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AN INNOVATIVE METHOD FOR SOLVING LINEAR AND NONLINEAR FRACTIONAL TELEGRAPH EQUATIONS

Mona Magzoub[1], Tarig M. Elzaki[1] and Mourad Chamekh[1,2]

[1]Mathematics Department
University of Jeddah
Jeddah, Saudi Arabia
e-mail: mmahmed@uj.edu.sa
tfarah@uj.edu.sa; tarig.alzaki@gmail.com

[2]University of Tunis El Manar
National Engineering School at Tunis
LAMSIN, 1002, Tunis, Tunisia
e-mail: chamekhmourad1@gmail.com

Abstract

This work investigates and solves the time-fractional telegraph equations (TFTEs) occurring in electromagnetism, which serve as mathematical models in several practically significant applied research

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domains. Elzaki transform (ET) is employed in this process. Caputo sense describes fractional derivatives. Solutions of TFTEs were found in an easy-to-understand, step-by-step way using ET. In addition, instances are given to show how the phrase can be applied and how valid it is for the problem-solving form. The exact solutions and the analytical solutions accord well for the tested problems. This work also discusses the convergence of the ET technique to the exact solution of TFTEs. Several examples of linear and nonlinear TFTEs are used to demonstrate the suggested methodology. The novel technique's results show that it is an effective way to solve TFTEs, and it makes the procedure easier.

1. Introduction

There are numerous applications for the integral transform in mathematics. Integro-differential equations, integral equations, and linear DEs can all be solved with ET. This method is not appropriate for solving nonlinear DEs due to the nonlinear variables. Nonlinear DEs can be solved using ET support for the homotopy perturbation approach, differential transform method, and any other methods. These days, nonlinear equations are very important. Applications of nonlinear phenomena are significant in engineering, physics, and applied mathematics. Finding new exact or approximate solutions to nonlinear PDEs requires creative thinking, which is challenging even in fields like applied mathematics and physics where precise solutions are crucial.

Several workers have focused on investigating the solutions of nonlinear PDEs using a variety of approaches in recent years. Numerous techniques have been attempted, such as the homotopy perturbation, differential transform, variational iteration, Laplace variational iteration, differential transform, ET, Laplace; double Laplace, and ET transforms [1-12]. Numerous analytic and numerical techniques, including the local fractional variational iteration approach, the Yang-Laplace transform, and others, have been developed to solve nonlinear PDEs with fractional derivatives.

While they have a long history in mathematics, fractional derivatives (FDs) were not used in scientific fields for a very long period. One reason why FDs are disliked could be the predominance of non-equivalent definitions. Another problem is that it is hard to interpret the geometric significance of FDs because they are nonlocal. Nonetheless, in the last 20 years, mathematicians and engineers have started to focus considerably more on fractional calculus. It was found that FDs may be used to simulate a variety of applications, especially those that are multidisciplinary. For example, FDs can be utilized to solve the fluid-dynamic traffic model's issue. Based on actual data, a number of researches suggest fractional PDEs and DEs with fractional order features for seepage flow in porous media. Over the last ten years, scientists have found that the best explanation for a variety of physical phenomena, including dumping laws and diffusion processes, comes from non-integer order derivatives. These findings stimulated interest in the study of fractional calculus in many fields, including natural philosophy, technology, and alchemy.

Heaviside developed telegraph equations in 1880, have been applied to many problems in a range of scientific fields. The telegraph equation [13] describes the difference and time in electric communications with current and voltage. Telegraph equations of fractional orders have been solved using a variety of numerical and analytical methods, such as the reduced differential transform technique [18], the Adomian technique [14], the homotopy perturbation technique [15], Laplace decomposition in conjunction with HPM [16], and the modified Adomian decomposition method (MADM) [17]. With less computation, the VIM used to examine the suggested model's solution produced the same outcome as the ADM [19]. Furthermore, the hyperbolic telegraph equation is studied using the Chebyshev tau technique [20]. An attempt has been made to gain a better understanding of the anomalous diffusion processes seen in blood flow investigations by taking up the fractional telegraph equations. Hyperbolic equations with analytic or analogous asymptotes, like the telegraph equation, can sometimes provide a better model of random motion when it comes to fitting the data obtained from certain blood flow tests. Specifically, the telegraph equation more accurately represents numerous experimental data than the heat equation, as reported in certain publications.

There are numerous uses for the telegraph partial differential equations in diverse domains. Finding the approximate solution of hyperbolic PDEs is one use for it in the mathematical modeling of transmission lines. Reaction-diffusion process representation in biological and technical fields is another use. In addition, the telegraph equation is applied to random walk, wave propagation, and signal analysis problems. It is also used to investigate how microwaves affect signal transmission in telecommunications water. In addition, the telegraph equation finds application in finance for non-linear conversions of traditional telegraph procedures, including option pricing.

This study will clearly provide and illustrate the new approach, which is predicated on the new integral transform (ET). In this work, we also explore the possible applications of this new transform with the recently developed approach to solving TFTEs. This method works well for impulse functions as well as functions with discontinuities.

Definition 1. The *R-L operator*, of the order $\alpha > 0$, of a function $W \in C_{\mu}, \mu \geq -1$, is

$$J^{\alpha}W(\kappa) = \frac{1}{\Gamma(\alpha)}\int_{0}^{\kappa}(\kappa - v)^{\alpha-1}W(v)\,dv, \quad \alpha > 0,$$

$$J^{0}W(\kappa) = W(\kappa). \tag{1}$$

Following are some necessary properties of J^{α}:

For $W^{n} \in C_{\mu}$, $n \in N$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$:

(1) $J^{\alpha}J^{\beta}W(\kappa) = J^{\alpha+\beta}W(\kappa)$,

(2) $J^{\alpha}W^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}\kappa^{\alpha+\gamma}$.

Definition 2. According to Caputo, the *FDs* of $W(\kappa)$, are as follows:

$$D^{\alpha}W(\kappa) = J^{l-\alpha}D^{l}W(\kappa)$$

or

$$ {}_{a}^{C}D_{\kappa}^{\alpha}W(\kappa) = \frac{1}{\Gamma(l-\alpha)}\int_{a}^{\kappa}(\kappa-\tau)^{l-\alpha-1}W^{(l)}(\tau)\,d\tau. \tag{2}$$

For $l-1 < \alpha \leq l,\ l \in N,\ \kappa > 0,$ and $W \in C_{-1}^{l},$ the following are the basic properties of the operator ${}_{a}^{C}D_{\kappa}^{\alpha}$:

(1) ${}_{a}^{C}D_{\kappa}^{\alpha}[c] = 0,$

(2) ${}_{a}^{C}D_{\kappa}^{\alpha}I_{\kappa}^{\alpha}[W(\kappa)] = W(\kappa),$

(3) ${}_{a}^{C}D_{\kappa}^{\alpha}[\kappa^{\beta}] = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\kappa^{\beta-\alpha},$

(4) $I^{\alpha}\,{}_{a}^{C}D_{\kappa}^{\alpha}[W(\kappa)] = W(\kappa) - \sum_{k=0}^{m-1}W^{(k)}(0)\dfrac{\kappa^{k}}{k!},$

where c, α and β are constants.

2. Elzaki Transform

Elzaki has demonstrated how to use the modified Sumudu transform, or ET, to solve PDEs, ODEs, and IEs. ET is a useful tool when Sumudu and Laplace transformations are not able to solve DEs with variable coefficients [21]. ET is a powerful tool in applied mathematics and engineering, see [22-26]. The essential ideas of this change are presented as follows:

The ET of $W(\kappa)$ is

$$E[W(\kappa)] = v\int_{0}^{\infty}W(\kappa)e^{-\frac{\kappa}{v}}d\kappa = T(v), \quad \kappa > 0. \tag{3}$$

If $T'(v)$ is the ET of the derivative of $W(\kappa)$, then

(a) $T'(v) = \dfrac{T(v)}{v} - vW(0),$

(b) $T^{(n)}(v) = \dfrac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} W^{(k)}(0),\ n \geq 1,$

where $T^n(v)$ is ET of the nth derivative of the function $W(\kappa)$, see [9].

A useful ET that is utilized in this paper is stated below.

Let $E[W(\kappa)] = T(v)$. Then

(1) $E[\kappa] = v^{\alpha+2}\Gamma(\alpha+1),\ \alpha > -1,$

(2) $E[W^{(n)}(\kappa)] = \dfrac{T(v)}{v^n} - \dfrac{W(0)}{v^{n-2}} - \dfrac{W'(0)}{v^{n-3}} - \cdots - vW^{(n-1)}(0).$

Let $E[W(r, \kappa)] = T(r, v)$. Then the ETs of partial derivatives of $W(r, \kappa)$ are

$$E\left[\frac{\partial W(r, \kappa)}{\partial \kappa}\right] = \frac{1}{v}T(r, v) - vW(r, 0),$$

$$E\left[\frac{\partial^2 W(r, \kappa)}{\partial \kappa^2}\right] = \frac{1}{v^2}T(r, v) - W(r, 0) - v\frac{\partial W(r, 0)}{\partial \kappa},$$

$$E\left[\frac{\partial W(r, \kappa)}{\partial r}\right] = \frac{d}{dr}[T(r, v)],$$

$$E\left[\frac{\partial^2 W(r, \kappa)}{\partial r^2}\right] = \frac{d^2}{dr^2}[T(r, v)],$$

$$E\left[\frac{\partial^n W(r, \kappa)}{\partial \kappa^n}\right] = \frac{T(r, v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} W^{(k)}(r, 0), \quad n \geq 1.$$

Lemma 1. *ET of R-L operator of order* $\alpha > 0$ *can be expressed as follows*:

$$E[J^{\alpha}W(\kappa)] = v^{\alpha}T(v). \tag{4}$$

Proof. We have, for $\alpha > 0$,

$$E[J^{\alpha}W(\kappa)] = E\left[\frac{1}{\Gamma(\alpha)}\int_0^{\kappa} (\kappa - \alpha)^{\alpha-1} W(\kappa)d\kappa\right]$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{v}T(v)G(v) = v^{\alpha}T(v),$$

where

$$G(v) = E[\kappa^{\alpha-1}] = v^{\alpha+1}\Gamma(\alpha).$$

Lemma 2. *ET of Caputo FDs for* $\alpha > 0,\ m - 1 < \alpha \le m,\ m \in N$ *is*

$$E[{}^{c}D_{\kappa}^{\alpha}W(\kappa)] = v^{m-\alpha}\left[\frac{T(v)}{v^{m}} - \frac{W(0)}{v^{m-2}} - \frac{W'(0)}{v^{m-3}} - \cdots - vW^{(m-1)}(0)\right]. \tag{5}$$

Proof. Since

$$E[{}^{c}D_{\kappa}^{\alpha}W(\kappa)] = E[J^{m-\alpha}W^{(m)}(\kappa)] = v^{m-\alpha}E[W^{(m)}(\kappa)],$$

we find the result by using equation (4).

Equation (5) can be written as

$$E[{}^{c}D_{\kappa}^{\alpha}W(\kappa)] = \frac{1}{v^{\alpha}}E[W(\kappa)] - \sum_{k=0}^{m-1} W^{(k)}(0)v^{2-\alpha+k}. \tag{6}$$

2.1. Mittag-Leffler functions (M-LFs)

The M-LFs are regular and significant in FDE solutions. Because of the growing interest in non-traditional models and pure and practical mathematics among researchers and academics, the scientific community is becoming more interested in M-LFs. By concentrating on the idea of M-LFs, we may explain a wide range of events in a range of processes that develop or decay too slowly to be adequately captured by conventional functions like the exponential function and its surroundings.

We now define the M-LFs and the generalized M-LFs, respectively,

$$\varepsilon_{\theta}(\kappa) = \sum_{n=0}^{\infty} \frac{\kappa^n}{\Gamma(n\theta + 1)}, \qquad \varepsilon_{\theta, \vartheta}(\kappa) = \sum_{n=0}^{\infty} \frac{\kappa^n}{\Gamma(n\theta + \vartheta)}.$$

For special values of $\theta, \vartheta,$ the M-LFs are given by the following:

(1) $\varepsilon_{\theta,1}(\kappa) = \varepsilon_{\theta}(\kappa),$

(2) $\varepsilon_{0,1}(\kappa) = \dfrac{1}{1-\kappa},$

(3) $\varepsilon_{1,1}(\kappa) = e^{\kappa},$

(4) $\varepsilon_{2,2}(\kappa^2) = \dfrac{\sinh \kappa}{\kappa},$

(5) $\varepsilon_{2,1}(-\kappa^2) = \cos \kappa,$

(6) $\varepsilon_{2,2}(-\kappa^2) = \dfrac{\sin \kappa}{\kappa}.$

We now establish the subsequent lemma, which is helpful in determining the function $W(\kappa)$ from its ET.

Lemma 3. *If* $\theta, \vartheta > 0,$ $a \in C$ *and* $\frac{1}{v^{\theta}} > |a|,$ *then the inverse ET is*

$$E^{-1}\left[\frac{v^{\vartheta+1}}{1 + av^{\theta}}\right] = \kappa^{\vartheta-1}\varepsilon_{\theta, \vartheta}(-a\kappa^{\theta}). \tag{7}$$

Proof. Noting that

$$\frac{v^{\vartheta+1}}{1+av^{\theta}} = v^{\vartheta+1}\frac{1}{1+av^{\theta}} = v^{\vartheta+1}\sum_{n=0}^{\infty}(-a)^n (v^{\theta})^n = \sum_{n=0}^{\infty}(-a)^n v^{n\theta+\vartheta+1},$$

we have

$$E^{-1}\left[\frac{v^{\vartheta+1}}{1+av^{\theta}}\right] = E^{-1}\left[\sum_{n=0}^{\infty}(-a)^{n}\, v^{n\theta+\vartheta+1}\right] = \sum_{n=0}^{\infty}\frac{(-a)^{n}\,\kappa^{n\theta+\vartheta-1}}{\Gamma(n\theta+\vartheta)}$$

$$= \kappa^{\vartheta-1}\sum_{n=0}^{\infty}\frac{(-a\kappa^{\theta})^{n}}{\Gamma(n\theta+\vartheta)} = \kappa^{\vartheta-1}\varepsilon_{\theta,\vartheta}(-a\kappa^{\theta}).$$

3. Analysis of Proposed Scheme

Using the subsequent nonlinear TFTEs and initial conditions, we demonstrate the fundamental concept of this approach:

$${}^{c}D_{\kappa}^{\theta}W(r,\ \kappa) = \frac{\partial^{2}W}{\partial r^{2}} - \frac{\partial W}{\partial \kappa} + NW(r,\ \kappa) + g(r,\ \kappa),\ r,\ \kappa \geq 0,\ 0 < \theta \leq 2,$$

$$W(r,\ 0) = \Upsilon_{1}(r),\quad W_{\kappa}(r,\ 0) = \Upsilon_{2}(r). \tag{8}$$

Using ET in equation (8), we obtain

$$E[{}^{c}D_{\kappa}^{\theta}W] = E\left[\frac{\partial^{2}W}{\partial r^{2}} - \frac{\partial W}{\partial \kappa} + NW(r,\ \kappa) + g(r,\ \kappa)\right].$$

Applying ET's property, we obtain

$$\frac{1}{v^{\theta}}E[W(r,\ \kappa)] - v^{2-\theta}W(r,\ 0) - v^{3-\theta}W_{\kappa}(r,\ 0)$$

$$= E\left[\frac{\partial^{2}W}{\partial r^{2}} - \frac{\partial W}{\partial \kappa} + NW(r,\ \kappa) + g(r,\ \kappa)\right]. \tag{9}$$

Equation (9), after two sides are each treated as the Elzaki inverse, yields the following result:

$$W(r,\ \kappa) = G(r,\ \kappa) + E^{-1}\left\{v^{\theta}E\left[\frac{\partial^{2}W}{\partial r^{2}} - \frac{\partial W}{\partial \kappa} + NW(r,\ \kappa)\right]\right\},$$

where $G(r,\ \kappa)$ represents the term that comes from all or some of the function $g(r,\ \kappa)$ and the initial conditions that are prescribed.

To solve the problem iteratively, make use of the following relations:

$$W_{n+1}(r,\ \kappa) = E^{-1}\left\{v^{\theta}E\left[\frac{\partial^2 W_n}{\partial x^2} - \frac{\partial W_n}{\partial t} + NW_n(r,\ \kappa)\right]\right\},$$

$$W_0(r,\ \kappa) = G(r,\ \kappa). \tag{10}$$

The following is thought to be the series form solution to equation (8):

$$W(r,\ \kappa) = \sum_{n=0}^{\infty} W_n(r,\ \kappa). \tag{11}$$

From equation (10), the following $W_0(r,\ \kappa),\ W_1(r,\ \kappa),\ W_2(r,\ \kappa),\ \ldots,$ can be inferred. The solution can then be found using equation (11).

3.1. Convergence analysis

The convergence of the ET method to the exact solution of TFTEs is discussed in this subsection.

Theorem 1. *In a Banach space B*, $\sum_{n=0}^{\infty} W_n$, *in equation* (11) *converges to* $a \in B$ *if* $\exists (0 \le \xi < 1)$, *s.t.* $\forall \tau \in \mathbb{N} \Rightarrow \| W_\tau \| \le \xi \| W_{\tau-1} \|$.

Proof. The partial sum sequence is described as $\{a_\tau\}_{\tau=0}^{\infty}$,

$$\begin{aligned} a_0 &= W_0 \\ a_1 &= W_0 + W_1 \\ a_2 &= W_0 + W_1 + W_2 \\ &\vdots \\ a_\tau &= W_0 + W_1 + \ldots + W_\tau. \end{aligned}$$

It is now required to demonstrate that $\{a_\tau\}_{\tau=0}^{\infty}$ is a Cauchy sequence in B. Note that

$$\| a_{\tau+1} - a_\tau \| = \left\| \sum_{n=0}^{\tau+1} W_n - \sum_{n=0}^{\tau} W_n \right\| = \| W_{\tau+1} \| \le \xi \| W_\tau \| \le \cdots \le \xi^{\tau+1} \| W_n \|,$$

for all $\tau, \lambda \in \mathbb{N}$. As $\tau \geq \lambda$,

$$\| a_\tau - a_\lambda \| = \| (a_\tau - a_{\tau-1}) + (a_{\tau-1} - a_{\tau-2}) + \cdots + (a_{\lambda+1} - a_\lambda) \|$$

$$\leq \| a_\tau - a_{\tau-1} \| + \| a_{\tau-1} - a_{\tau-2} \| + \cdots + \| (a_{\lambda+1} - a_\lambda) \|$$

$$\leq \xi^\tau \| W_0 \| + \xi^{\tau-1} \| W_0 \| + \cdots + \xi^{\lambda+1} \| W_0 \|$$

$$\leq \xi^{\lambda+1} \| W_0 \| (\xi^{\tau-\lambda-1} + \xi^{\tau-\lambda-2} + ... + \xi) = \frac{1 - \xi^{\tau-\lambda}}{1 - \xi} \xi^{\lambda+1} \| W_0 \|.$$

Then, $(\xi^{\tau-\lambda-1} + \xi^{\tau-\lambda-2} + ... + \xi)$ is a geometric series and $0 \leq \xi < 1$. Thus, $\lim_{\tau, \lambda \to \infty} a_\tau - a_\lambda = 0.$

This means that the series solution $W = \sum_{n=0}^{\infty} W_n$, is as given in equation (12), converges, and we have the required.

3.2. Numerical applications

We now consider the recommended course of action to maximize the impact of the TFTEs. Better numerical results are obtained with the recommended system. The following examples show the efficacy.

Example 1. Consider the following linear TFTE:

$$^c D_\kappa^\theta W(r, \kappa) = \frac{\partial^2 W}{\partial r^2} - 2 \frac{\partial W}{\partial \kappa} - W, \quad \kappa \geq 0, \quad 0 < \theta \leq 2,$$

$$W(r, 0) = e^r, \quad W_\kappa(r, 0) = -2e^r. \tag{12}$$

Using ET from equation (12), we obtain

$$\frac{1}{v^\theta} E[W(r, \kappa)] - v^{2-\theta} W(r, 0) - v^{3-\theta} W_t(r, 0) = E\left[\frac{\partial^2 W}{\partial r^2} - 2 \frac{\partial W}{\partial \kappa} - W\right],$$

$$E[W(r, \kappa)] = v^2 e^r - 2v^3 e^r + v^\theta E\left[\frac{\partial^2 W}{\partial r^2} - 2 \frac{\partial W}{\partial \kappa} - W\right].$$

By inverse ET,

$$E^{-1}[E[W(r, \kappa)]] = E^{-1}\left[v^2 e^r - 2v^3 e^r\right] + E^{-1}\left[v^\theta E\left[\frac{\partial^2 W}{\partial r^2} - 2\frac{\partial W}{\partial \kappa} - W\right]\right].$$

The iteration formula that makes use of a first approximation is shown in the below diagram:

$$W_{n+1}(r, \kappa) = E^{-1}\left[v^\theta E\left[\frac{\partial^2 W_n}{\partial r^2} - 2\frac{\partial W_n}{\partial \kappa} - W_n\right]\right],$$

$$W_0(r, \kappa) = e^r - 2\kappa e^r = e^r(1 - 2\kappa). \tag{13}$$

Equation (13) gives

$$W_1(r, \kappa) = E^{-1}[v^\theta E[4e^r]] = e^r E^{-1}[4v^{\theta+2}] = \frac{2^2 e^r \kappa^\theta}{\Gamma(\theta + 1)},$$

$$W_2(r, \kappa) = -\frac{2^3 e^r \kappa^{2\theta-1}}{\Gamma(2\theta)},$$

$$W_3(r, \kappa) = \frac{2^4 e^r \kappa^{3\theta-2}}{\Gamma(3\theta - 1)},$$

$$W_4(r, \kappa) = -\frac{2^5 e^r \kappa^{4\theta-3}}{\Gamma(4\theta - 2)}, \dots .$$

Then,

$$W(r, \kappa) = e^r - 2\kappa e^r + \frac{2^2 e^r \kappa^\theta}{\Gamma(\theta+1)} - \frac{2^3 e^r \kappa^{2\theta-1}}{\Gamma(2\theta)} + \frac{2^4 e^r \kappa^{3\theta-2}}{\Gamma(3\theta-1)} - \frac{2^5 e^r \kappa^{4\theta-3}}{\Gamma(4\theta-2)}, \dots .$$

If $\theta = 2$, then $W(r, \kappa) = e^{r-2\kappa}$.

We have presented in Figure 1, the exact solution with the numerical solution obtained after 4 iterations of the proposed method.

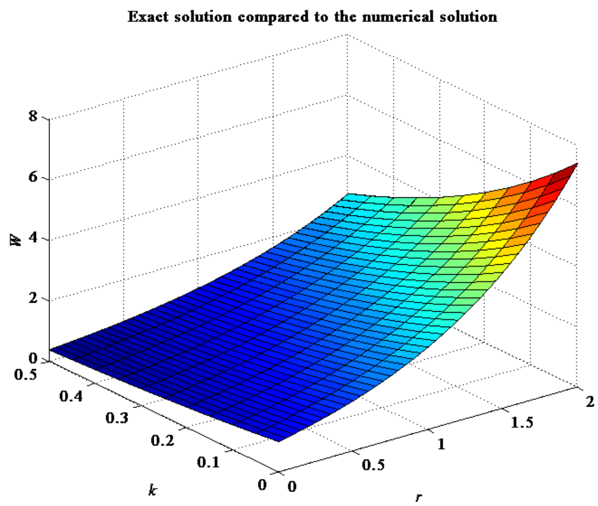

Figure 1. The exact and numerical solutions of Example 1.

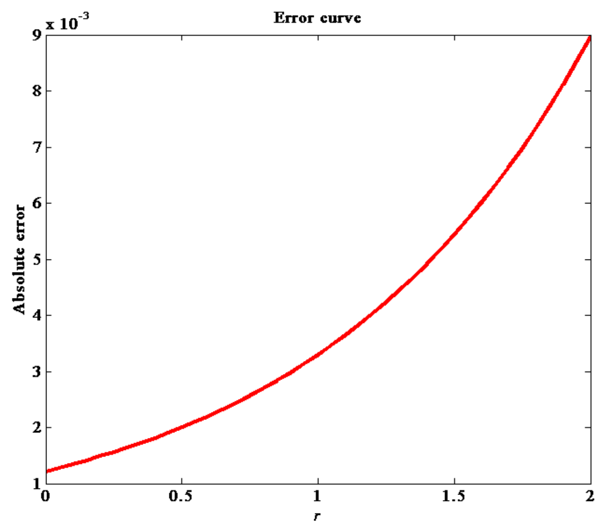

Figure 2. The absolute error curve.

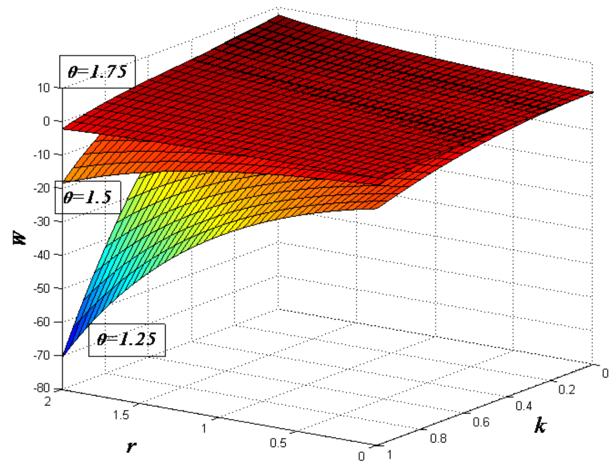

Figure 3. Different solutions of Example 1 depend on the values of θ.

Example 2. Consider the following linear TFTE:

$$ {}^{c}D_{\kappa}^{\theta}W(r,\, s,\, \kappa) = \frac{\partial^2 W}{\partial r^2} + \frac{\partial^2 W}{\partial s^2} - 3\frac{\partial W}{\partial \kappa} - 2W, \quad \kappa \geq 0,\ \ 0 < \theta \leq 2, $$

$$ W(r,\, s,\, 0) = e^{r+s}, \quad W_{\kappa}(r,\, s,\, 0) = -3e^{r+s}. \tag{14} $$

Using ET, we get

$$ \frac{1}{v^{\theta}}E[W(r,\, s,\, \kappa)] - v^{2-\theta}W(r,\, s,\, 0) - v^{3-\theta}W_{\kappa}(r,\, s,\, 0) $$

$$ = E\left[\frac{\partial^2 W}{\partial r^2} + \frac{\partial^2 W}{\partial s^2} - 3\frac{\partial W}{\partial \kappa} - 2W\right] $$

$$ \Rightarrow E[W(r,\, s,\, \kappa)] = v^2 e^{r+s} - 3v^3 e^{r+s} + v^{\theta}E\left[\frac{\partial^2 W}{\partial r^2} + \frac{\partial^2 W}{\partial s^2} - 3\frac{\partial W}{\partial \kappa} - 2W\right]. $$

By the technique used in Example 1, we acquire the recurrent link in the subsequent manner:

$$ W_{n+1}(r,\, s,\, \kappa) = E^{-1}\left[v^{\theta}E\left[\frac{\partial^2 W_n}{\partial r^2} + \frac{\partial^2 W_n}{\partial s^2} - 3\frac{\partial W_n}{\partial \kappa} - 2W_n\right]\right], $$

$$ W_0(r,\, s,\, \kappa) = e^{r+s}(1 - 3\kappa). \tag{15} $$

Then

$$W_1(r, s, \kappa) = E^{-1}\left[v^{\theta} E\left[\frac{\partial^2 W_0}{\partial r^2} + \frac{\partial^2 W_0}{\partial s^2} - 3\frac{\partial W_0}{\partial \kappa} - 2W_0\right]\right]$$

$$= E^{-1}[v^{\theta} E[9e^{r+s}]] = \frac{3^2 \kappa^{\theta} e^{r+s}}{\Gamma(\theta+1)},$$

$$W_2(r, s, \kappa) = -\frac{3^3 \kappa^{2\theta-1} e^{r+s}}{\Gamma(2\theta)},$$

$$W_3(r, s, \kappa) = \frac{3^4 \kappa^{3\theta-2} e^{r+s}}{\Gamma(3\theta-1)},$$

$$W_4(r, s, \kappa) = -\frac{3^5 \kappa^{4\theta-3} e^{r+s}}{\Gamma(4\theta-2)}, \dots .$$

Therefore,

$$w(r, s, \kappa) = e^{r+s}\left(1 - 3\kappa + \frac{3^2 \kappa^{\theta}}{\Gamma(\theta+1)} - \frac{3^3 \kappa^{2\theta-1}}{\Gamma(2\theta)} + \frac{3^4 \kappa^{3\theta-2}}{\Gamma(3\theta-1)} - \frac{3^5 \kappa^{4\theta-3}}{\Gamma(4\theta-2)}, \dots\right).$$

If $\theta = 2$, then $W(r, s, \kappa) = e^{r+s-3\kappa}$.

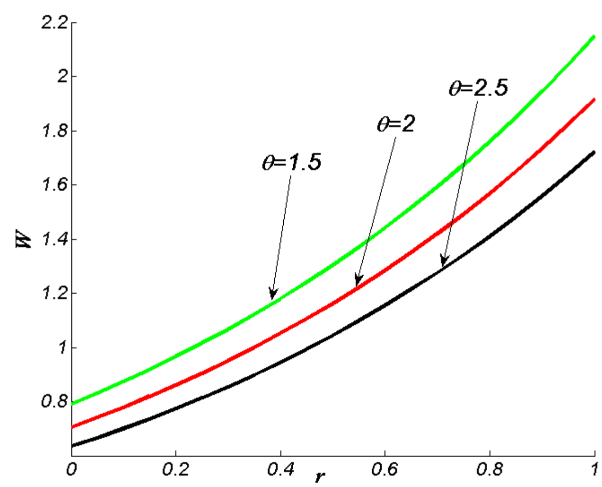

Figure 4. Plots of solutions to equation (14) obtained for $s = 0.1$ and $\kappa = 0.15$.

Table 1. Solution for the first five approximations with exact solution of equation (14)

r	$\theta = 1.75$	$\theta = 1.85$	$\theta = 1.95$	$\theta = 2$	Exact
0.	1.2832034304119277	1.255282057546795	1.2315710995754274	1.2214011577765744	1.2214027581601699
0.25	1.6476658194298148	1.6118140670021688	1.581368594312918	1.568310130556956	1.5683121854901687
0.5	2.1156447903555167	2.069610229005623	2.0305174682495526	2.0137500688840015	2.0137527074704766
0.75	2.716541683499492	2.657432136680157	2.6072360382608695	2.5857062713037484	2.585709659315846
1.	3.488108567105054	3.4122104066201335	3.347757340431209	3.3201125724429015	3.3201169227365472

Example 3. Consider the nonlinear TFTE:

$$
{}^{c}D_{\kappa}^{\theta}W(r,\ \kappa) = \frac{\partial^2 W}{\partial r^2} + \frac{\partial W}{\partial \kappa} - W^2 + rW\frac{\partial W}{\partial r},\ \ \kappa,\ r \geq 0,\ \ 1 < \theta \leq 2,
$$

$$
W(r,\ 0) = W_{\kappa}(r,\ 0) = r. \tag{16}
$$

We obtain the recurrence relationship in the following using the same technique as in Example 1:

$$
W_{n+1}(r,\ \kappa) = E^{-1}\left[v^{\theta}E\left[\frac{\partial^2 W_n}{\partial r^2} + \frac{\partial W_n}{\partial \kappa} - W_n{}^2 + rW_n\frac{\partial W_n}{\partial r}\right]\right],
$$

$$
W_0(r,\ \kappa) = r + r\kappa. \tag{17}
$$

Then

$$
W_1(r,\ \kappa) = E^{-1}[v^{\theta}E[r]] = \frac{r\kappa^{\theta}}{\Gamma(\alpha+1)},
$$

$$
W_2(r,\ \kappa) = E^{-1}\left[v^{\theta}E\left[\frac{\theta r\kappa^{\theta-1}}{\Gamma(\theta+1)}\right]\right] = \frac{r\kappa^{2\theta-1}}{\Gamma(2\theta)},
$$

$$
W_3(r,\ \kappa) = \frac{r\kappa^{3\theta-2}}{\Gamma(3\theta-1)},
$$

$$
W_4(r,\ \kappa) = \frac{r\kappa^{4\theta-3}}{\Gamma(4\theta-2)}, \ldots .
$$

The solution is thus

$$W(r,\ \kappa) = r\left[1 + \kappa + \frac{\kappa^{\theta}}{\Gamma(\theta+1)} + \frac{\kappa^{2\theta-1}}{\Gamma(2\theta)} + \frac{r\kappa^{3\theta-2}}{\Gamma(3\theta-1)} + \frac{r\kappa^{4\theta-3}}{\Gamma(4\theta-2)} \dots\right].$$

If $\theta = 2$, then $W(r,\ \kappa) = re^{\kappa}$.

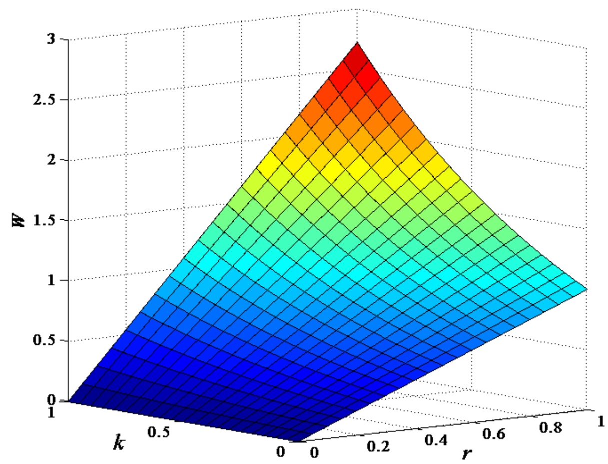

Figure 5. The exact and numerical solution of Example 3 for $\theta = 2$.

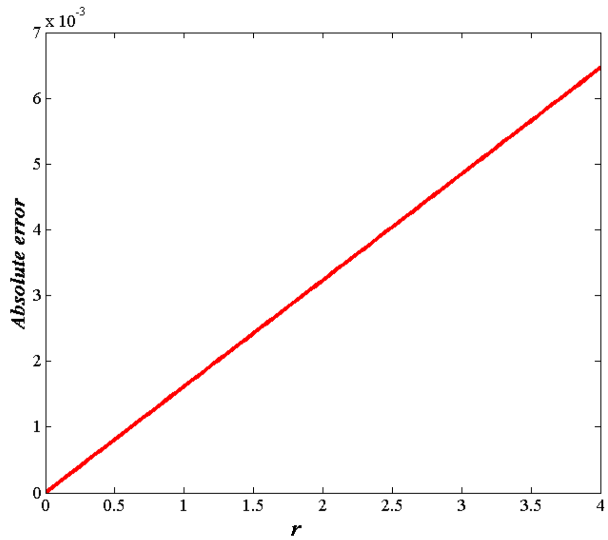

Figure 6. The absolute error for the nonlinear solution of Example 3.

4. Results Numerical Analysis

For $\theta = 2,$ we notice from the calculated values of the solution with variable $(\kappa, r) \in \left[0, \frac{1}{2}\right] \times [0, 2]$ in Figure 1 and Figure 6 that the numerical solution curve obtained after 4 iterations is confused with the exact one of the linear case (problem of Example 1) and the nonlinear problem of Example 3. Precisely, we calculated the absolute error committed by the method for a value of $\kappa = 0.5.$ Note that 9×10^{-3} is the maximum error, as shown in the plot for Figure 2.

Figure 3 illustrates the variations of the solutions obtained for the fractional values of $\theta = 1.25, 1.5$ and $1.75.$

For the nonlinear case of Example 3, we noticed that the absolute error takes an almost linear curve.

We have shown in this calculation that this method can be useful due to its simplicity in obtaining a practical and reasonable solution in terms of accuracy. It only takes a few iterations to obtain a digital solution without advanced programming.

5. Discussion and Conclusion

This article has discussed the derivation, convergence, and application of the ET method to both linear and nonlinear TFTEs. We used the first approximation in order to solve the problem exactly. The exact solutions of TFTEs converged when the ET was applied. Three instances have been successfully resolved using this method. We think that our method for solving TFTEs will be useful in solving other nonlinear equations. The investigation also demonstrated that the figures and table in this paper attest to the method’s efficacy in solving TFTEs. Because of how useful and simple it is, we also plan to apply it to other fractional PDEs in the future.

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