,\ z(\tau,\ s;\ \varepsilon)u(\tau,\ s)+\Delta u(\tau,\ s;\ \varepsilon))]$$

$$\times[-f(t,\ x,\ z(\tau,\ s),\ u(\tau,\ s))]dsd\tau. \quad (3.4)$$

Let us assume

$$h(t, x) = (h_1(t, x), \ldots, h_n(t, x)) = \lim_{\varepsilon \to 0} \frac{z(t, x; \varepsilon) - z(t, x)}{\varepsilon}.$$

From the estimate given, for example, in [3, 4], we obtain that

$$\| \Delta z(t, x; \varepsilon) \| = \| z(t, x; \varepsilon) - z(t, x) \| \leq L_1 \varepsilon, \quad (t, x) \in D, \tag{3.5}$$

where L_1 is a positive constant.

From (3.5), it is clear that $t \in [t_0, \theta)$, $x \in [x_0, \xi)$, $h(t, x) = 0$.

Taking into account the estimate (3.5) from (3.4) and using the fact that the vector function $f(t, x, z, u)$ has directional derivatives with respect to z, we obtain that for $t > \theta$ and $x > \xi$,

$$h(t, x)$$

$$= \int_\theta^t \int_\xi^x \left[A(\tau, s) \cdot h_\tau(\tau, s) + B(\tau, s) h_s(\tau, s) + \frac{\partial f(\tau, s, z(\tau, s), u(\tau, s))}{\partial h(\tau, s)} \right] ds d\tau$$

$$\times [f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi))]. \tag{3.6}$$

The integral equation (3.6) is equivalent to the following partial differential equation:

$$h_{tx}(t, x) = \frac{\partial f(t, x, z(t, x), u(t, x))}{\partial h(t, x)} + A(t, x) h_t(t, x)$$

$$+ B(t, x) h_x(t, x), \; t \geq \theta, \; x \geq \xi, \tag{3.7}$$

with boundary conditions

$$h(\theta, x) = f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi)), \tag{3.8}$$

$$h(t, \xi) = f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi)). \tag{3.9}$$

By assumption, the function $\Phi(z)$ is differentiable in any direction and satisfies the Lipschitz condition. Therefore, using the formula

$$z(t, x; \varepsilon) = z(t, x) + \varepsilon h(t, x) + o(\varepsilon; t, x),$$

we get

$$S(u(t,\ x;\ \varepsilon)) - S(u(t,\ x))$$
$$= \Phi(z(t_1,\ x_1) + \varepsilon h(t_1,\ x_1) + o(\varepsilon;\ t_1,\ x_1)) - \Phi(z(t_1,\ x_1))$$
$$= \Phi(z(t_1,\ x_1) + \varepsilon h(t_1,\ x_1) + o(\varepsilon;\ t_1,\ x_1)) - \Phi(z(t_1,\ x_1) + \varepsilon h(t_1,\ x_1))$$
$$+ \Phi(z(t_1,\ x_1) + \varepsilon h(t_1,\ x_1)) - \Phi(z(t_1,\ x_1)). \tag{3.10}$$

By virtue of the Lipschitz condition, we obtain that

$$|\,\Phi(z(t_1,\ x_1) + \varepsilon h(t_1,\ x_1) + o(\varepsilon;\ t_1,\ x_1)) - \Phi(z(t_1,\ x_1) + \varepsilon h(t_1,\ x_1))\,| \le L_2\varepsilon,$$

where L_2 is a positive constant.

Therefore, from (3.10), we get that

$$S(u(t,\ x;\ \varepsilon)) - S(u(t,\ x)) = \varepsilon \frac{\partial \Phi(z(t_1,\ x_1))}{\partial h(t_1,\ x_1)} + o(\varepsilon).$$

It follows from the resulting expansion.

Theorem 3.1. *For an admissible control* $u(t,\ x)$ *to be an optimal control*, *it is necessary that the inequality*

$$\frac{\partial \Phi(z(t_1,\ x_1))}{\partial h(t_1,\ x_1)} \ge 0, \tag{3.11}$$

holds for all variations $h(t_1,\ x_1)$ *of the state* $z(t_1,\ x_1)$.

From the obtained general optimality condition (3.11), we can obtain more constructively verifiable necessary optimality conditions, in particular Pontryagin's maximum principle.

Let the functions $f(t,\ x,\ z,\ u)$ and $\Phi(z)$ be continuously differentiable with respect to z. Then the boundary value problem (3.7)-(3.9) takes the form

$$h_{tx}(t,\ x) = A(t,\ x)h_t(t,\ x) + B(t,\ x)h_x(t,\ x) + \frac{\partial f(t,\ x,\ z(t,\ x),\ u(t,\ x))}{\partial h(t,\ x)}, \tag{3.12}$$

$$h(\theta, x) = f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi)), \tag{3.13}$$

$$h(t, \xi) = f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi)). \tag{3.14}$$

Based on the formula for the integral representation of solutions to the linear Goursat-Darboux boundary value problem (see, for example, [5]), the problem (3.12)-(3.14) admits the representation

$$h(t, x) = R(t, x, t_0, x_0)[f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi))]$$

$$- \int_\theta^t R(t, x, \tau, \xi) B(\tau, \xi)$$

$$\times [f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi))] d\tau$$

$$- \int_\xi^x R(t, x, \theta, s) A(\tau, \xi)$$

$$\times [f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi))] ds.$$

Note that $R(t, x, \tau, s)$ is an $(n \times n)$ matrix function, which is a solution to the Volterra integral equation

$$R(t, x, \tau, s) = \int_s^x R(t, x, \tau, \beta) A(\tau, \beta) d\beta$$

$$+ \int_\tau^t R(t, x, \alpha, s) B(\alpha, s) d\alpha$$

$$+ \int_\tau^t \int_s^x R(t, x, \alpha, \beta) f_z(\alpha, \beta, z(\alpha, \beta), u(\alpha, \beta)) d\alpha d\beta. \tag{3.15}$$

By entering the notation

$$K(t, x) = R(t, x, t_0, x_0) - \int_\theta^t R(t, x, \tau, \xi) B(\tau, \xi) d\tau$$

$$- \int_\xi^x R(t, x, \theta, s) A(\tau, \xi) ds, \tag{3.16}$$

from (3.15), we get that

$$h(t_1, x_1) = K(t_1, x_1)[f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi))]. \quad (3.17)$$

Taking into account (3.17) from the inequality (3.11), we obtain that

$$\frac{\partial \Phi'(z(t_1, x_1))}{\partial z} K(t_1, x_1)$$

$$\times [f(\theta, \xi, z(\theta, \xi), v) - f(\theta, \xi, z(\theta, \xi), u(\theta, \xi))] \geq 0. \quad (3.18)$$

Assuming that

$$\psi(\theta, \xi) = -K'(t_1, x_1) \frac{\partial \Phi(z(t_1, x_1))}{\partial z},$$

$$H(\theta, \xi, z(\theta, \xi), u(\theta, \xi), \psi(\theta, \xi)) = \psi'(\theta, \xi) f(\theta, \xi, z(\theta, \xi), u(\theta, \xi)),$$

from the inequality (3.18), due to the arbitrariness of $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$ and $v \in U$, we arrive at the following statement:

Theorem 3.2 (An analogue of Pontryagin's maximum principle)**.** *If the functions* $f(t, x, z, u)$ *and* (z) *are continuously differentiable with respect to z, then for optimality of the admissible control* $u(t, x)$, *it is necessary that the inequality*

$$H(\theta, \xi, z(\theta, \xi), v, \psi(\theta, \xi)) - H(\theta, \xi, z(\theta, \xi), u(\theta, \xi), \psi(\theta, \xi)) \leq 0,$$

holds for all $v \in U$, $(\theta, \xi) \in [t_0, t_1) \times [x_0, x_1)$.

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